CP-violating Reflection of High Energy Fermions during a first order Phase Transition

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Abstract

We study the high energy behaviour of fermions hitting a general wall caused by a first-order phase transition. The wall profile is introduced through general analytic and non-analytic functions. The reflection coefficient is computed in the high energy limit and its connection with the analytic properties of the wall profile function is shown. The high energy behaviour of the fermions hitting the wall is determined either by the leading singularity, i.e. the closest pole to the real axis, when the profile function is analytic, or by the first non-continuous derivative on the real axis, in the non-analytic case. CP-violating wall profiles are studied and it is shown that the respective symmetry properties of the CP-conserving and CP-violating profile functions plays an important role on the size of the CP asymmetry.

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1 Introduction

In ref. [1] we studied the problem of transmission and reflection of fermions through a wall and established the relationship which connects the complex plane poles of the wall profile function with the high energy reflection coefficient. The aim of the present paper is to study in more details this previous result and to go further into its consequences, mainly CP violation. The interest in the above-mentioned old problem has been renewed in the last years by the proposal that the baryon asymmetry of the universe might have been produced if a first order $SU(2) \times U(1)$ phase transition took place during the cosmological evolution. This electroweak phase transition can be described in terms of bubbles of “true” vacuum with an inner expectation value of the Higgs field $v \neq 0$, i.e. a spontaneously non-symmetric phase, appearing and expanding in the preexisting “false” vacuum with $v = 0$. If the weak phase transition is first order the standard model of weak interactions has qualitatively all the ingredients to produce the cosmological baryon asymmetry [2], i.e. baryon number violation, C and CP-violation and out-of-equilibrium processes [3]. The question is whether it can produce a large enough asymmetry to account for the observed baryon number of the universe: $n_B/s \sim (4 - 6)10^{-11}$. CP-violation in the Kobayashi-Maskawa scheme [4] has been claimed to be large enough to generate alone the observed baryon asymmetry [5]. However a zero temperature estimate [6] of the GIM suppression of electroweak C and CP effects, as well as a finite temperature study stressing the effect of gluonic thermal fluctuation [7], allows to argue that the SM scenario [5] produces a CP asymmetry more than ten order of magnitudes too small, $n_B/s < 10^{-20}$.

Simple extensions of the SM may produce much larger CP asymmetries, such as the two-Higgs-doublet [8], [9], [10] and [11], left-right symmetric models [12], SUSY [13] or models with heavy leptons [14], [15]. These models contain several Higgs fields and a relative phase between the Higgs vacuum expectation values may generate a new source of CP violation. The latter CP violation may be much larger than in the standard model for two reasons: i) CP violation often appears at the tree level while in SM it is a loop effect; ii) there is more freedom in the parameters than in the SM. Even so, the parameters in the model of ref. [15] are too constrained to allow for enough CP violation, but the discrepancy is only two orders of magnitudes as compared to the ten orders of magnitude of the SM.

We will therefore concentrate on tree level CP violation due to vacuum expectation values of Higgs fields beyond the standard model. These CP-violating processes do not directly violate baryon number nor fermion number. But they can create local inhomogeneities in the baryon (lepton) number, for example when the fermions in the plasma hit a bubble wall: an excess of antibaryons (baryons) may be created outside (inside) the bubble, which may subsequently be converted [16], via anomalous weak interaction[17], into a baryon asymmetry in the universe.

We thus need to study the problem of particles, propagating in the plasma,
say from outside the bubble, penetrating the wall and being eventually reflected outside or transmitted inside the bubble. This is a difficult problem since the fermions interact not only with the wall (i.e. the Higgs vacuum expectation value) but also with the particles in the surrounding plasma. It involves solving a quantum Fokker-Planck equation taking into account CP-violation and baryon anomaly. A useful simplifying assumption has been proposed, the so-called “charge transport” mechanism [8], in which the anomalous weak interaction happens at a distance from the bubble wall. This assumption consists in decomposing the process into two steps, one describing the production of the CP asymmetry when the quarks/antiquarks are reflected/transmitted on the wall, the second describing the transport and eventual transformation of this CP asymmetry via the baryon number anomaly. We will concentrate on the first step, the interaction of the incoming fermion with the wall. It may be assumed that during this first step the diffusion corrections are relatively minor ones to the scattering from the bubble wall, which is driven by the Higgs expectation value (during the second step they are, of course, crucial). The thermal effects are only considered via the introduction of the Higgs field effective potential, which takes into account the thermal bath temperature. For simplicity we will thus describe the reflection/transmission mechanism of fermions at zero temperature on a wall generated by a Higgs expectation value which depends on $\vec{x}$. The thermal effects could also be present via the thermal modification of the fermion spectrum (thermal masses). We will not consider them, although we believe that our conclusions in the present paper can easily be extended to incorporate them. Furthermore, the information given for the solution of the academic $T=0$ case can also be of use in other problems of physics, in solid state physics, in astrophysics, etc.

The study of the reflection and transmission of fermions on a bubble surface is thus crucial to elucidate whether it may produce a large enough CP-asymmetry to generate the correct baryon asymmetry of the universe. The size of the bubble is in general very large compared to the typical size of the CP violating process under study. Therefore we will approximate the bubble surface by a flat wall perpendicular to the $z$ direction. The Higgs expectation values depend on $z$, and the resulting effective mass of the fermion is a function of $z$. This function depends on the above mentioned Higgs field effective potential. The profile obtained by solving the equation of motion is therefore rather complex and depends on many coupling constants [8].

A frequently used simplifying assumption is to represent the wall profile by a step function considering an extremely sharp phase transition which allows to compute, in an exact way, the two-point Green function for a free fermion in the presence of this thin wall [6]. However, it is known and will clearly appear later in this paper that in such a thin wall approximation all CP asymmetries due to Higgs expectation value at the tree level do vanish. The CP violating phase can just be rotated away by a chiral rotation. Therefore we will now concentrate on thicker walls, although the wall thickness must be smaller enough than the mean free path of the fermions in the plasma to neglect the diffusion corrections.
In general, for more complex wall profiles the task of obtaining the two-point Green function is very difficult and analytic results for this Green function may not exist in most cases or be in practice unreachable. One may try numerical calculations, but, since Green functions are not really necessary at tree level we will not insist in this direction. Still, the first step is to find an orthonormal and complete set of eigenfunctions. The orthogonality of these eigenfunctions is a crucial question which can be considered in a general way. In appendix B, we show the orthogonality of the two different types of eigenstates built by requiring appropriate asymptotic behaviours far from the wall as in refs. [18], [19]. One describes the wall by an ansatz that simulates the dynamics of the phase transition [8] [18] [19]. There exists one such ansatz for which, in refs. [18] and [19], an analytical solution is given and a numerical calculation has been performed in [8]. In both cases a new net effect of CP-violation is obtained at tree level, that cannot be obtained for the thin wall.

It is known that observable CP asymmetries can only be obtained if, besides a CP violating (CP-odd) phase, a CP-even phase (a phase which is equal for particles and anti-particles) exists and interferes with the CP-odd one. In the thin wall approximation a CP-even phase only appears in the total reflection domain. This is no longer true in the thick wall case, and CP-violating effects may exist outside the total reflection region, up to infinite energy of the incoming fermion.

A completely general solution of the thick wall problem is difficult. However, there is a domain in which a few exact results can be derived: when the energy of the incoming fermion is large. In the present paper we develop a general method to calculate in the high energy limit the reflection coefficient of fermions hitting a wall, establishing a simple relationship between the analytic properties of the profile function and the high energy behaviour. The quantum correction to the expected classical behaviour is obtained and its importance depending on the above mentioned analytic properties is showed. As we will point out, the range of energy for the quarks in the electroweak phase transition is appropriate to consider this limit, except for the top. The CP-violating effects are incorporated by means of an appropriate imaginary mass term and the basic quantity $|R(E)|^2 - |\bar{R}(E)|^2$ is obtained. Interesting general theoretical consequences for the fermions quantum scattering and for the CP-violating effects will be derived from this connection of the analytic properties of the profile function and the high energy behaviour. Apart of the clear interest of this problem in cosmology, it is possible to apply the formalism to systems of relativistic fermions that can suffer a phase transition. Two examples can be condensed matter under extreme external conditions or certain stages of the quark-gluon plasma formation process.

In section 2 we formulate the problem and obtain an integral expansion in the parameter $m_0/E$, where $m_0$ is the height of the wall. We consider separately analytic and non-analytic functions as wall profiles. Section 3 is devoted to the study of functions defined as zero outside a certain region, in which the wall is located, where the potential varies. These functions must be non-analytic on the real axis as will be
seen. In section 4 general analytic functions are treated. An imaginary, chirally odd
and CP violating mass term is incorporated into the formalism in section 5. Finally,
in section 6 we summarize and conclude.

2 The integral expansion for the Dirac equation

solutions

As usual in this kind of problems, we work in the rest frame of the wall, which is
taken to be planar and parallel to the \(x-y\) plane. Thus, the mass will be character-
ized by a \(z\)-dependent function, the wall profile function. The approximation for a
planar interface for the bubble wall should be valid for large bubbles compared to
the microscopic size scale. In the electroweak phase transition this is valid for most
of the evolution of bubbles [20]. For simplicity we study the Dirac equation for one
flavour. In order to calculate the reflection coefficient we need only the plane wave
solution for particles moving along the \(z\)-axis. In any case, general solutions for other
incoming directions can be obtained by performing the appropriate Lorentz boost in
the \(x-y\) plane.

Following Nelson et al. [8] [14], we work in the chiral basis and factor the Dirac
operator into \(2 \times 2\) blocks. Thus the Dirac equation can be expressed as

\[
\begin{pmatrix}
    i\partial_z + i\partial_t & -m^*(z) \\
    m(z) & i\partial_z - i\partial_t \\
    0 & 0 \\
    0 & 0 \\
    \end{pmatrix}
\begin{pmatrix}
    \psi_I \\
    \psi_{II} \\
    \end{pmatrix}
= 0
\]

and using the following ansatz for solutions with positive energy \(E\)

\[
\Psi = \begin{pmatrix}
    \psi_I \\
    \psi_{II} \\
\end{pmatrix}
= e^{-iEt} \begin{pmatrix}
    \psi_1 \\
    \psi_2 \\
\end{pmatrix}
\]

we obtain

\[
(i\partial_z + Q(z)) \psi_I = 0 \\
(i\partial_z + \overline{Q}(z)) \psi_{II} = 0
\]

where

\[
Q(z) = \begin{pmatrix}
    E & -m^*(z) \\
    m(z) & -E \\
\end{pmatrix}, \quad \overline{Q}(z) = \begin{pmatrix}
    E & -m(z) \\
    m^*(z) & -E \\
\end{pmatrix}.
\]

Taking into account that the eigenvectors of the chiral basis have been reordered to
obtain \(\gamma^5 = \begin{pmatrix}
    \sigma_3 & 0 \\
    0 & -\sigma_3 \\
\end{pmatrix}\), the eigenvalues of this matrix, the chirality, will be + 1 for
\(\psi_1\) and \(\psi_4\), and - 1 for \(\psi_2\) and \(\psi_3\).
The solutions of the equation (3) can be written as follows

\[ \psi_I(z) = P e^{i \int_{z_0}^{z} d \tau Q(\tau)} \psi_I(z_0) , \]
\[ \psi_{II}(z) = P e^{i \int_{z_0}^{z} d \tau \overline{Q}(\tau)} \psi_{II}(z_0) . \]

Where \( P \) indicates a path ordered product and \( \tau \) is the position variable along the \( z \)-axis. The Dirac equation solutions are determined by

\[ \Omega(z, z_0) = P e^{i \int_{z_0}^{z} d \tau Q(\tau)} \]
\[ \overline{\Omega}(z, z_0) = P e^{i \int_{z_0}^{z} d \tau \overline{Q}(\tau)} , \]

which, in a matrix way, can be written as

\[ \Omega(z, z_0) = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} , \quad \overline{\Omega}(z, z_0) = \begin{pmatrix} \overline{\omega}_{11} & \overline{\omega}_{12} \\ \overline{\omega}_{21} & \overline{\omega}_{22} \end{pmatrix} . \]

It is very easy to show that \( \omega_{21} = \omega_{12}^* \) and \( \omega_{22} = \omega_{11}^* \) and analogously for \( \overline{\Omega}(z, z_0) \).

