ℓ^p INTERPOLATION AND OPTIMIZED BOUNDS ON PAIRWISE INTERACTING FERMION SYSTEMS

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ABSTRACT

We derive a set of inequalities which relate the translation invariant problem of N identical particles with pairwise interactions to an independent particle problem. These inequalities apply to attractive power law potentials $V(r) = r^q$, $1 \leq q \leq \infty$, and superpositions of such potentials; they become identities in the harmonic oscillator case $q = 2$. We use the inequalities to derive new upper and lower bounds for the ground state energies of fermion systems, which interact through these potentials. These bounds improve all previous results in the range $1 < q \leq \infty$, they reduce to the exact answer in the harmonic oscillator case.

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Since the pioneering works of Fischer and Ruelle on the thermodynamic limit [1], of Dyson and Lenard on the stability of matter [2], and of Lévy-Leblond on the non-saturation of gravitational forces [3], a great deal of interest has been devoted to the derivation of lower bounds for the ground state energies of many particle systems in terms of two particle spectra.

In Ref. [4] we have shown that for boson systems, or particles of different masses, previous bounds could be considerably improved by a simple technique. The improved bounds coincide with the exact answer for harmonic interactions. This has a number of applications, for instance in relating baryon to meson masses in the quark model.

Pairwise interacting N fermion systems present a variety of interesting applications. In that case, however, the methods introduced in Ref. [4] are not able to produce any improvements to the bounds obtained by the Fischer-Ruelle-Dyson-Lenard-Lévy-Leblond (FRDL3) [1]-[3] technique. This is quite frustrating in the harmonic oscillator case which can be solved exactly, and where the FRDL3 bound differs from the exact result by a factor of $\sqrt{2}$. Similarly, the attractive Coulomb case can be calculated for large N with the Thomas-Fermi method. In that case, the FRDL3 result is off by a factor of ~ 3 .

One reason for this difficulty in improving the FRDL3 bounds can be traced back to the known difficulty of constructing an orthogonal set of N particle wave functions which satisfies the three conditions of being completely antisymmetric, translationally invariant, and of being eigenfunctions of an independent particle Hamiltonian. This can only be done in the harmonic case.

In this paper, we present a new approach to the N fermion problem with pairwise interactions which deals with the particular class of potentials

$$V(r) = \int_1^\infty r^q \mu(q) dq \quad \mu \geq 0$$

Although, this class does not include some cases of practical interest such as the Coulomb and Yukawa potentials, or the quark-quark confining potentials, the results are nevertheless very interesting insofar as 1) they systematically improve the FRDL3 results; 2) they produce both upper and lower bounds; and 3) they reduce to the exact answer in the harmonic case.

The derivation of these new bounds relies on the existence of a set of inequalities which are interesting as such. The proof of these inequalities is non-trivial, it requires the techniques of ℓ^p interpolation, and the Riesz-Thorin interpolation theorem. For the class of potentials under consideration, these inequalities allow to relate an N -body problem with pairwise interactions and translation invariance, to an independent particle problem whose solution is known.

In Section 1, we state the problems under consideration and recall previous bounds. In Section 2, we use the inequalities to derive the upper and lower bounds on fermion systems, and we draw our first conclusions. Finally, in Section 3, we give the derivation of the inequalities themselves.

1 Position of the Problem

Consider a system of N identical particles with pairwise interactions $V_{ij} \equiv V(r_i - r_j)$, of Hamiltonian

$$H^N = \sum_{i=1}^N p_i^2 + \sum_{i=1}^N \sum_{j=i+1}^N V_{ij} \quad (1)$$

for simplicity we have set the mass equal to $1/2$.

