Five-Dimensional Supersymmetric Gauge Theories and Degenerations of Calabi–Yau Spaces

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We discuss five-dimensional supersymmetric gauge theories. An anomaly renders some theories inconsistent and others consistent only upon including a Wess–Zumino type Chern–Simons term. We discuss some necessary conditions for existence of nontrivial renormalization group fixed points and find all possible gauge groups and matter content which satisfy them. In some cases, the existence of these fixed points can be inferred from string duality considerations. In other cases, they arise from M-theory on Calabi–Yau threefolds. We explore connections between aspects of the gauge theory and Calabi–Yau geometry. A consequence of our classification of field theories with nontrivial fixed points is a fairly complete classification of a class of singularities of Calabi–Yau threefolds which generalize the “del Pezzo contractions” and occur at higher codimension walls of the Kähler cone.

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1. Introduction

It has recently been argued that string theory duality predicts the existence of new, nontrivial fixed points of the renormalization group for a variety of field theories in a variety of dimensions. The field theories arise on D-brane probes in string theory, whose dynamics must reflect aspects of string duality [1,2,3]. For example, in [4] it was thus argued that there are nontrivial fixed points in five-dimensional supersymmetric gauge theories. In particular, it was argued in [4] that five-dimensional supersymmetric $SU(2)$ gauge theory has strongly coupled nontrivial fixed points for $n_F \leq 7$ quarks in the fundamental representation. In [4,5] it was conjectured, more generally, that nontrivial fixed points exist when a certain one-loop quantity has a particular sign, a generalization to higher dimensions of the notion of asymptotic freedom. We will discuss this quantity and the conditions for a nontrivial fixed point for general five-dimensional supersymmetric gauge theories. We provide a rather complete classification of all possible nontrivial five-dimensional fixed points of gauge-theoretic origin.

As discussed in [6,7], there is a direct correspondence between the nontrivial fixed points which exist for five-dimensional $SU(2)$ gauge theory with $n_F \leq 7$ quarks and the mathematical classification of a particular kind of singularity which can occur in Calabi–Yau threefolds at codimension-one boundary walls of the Kähler cone: the collapse of a “del Pezzo surface” with $n_F \leq 7$ blown-up points. The correspondence follows via compactification of M-theory to five dimensions. The generalization to higher codimension boundary walls involves the study of the class of singularities known as isolated canonical singularities. Although quite a bit is known about the structure of these singularities [8], to give a complete classification would appear to be a very challenging mathematical problem. However, for those isolated canonical singularities which arise from contraction of a curve of singularities (and so correspond to the strong coupling limit of a gauge theory) we can and do give a fairly complete classification. Our classification omits cases such as the $E_0$ theory of [6] which do not arise in this way.

In F-theory on a Calabi–Yau threefold to six dimensions there are strings which couple to two-form gauge fields $B_{\mu\nu}$ and can become tensionless at singularities on the moduli space of the Calabi–Yau [9,10]. For example, this is the situation with the small $E_8$ instanton [11,9]. Upon reduction to five dimensions of the small $E_8$ instanton, the two-form $B_{\mu\nu}$ yields a gauge field $A_\mu$, which is part of an $SU(2)$ gauge group which has $n_F = 7$ flavors. In other cases, there can be, already in six dimensions, non-Abelian gauge fields in
addition to the two-form $B_{\mu\nu}$. It is an interesting possibility that these gauge fields could combine in five dimensions with that obtained upon reduction of the $B_{\mu\nu}$ into a larger non-Abelian gauge group, leading to fixed points of the type considered in this paper.

In the next section, we discuss general aspects of five-dimensional supersymmetric gauge theories and argue that there is an anomaly which renders some theories inconsistent and other theories consistent only upon including a Chern–Simons term.

In sect. 3 we present a general expression for the quantum prepotential on the Coulomb branch. We discuss the general necessary condition for there to be a nontrivial renormalization group fixed point: the prepotential should be a convex function over the entire Coulomb branch. This is the generalization to higher-rank gauge groups of the condition discussed in [4] for the $SU(2)$ case.

In the following few sections we classify the various gauge groups and matter content which satisfy this condition.

In sect. 4 we consider $Sp(N)$ gauge theory. Because the fundamental representation is pseudo-real, classically half-hypermultiplets in the fundamental are possible. However, the anomaly implies that the theory is only consistent when the number of hypermultiplets is integral. We argue that there can be nontrivial fixed points only with matter in the antisymmetric tensor and/or fundamental representations, with $n_A \leq 1$ antisymmetric tensors; for $n_A = 1$, the number of fundamentals is $n_F \leq 7$, while for $n_A = 0$ it is $n_F \leq 2N + 4$. The existence of nontrivial fixed points for $n_A = 1$ or $n_A = 0$ with $n_F \leq 7$ follows from stringy probe considerations, as in [4]. The existence of the other cases is demonstrated in sect. 9.