If we consider \( m(\tau) = 0 \) for \( \tau < z_0 \), then \( \psi_1 \) and \( \psi_2 \) correspond to right-moving right-handed particles and left-moving left-handed particles respectively, as well as \( \psi_3 \) and \( \psi_4 \) to right-moving left-handed and left-moving right-handed particles, in this region. If \( m(\tau) \) is considered to be a certain constant, \( m \), for \( \tau > z \), \( \Omega(\tau, z) \) and \( \overline{\Omega}(\tau, z) \) can be immediately diagonalized in order to identify the right-moving and the left-moving flux of particles. Thus, by requiring that the left-moving flux is zero for \( \tau > z \), i.e. allowing only a transmitted flux beyond the wall, a left-handed reflected flux is obtained from the right-handed incident one and vice versa. Therefore

\[ R_{R \rightarrow L} = |R(E)|^2 \quad \text{and} \quad R_{L \rightarrow R} = |\overline{R}(E)|^2 \]

where \( R_{R \rightarrow L} \) represents the probability for a reflected left-handed flux from an incident right-handed one and \( R_{L \rightarrow R} \) that for a reflected right-handed flux from an incident left-handed one. From the CPT theorem we have that \( R_{L \rightarrow R} = R_{R \rightarrow L} \) and \( R_{R \rightarrow L} = R_{L \rightarrow R} \), where \( R \) and \( \overline{R} \) label left and right antifermions, respectively[8], [6].

We obtain for the reflection coefficients

\[ R(E) = -e^{2iEz_0} \frac{\omega_{12} - \frac{E-P}{m_0} \omega_{11}}{\omega_{11} - \frac{E-P}{m_0} \omega_{12}} , \]
\[ \overline{R}(E) = -e^{2iEz_0} \frac{\overline{\omega}_{12} - \frac{E-P}{m_0} \overline{\omega}_{11}}{\overline{\omega}_{11} - \frac{E-P}{m_0} \overline{\omega}_{12}} ; \]

with \( p = +\sqrt{E^2 - m_0^2} \). A more detailed derivation of (9) may be found in refs. [8] and [14]. Now the task is to evaluate \( \Omega(z, z_0) \) and \( \overline{\Omega}(z, z_0) \). We consider at first the
real mass case whence from (4) that \( Q(\tau) = Q(\tau) \). Thus, \( \Omega(z, z_0) = \Omega(z, z_0) \) and consequently \( \mathcal{R}_{R \rightarrow L} = \mathcal{R}_{L \rightarrow R} \). Therefore, it is obvious that if the mass is real in (4) no CP-violating effect exists.

In this case \( Q(\tau) \) can be expressed as: \( Q(\tau) = \sigma_3 E - i \sigma_2 m_0 f(\tau) \), with \( m(\tau) = m_0 f(\tau) \) and \( \sigma \) are the Pauli matrices. We require \( f(\infty) = 1 \) and \( f(\tau) = O[1] \). In what follows the characteristic energy and length, \( m_0 \) and \( \frac{1}{m_0} \), are used to get dimensionless quantities, so that eq. (10) must be written as:

\[
Q^*(\tau) = \sigma_3 E^* - i \sigma_2 f(\tau) ,
\]

where \( Q^*(\tau) = \frac{Q(\tau)}{m_0} \) and \( E^* = \frac{E}{m_0} \). We do not mark the dimensionless quantities with an asterisk in what follows.

As shown in appendix A, taking into account the anti-commutation properties of the Pauli matrices and the definition of \( \Omega(z, z_0) \) as a path ordered product, the latter can be expanded as follows

\[
\Omega(z, z_0) = \left( \sum_{n=0}^{\infty} \Gamma_n(E) \left( \frac{1}{E} \right) \right)^n e^{i \sigma_3 \int_{z_0}^{z} d\tau \ p(\tau)}
\]

where

\[
p(z) = + (E^2 - [f(\tau)]^2)^{1/2}
\]

and the first terms of the expansion can be written as

\[
\begin{align*}
\Gamma_0 &= 1 , \\
\Gamma_1(E) &= \sigma_2 \int_{z_0}^{z} d\tau \ p(\tau) \ f(\tau) \ e^{-2i \sigma_3 \int_{z_0}^{z} d\xi p(\xi)} , \\
\Gamma_2(E) &= \int_{z_0}^{z} d\tau \int_{x_0}^{x} d\xi \ p(\tau) \ p(\xi) \ f(\tau) \ f(\xi) \ e^{-2i \sigma_3 \int_{\xi}^{x} d\chi p(\chi)} + i \sigma_3 \int_{z_0}^{z} d\tau \ p(\tau) (f(\tau))^2 , \\
\Gamma_3(E) &= \\
&= \sigma_2 \left\{ \int_{z_0}^{z} d\tau \int_{x_0}^{x} d\xi \ p(\tau) \ p(\xi) \ f(\tau) \ f(\xi) \ f(\chi) e^{-2i \sigma_3 \int_{\xi}^{x} d\rho p(\rho)} e^{-2i \sigma_3 \int_{\xi}^{x} d\rho p(\rho)} \right. \\
&+ i \sigma_3 \int_{z_0}^{z} d\tau \int_{x_0}^{x} d\xi \left[ f(\tau)(f(\xi))^2 e^{-2i \sigma_3 \int_{\xi}^{x} d\chi p(\chi)} - f(\xi)(f(\tau))^2 e^{-2i \sigma_3 \int_{\xi}^{x} d\chi p(\chi)} \right] p(\tau) p(\xi) + \\
&\left. + \frac{1}{2} \int_{z_0}^{z} d\tau \ p(\tau) \ (f(\tau))^3 e^{-2i \sigma_3 \int_{\xi}^{x} d\chi p(\chi)} \right\} .
\end{align*}
\]

In principle, the expansion (11) must be convergent as a consequence of the definition of \( \Omega(z, z_0) \). Nevertheless, an explicit proof is not easy. In any case, it can be directly shown [21], although tediously, that (11) is a good asymptotic expansion by using a diagrammatic notation and certain prescriptions to obtain the mentioned expansion in a general way. This asymptotic character is enough for our high energy considerations.
In next sections we will study separately non-analytical profile functions for a wall with finite real thickness and analytical profile functions for a wall extending formally to infinity, although with a finite effective thickness because the mass goes exponentially to zero when $\tau \to -\infty$ and to $m_0$ when $\tau \to +\infty$.

3 The non-analytical case

We consider that the profile functions describes a finite domain wall which extends from $\tau = -\delta_w$ to $\tau = \delta_w$ (obviously $\delta_w$ is the half of the total wall thickness). In general, a certain function in the region $(-\delta_w, \delta_w)$ is matched onto $f(\tau) = 0,1$ for $\tau < -\delta_w$ and $\tau > \delta_w$, respectively. There is no analytical function able to describe the wall structure from $z_0 < -\delta_w$ to $z > \delta_w$ because at least a derivative of a certain order does not exist at the points $\tau = -\delta_w, \delta_w$. All the derivatives defined to the left of $\tau = -\delta_w$ or to the right of $\tau = \delta_w$ are zero,

$$\lim_{\tau \to -\delta_w} f^{(n)}(\tau) = \lim_{\tau \to \delta_w^+} f^{(n)}(\tau) = 0 \quad \forall \ n > 0 ;$$

nevertheless, there must be certain $k_0, k_1$ that verify

$$\lim_{\tau \to -\delta_w} f^{(k_0)}(\tau) \neq 0 \quad \lim_{\tau \to \delta_w^+} f^{(k_1)}(\tau) \neq 0 .$$

The $k_0$-derivative and the $k_1$-derivative are not continuous functions on $\tau$, they are not well-defined for $\tau = -\delta_w, \delta_w$, respectively and $f(\tau)$ is not analytical on the real axis.

In order to study this case it is convenient to introduce the new integration variable $x = \frac{\tau}{\delta_w}$ and the parameter $a = 2E\delta_w$. We consider the evolution inside the domain wall (i.e. from $z_0 = -\delta_w$ to $z = \delta_w$). In principle we take $a$ and $E$ as two different parameters in the expansion. Following these prescriptions and expanding $p(\tau)$ in powers of $(1/E)$ we can derive from (13)

$$\Gamma_1(E) = \sigma_2 e^{-i\sigma_3 a} \left\{ \frac{a}{2} \int_{-1}^{1} dx \ F(x) \ e^{i\sigma_3 ax} - \left( \frac{1}{E} \right)^2 \frac{a}{4} \int_{-1}^{1} dx \ (F(x))^2 \ e^{i\sigma_3 ax} \right. $$

$$+ i\sigma_3 \left( \frac{1}{E} \right)^2 \frac{a^2}{4} \int_{-1}^{1} dx \ \int_{x}^{1} dy (F(y))^2 F(x) \ e^{i\sigma_3 ax} \right\}$$

$$\Gamma_2(E) = \frac{a^2}{4} \int_{-1}^{1} dx \ \int_{-1}^{x} dy \ F(x) \ F(y) \ e^{-i\sigma_3 ax} \ e^{i\sigma_3 ay} + \frac{i\sigma_3}{4} a \int_{-1}^{1} dx (F(x))^2$$

$$\Gamma_3(E) = \sigma_2 e^{-i\sigma_3 a} \left\{ \frac{a^3}{8} \int_{-1}^{1} dx \ \int_{-1}^{x} dy \ \int_{y}^{1} dt \ F(x) \ F(y) \ F(t) \ e^{i\sigma_3 ax(y-t)} \right.$$ 

$$+ i\sigma_3 \frac{a^2}{8} \int_{-1}^{1} dx \ \int_{-1}^{x} dy \ \left[ F(x)(F(y))^2 e^{i\sigma_3 ax} - F(y)(F(x))^2 e^{i\sigma_3 ay} \right] $$

$$+ \frac{a}{4} \int_{-1}^{1} dx (F(x))^3 \ e^{i\sigma_3 ax} \right\} . \quad (16)$$