We denote $\epsilon_n(\mu)$ the energy levels of a particle of mass $\mu/2$ placed in the potential $V(r)$ centered at the origin, of Hamiltonian $h = p^2/\mu + V(r)$, by g_n their degeneracies, and by $f_N(\mu)$ the ground state energy of N independent fermions, of spin s and mass $\mu/2$, each placed in this potential, of Hamiltonian

$$\tilde{H}^N = \sum_{i=1}^N h_i = \sum_{i=1}^N (p_i^2/\mu + V(r_i)) \quad (2)$$

We have

$$f_N(\mu) = (2s+1) \sum_{n=1}^k g_n \epsilon_n(\mu) + \tau \epsilon_{k+1}(\mu), \quad (3)$$

where the integer k is defined in terms of N by

$$(2s+1) \sum_{n=1}^k g_n + \tau = N < (2s+1) \sum_{n=1}^{k+1} g_n, \quad \tau \geq 0. \quad (4)$$

In the Coulomb case, $V = -1/r$ we have $\epsilon_n(\mu) = -\mu/4n^2$, $g_n = n^2$, and, for large N , $f_N(\mu) \approx -(\mu/4) (2s+1)^{2/3} (3N)^{1/3}$.

In the three-dimensional harmonic oscillator $V = r^2$, $\epsilon_n(\mu) = (2n+1)/\sqrt{\mu}$, $g_n = n(n+1)/2$ and, for large N , $f_N(\mu) \approx (6N)^{4/3} / ((2s+1)^{1/3} 4\sqrt{\mu})$.

Following an argument due originally to Fischer and Ruelle [1], one can obtain a lower bound on the N -particle ground state energy E^N , in terms of the two-particle spectrum, by rewriting the Hamiltonian (1) as a sum of N terms

$$H^N = \frac{1}{2} \sum_{i=1}^N H_i \quad (5)$$

$$H_i = \sum_{j=1, j \neq i}^N \left(\frac{2p_{ij}^2}{(N-1)} + V_{ij} \right) \quad (6)$$

If $|\Omega\rangle$ is the N particle ground state, $E^N = \langle \Omega | H^N | \Omega \rangle$, and therefore

$$E^N = \frac{N}{2} < \Omega | H_i | \Omega > \quad (7)$$

In eq. (6), H_i appears as the Hamiltonian of $N-1$ independent particles of mass $\mu/2 = (N-1)/4$, placed in the field of particle i considered as infinitely massive. Since $\langle H_i \rangle$ is larger than the ground state energy of this system, one obtains the desired lower bounds.

For bosons, we have

$$E_B^N \geq \frac{N(N-1)}{2} \epsilon_1(\mu = (N-1)/2), \quad (8)$$

which yields in the Coulomb and harmonic cases

$$E_B^N \geq 3N \sqrt{(N-1)/2} \quad (V = r^2) \quad (9a)$$

$$E_B^N \geq -N(N-1)^{3/2}/16 \quad (V = 1/r). \quad (9b)$$

In the case of fermions, the antisymmetry of $|\Omega\rangle$ imposes to retain only those levels of H_i which correspond to antisymmetric states in the $N-1$ particles $j = 1 \dots N, j \neq i$. The minimal requirement is therefore to distribute evenly the $N-1$ particles over the $N-1$ lowest eigenstates of the one-particle Hamiltonian. One thus obtains

$$E_F^N \geq \frac{N}{2} f_{N-1} \quad (\mu = (N-1)/2) \quad (10)$$

For large N , in the harmonic and Coulomb cases respectively, one obtains

$$E_F^N \geq \frac{6^{4/3}}{2^{1/24} (2s+1)^{1/3}} N^{11/6} \quad (V = r^2) \quad (11a)$$

$$E_F^N \geq -\frac{(2s+1)^{2/3} 3^{1/3}}{16} N^{7/3} \quad (V = -1/r) \quad (11b)$$

These bounds are off by factors of $\sqrt{2}$ (harmonic case which is exactly soluble) and ≈ 3 (Coulomb case which can be treated by the Thomas-Fermi method for large N).

In the case of bosons (more generally, non-identical particles), we have improved the above bounds [4] in what we believe is an optimal way.