In sect. 5 we consider $SU(N)$ gauge theory. Here we can, and sometimes must, add a bare cubic prepotential with coefficient $c_{cl}$. The anomaly restricts $c_{cl}$; for example, with an odd number of matter flavors in the fundamental representation, $c_{cl}$ must be half of an odd integer, and thus can not be zero. We classify the theories which satisfy the necessary convexity condition for a non-trivial fixed point. For general $N$, there can only be matter in the fundamental representation, with the number of flavors and $c_{cl}$ restricted by $n_F + 2|c_{cl}| \leq 2N$. For $N \leq 8$ there can also be fixed points with $n_A = 1$ matter field in the antisymmetric tensor representation and $n_F$ in the fundamental provided $n_F + 2|c_{cl}| \leq 8 - N$. For $N = 4$ there can also be a fixed point with $n_A = 2$ matter fields in the antisymmetric tensor representation and $n_F = c_{cl} = 0$.

In sect. 6 we consider $Spin(M)$ gauge theories, showing that there can be fixed points with $n_V \leq M - 4$ matter fields in the vector representation. For $M \leq 12$ there can also be
fixed points with \( n_S \leq 2^{6-\frac{1}{2}M} \) (\( n_S \leq 2^{5-\frac{1}{2}(M-1)} \)) matter fields in the spinor representation for \( M \) even (odd) and \( n_V \leq M - 4 \).

In sect. 7 we consider the exceptional gauge groups. For \( G_2 \) there can be a nontrivial fixed point for \( n_7 \leq 4 \) fundamentals. For \( F_4 \), there can be a fixed point for \( n_{26} \leq 3 \) fundamentals. For \( E_6 \), there can be a fixed point for \( n_{27} \leq 4 \) fundamentals. For \( E_7 \), there can be a fixed point for \( n_{56} \leq 3 \) (\( n_{56} \) can be half-integral) fundamentals. For \( E_8 \), there can only be a fixed point for the theory without matter fields.

In sect. 8 we discuss the Calabi–Yau interpretation of these results. The connection is made via \( M \) theory. Determining a detailed correspondence between configurations of surfaces on Calabi–Yau threefolds and gauge theories with specified matter content occupies us for the bulk of this section. We compute the prepotential from the Calabi–Yau side, and check that it agrees with the one calculated in gauge theory. This leads to some miraculous formulas for the cubic intersection form among certain divisors on Calabi–Yau threefolds, as anticipated in [12].

Finally, in sect. 9 we study strong coupling limits of the Calabi–Yau theories. We are not able to settle the question of the existence of a strong coupling limit in all cases, but we do find examples in which such limits exist, enabling us to conclude that many of the exotic strong-coupling points—such as \( Sp(N) \) with \( n_A = 0 \) and \( n_F \leq 2N + 4 \)—do in fact occur.

2. General Aspects of Supersymmetric Gauge Theories in Five Dimensions

On the Coulomb branch of the moduli space, the real scalar \( \Phi \) of the vector multiplet \( A \) gets expectation values in the Cartan sub-algebra of the gauge group \( G \), breaking \( G \) to the Cartan \( U(1)^r \), with \( r = \text{rank}(G) \). The Coulomb branch is thus a Weyl chamber, which is a *wedge* subspace of \( \mathbb{R}^r \) parameterized by Cartan scalars \( \phi^i \) in \( \mathbb{R}^r / \mathcal{W} \) (\( \mathcal{W} \) is the Weyl group of \( G \)), the expectation values of the massless Cartan \( U(1)^r \) vector multiplets \( A^i \).

Away from the origin on the Coulomb branch the low-energy effective Abelian theory is characterized by a prepotential \( \mathcal{F}(A^i) \), which is at most cubic locally in the \( A^i \). There is enhanced gauge symmetry on the walls of the Weyl chamber.