7
For convenience we define $F(x) = f(\delta w x)$. By assuming that $E^2 \gg a^2$, the exponentials containing integrals of $F(x)$ have been expanded in $(\frac{a}{E^2})$. In (16) the expansion is considered up to terms $O\left(\frac{1}{E^3}\right)$. In general, the comparison between (7) and (11) gives expressions for $\omega_{11}$ and $\omega_{12}$ depending on the functions $\Gamma_n(E)$, the former is written as an expansion for $n$ even and the latter for $n$ odd. Follows from (16) and after some calculations up to order $O\left[\frac{1}{E^3}\right]$.

$$\omega_{11} = \left[1 + \frac{1}{4} \left(\frac{1}{E}\right)^2 (R_{2,2}(a) + iR_{2,1}(a)) + o\left[\left(\frac{1}{E}\right)^3\right]\right] e^{i\phi}$$

$$\omega_{12} = -ie^{ia}\left[\frac{1}{2} \left(\frac{1}{E}\right) R_{1,1}^*(a) + \frac{1}{8} \left(\frac{1}{E}\right)^3 \left(R_{3,3}(a) - iR_{1,1}^*(a)R_{2,1}(a)\right) + o\left[\left(\frac{1}{E}\right)^3\right]\right] e^{-i\phi},$$

where

\[
R_{1,1}(a) = a \int_{-1}^{1} dx \ F(x) \ e^{iax} \\
R_{2,1}(a) = a \int_{-1}^{1} dx (F(x))^2 \\
R_{2,2}(a) = a^2 \int_{-1}^{1} dx \int_{-1}^{x} dy \ F(x) \ F(y) \ e^{ia(y-x)} \\
R_{3,3}(a) = a^3 \int_{-1}^{1} dx \int_{-1}^{x} dy \int_{-1}^{y} dt \ F(x) \ F(y) \ F(t) \ e^{ia(t-y+x)}.
\]

$\phi$ is defined as the classical path, $\phi = \int_{z_0}^{z} d\tau \ p(\tau)$, and for convenience the expansion of the exponentials of $\phi$ will be considered only at the end.

From $\omega_{11}$ and $\omega_{12}$ we can obtain, eq. (9), a general expansion for the reflection coefficient $R(E)$. It can be seen that the numerator of (9) only contains odd powers of $\frac{1}{E}$, while the even ones only are in the denominator. As a consequence of this fact, by expanding the denominator, $R(E)$ can be written as a series of only odd powers in $\frac{1}{E}$. It is convenient to write

$$R(E) = e^{2i(\phi - a)} R_0(E),$$

where

$$R_0(E) = \sum_{n=1}^{\infty} C_n(a) \left(\frac{1}{E}\right)^n.$$  

It is interesting to stress that the phase factor $e^{2i(\phi - a)}$ which multiplies $R_0(E)$ depends on the difference between the classical path and the classical free path neglecting the mass function. This phase will be of the order $O\left[\frac{1}{E^2}\right]$.

From eq.(18), the first coefficients for (20) can be identified. We have

$$C_1(a) = \frac{1}{2} \left(e^{ia} - iR_{1,1}(a)\right)$$
\[ C_3(a) = \frac{1}{8} \left[ e^{ia} + iR'_{3,3}(a) + \left( e^{ia} - iR_{1,1}(a) \right) \left( 2R_{2,1}(a) - e^{ia}R'_{1,1}(a) + 2iR'_{2,2}(a) - i|R_{1,1}(a)|^2 \right) \right], \]

(21)

where the functions \( R_{1,1}(a), R_{2,2}(a) \) and \( R_{2,1}(a) \) are defined in (18). In order to express the final result in a more compact way, \( R'_{33} \) can be written in terms of \( R'_{33} \) plus another term which will be canceled during the calculation. \( R'_{33} \) defined as follows

\[ R'_{3,3}(a) = a^3 \int_{-1}^{1} dx \int_{-1}^{x} dy \int_{-1}^{y} dt \ F(x) \ F(y) \ F(t) \ e^{ia(t-y+x)}. \]

(22)

The validity of (21) is general provided that the profile function is strictly equal to 0 for \( \tau < -\delta_w \) and 1 for \( \tau > \delta_w \). We will now check this result by considering a simple example as the step profile function, \( f(\tau) = \theta(\tau) \). Then \( F(x) = \theta(x) \) from \( x = -1 \) to \( x = 1 \). Obviously the parameter \( \delta_w \) lacks any physical meaning and we hope that the final result is independent on \( a \). In fact, the exact reflection coefficient has been directly calculated for this case (see for example [6]) and we have

\[ R(E) = \frac{1}{p + E}, \]

(23)

where \( p \) has been defined in (12). By expanding (23) in powers of \( \frac{1}{E} \), we obtain

\[ R(E) = \frac{1}{2} \left( \frac{1}{E} \right) + \frac{1}{8} \left( \frac{1}{E} \right)^3 + o \left[ \left( \frac{1}{E} \right)^3 \right]. \]

(24)

On the other hand, from (18), (21) and (22) applied with the step function \( F(x) = \theta(x) \) we obtain

\[ C_1(a) = 1/2, \]
\[ C_3(a) = \frac{1}{8} (1 + 2ia). \]

(25)

Therefore, taking into account that \( \phi = a - a^2 \left( \frac{1}{E} \right)^2 + o \left[ \left( \frac{1}{E} \right)^3 \right] \) in this case, it follows from (19) and (20)

\[ R(E) = e^{-ia} \left( \frac{1}{E} \right)^{2+o} \left[ \left( \frac{1}{E} \right)^3 \right] \left( \frac{1}{2} \left( \frac{1}{E} \right) + \frac{1}{8} \left( \frac{1}{E} \right)^3 + o \left[ \left( \frac{1}{E} \right)^3 \right] \right) = \]
\[ = \frac{1}{2} \left( \frac{1}{E} \right) + \frac{1}{8} \left( \frac{1}{E} \right)^3 + o \left[ \left( \frac{1}{E} \right)^3 \right]. \]

(26)

Thus, we have a positive check for (21).

Now, we study the problem in a general way. By considering the arguments exposed in the beginning of this section, we introduce a certain function \( F(x) \) between
\(x = -1\) and \(x = 1\), in such a way that the function and all its \(k\)-derivatives up to \(k = h - 1\) are matched onto \(F(x) = 0, 1\) for \(x < -1\) and \(x > 1\), respectively. Therefore, \(F(x)\) must verify

\[
\begin{align*}
F^{(k)}(1) &= F^{(k)}(-1) = 0 \quad , \quad 0 < k < h \quad ; \\
F(1) &= 1 \\
F(-1) &= 0 \
\end{align*}
\]  
(27)

With these requirements it follows from (18) and (21) that

\[
C_1(a) = \sum_{k=h}^{\infty} \frac{i^k}{a^k} A_k e^{i \varphi_k}
\]  
(28)

where

\[
A_k = \frac{1}{2} \left[ \left( F^{(k)}(1) \right)^2 + \left( F^{(k)}(-1) \right)^2 - 2 \cos 2a F^{(k)}(1) F^{(k)}(-1) \right]^{1/2} ,
\]

\[
\varphi_k = \pi + \arctan \left[ \frac{F^{(k)}(1) + F^{(k)}(-1)}{F^{(k)}(1) - F^{(k)}(-1) \tan a} \right] .
\]  
(29)

To obtain (28), the following result has been used

\[
\int_0^x d\tau \ F(\tau)e^{ia\tau} = \frac{e^{iax}}{ia} \sum_{k=0}^{\infty} \left( \frac{i}{a} \right)^k F^{(k)}(x) .
\]  
(30)

Last eq. (30) can be usually found for any function \(F(x)\) which can be written as a finite power series\([22]\). Nevertheless, it is possible to prove it to be valid more generally for any power series provided that the r.h.s. of eq. (30) is absolutely convergent\([21]\). Moreover, by successively integrating by parts, each one of the different orders in \(1/a\) appears and the asymptotic character of such series, which is the only requirement we need, can be proven provided that \(F(x)\) is not a quasi-periodic function (i.e. it cannot be written as the product of a positive function times a periodic one). We will therefore concentrate on non quasi-periodic profiles, although an interesting physical consequence of the quasi-periodic ones, which can be well understood in our formalism, must be shown before. The decreasing with the energy of the reflection coefficient and the resulting CP asymmetry (as will be seen in section 5) is due to the position dependent optical phase, \(e^{iax}\), in our formulas. A larger energy produces oscillations of a minor period for the integrand in the left hand-side of eq. (30), in such a way that the integral goes to zero, decreasing as energy increases. Nevertheless, if the profile function is quasi-periodic positive interferences can take place. In fact, when the energy is taken so that both periods, the one for the oscillations due to the optical phase and the one of the quasi-periodic profile, are the same, the integrand is defined positive and a peak is expected around this value. Moreover, by taking
into account that \( a = 2E\delta_w \), this peak shifts towards larger energies when either a thinner wall is considered or the period of the profile decreases. For the latter case, a minor period will also generate a concentration of the asymmetry around the peak, since the effect of the oscillations due to the profile outside the energy range of the peak will be the same as those of the high energy optical phase. These effects were pointed out in ref. [23] for a particular quasi-periodic profile wall.

Until now, \( E \) and \( a \) have been considered as different parameters in the expansion. In fact, we expand in \( \frac{1}{E} \) for large \( E \) and consider \( a \) fixed. In this case (28) gives the leading term in the asymptotic behaviour of the reflection coefficient. The convergence of the sum depends in general on the convergence of the right-hand side of (30), and the good asymptotic behaviour of the latter guarantees that of the former. Nevertheless, \( a \) fixed for \( E \) large implies \( \delta_w \) small and in this way the profile function considered is representing a thin wall, thinner for larger \( E \). By comparing the reflection coefficient obtained from (20) and (28) with (26), we can see that a power law for the fall with the energy is obtained in both cases, and in the limit \( a \rightarrow 0 \) the result for the step case is recovered. It is easy to show from (18) and (21) that

\[
C_1(a) = \frac{1}{2} + O[a] , \\
C_3(a) = \frac{1}{8} + O[a] .
\]

It results that the validity of the step profile functions demands, beyond the obvious condition \( \delta_w \ll 1 \), the additional one \( a \ll 1 \). With the appropriate dimensions, these conditions are \( \delta_w \ll \hbar/m_0c \) and \( E \ll hc/\delta_w \), respectively. The latter giving the range of energy where the fermions are not sensitive to the real functional form of the wall.