We use the identity

$$\sum_{i=1}^N p_i^2 = \sum_{1 \leq i < j \leq N} (p_i - p_j)^2 + \sum_{i=1}^N p_i^2 \quad (12)$$

valid for any set of N vectors p_i , $i = 1, \dots, N$ of R^k , to separate out the centre-of-mass motion in (1) as

$$H^N = H_{\text{rel}}^N + P^2/N \quad (13)$$

where the relative Hamiltonian is

$$H_{\text{rel}}^N = \sum_{1 \leq i < j \leq N} \left(\frac{1}{N} (p_i - p_j)^2 + V_{ij} \right) \quad (14)$$

and $P = \sum p_i$ is the total momentum.

In the N body centre-of-mass frame $<P> = 0$ (this can be made rigorous by adding, e.g., a term $\alpha^2(\sum r_i/N)^2$ which commutes with H_{rel}^N and letting α^2 tend to zero), and therefore

$$E_F^N = \frac{N(N-1)}{2} < \Omega \left| \left(\frac{4}{N} \left(\frac{p_i - p_j}{2} \right)^2 + V_{ij} \right) \right| \Omega > \quad (15)$$

Since $(p_i - p_j)/2$ and $(r_i - r_j)$ satisfy canonical commutation relations, one obtains, for bosons

$$E_F^N \geq \frac{N(N-1)}{2} \epsilon_1(\mu = N/4) \quad (16)$$

which improves (8) since $\epsilon_n(\mu)$ increases with $1/\mu$.

This new lower bound becomes the exact answer for a harmonic interaction: $E_F^N = 3(N-1)\sqrt{N}$, and we have shown [4] that it is very close to the exact answer for the power-law potentials $V = r^q$, $-1 \leq q \leq 5$, including the Coulomb case (we thank J.M. Richard for making new tests for $q > 3, N = 3$).

In fermion systems, however, this procedure does not help. We can of course perform a decomposition of H^N similar to (5), (6)

$$H^N = \frac{1}{2} \sum_{i=1}^N H_i \quad (17)$$

$$H_i = \sum_{j=1, j \neq i}^N \left(\frac{4}{N} \left(\frac{p_j - p_i}{2} \right)^2 + V_{ij} \right) \quad (18)$$

but H_i is no longer an independent particle Hamiltonian, and it cannot be diagonalized in general. In fact, although $(p_j - p_i)/2$ and $(r_j - r_i)$ have canonical commutation relations, $(p_j - p_i)$ and $(r_k - r_i)$ do not commute.

H_i can be viewed as describing the motion of $N-1$ independent particles of mass $N/2$, placed in the field of a much lighter particle i of mass $N/(2(N+1))$. This problem has no simple solution in general. The difficulty we meet here is also related to the known difficulty of constructing fully antisymmetric N -particle wave functions which are both eigenstates of an $N-1$ independent particle Hamiltonian and translationally invariant.

No recasting of the kinetic term, such as (12) or the many analogous formulas one can write, can do the job, and it seems impossible to improve the fermionic bound (10) in a simple way without further specifying the potential.

This situation is rather frustrating with regards to the harmonic interaction, where we know the exact answer and the bound (10) is off by a factor of $\sqrt{2}$.

A naïve conjecture, following from the inspection of Eq. (14), and assuming the first $N-1$ levels of the Hamiltonian $(4/N)p^2 + V(r)$ are equally populated, such as what would occur in

a Slater determinant, would lead to

$$E_F^N \geq \frac{N}{2} f_{N-1} \quad (\mu = N/4) \quad (19)$$

which, in the harmonic case, reduces to $\sqrt{N} f_{N-1}(1)$, giving the right answer asymptotically in N . However, it does not seem possible to prove such a conjecture in a simple way.

2 Derivation of new fermionic bounds

Improved bounds on fermion systems may be obtained by further specifying the potential.

In what follows, we restrict ourselves to power-law potentials

$$V(r) = r^q \quad 1 \leq q \leq \infty \quad (20)$$

which include the harmonic case $q = 2$ and the infinite square well $q = \infty$, and superpositions of such potentials.