A more interesting situation is when \( \delta_w \) is kept fixed and \( a \) increases with \( E \). The expansion (20) must be rewritten in terms of \( E \) and \( \delta_w \). We obtain

\[
R_0(E) = \sum_{k=h} D_k(\delta_w, E) \left( \frac{1}{E} \right)^{k+1} ,
\]

where

\[
D_h(\delta_w, E) = \frac{i^h}{(2\delta_w)^h} A_h e^{i\varphi_h} .
\]

The remaining dependence on \( E \) of the coefficients \( D_k \) in (32) is due to the factors \( e^{\pm ia} \) which appear in the integrals of (18) and they are not expandable in powers of \( \frac{1}{E} \). It is worth to stress that the exponential factor \( e^{2i(\phi - a)} \) in (19) can be expanded in powers of \( (1/E) \) because, in this term, the phase goes to 0 as \( E \rightarrow \infty \). This energy dependence of \( D_k \) does not affect the good asymptotic properties of the expansion (32), i.e. \( D_h \left( \frac{1}{E} \right)^{h+1} \) gives the high energy behaviour of the reflection coefficient, since the factors \( e^{\pm ia} \), and therefore the coefficients \( D_k \), remain bounded when \( E \rightarrow \infty \).
These factors $e^{\pm ia}$ are crucial for the study of the CP-violating process, as will be seen in section 5, because they provide a non-trivial CP-even phase. Eqs. (32) and (33) are derived from (20) and (28) provided that the expansion in eq. (20) keeps its asymptotic character as the dependence on $E$ of the coefficients $C_n(a)$ through the parameter $a$ is considered. The latter happens when

$$\lim_{E \to \infty} \frac{1}{E} \frac{k+2}{k} |C_{k+2}(2\delta_w E)| = 0$$

for all odd $k$. The observance of this condition can be proven although it is not an immediate consequence of the asymptotic character of the expansion in (11) because the profile function $f(\tau)$ is not zero outside the domain wall: $f(\tau) = 1$ for $\tau \geq \delta_w$. It follows from (21) that (34) is satisfied for $k = 1$. A general proof may be found in [21].

Therefore, for the functions introduced in this section as mathematical representation of a strictly finite physical wall, we obtain a fall with the energy as the power $-(h+1)$, where $h$ is the order of the first derivative which is not continuous (or does not exist) at any of the two point $\tau = 0, 1$, or at both. This result can be easily generalized in order to assert that the dominant high energy behaviour, when the profile function is $h$ times derivable on the real axis, its $h$-derivative, being discontinuous, is $\sim 1/E^{h+1}$ as in eq. (32). This is the main result of this section.

4 Analytic profile functions

We consider now the case where the profile function is analytic on the real axis. By considering the arguments we presented in the previous section it follows immediately that an analytic function cannot describe a wall profile which is constant outside the domain wall. Therefore, these profile functions can only be asymptotically equal to 0, 1, as $\tau \to -\infty, +\infty$, respectively. Thus, the wall extends formally from $\tau = -\infty$ to $\tau = +\infty$. Nevertheless, this wall will be characterized in general by an effective thickness, $\sigma$, which allows to define a certain domain wall. The particular criterion considered in order to define $\sigma$ is not important. For instance, $|f(\tau) - 1| < 0.1$ for $\tau > \sigma$ and $|f(\tau)| < 0.1$ for $\tau < -\sigma$. However, in general we can write $f(\tau) = F(\frac{\tau}{\sigma})$, where we will assume

$$F(x) \sim e^{xC_-} \quad , \quad F(x) \sim 1 - e^{-xC_+} \quad (35)$$

being $C_+ > 0, C_- > 0$, parameters depending on the particular profile wall. The asymptotic functional behaviour assumed in eq. (35) is in principle the simplest that can be demanded to a profile function verifying the initial asymptotic conditions as
\[ \tau \rightarrow -\infty, +\infty \text{ introduced above. Furthermore, it is not easy to find functional} \]
\[ \text{behaviour other than that of (35), with good analytic properties on the real axis,} \]
\[ \text{and, on the other hand, a naive derivation of the wall profile from the corresponding} \]
\[ \text{equation of motion as done in [18] agrees with this assumption}^{4}. \]
\[ \text{We consider the} \]
\[ \text{evolution from } \tau = -\delta_w \text{ to } \tau = \delta_w, \text{ but this parameter is now defined in such a way} \]
\[ \text{that } f(\delta_w x) = F(\lambda x), \text{ where } \lambda = \frac{\delta_w}{\sigma} \text{ verifies } 1 \ll \lambda \ll \frac{E}{\sigma}. \]
\[ \text{Thus, analogously to the} \]
\[ \text{previous section, it follows from (11) and (13)} \]
\[ \Omega(\lambda \sigma, -\lambda \sigma) = \left\{ 1 + \frac{1}{E} \sigma_2 a \int_{-1}^{1} dx \, F(\lambda x) \, e^{-i\sigma_3 a(1-x)} + o \left[ \left( \frac{1}{E} \right)^2 \right] \right\} e^{i\sigma_3 \phi}; \quad (36) \]
\[ \text{where all the previous definitions are kept.} \]
\[ \text{Again, we define } R(E) = e^{2i(\delta-\alpha)} R_0(E) \text{ and obtain} \]
\[ R_0(E) = \frac{1}{2} \left( \frac{1}{E} \right) \left( e^{ia} - ia \int_{-1}^{1} dx \, F(\lambda x)e^{iax} \right) + o \left[ \left( \frac{1}{E} \right)^2 \right]. \quad (37) \]
\[ \text{In order to derive (37) from (36), the first of the equations (9) must be used, and} \]
\[ \text{therefore the profile function is modified outside the region } (-\delta_w, \delta_w) \text{ to describe} \]
\[ \text{a situation with zero and constant mass for } \tau < -\delta_w, \tau > \delta_w. \]
\[ \text{This technical} \]
\[ \text{assumption, necessary to identify the incident, reflected and transmitted particle} \]
\[ \text{fluxes outside the region considered, introduces non-negligible effects for the high} \]
\[ \text{energy behaviour which must be separated from the real ones caused by the evolution} \]
\[ \text{through the wall region considered. In fact, the numerical calculation performed in} \]
\[ [8] \text{requires this technical assumption and as a consequence the authors consider a} \]
\[ \text{wall strictly located in the region between } z = 0 \text{ and } z = z_0 \text{ despite they work with} \]
\[ \text{an analytical ansatz which goes exponentially to zero and one when } z \rightarrow -\infty, +\infty, \]
\[ \text{respectively. However, this modification has no practical consequence except for the} \]
\[ \text{high energy behaviour. It should be noted that the hypothesis } E \gg \delta_w, \text{ is necessary to} \]
\[ \text{obtain (36), therefore we cannot take the limit } \delta_w \rightarrow \infty \text{ directly in order to calculate} \]
\[ \text{the reflection coefficient. When the profile function varies from } \tau = -\infty \text{ to } \tau = +\infty, \]
\[ \text{the coefficients must be formally defined by using the asymptotic flux at } \tau = +\infty, -\infty. \]
\[ \text{Obviously, it should be done before obtaining eq. (36), by considering the} \]
\[ \text{evolution of the fermions from } \tau = -\infty \text{ to } \tau = +\infty, \text{ without the introduction of} \]
\[ \text{the parameter } \delta_w \text{ and without any additional condition for the energy beyond the} \]
\[ \text{initial one, } \delta_w \gg 1. \]
\[ \text{Nevertheless, in this case we cannot expand the exponentials of (13) to derive an analogous expression to (16), the integrals we obtained for the} \]
\[ \text{expansion of the reflection coefficient are then much more complex and practically} \]
\[ \text{it is not possible to analyze them in a general way. This is why we approach the} \]
\[ \text{problem in a different way: we study in practice the evolution from } -\delta_w \text{ to } \delta_w \text{ in} \]
\[ \text{order to find the total contribution from } -\infty \text{ to } +\infty \text{ and other terms depending on} \]
\[ \text{which must be in principle negligible by considering the requirement (35) for the} \]
\[ \text{profile function.} \]

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By rewriting appropriately the integral in (37) as the sum of two terms, the first giving the contribution of the whole integration domain and the second due to contributions of $x \notin [-1, 1]$, we obtain after some calculation, for the leading term,

$$R^{(1)}_0(E) = -i \left( \frac{1}{E} \right) \frac{a}{2} \left\{ \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} dx \ F(\lambda x) \ e^{(i\alpha-\varepsilon)x} \right\} + \left( \frac{1}{E} \right) R_{N-C}(a) \quad (38)$$

where, from (35) and $\lambda \gg 1$

$$R_{N-C}(a) = \frac{i}{2} \left\{ e^{-\lambda C_+} \ e^{i\alpha} \ \frac{a}{ia - \lambda C_+} + e^{-\lambda C_-} \ e^{-i\alpha} \ \frac{a}{ia + \lambda C_-} \right\} . \quad (39)$$

It can be seen that the first term of the right-hand side of (38) does not depend on $\delta_w$ (Although it seems to have a dependence on $\delta_w$ through $a$, if we change the integration variable $x$ by $x' = \lambda x$, we see that this dependence disappears). Now, we analyze its functional dependence on the energy and then we turn to the question whether $R_{N-C}(a)$ is negligible with regard to this term.

If the Laurent expansion for $F(z)$, the analytic complex extension of $F(x)$, in the pole of order $\nu_j$, $z = z_j$, is $\sum_{n=-\nu}^{+\infty} a_j^n (z - z_j)^n$ the following result can be shown by applying the Cauchy theorem

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} dx \ F(\lambda x) \ e^{(i\alpha-\varepsilon)x} = 2\pi i \sum_{j=1}^{N} e^{-\frac{2\pi}{\lambda} y_j} \ e^{i\frac{2\pi}{\lambda} x_j} \sum_{n=1}^{\nu_j} a_j^n \left( \frac{(ia)^{n-1}}{\lambda^n (n-1)!} \right) . \quad (40)$$

Where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, ..., $z_N = x_N + iy_N$, are all the poles of $F(z)$ with positive imaginary part we have picked when the integration contour is adequately closed. As the profile function, $F(x)$, is analytic on the real axis, it is obvious that $y_j \neq 0$ and therefore we obtain an exponential dependence on $a$. Nevertheless, in this case the parameter $a$ has no physical meaning because $\delta_w$ is arbitrary. The distance $\delta_w$ is taken as several times the parameter $\sigma$ in order to apply eq. (35). Moreover, $\sigma$ is defined after an arbitrary criterion. The right-hand side of eq. (40) may however be rewritten as follows

$$\frac{2\pi i}{\lambda} \sum_{j=1}^{N} e^{-2E\sigma y_j} \ e^{2iE\sigma x_j} \sum_{n=1}^{\nu_j} a_j^n \left( \frac{(2iE\sigma)^{n-1}}{(n-1)!} \right) , \quad (41)$$

where the dependence on $\delta_w$, through the parameter $a$, is replaced by the one on $\sigma$. The final result will be expressed, (47), in terms of the singularities of $f(z)$, instead $F(z)$, eliminating any dependence on our arbitrary parameter $\sigma$ [1].

Now, the discussion of (41) is very similar to the previous one for the non-analytic case. If the range of energy and the effective thickness, $\sigma$, allows to consider that the factor $E\sigma$ is not large, then the term $\frac{1}{E} R_{N-C}$ is negligible with regard to the first one of eq. (38),

$$\sigma E \ e^{-2E\sigma y_j} \gg e^{-\lambda C_+} , \quad e^{-\lambda C_-} ; \quad (42)$$
since $\lambda$ is large the latter gives then the asymptotic behaviour of the reflection coefficient. Nevertheless, by considering the energy dependence of (41) and $R_{N-C}(a)$, it is obvious that for large enough energy the first term of (38) is smaller than $\frac{1}{E}R_{N-C}$ since (42) is not satisfied. This result is at first surprising and seems to disagree with our requirements about the profile function. But, now we must consider the above mentioned technical assumption since as a consequence of that, the profile function we are studying in practice is non-continuous at $\tau = -\delta_w$ and $\tau = \delta_w$. Therefore, a contribution of the order $\frac{1}{E}$ to some power follows from the results of the previous section which obviously must depend on $\delta_w$. It is not easy to obtain exactly these contributions because the result (30) cannot be in general applied to these profile functions, except in the large $a$ limit where the asymptotic behaviour can be determined. It can be proven that the asymptotic behaviour for large $E$, and therefore $a$, of the integral in the left-hand side of eq. (30) is indeed given by the first term in the right-hand side of eq. (30), in a completely general way for any function $F(x)$. Thus, the contribution of the non-continuities can be identified. Moreover, if we take the limit $\sigma \to 0$ then $R_{N-C} \to 0$, which is the behaviour we hope for the contribution of non-continuities: the profile function gives the step function, in this limit, and it can also be proven that the contribution of the right-hand side’s first term of eq. (38) is exactly the reflection coefficient for the step case [24]. The energy dependence of the term $\frac{1}{E}R_{N-C}$ is explained in this way, as well its surprising large energy dominance. However we are interested in the high energy contribution of the total evolution from $\tau = -\infty$ to $\tau = +\infty$, and all the previous discussions allow to assume that it is given by the first term of eq. (38). By using eq. (41) we obtain for this high energy behaviour of the reflection coefficient

$$R(E) = 2\pi\sigma \sum_{j=1}^{N} e^{-2E\sigma y_j} e^{2iE\sigma x_j} \sum_{n=1}^{\nu_j} a_{j,n} \frac{(2iE\sigma)^{n-1}}{(n-1)!}. \quad (43)$$

Besides, if we are in the range of energy where $E\sigma$ is large, the leading contribution is given by the pole of $F(z)$ with the smallest imaginary part, i.e. the closest to the real axis. Calling $z_k = x_k + iy_k$ this prevailing pole we obtain

$$R(E) = 2\pi\sigma e^{-2E\sigma y_k} a_{-\nu_k}^{k} \frac{(2iE\sigma)^{\nu_k-1}}{(\nu_k-1)!} e^{2iE\sigma x_k}. \quad (44)$$

In this case there is no proof of a result analogous to (34) (even the particular checks are difficult). However, by studying directly (13) as well as the prescription to obtain the coefficients of the general expansion (11) which is given in ref. [21], it can be reasonably guessed that the asymptotic character of eq. (20) is given by

$$\lim_{E \to \infty} \left( \frac{1}{E} \right)^{k+2} \frac{|C_{k+2}(2\sigma E)|}{|C_k(2\sigma E)|} \lesssim \sigma^2. \quad (45)$$
Looking at (13) and taking into account the expansion of eq. (12) for $p(\tau)$ in powers of $(1/E)$, it is easy to see that $\Gamma_3(E)$ contains terms of the order $E^3\sigma^3$, because the first of the three integrals of $\Gamma_3(E)$ in eq. (16) appears multiplied by $(\sigma E)^3$ when the change of integration variable, $x = \tau/\sigma$, is performed after the introduction of the function $F(\tau) = f(\sigma \tau)$. If we conjecture that this result can be generalized to $\Gamma_n(E)$ containing terms of the order $E^n\sigma^n$ (this conjecture is supported by the study of the general expansion given by eq. (11) in ref. [21]), we get the ratio given by eq. (45).

If the particular profile function considered generates an expansion for the reflection coefficient which verifies (45), then (43) and (44) will be valid in any range of energy satisfying $E \gg 1$ and $\sigma \ll 1$ (obviously (44) is only valid for $\sigma E$ large). As can be seen above, these conditions are suitable for studying quarks propagating through a wall except for the case of the top.

We worked by convenience with the function $F(z)$, obtained through the change of variables performed at the beginning of the present section. Let us now return to the physical variables and express the final result as a function of the singularities of $f(z)$ instead of $F(z)$ in order to avoid the apparent arbitrariness arising from the dependence on $\sigma$. If we consider the Laurent expansion $\sum_{n=-\nu_j}^{+\infty} a_n' (z - z_j')^n$, for $f(z)$ in the pole of the order $\nu_j$, $z = z_j'$ and take into account that $f(z) = F(\frac{z}{\sigma})$, we obtain

$$z_j' = \sigma z_j,$$
$$a_n' = \frac{a_n}{\sigma^n}. \quad (46)$$

Eq. (44) can thus be re-written as follows

$$R(E) = 2\pi e^{-2E y'_k} \frac{a_{-\nu_k}'}{a_{-\nu_k}} \frac{(2iE)^{\nu_k-1}}{(\nu_k - 1)!} e^{2iEy'_k}; \quad (47)$$

where the former condition $\sigma E \gg 1$ must be re-formulated by requiring that $e^{2E(y'_k - y'_h)}$ may be neglected, $y'_k$ ($y'_h$) being the imaginary parts of the (next to) closest pole to the real axis. It is also interesting to notice that the wall thickness we introduced through the parameter $\sigma$ is now incorporated without arbitrariness into the profile function. From eq. (46), the connection between this thickness and the distance between the poles of $f(z)$ is apparent.

Now, we check the present results by using the particular analytic solution obtained in references [18] and [19] for the ansatz

$$f(\tau) = \frac{1 + \tanh(\tau/\sigma)}{2}. \quad (48)$$

For this profile function it follows from (43) that

$$R(E) = \frac{\pi \sigma}{2 \sinh(\pi E \sigma)}, \quad (49)$$

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which agrees with the result of [18] and [19] in the above mentioned conditions.

Therefore, with this positive check, we conclude that (43) in general and (44) for $E\sigma$ large give the high energy behaviour of the reflection coefficient, for the analytic profile functions: at high energy the reflection coefficient decreases exponentially, the coefficient in the exponential being the smallest positive imaginary part of the complexified profile function. This is the main result of this section.

It is interesting to emphasize the exponential dependence on the energy in this case instead the power dependence in the previous one. Apparently, there is no connection between power law energy dependence derived in in the preceding section for non-analytic profiles, and the exponential law for analytic profiles derived in the present section. Nevertheless it can be proven that the results for non-analytic profiles can be understood through the properties in the complex plane of analytic extensions for these profile functions by using eq. (43) [24].

5 CP-violating wall profiles

As pointed out in the introduction, our effort to study CP-violating effects concentrates on CP violation at the tree level due to the expectation value of the Higgs fields beyond the Standard Model. The time independent Dirac Hamiltonian incorporating CP-violation can be written as follows

$$H = \bar{\alpha}\vec{p} + \beta m_R(z) + i\beta\gamma^5 m_I(z).$$

(50)

As stated in ref. [6], $m_R(z)$ and $m_I(z)$ are real because of the hermiticity of the Hamiltonian. The CP violation comes from the term $\propto m_I(z)$, when it cannot be rotated away through a chiral rotation. Therefore, we must go beyond the SM in order to find CP violation at tree level. In the SM, with one Higgs field $m_I(z)/m_R(z)$ is a constant independent of $z$ and therefore this source of CP violation can be rotated away. In the other models, with more than one Higgs field, $m_I(z)/m_R(z)$ depends in general on $z$ and the matrix $m_I(z)$ will remain.

Following the notations and conventions of section 2, the time independent Dirac equation can be written as in eqs. (3) and (4), where $m(z) = m_R(z) + im_I(z)$. The CP-violating effects are so incorporated by assigning a complex mass to the fermion inside the domain wall.

Analogously to (10), we define $m_R(\tau) = m_0 f(\tau)$ and $m_I(\tau) = \varepsilon m_0 g(\tau)$, where $f(\tau)$ and $g(\tau)$ are real functions of order 1 which characterize the CP-violating profile, and $\varepsilon$ is a parameter associated to CP-violation. We express

$$Q(\tau) = \sigma_3 E - i (\sigma_2 f(\tau) - \varepsilon \sigma_1 g(\tau)),$$

$$\bar{Q}(\tau) = \sigma_3 E - i (\sigma_2 f(\tau) + \varepsilon \sigma_1 g(\tau)).$$

(51)
We keep the same dimensionless quantities, the same definitions of the previous sections and assume that \( g(\tau) \) goes exponentially to zero at both, \( \tau \to +\infty, \tau \to -\infty \), for an analytic profile or that it is strictly zero outside the domain wall for a non-analytic one. This choice is always possible by an appropriate chiral rotation which rotates away the imaginary part \( m_I(+) \), i.e. which corresponds to a real mass in the broken phase. We have checked that the same result is obtained with another choice.

Thus, the high energy leading term for the reflection coefficient can be expressed in a general way as

\[
\left\{ \frac{R^{(1)}(E)}{R^{(1)}(-E)} \right\} = \frac{1}{2} \left( \frac{1}{E} \right) \left( e^{ia} - ia \int_{-1}^{1} dx \, F(\lambda x) \, e^{iax} \pm 2a \varepsilon \int_{-1}^{1} dx \, G(\lambda x) \, e^{iax} \right)
\]

(52)

where \( G(x) \) is defined as \( F(x) \) in the previous section, \( x \) being the same dimensionless variable which was then introduced, and \( \lambda = 1, \delta_w/\sigma \) for non-analytic, analytic profiles, respectively. The sign \(-\) is for \( R(E) \) and \(+\) for \( R(-E) \).

For both, analytic and non-analytic profiles, the difference \( R_{R \to L} - R_{L \to R} \), which is a quantitative measure of the CP-violating effects induced by the wall, can be computed. For the sake of simplicity we concentrate on the non-analytic ones:

\[
R_{R \to L} - R_{L \to R} = -4\varepsilon \left( \frac{1}{E} \right)^{h+\ell+2} \frac{A_h B_{\ell}}{(2\delta_w)^{h+\ell}} \sin(\Delta \varphi)
\]

(53)

where the \( \ell \)-derivative of \( G(x) \) has been considered as the first non-continuous one, where \( B_{\ell} \) is defined from \( G(x) \) as \( A_h \) in (29) from \( F(x) \), and where

\[
\Delta \varphi = \alpha_{\ell} - \varphi_h + (\ell - h) \frac{\pi}{2}
\]

(54)

with \( \varphi_h \) defined in (29) from \( F(x) \) and \( \alpha_{\ell} \) analogously from \( G(x) \). When \( g(\tau) \) vanishes, \( B_{\ell} = 0 \) and as expected the r.h.s of (53) vanishes.

The former requirement, \( g(\tau) \) vanishing outside the domain wall, is imposed as above mentioned in order to work in the physical basis, in which the masses are real. For technical reasons, we will also consider the chirally rotated convention in which \( g(\tau) = 1 \) in the broken phase outside the domain wall, i.e. \( m_I = \varepsilon m_0 \).