It is instructive to recall the solution of the harmonic case. Consider the independent particle Hamiltonian

$$\tilde{H}^N = \sum_{i=1}^N p_i^2 + N \sum_{i=1}^N r_i^2 \quad (21)$$

the ground-state energy of the totally antisymmetric solution is

$$\tilde{E}^N = \sqrt{N} f_N \quad (22)$$

[from now on we write f_N instead of $f_N(1)$ since we know that the eigenvalues of $ap^2 + br^q$ scale as $a^q/(q+2)p^{2/(q+2)}$, and therefore $f_N(\mu)$ can be expressed in terms of $f_N(1)$].

Using the identity (12) both for the kinetic and potential terms, we obtain

$$\tilde{H}^N = H_{\text{rel}}^N + \left(\frac{P^2}{N} + N^2 X^2 \right) \quad (23)$$

where $X = \sum r_i/N$ is the centre of gravity of the system. The second term on the right-hand side of (23) is a one-particle Hamiltonian which commutes with H_{rel}^N . Therefore the ground state energy of $\tilde{H}^N$ is the sum of ground state energies $\tilde{E}^N = E^N + \sqrt{N} \epsilon_1$, that is to say

$$E^N = \sqrt{N} (f_N - \epsilon_1) \quad (24)$$

In order to derive upper and lower bounds for fermion systems with $V(r) = r^q$ potentials, we will follow a similar procedure, and we will make use of the following inequalities.

Consider N vectors $r_i, i = 1, \dots, N$ of R^3 , and $1 \leq q \leq \infty$, then

$$4 \sum_{1 \leq i < j \leq N} \left| \frac{r_i - r_j}{2} \right|^q + N^2 \left| \frac{\sum_{i=1}^N r_i}{N} \right|^q \geq N \sum_{i=1}^N |r_i|^q \quad (25)$$

$$\sum_{1 \leq i < j \leq N} |r_i - r_j|^q + \left| \sum_{i=1}^N r_i \right|^q \leq N \sum_{i=1}^N |r_i|^q \quad (26)$$

where the upper inequality holds for $1 \leq q \leq 2$, and the lower one for $2 \leq q \leq \infty$; $|r|$ is the Euclidean norm of $r \in R^3$.

The proof of these inequalities is given in the next section. Notice that both of them become identities for $q = 2$ since they reduce to (12).

The physical interest of such inequalities lies in the fact that they allow us to relate a problem of translationally invariant pairwise interaction to an independent particle problem.

Indeed, both inequalities are of the form

$$\sum_{1 \leq i < j \leq N} |r_i - r_j|^q + a |X|^q \geq b \sum_{i=1}^N |r_i|^q \quad (27)$$

where $X = \sum r_i/N$ is the position of the centre of gravity. We therefore have the operator inequalities

$$\sum_{i=1}^N p_i^2 + \sum_{i=1}^N |r_i - r_j|^q + a |X|^q \geq \sum_{i=1}^N (p_i^2 + b |r_i|^q) \quad (28)$$

The right-hand side of (28) is an independent-particle Hamiltonian whose (antisymmetric) ground state energy is

$$\tilde{E}_f^N = b \frac{N^2}{4} f_N \quad (29)$$

Using (12) the left-hand side can be rewritten as the sum of two commuting Hamiltonians, the Hamiltonian we are interested in H_{rel}^N and a one-particle Hamiltonian

$$h = \frac{p^2}{N} + a |X|^q \quad (30)$$

whose ground state energy is $a^{2/(q+2)} N^{-q/(q+2)} \epsilon_1$. H_{rel}^N and h commute, their eigenstates factorize, and their eigenvalues add.

Since (28) is an operator inequality we therefore have the bounds

$$E_f^N + a^{2/(q+2)} N^{-q/(q+2)} \epsilon_1 \geq b^{2/(q+2)} f_N. \quad (31)$$

Inserting the actual values of the coefficients a and b , as can be read of from Eqs. (25) and (26), we obtain the following upper and lower bounds for the ground state energy E_1^N of a N fermion system with pairwise interactions $|r_i - r_j|^q$. For $1 \leq q \leq 2$:

$$2 \frac{2(q-2)}{q+2} \left[N \frac{2}{q+2} f_N - N \frac{1}{q+2} \epsilon_1 \right] \leq E_1^N \leq N \frac{2}{q+2} f_N - N \frac{1}{q+2} \epsilon_1 \quad (32)$$

and, for $2 \leq q \leq \infty$, the inequalities are reversed. At $q = 2$, we recover the exact harmonic oscillator results.