One can easily check from (51) that the imaginary mass in the broken phase will be removed by the chiral rotation \( \exp(i\eta \gamma_5) \) with \( \varepsilon = \tan(2\eta) \), and the real mass in this physical region will become \( m = m_0|\sqrt{1 + \varepsilon^2}| \). As a matter of the fact, in that region the eigenvalues of the matrix \( Q(z) \) and \( \bar{Q}(z) \) will be \( \pm p \), where \( p = |\sqrt{E^2 - m^2}| \), \( m \) obviously being the physical mass. Therefore, the dimensionless quantities we introduced in previous sections should be now built by using \( m \) instead of \( m_0 \). Then, eq. (52) must be replaced by

\[
\left\{ \frac{R^{(1)}(E)}{R^{(1)}(-E)} \right\} = \frac{1}{2} \left( \frac{1}{E} \right) \left( e^{ia} \frac{1 \pm \varepsilon}{\sqrt{1 + \varepsilon^2}} e^{iax} - a \varepsilon \int_{-1}^{1} dx \, F(\lambda x) \, e^{iax} \right)
\]

(55)
\[ \pm \frac{a \varepsilon}{\sqrt{1 + \varepsilon^2}} \int_{-1}^{1} dx \, G(\lambda x) \, e^{i \lambda x} \]

and it follows that
\[ \mathcal{R}_{R \to L} - \mathcal{R}_{L \to R} = -\frac{4\varepsilon}{1 + \varepsilon^2} \left( \frac{1}{E} \right)^{h+\ell+2} \frac{A_h B_\ell}{(2\delta_w)^{h+\ell}} \sin(\Delta \phi) \]

(56)

instead of eq. (53). By comparing eqs. (53) and (56) it is very easy to see that all physical consequences and in particular the discussion which follows will be valid for both cases. For example, the vanishing of the r.h.s. of (56) when no CP violation is present is now due to \( \Delta \phi = 0 \). An amusing remark is that eq. (56) shows that the highest chiral asymmetry originated by the CP-violating wall profile appears for the case \( \varepsilon = 1 \). In other words, by taking into account that as well \( f(\tau) \) as \( g(\tau) \) are required to be \( O[1] \), the most effective situation in what concerns the production of a net baryon number left in the broken phase for the high energy range arises then from requiring that real and pure imaginary mass terms are of the same order. In fact, Eq. (56) remains unchanged if we make \( \varepsilon \to \frac{1}{\varepsilon} \), this symmetric behaviour for \( \varepsilon > 1 \) and \( \varepsilon < 1 \) being coherent with the fact that real and pure imaginary masses in the broken phase can be turned into each other through a chiral rotation. As already stated, the imaginary mass in the broken phase can be rotated away by a chiral rotation to return from the latter convention, \( g(\infty) = 1 \), to the former one in which \( g(\tau) \) vanishes outside the domain wall. The chiral rotation implies a redefinition of the new real and pure imaginary functions, \( \mathcal{F}(\tau) \) and \( \mathcal{G}(\tau) \), as a function of the old ones, \( f(\tau) \) and \( g(\tau) \). If we then apply the new functions, \( \mathcal{F}(\tau) \) and \( \mathcal{G}(\tau) \), which behave now appropriately outside the domain wall, to eq. (52) and obtain the first non-null contribution to the chiral asymmetry, eq. (56) will be re-obtained. This expected result is not immediate to obtain explicitly, because the contribution to the chiral symmetry given by eq. (53), which is in general non-null, gives zero for \( \mathcal{F}(\tau) \) and \( \mathcal{G}(\tau) \) if \( h \neq \ell \). Therefore upper order terms must be considered to re-obtain eq. (56).

As well known, a CP-violating effect is always generated by the interference between a CP-even phase and a CP-odd one. In fact, it can be seen from eqs. (52), (53) and (54) that a non-zero leading term first needs the CP-odd phase arising from the reversing of the sign for the pure imaginary mass term in (52), but also a non-zero angle \( \Delta \phi \), i.e. a net CP-even phase. Notice that working at high energy \( (E \gg 1) \) we are far off the total reflection domain \( (E \leq 1) \), and that due to the thickness of the wall nothing prevents a CP-even phase. In fact the CP-even phase is generated by the interference of the phase of the mass term \( m_I(z)/m_R(z) \) at different values of \( z \). In the step approximation, it is always possible to make the mass term real by rotating away the phase through a chiral rotation, but if we have a \( z \)-dependent complex mass term inside a certain region, this rotation is not possible in general and the imaginary term generally remains. Tree level effects can thus appear. Nevertheless, if the \( z \)-dependence of the imaginary term is proportional to the real one, we can,
again through a chiral rotation, turn the complex mass into real. When we take the convention \( g(\infty) = 0 \) this situation corresponds to \( g(z) = 0 \) for all \( z \). This is stressed in references [8] and [19] and it is obvious from (29) that in this case \( h = \ell, \varphi_h = \alpha_\ell \) and therefore \( \Delta \varphi = 0 \), which lead to \( R_{R \to L} - R_{L \to R} = 0 \). Furthermore, the leading term will be zero if we assume a weaker condition than the proportionality of real and pure imaginary terms.

Assuming \( F(x) = C + F_0(x) \) and \( G(x) = K + G_0(x) \), where \( C, K \) are simple constants and \( F_0(-x) = -F_0(x), G_0(-x) = -G_0(-x) \), we obtain

\[
F^{(h)}(1) = (-1)^{h+1} F^{(h)}(-1),
G^{(\ell)}(1) = (-1)^{\ell+1} F^{(\ell)}(-1); \tag{57}
\]

for \( h, \ell > 0 \). It follows from (57) and (54) that under these conditions

\[
\Delta \varphi = n\pi, \tag{58}
\]

where \( n \) is an integer. Analogously we obtain (58) if both functions have even symmetry with regard to the axis \( x = 0 \) (It is obvious that our requirement \( F(\infty) = 1 \) forbids an even symmetry for \( F(x) \), but our results can be applied to general situations where \( F(\infty) = 0 \)). We conclude therefore that this weaker condition implies a null CP-violating leading term as it follows from (53). This does not lead to a strict suppression of CP-violating effects as the stronger requirement of proportionality does, but it can be stated that CP-violation will be of a lower order in the high energy range. Whether these symmetry conditions involve any similar consequence in a lower energy range is an interesting question, but it is out of the scope of the present work.

It is also interesting to stress that the most favourable situation is given by the opposite symmetry properties of \( F(x) \) and \( G(x) \) with regard to the axis \( x = 0 \). If we assume that the functions \( F_0(x) \) and \( G_0(x) \), above defined, are even and odd symmetric, respectively, or vice versa, we immediately obtain

\[
\Delta \varphi = \frac{2n + 1}{2}\pi, \tag{59}
\]

\( n \) being an integer. The highest value for \( \sin(\Delta \varphi) \) is thus given by imposing these symmetry requirements.

An expression analogous to (53) can be found for the analytic case by using (44), and a similar discussion may be done. The crucial point for such a discussion is that the integral \( \int_{-\infty}^{\infty} dx F(\lambda x)e^{i(ax - \varepsilon)x} \), which is obtained from \( \int_{-1}^{1} dx F(\lambda x)e^{iax} \), is pure imaginary if \( F(x) \) differs by any constant from an odd symmetric function, and real if it is even symmetric. It is very easy to see that, for instance, the analytic ansatz (48) is odd symmetric with regard to the axis \( x = 0 \) up to a constant, its contribution to the reflection coefficient has indeed no CP even phase factor (eq. (49)). If a function \( F(x) \) for the real mass term and a function \( G(x) \) for the imaginary one have both
odd symmetry, the angle $\Delta \varphi$ vanishes. No CP-even phase is obtained and no CP-violating effects at tree-level and at first order in the high energy limit arise from this case. Nevertheless, if a function $G(x)$ defined as $\frac{dF(x)}{dx}$, which is even symmetric, is taken, a net CP-even phase and therefore CP-violating effects at the first order are obtained. This simple ansatz for $G(x)$ illustrates our statement about the symmetry properties.

In ref. [19] numerical calculations are performed by considering the CP-violating term as a perturbation to the CP conserved Dirac equation, for the particular real profile function above mentioned, $F(x) = 1/2(1 + \tanh x)$, and for the two following pure imaginary ones

$$G_1(x) = \frac{dF(x)}{dx}; \quad G_2(x) = (F(x))^2.$$  \hspace{1cm} (60)  

The quantity $\Delta^{CP} = R_{R \to L} - R_{L \to R}$ for both particular cases can be immediately obtained in the high energy range from the general results given by eqs. (44) and (52),

$$\Delta_1^{CP} = 8\pi^2 \varepsilon \sigma^3 E e^{-2\pi E \sigma}, \quad \Delta_2^{CP} = -4\pi^2 \varepsilon \sigma^3 E e^{-2\pi E \sigma}.$$  \hspace{1cm} (61)  

These results are valid obviously for $E \gg 1$ and $\sigma \ll 1$ as we explained in section 4, but the requirement $E \sigma \gg 1$ is also necessary to isolate the contribution of the leading singularity [24]. Nevertheless, the contribution for all the poles can be resummed and we obtain, for instance, in the first of the two former cases

$$\Delta_1^{CP} = 2\varepsilon \sigma^3 E \left(\frac{\pi}{\sinh(\pi \sigma E)}\right)^2,$$  \hspace{1cm} (62)  

which is a valid result for any $E \sigma$.

The energy range we are interested in is not explored in the numerical results given in ref. [19], although a few results for $E = 5.0$ are presented amusingly in relatively good agreement with the ones we obtain from our analytic expression [6]. We also checked in section 4 that the unperturbed reflection coefficient which is obtained in that work for the above expressed particular ansatz completely agrees in the appropriate limit with our general result for a non CP-violating wall profile. Furthermore, we obtain the ratio $\Delta_1^{CP}/\Delta_2^{CP} = -2$ which is observed for the results given by Funakubo et al. [19], even for $E \sim 1$.

Several interesting remarks must be finally done. The function $G_2(x) = |F(x)|^2$ does not present (up to a constant) any well defined symmetry with respect to the axis
$x = 0$, while $G_1[x] = \frac{dF(x)}{dx}$ is even symmetric; a higher CP asymmetry for the latter than for the former is thus not surprising in view of our conclusions about symmetry. Moreover, the dependence on $\sigma$ of eq. (62) also agrees with the well-known fact that no observable CP-asymmetry can appear in the thin wall case at tree level, i.e. in the limit $\sigma \to 0$. Funakubo et al state that the degree of decreasing of $\Delta^{CP}$ seems to be larger as the wall thickness increases after studying two particular cases for several values of the wall thickness. Regarding eq. (53) we can state a more precise conclusion for any non-analytic wall profile in the high energy range which confirms the trend of the CP asymmetry to decrease when thickness increases pointed out for those authors. The quantity $\Delta^{CP}$ generally is an oscillating function on the thickness parameter because the angle $\Delta \varphi$ depends on it, but those oscillations are modulated by a factor decreasing with larger thickness as eq. (53) shows ($\Delta^{CP}$ as a function of the thickness has in general an infinite number of zeros, the first of them being in fact as the thickness is zero). The functional behaviour of $\Delta^{CP}$ given in eqs. (61) and (62) for analytic wall profiles seems also confirm that trend, but in that case our results are only valid for $\sigma \ll 1$.