Since, for $V = r^q$, $f_N \sim N^{(q+5)/(q+6)}$ as $N \rightarrow \infty$ in three dimensions, as it can be shown from semi-classical estimates valid in the large N limit for power potentials, the f_N terms dominate over the ϵ_1 terms at large N , and, for simplicity, we concentrate on this limit for the discussion:

$$2 \frac{2(q-2)}{q+2} \leq \lim_{N \rightarrow \infty} \left[E_f^N / N \frac{2}{q+2} f_N \right] \leq 1 \quad (33)$$

the upper and lower inequalities corresponding respectively to $1 \leq q \leq 2$ and to $2 \leq q \leq \infty$.

For $q = 1$, the upper and lower bounds differ by a factor $2^{2/3} = 1.59$; the lower bound is the same as the previous result (10). For $q = \infty$ (infinite square well) the upper and lower bounds differ by a factor of 4 and again the lower bound coincides with (10).

In all the range $1 < q < \infty$, the previous result (10) is improved. The maximal improvement occurs for the harmonic case $q = 2$, where these bounds reduce to the exact answer.

This approach provides both upper and lower bounds.

We shall exploit this fact in turning to the more general class of potentials which are linear superpositions of power law potentials (20)

$$V(r) = \int_1^\infty r^q \mu(q) dq \quad (34)$$

In principle, since we have both upper and lower inequalities in the two ranges $1 \leq q \leq 2$ and $2 \leq q \leq \infty$ we should be able to write upper and lower bounds for this class of potentials.

Consider first the simplest case which corresponds to linear superpositions (34) with positive weights and $q \geq 2$

$$V(r) = \int_2^\infty r^q \mu(q) dq \quad \mu(q) \geq 0 \quad (35)$$

a particular case of which is entire functions with positive expansion coefficients and no linear term (a constant can always be subtracted out).

From the inequalities (25) and (26), we readily obtain, writing down (35) on both sides,

$$\sum_{1 \leq i < j \leq N} V(r_i - r_j) + V\left(\sum_{i=1}^N r_i\right) \geq N \sum_{i=1}^N V(r_i) \quad (36)$$

$$\sum_{i=1}^N V(r_i - r_j) + \frac{N^2}{4} V\left(\frac{2 \sum_{i=1}^N r_i}{N}\right) \leq \frac{N}{4} \sum_{i=1}^N V(2r_i) \quad (37)$$

Adding the kinetic terms and taking into account that the eigenvalues of $ap^2 + bV = b(a/b p^2 + V)$ are directly expressed in terms of those of V : $\epsilon_n(\mu = b/a)$, and that multiplying r by a constant is equivalent to rescaling the mass by the inverse squared of this constant, one obtains the bounds

$$E_f^N \geq N f_N(\mu = N) - \epsilon_1(\mu = 1/N) \quad (38)$$

$$E_F^N \leq \frac{N}{4} f_N(\mu = \frac{N}{16}) - \frac{N^2}{4} \epsilon_1(\mu = \frac{N}{16}) \quad (39)$$

For potentials of the class

$$V(r) = \int_1^2 r^q \mu(q) dq \quad \mu \geq 0 \quad (40)$$

the signs of these two bounds are simply reversed.

The bounds (38) and (39) are quite remarkable in that they hold for any potential of the classes (35) and (40), and are expressed in terms of the two-particle spectrum. They reduce to the particular cases (32) for power law potentials, and become exact for a harmonic potential.

We notice that, although (38) reproduces the harmonic result, it is different from the naïve conjecture (19). It is only for potentials of the class (35) that (38) is an improvement over the result (10).