6 Summary and conclusions

Rigorous and completely general conclusions about the behaviour of fermions hitting a wall with an arbitrary profile are very difficult. Obviously, the fermion scattering is determined by the particular profile function which characterizes the wall we are treating. The crucial point in order to get general answers in this problem is to show relations between the functional properties of the wall profile and the reflection coefficients, etc, which characterize the behaviour of the fermions hitting the wall. A prescription to solve the Dirac equation in the presence of CP-violating electroweak bubble wall is presented in [19]. Nevertheless, the results depend on certain functions of the energy which must be obtained solving a second-order differential equation. This prescription, applied to the particular ansatz studied in [18], leads to a hypergeometric differential equation, but in general it is a difficult problem which does not allow to relate in a direct and useful way the wall profile function to the fermion behaviour.

A formalism to study the scattering of the high energy fermions in the presence of a general wall is proposed in the present work. High energy fermions mean $E/m_0 \gg 1$. If we consider that the temperature during the supposedly electroweak first order phase transition is about 100 GeV [16], the Boltzmann thermal distribution gives an average energy for the quarks which verifies $\frac{m_0}{E} = O[10^{-2}]$ except for the top. Thus, the range of energy considered is interesting in the cosmological problem, although for the top we only study a high energy tail of the thermal distribution. The quantum corrections to the classical behaviour expected for this range have been
obtained for two different types of mathematical representations of the wall. The first describes walls with profile functions which differ from the asymptotic values only in a finite domain. Whether they are realistic is a difficult question. They are modeling situations involving different length scales of variation for the profile function and its derivatives, i.e. so that to a given accuracy, some derivatives can be considered as discontinuous. In the second case, when we assume that the wall profile function differs from the asymptotic values up to infinity, edge effects related with different length scales of variation are neglected (quantum, thermal and statistics fluctuation are not considered, for example). However, we study the quantum corrections in both cases and show how the high energy fermions are sensitive to these facts. As can be expected, the quantum corrections are more important as the profile becomes sharper. In fact, if the \( h \)-th derivative of the profile function is the first non continuous one, i.e. it presents a certain step, the high energy behaviour of the reflection coefficient is characterized by the following power law

\[
R(E) \propto m_0 e^2(\bar{h}c/\delta_w)^h(1/E)^{h+1},
\]

against the exponential behaviour

\[
R(E) \propto \exp\left[-y'_kE/(\bar{h}c)\right],
\]
given by a profile function analytic on the real axis. The physical dimensions are restored in eqs. (63) and (64). \( \delta_w \) in the first equation is the wall thickness parameter defined in section 3 and the coefficient \( y'_k \) in the latter one is the smallest positive imaginary part among the poles of the analytic complex extension of the profile function \( f(\tau) \). It worth to stress that \( y'_k \) has the dimension of a length, since the variable \( \tau \) for \( f(\tau) \) characterizes the position in the normal axis to the planar wall. Notice also that, while in the former case the reflection coefficient is proportional to some power in \( \bar{h} \) as expected for a quantum effect, in the latter case the reflection coefficient is not even analytic in \( \bar{h} \), similar to a tunnelling effect (for example, the term given by eq. (64) has an analogous dependence on \( \bar{h} \) to that of the so-called Gamow factor, \( e^{-2\pi ZZ'e^2/\bar{h}v} \), which gives the dominant behaviour for processes involving nuclei with charge \( Z'e \) and velocity \( v \) tunneling through the Coulomb barrier due to a charge \( Ze[25] \)).

Finally, CP-violating effects have been incorporated by introducing a complex mass, as we explained in section 5. The reflection coefficients, \( R(E) \) and \( \bar{R}(E) \), are expressed as a function of both, real, chirally-even, and pure imaginary, chirally odd mass functions, \( f(z) \) and \( g(z) \). We first perform a chiral rotation to have a real mass for \( z \to \infty \), deeply in the broken phase.

If \( g(z) \) does not identically vanish, the phase of the mass depends on the chirality of the incoming flux. The resulting chiral asymmetry leads to a CP-asymmetry through the CPT theorem. This CP-violating term introduced in the Dirac Hamiltonian produces a CP-odd phase. This CP-odd phase is not enough to generate an
observable CP asymmetry. It is known that a CP-even phase should exist and interfere with the CP-odd one. That effect is clearly shown in our formalism through the dependence of the relevant quantity \( \Delta^{CP} = R_{R\rightarrow L} - R_{L\rightarrow R} \) on the angle \( \Delta \phi \). A non-zero angle is due to the CP-even phases arising from the integration of the real and pure imaginary terms of the profile function, although the asymptotic regime considered here is far off the total reflection. We found that the respective symmetry properties of the \( f(z) \) and \( g(z) \) play an important role, and we obtain the following general conclusions: i) CP-even phases and therefore CP-violating observable effects at tree level will be found in the thick wall case, provided that the symmetry properties are not the same for both, \( f(z) \) and \( g(z) \). ii) CP-violating effects will be of lower order in \( 1/E \) (if not vanishing) when the symmetry properties are the same for both functions. iii) the most favourable situation in order to generate CP-asymmetry in this high energy range is given by a profile where real and pure imaginary functions present opposite symmetries.

It is finally very important to remark that the axis \( z = 0 \) is an arbitrary reference for the symmetry properties of functions. As expected our results are obviously independent of any shift on \( z \), the physical conclusions thus depending on the respective symmetry properties of the real and imaginary mass terms. It is also important to notice that the conclusion about symmetry properties are valid even if the symmetry properties appear after a chiral rotation which in general leads to a non vanishing \( g(+\infty) \).

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A The integral expansion

In this appendix we derive the integral expansion given in equations (11) and (13).

The starting point is the definition given in (6) for \( \Omega(z, z_0) \) which leads to the solutions of the time-independent Dirac equation as shown in (5). We had

\[
\Omega(z, z_0) = \mathcal{P} \ e^{i\int_{z_0}^{z} d\tau \ Q(\tau)} ,
\]

where \( Q(\tau) \) is expressed by means of the usual Pauli’s matrices in (10). By considering a certain path partition \((z_0, z_1, \ldots, z_{N-1}, z_N, z)\), we can write
\[ \mathcal{P} e^{i \int_{z_0}^{z} \! Q(\tau) \, d\tau} = \mathcal{P} e^{i \int_{z_0}^{z} \! Q(\tau) \, d\tau} \mathcal{P} e^{i \int_{z_0}^{z} \! Q(\tau) \, d\tau} \ldots \mathcal{P} e^{i \int_{z_0}^{z} \! Q(\tau) \, d\tau}. \]  

(66)

If we take \( Q(\tau) \) as constant in each interval \((z_j, z_{j+1})\), defining for each one \( f_j \) as the following integral average value

\[ f_j = \frac{1}{\Delta_j} \int_{z_j}^{z_{j+1}} \! d\tau \, f(\tau), \]  

(67)

where \( \Delta_j = z_{j+1} - z_j \), it can be written by using (10)

\[ \mathcal{P} e^{i \int_{z_j}^{z_{j+1}} \! Q(\tau) \, d\tau} = e^{(i \sigma_3 E + \sigma_2 f_j) \Delta_j}. \]  

(68)

Taking into account the following result for two operators verifying \( \left\{ A, B \right\} = 0 \) and \( A^2 = B^2 = 1 \)

\[ e^{\alpha A + \beta B} = \cosh \left( \left( \alpha^2 + \beta^2 \right)^{1/2} \right) + \frac{\alpha A + \beta B}{\left( \alpha^2 + \beta^2 \right)^{1/2}} \sinh \left( \left( \alpha^2 + \beta^2 \right)^{1/2} \right), \]  

(69)

which can be easily proven, it follows from (68) that:

\[ \mathcal{P} e^{i \int_{z_j}^{z_{j+1}} \! Q(\tau) \, d\tau} = e^{i \sigma_3 p_j \Delta_j} + \left\{ \frac{1}{E} \sigma_2 f_j + \frac{1}{2} \sigma_3 f_j^2 + \frac{1}{E} \frac{3}{2} \sigma_2 f_j^3 + o \left( \frac{1}{E} \right)^3 \right\} \sin(p_j \Delta_j). \]  

(70)

where \( p_j = +(E^2 - f_j^2)^{1/2} \). In order to obtain (70) \( \frac{E}{p_j} \) and \( \frac{1}{p_j} \) have been expanded in powers of \( \frac{1}{E} \). Eq. (69) is also valid for any set of unitary operators anticommuting with each other and consequently analogous result to the following can be obtained for the complex mass case.

By substituting the path ordered products of the right-hand in (66) by (70), multiplying and reordering the terms, we obtain

\[ \Omega(z, z_0) = e^{i \sigma_3 \sum_{j=0}^{N} p_j \Delta_j} + \frac{1}{E} \sigma_2 \sum_{k=0}^{N} \sin(p_k \Delta_k) f_k e^{-i \sigma_3 \sum_{j=k+1}^{N} p_j \Delta_j} e^{i \sigma_3 \sum_{j=0}^{k-1} p_j \Delta_j} \]

\[ + \left( \frac{1}{E} \right)^2 \left\{ \sum_{k=0}^{N} \sum_{m=0}^{k-1} \sin(p_k \Delta_k) \sin(p_M \Delta_m) f_k f_m e^{i \sigma_3 \sum_{j=k+1}^{N} p_j \Delta_j} e^{-i \sigma_3 \sum_{j=m+1}^{k} p_j \Delta_j} e^{i \sigma_3 \sum_{j=0}^{m-1} p_j \Delta_j} \right. \]

\[ \left. + i \frac{\sigma_3}{2} \sum_{k=0}^{N} \sin(p_k \Delta_k) f_k^2 e^{i \sigma_3 \sum_{j=0}^{N} p_j \Delta_j} \right\} \]

25
Thus, in order to label the Hamiltonian eigenstates the conserved quantum numbers $x$ also invariant under Lorentz boost parallel to the $\Delta e^A$ than the upper. The general result for two operators where the sum in the exponential is taken to be zero if the lower index is bigger that $f$ phase correspond to eigenstates of $J_p$ obtain $z$ that implies conservation of total angular momentum in the $\Delta$ phase the quark. For a static wall the particle energy, $E$, one flavour quark propagation problem. Let (framework presented in section 2, assuming a $z$-dependent mass in order to study the of different Dirac Hamiltonian eigenstates in the presence of a wall. We keep the same

In the present appendix we consider in a general way the problem of the orthogonality of different Dirac Hamiltonian eigenstates in the presence of a wall. We define the eigenstates $\psi_n$ by

$$H\psi_n = (-1)^n E_n \psi_n,$$

(72)

where the sum in the exponential is taken to be zero if the lower index is bigger than the upper. The general result for two operators $A$, $B$, which verify $\{A, B\} = 0$, $e^B A = A e^{-B}$ has been used. We know that $\sin(p_j \Delta_j) = p_j \Delta_j + O[\Delta_j^2]$ and in the limit $\Delta_j \rightarrow 0$, $\sum \Delta_j \rightarrow f \, d\tau$ with $f_j \rightarrow f(\tau)$ and $p_j \rightarrow p(\tau)$. These replacements can be understood if we consider the definition of $f_j$ as an integral average value, assuming that $f_j = f(\tau_j)$ with $\tau_j \in [z_j, z_{j+1}]$, provided that $f(\tau)$ is continuous. Following the last prescriptions, Eq. (11) and (13) can be immediately derived from (71).