For the more general superposition (34) with positive weights, one can obtain similar formulas but these are more complicated to write. The N -body results are expressed in terms of two-body spectra corresponding not to the original potential but to a modified one:

$$\begin{aligned} \tilde{V}(r) &= \int_1^2 \frac{(2r)^q}{4} \mu(q) dq + \int_2^\infty r^q \mu(q) dq \\ &= V(r) - \int_1^2 \left(1 - \frac{2^q}{4}\right) r^q \mu(q) dq \end{aligned} \quad (41)$$

For weights of alternating signs, sticking to the class (35) for simplicity, one can write the potential $V(r)$ as

$$V(r) = V_+(r) - V_-(r) \quad (42)$$

with

$$V_\pm(r) = \int_2^\infty r^q \mu_\pm(q) dq \quad \mu_\pm(q) \geq 0 \quad (43)$$

In order to write lower bounds, for instance, one uses inequality (36) for V_+ and inequality (37) for V_- in order to obtain

$$H^N + \frac{P^2}{N} + V_+(NX) - \frac{N^2}{4} V_-(2X) \geq \sum_{i=1}^N \left[p_i^2 + N V_+(r_i) - \frac{N}{4} V_-(2r_i) \right] \quad (44)$$

Leaving aside the one-particle Hamiltonian for the centre-of-mass variable X in the left-hand side, this inequality allows us to relate H^N to an independent-particle Hamiltonian where the potential is no longer $V(r)$ but the effective potential

$$\tilde{V}(r) = V_+(r) - \frac{1}{4} V_-(2r) \quad (45)$$

In general, however, this is of no use in deriving lower bounds because in many cases of interest $\tilde{V}(r)$ is unbounded from below. Consider, for instance

$$V = -\lambda r^2 \exp(-r) \quad (46)$$

which possesses two-body bound states for λ large enough, and pertains to the class under consideration. For this potential, one has

$$\begin{aligned} V_+(r) &= r^2 \sinh r \\ V_-(r) &= r^2 \cosh r \end{aligned} \quad (47)$$

therefore the effective potential $\tilde{V}(r)$ entering in the right-hand side of (44) is:

$$\tilde{V}(r) = r^2 \sinh r - r^2 \cosh 2r \quad (48)$$

which is unbounded from below.

Upper bounds can nevertheless be obtained in such cases.

Lower bounds can only be obtained if $\mu(q)$ oscillates in a bounded region in q (for instance if V is a polynomial in r).

3 Proof of the Inequalities

We now give the proof of the inequalities (25) and (26). The keypoint in this proof is due to Jean-Michel Bony.

Consider N_1 vectors x_i , $i = 1, \dots, N_1$ of R^3 (or R^k in general) and let $|x_i|$ denote the usual Euclidean norm of x_i .

The set $X = \{x_1, x_2, \dots, x_{N_1}\}$ can be considered as a point in $(R^3)^{N_1}$. In this latter space one defines the ℓ^p norm of X , noted $\|X\|_p$, $p \geq 1$, associated with the positive coefficients $\gamma_1, \gamma_2, \dots, \gamma_{N_1}$ as

$$\|X\|_p = (\gamma_1 |x_1|^p + \gamma_2 |x_2|^p + \dots + \gamma_{N_1} |x_{N_1}|^p)^{1/p} \quad (49)$$

Consider a point $Y = \{y_1, y_2, \dots, y_{N_2}\}$ of $(R^3)^{N_2}$ with the ℓ^p norm $\|Y\|_p = (\sum_{i=1}^{N_2} \delta_i |y_i|^p)^{1/p}$, δ_i positive, and an injective linear transformation L from $X \in (R^3)^{N_1}$ to $Y \in (R^3)^{N_2}$ noted $Y = LX$, such that, for $1 \leq p_2 \leq p_1 \leq +\infty$

$$\|LX\|_{p_1} \leq c_1 \|X\|_{p_1} \quad (50a)$$

$$\|LX\|_{p_2} \leq c_2 \|X\|_{p_2} \quad (50b)$$

The Riesz-Thorin interpolation theorem [5] states that for all p between p_2 and p_1

$$p_1 \geq p \geq p_2 \quad \|LX\|_p \geq C_p \|X\|_p \quad (51)$$

with

$$C_p = C_2^{1-\theta} C_1^\theta \quad (52)$$

θ being given in terms of p, p_1 and p_2 by

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad (53)$$

Returning to our problem, we first consider $X = \{x_1, x_2, \dots, x_N\} \in (R^3)^N$ with the ℓ^p norm

$$\|X\|_p = \left(N \sum_{i=1}^N |x_i|^p \right)^{1/p} \quad (54)$$

and $Y = \{v_{ij}\}, w \in (R^3)^{N'}$ $N' = N(N-1) + 1$ where

$$v_{ij} = \frac{x_i - x_j}{2} \quad w = \sum_{i=1}^N \sum_{j=1}^N \frac{x_i}{N} \quad (55a)$$

the inverse transformation being

$$x_i = \frac{(w + 2 \sum_{j=1}^N v_{ij})}{N} \quad (55b)$$

Equations (55a) and (55b) provide a bijection of $(R^3)^N$ and its image $\mathcal{J}$ in $(R^3)^{N'}$ which is such that

$$v_{ij} + v_{ji} = 0 \quad v_{ij} + v_{jk} + v_{ki} = 0$$

The ℓ^p norm of Y is defined by

$$\|Y\|_p + \left(2 \sum_{i=1}^N \sum_{j=1}^N \sum_{i \neq j} |v_{ij}|^p + N^2 |w|^p \right)^{1/p} \quad (56)$$

For $p = 2$, these norms are equal since they reduce to the identity (12): $\|X\|_2 = \|Y\|_2$.

For $p = 1$, using the identity

$$x_i = \frac{1}{N} \left(\sum_{j=1}^N (x_i - x_j) + \sum_{j=1}^N x_j \right)$$

one has

$$|x_i| \leq \frac{1}{N} \left(\sum_{j=1}^N |x_i - x_j| + \sum_{j=1}^N |x_j| \right)$$

Therefore, by summing over i

$$N \sum |x_i| \leq 2 \sum_{i=1}^N \sum_{j=1}^N \sum_{i \neq j} \left| \frac{x_i - x_j}{2} \right| + N^2 \left| \frac{\sum x_i}{N} \right| \quad (57)$$

In other words, $\|X\|_1 \leq \|Y\|_1$, and by the Riesz-Thorin interpolation theorem

$$\|X\|_p \leq \|Y\|_p \quad 1 \leq p \leq 2$$

which proves the inequality (25) in this case.

For $p > 2$, the inequality is reversed. Indeed

$$\|X\|_\infty \geq \|Y\|_\infty \quad (58)$$

which can be seen as follows. Let $|x_i| = \sup_j |x_j|$. It is obvious that $|x_i| \geq |x_j - x_k|/2 \quad \forall j, k$ and $|x_i| \geq |\sum x_i|/N$.

So that $\sup_j |x_j| \geq \sup(|v_{ij}|, |w|)$.

The only case we need to consider is when all vectors are collinear and of equal moduli. If all vectors are equal, both sides of (58) are equal. If p vectors are equal to x and $(N-p)$ to $-x$, one easily checks that $N \sum |x_i| \geq 2 \sum \sum |v_{ij}|$, this becoming an inequality for $p = N/2$.

Therefore, (58) holds and, since $\|X\|_2 = \|Y\|_2$

$$\|X\|_p \geq \|Y\|_p \quad \text{for } p \geq 2$$

which completes the proof of inequality (25). This inequality is saturated, i.e., it becomes an equality, when either all x_i are equal, or $N/2$ of the x_i are equal to x and the other $N/2$ are equal to $-x$ (N even).

Turning to inequality (26), we now replace (55) by

$$v_{ij} = (x_i - x_j) \quad w = \sum_{i=1}^N x_i \quad (59)$$

and we define the ℓ^p norm of Y by

$$\|Y\|_p = \left(\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{i \neq j} |x_i - x_j|^p + \left| \sum_{i=1}^N x_i \right|^p \right)^{1/p} \quad (60)$$

$\|X\|_p$ being defined as previously.