B General Orthogonality properties

In the present appendix we consider in a general way the problem of the orthogonality of different Dirac Hamiltonian eigenstates in the presence of a wall. We keep the same framework presented in section 2, assuming a $z$-dependent mass in order to study the one flavour quark propagation problem. Let $(E, p_x, p_y, p_z)$ be the four-moment of the quark. For a static wall the particle energy, $E$, is conserved. In the unbroken phase $E = p_z$. The system is symmetric with respect to rotations around the $z$-axis that implies conservation of total angular momentum in the $z$ direction, $J_z$. It is also invariant under Lorentz boost parallel to the $x - y$ plane as we stressed above. Thus, in order to label the Hamiltonian eigenstates the conserved quantum numbers $E, p_x, p_y$ and $j_z$ (the eigenvalue of $J_z$) can be used. If we boost the reference frame to obtain $p_x = p_y = 0$, the helicity states of the incoming plane waves in the unbroken phase correspond to eigenstates of $J_z$ and this helicity may be also used to label them.

We define the eigenstates $\psi_n$ by

$$H\psi_n = (-1)^n E_n \psi_n,$$
where we choose \( n = p_x, p_y, j_z, E_n, r \). \( r = 2, 1 \) label the positive and negative energy states, respectively. In [5], for the Hamiltonian

\[
H = \bar{\alpha} \vec{p} + \beta m_0 \theta(z),
\]

(73)

it is stressed that the dimension of the eigenspace is two outside the total reflection energy range. In this case the following two set of eigenstates can be built

\[
\psi_{\text{inc}}^n(\vec{x}) = \left( u_h(\vec{p}_{\text{inc}})e^{i\vec{p}_{\text{inc}} \cdot \vec{x}} + R u_h(\vec{p}_{\text{inc}})e^{i\vec{p}_{\text{out}} \cdot \vec{x}} \right) \theta(-z) + (1 + R)u_h(\vec{p}_{\text{inc}})e^{i\vec{p}_{\text{out}} \cdot \vec{x}} \theta(z), \quad h = j_z.
\]

(74)

where \( u_h(\vec{p}_{\text{inc}}) \) is a solution of the Dirac equation in the unbroken phase and the reflection matrix \( R \) is given by

\[
R = \frac{m \gamma^3}{p_z + p_z'};
\]

(75)

and

\[
\psi_{\text{br}}^n(\vec{x}) = \sqrt{\frac{p_z}{p_z'}} \left[ \left( u_s(\vec{p}_{\text{br}})e^{i\vec{p}_{\text{br}} \cdot \vec{x}} + J u_s(\vec{p}_{\text{br}})e^{i\vec{p}_{\text{out}} \cdot \vec{x}} \right) \theta(z) + (1 + J)u_s(\vec{p}_{\text{br}})e^{i\vec{p}_{\text{out}} \cdot \vec{x}} \theta(-z) \right].
\]

(76)

where \( J \) is the reflection matrix when the particle is coming from the broken phase, given by

\[
1 + J = \frac{p_z'}{p_z} (1 + R),
\]

(77)

and \( s \) is a spin index dependent on \( j_z \), such that \((1 + J)u_s(\vec{p}_{\text{br}}) = u_h(\vec{p}_{\text{out}})\) is a massless spinor with helicity \( h = -j_z \); \( u_s(\vec{p}_{\text{br}}) \) satisfies the Dirac equation in the broken phase.

The incoming, outgoing, transmitted and broken incoming four-moment are defined as follows

\[
p_{\text{inc}} = (E, p_x, p_y, p_z), \quad p_{\text{out}} = (E, p_x, p_y, -p_z), \quad p_{\text{tr}} = (E, p_x, p_y, p_z'), \quad p_{\text{br}} = (E, p_x, p_y, -(p_z)^*).
\]

(78)

The wave functions \( \psi_{\text{inc}}^n \) and \( \psi_{\text{br}}^n \) are orthogonal and allow to build an orthonormal basis of the eigenspace beside the analogous antiparticle wave functions.
(for $r = 1$ the spinor solution of the Dirac equation, $u_h$, must be replaced by the negative energy ones, $v_h$. In what follows our conclusions are valid as well for the $r = 1$ and $r = 2$ eigenstates). For general walls the argument about the dimension of the eigenspace can be generalized and the wave functions obtained to verify

$$
\psi_{inc}^n(\vec{x}) \sim \begin{cases} 
  u_h(\vec{p}^{inc})e^{i\vec{p}^{inc}\vec{x}} + Ru_h(\vec{p}^{inc})e^{i\vec{p}^{out}\vec{x}} & -z \gg \delta_w \\
  (1 + R)u_h(\vec{p}^{inc})e^{i\vec{p}^{in}\vec{x}} & z \gg \delta_w;
\end{cases} 
$$

(79)

where $R$ is the reflection matrix in each particular case and $\delta_w$ defines the characteristic wall thickness as can be seen above. Analogously for $\psi_{br}^n$. Thus, $\psi_{inc}^n$ describes in the general case an incident plane wave far from the wall coming from the unbroken phase, bouncing on the wall, and generating a reflected plane wave in the unbroken phase and a transmitted one in the broken phase. $\psi_{br}^n$ describes the same process although coming from the broken phase. In [18], wave functions like $\psi_{inc}^n$ and $\psi_{br}^n$ are obtained for the particular wall presented in section 5. In this work the authors calculate explicitly the overlap integral of these wave functions obtaining a non-zero result, concluding that they are not orthogonal. Nevertheless, a general argument allows to assert that $\psi_{inc}^n$ and $\psi_{br}^n$ must be orthogonal as we will see. In [21] the error in the overlap integral calculation of ref. [18] is shown and it is explicitly proven that the overlap integral is zero for this particular wall profile. (the fact that the moment eigenvalues are different in the broken and unbroken phases introduce a factor $p_z'$ which is forgotten in ref. [18]). Now we will present the general argument.

The ultimate reason for the orthogonality of the eigenstates considered follows from the different physical processes to which they are related. The real processes are described by wave packets coming from the unbroken (broken) phase, bouncing on the wall, and generating reflected and transmitted packets. The solutions which are asymptotically plane waves have no real physical meaning, but they allow to build up localized wave packets. As we will see, the solutions named $\psi_{inc}^n$ are associated to wave packets evolving from the unbroken phase to the broken phase and those named $\psi_{br}^n$ describe the same but in the opposite direction, from broken to unbroken phase. Since these wave packets do not overlap in the past, by unitarity they will not overlap in all their evolution. Let us give a few more details.

The $x$ and $y$ components of the wave packet are not modified by the wall, and will be ignored in what follows. We consider the following incoming wave packet, approaching the wall from the unbroken phase,

$$
P(\vec{p}^{inc}, z, t) = N \int dk_z e^{-(k_z - p_z^{inc})^2 d^2/2} e^{ik_z\tau} u_h(\vec{k}) 
\sim e^{-(\tau/d)^2} e^{ip_z^{inc}\tau} u_h(\vec{p}^{inc}),
$$

(80)

which has the helicity, $h$, fixed. $N$ is a normalization constant, the quantity $\tau = z + Z - t - T$ has been introduced, where $Z/d >> 1$ and $T/d >> 1$. $d$ denotes the
spatial extension of the wave packet which is located at the time \( t \approx -T \) around the position \( z \approx -Z \) \((Z > 0)\). \( d \) is introduced in such a way that \( p_{z}^{inc}d \gg 1 \). It can be easily proven that at \( t \approx -T \),

\[
P(p_{z}^{inc}, z, t) = N' \int dk_{z} A(k_{z}, p_{z}^{inc}) \psi_{n}^{inc}(z),
\]

where \( N' \) is a new normalization constant and \( n = k_{x}, k_{y}, j_{z}, E_{k}, r \). When the terms exponentially suppressed by \( e^{-(p_{z}^{inc}d)^2} \) are neglected as in the second line of (80), we obtain

\[
A(k_{z}, p_{z}^{inc}) = e^{-(p_{z}^{inc}-k_{z})^2 \frac{d^2}{2} e^{i(Z-t-T)k_{z}}.}
\]

In other words, an incoming wave packet located far away from the wall in the unbroken phase, with group velocity pointing towards the wall, totally expands on the eigenstates called incoming. In [6] this general conclusion for any wall profile is stated for the particular case of the thin wall (step profile function). Analogous conclusions hold for the wave packets coming from the broken phase and for the eigenstates we called broken incoming \((\psi_{n}^{br})\). Obviously the superposition of these two incoming wave packets located far from the wall, to the left and to the right, is zero at initial time \( -T \). As they evolve in the time, for \( t = 0 \), they coincide in the same spatial region but their overlap integral keeps on being zero, since the Hamiltonian being hermitian the time evolution operator is unitary. It follows that the superposition of the two wave packets located in a certain region centered at \( z = 0 \), which extends a distance of the order of \( d \), is zero. If the limit \( d \to \infty \) is taken now, the gaussian with the appropriate normalization constant gives a delta function in (82), and the overlap of the two wave packets becomes the overlap of \( \psi_{n}^{inc} \) and \( \psi_{n}^{br} \). Obviously, \( d \) and \( Z \) go to \( \infty \) so that \( Z/d \gg 1 \), our argument being valid in this limit. From the vanishing of that overlap we conclude to the orthogonality of these eigenfunctions.
References


FOOTNOTES

1 In these references the final expressions for $R(E)$ and $\overline{R}(E)$ are equivalent to (9), although the notation is different. Moreover, in these references $z_0$ is taken to be 0.

2 This is a reasonable assumption. By considering the appropriate dimensions we have that $\frac{\hbar c}{\delta_w} = O[10\text{GeV}]$ (see for example [18],[5]) and therefore $a = \frac{2E\delta_w}{\hbar c} = O[10]$ in the electroweak phase transition energy scale. For this range of energy $(\frac{E}{m_0})^2 \gg a$ except for the top mass.

3 As known the function $\arctan(x)$ defined from $R$ to $(0, 2\pi)$ is bivalued. The right-hand of the second equation in (29) is well-defined by considering the sign of $(F^{(k)}(1) + F^{(k)}(-1))\sin a$ to determinate if the angle belongs to $(0, \pi)$ or to $(\pi, 2\pi)$. If it is zero, $(F^{(k)}(1) - F^{(k)}(-1))\cos a$ must be considered in order to determine if the angle is 0 or $\pi$.

4 In ref. [24] we use analytic profiles which behave as $e^{C_-x^{2n+1}}$, $e^{-C_+x^{2n+1}}$, for $x << -1$, $x >> 1$, respectively. The results which will be derived in what follows can be, in the same way, obtained for these profiles.

5 The position of the closest pole to the real axis, $y_j$, gives us the smallest value of $\lambda$ we can consider in order to be able to neglect the contribution of $R_{N-C}$. We have $\lambda \gg \frac{2E\sigma}{C_+y_j}$, $\frac{2E\sigma}{C_-y_j}$. As it will be seen the term $R_{N-C}$ is associated to the fact of considering the evolution in the region $(-\delta_w, \delta_w)$ and introducing the technical assumption mentioned above.

6 In order to compare results the relationship $a = 1/\sigma$, where $a$ is the energy parameter used in ref. [19], must be taken into account.