For $p = 2$ we still have $\|X\|_2 = \|Y\|_2$ of course.

The proof that $\|Y\|_1 \geq \|X\|_1$ is straightforward, we have

$$\begin{aligned} |x_i - x_j| &\leq |x_i| + |x_j| \\ |\sum x_i| &\leq \sum |x_i| \end{aligned}$$

Therefore

$$\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{i \neq j} |x_i - x_j| + \left| \sum_{i=1}^N x_i \right| \leq N \sum_{i=1}^N |x_i|$$

and

$$\|Y\|_p \leq \|X\|_p \quad (61)$$

for $1 \leq p \leq 2$, which proves (26) in that range of values of p .

Notice that we also have for any $\lambda \geq 0$

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N |x_i - x_j|^2 + \lambda^2 \left| \frac{\sum x_i}{\lambda} \right|^2 &\leq (N-1+\lambda) \sum_{i=1}^N |x_i|^2 \\ \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N |x_i - x_j|^2 + \lambda^2 \left| \frac{\sum x_i}{\lambda} \right|^2 &\leq N \sum_{i=1}^N |x_i|^2 \end{aligned}$$

which lead to a set of λ -dependent inequalities valid for $1 \leq p \leq 2$ where one makes full use of the Riesz-Thorin interpolation theorem, in particular Eq. (52).

In practice, we have found out that, by optimizing in λ , one does improve the upper bound in the range $1 \leq p \leq 2$, but not asymptotically in N . One can readily check that, for $p = 1$, the upper bound can be brought down to $E_F^N \leq (N-1)^{2/3} (f_N^2 - N\epsilon_1^2)^{1/3}$ instead of $E_F^N \leq N^{2/3} f_N - N^{1/3}\epsilon_1$. The improvement is only felt for small values of N .

Finally, for $p \rightarrow +\infty$ we want to prove that

$$\left(\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N |x_i - x_j|^p + \left| \sum_{i=1}^N x_i \right|^p \right)^{1/p} \geq N^{1/p} \left(\sum_{i=1}^N |x_i|^p \right)^{1/p} \quad (62)$$

Let x_1 be one of the vector of largest norm $|x_1| \geq |x_i|$ $i = 2, \dots, N$. If one other vector, x_2 , of non-vanishing norm points in the backward hemisphere of x_1 , that is to say $x_1 \cdot x_2 / |x_1| |x_2| < 0$, then $|x_1 - x_2| \geq |x_1|$ and the inequality holds.

If all vectors x_i , $i \neq 1$ point in the forward hemisphere of x_1 and at least one is of non-zero norm, then $|\sum x_i| > |x_1|$ and the inequality holds.

If all vectors vanish except x_1 , both sides of (62) are equal.

Therefore, $\|X\|_\infty \leq \|Y\|_\infty$, and $\|X\|_p \leq \|Y\|_p$ for $p \geq 2$, which completes the proof of inequality (26). This inequality is saturated when all x_i vanish except one of them.

One can derive, in a similar manner, a variety of other inequalities, for instance by using Jacobi variables.

The Riesz-Thorin interpolation theorem actually applies to a much larger class of problems since it applies to measure spaces, with positive measure, in general. Notice for instance, by dividing Eq. (25) by N^2 , one can rewrite it in integral form as

$$2 \iint \left| \frac{x(u) - x(v)}{2} \right|^p d\mu(u) d\mu(v) + \left| \int x(u) d\mu(u) \right|^p \geq \int |x(u)|^p d\mu(u)$$

where we have introduced a positive measure such that $\int d\mu(u) = 1$. In order to recover (25) we set

$$d\mu(u) = \frac{1}{N} \sum_{i=1}^N \delta(u - x_i) du.$$

In this more general formulation, these inequalities may turn out to be useful in dealing with systems with an infinite number of degrees of freedom.

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