There can also be matter hypermultiplets, “quarks,” in a representation \( \oplus_f r_f \), where \( r_f \) are irreducible representations of \( G \). In addition to the Coulomb branch, there can be a Higgs branch associated with expectation values of the scalars in the matter hypermultiplets. The Higgs branch is hyper-Kähler and generally intersects the Coulomb branch.
along a subspace where certain $\phi^i$, those associated with photons which get Higgsed, are set to zero. And, as in four dimensions [13], there is a lack of neutral couplings between vector multiplets and hypermultiplets which implies that the gauge coupling eigenvalues for the remaining $\phi^i$ are independent of the expectation value on the Higgs branch. Also, as in four dimensions [14], because the gauge coupling can be regarded as a background expectation value of a vector multiplet, the Higgs branch does not receive quantum corrections. We thus focus on the Coulomb branch.

An important note about kinematics in five dimensions is in order here. One symmetry of our problem is parity $P: x^\mu \rightarrow -x^\mu$, combined with changing the sign of the scalars (and mass terms). When these theories are considered as dimensional reductions of the six-dimensional theory, this symmetry is understood as part of the six-dimensional Lorentz group. Also, there is a charge conjugation symmetry, $C$. Mass terms, which are like background gauge fields, change sign under charge conjugation. Thus, with non-zero masses, only the product $CP$ is a symmetry of the classical Lagrangian.

Finally, we note that for the particular case of gauge group $G = Sp(N)$ with no massless matter fields in the fundamental representation, there is an additional discrete parameter needed to specify the quantum theory. As in [7], this parameter can be interpreted as a $Z_2$ valued theta angle associated with $\pi_4(Sp(N)) = Z_2$; this is analogous to the theta angle of 4d gauge theories, which is associated with $\pi_3(G)$. When there are massless matter fields in the fundamental representation of $Sp(N)$, the discrete parameter is not physical, as the theta angle can then be rotated away. When the matter fields in the fundamental representation are massive we can replace this parameter with the sign of the determinant of the mass matrix. For gauge groups other than $Sp(N)$, $\pi_4$ is trivial and thus there is no theta parameter.

2.1. Abelian gauge theory and the Wess–Zumino term

The prepotential of a $U(1)^r$ gauge theory in five dimensions is a cubic polynomial in $\phi^i$. In every open set in the moduli space it is of the form

$$F = \frac{t^{(0)}_{ij}}{2} \phi^i \phi^j + \frac{c_{ijk}}{6} \phi^i \phi^j \phi^k.$$  \hspace{1cm} (2.1)

In different open sets the coefficients $t^{(0)}$ and $c$ can be different but

$$a_{D_i} = \partial_i F$$
$$t(\phi)_{ij} = \partial_i \partial_j F$$  \hspace{1cm} (2.2)

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must be continuous. $\mathcal{F}$ leads to an effective gauge coupling proportional to

$$ t(\phi)_{ij} F^i F^j, \quad (2.3) $$

a metric on the moduli space proportional to

$$ (ds)^2 = t(\phi)_{ij} d\phi^i d\phi^j, \quad (2.4) $$

and a Chern–Simons term

$$ \frac{c_{ijk}}{24\pi^2} A^i \wedge F^j \wedge F^k. \quad (2.5) $$

Our definition of $c_{ijk}$ here differs by a factor of two from the conventions in [4,6,15].

Gauge invariance restricts the coefficients $c_{ijk}$ because (2.5) is not gauge invariant but the integral of (2.5) should be well defined modulo $2\pi i \ell$, with $\ell \in \mathbb{Z}$. For the special case of $r = 1$ (one-dimensional moduli space) gauge invariance on a generic five-manifold thus quantizes $c \in 6\mathbb{Z}$. However, on the special manifolds which occur in M-theory [16]

$$ c \in \mathbb{Z}. \quad (2.6) $$

For any $r$, the condition of $c_{ijk}$ can be obtained by writing the integral of (2.5) over a five-manifold $W$ as the integral over a six-manifold $Y$ with $W = \partial Y$,

$$ 2\pi \frac{c_{ijk}}{6} \int_{Z_6} (\frac{F^i}{2\pi}) \wedge (\frac{F^j}{2\pi}) \wedge (\frac{F^k}{2\pi}) \equiv 2\pi i \frac{c_{ijk}}{6} c_1(L_i)c_1(L_j)c_1(L_k), \quad (2.7) $$

which should be $2\pi i \ell$, with $\ell \in \mathbb{Z}$. The quantization condition (2.6) follows because $c_1(L)^3$ is divisible by six for the special manifolds which occur in M-theory [16]. More generally, for $U(1)^r$, $c_1(L)^3$ is divisible by six for any $L$; writing $L = L_i + L_j + L_k$ and expanding, it follows that $c_1(L_i)^3$ is divisible by six, $c_1(L_i)^2 c_1(L_j)$ is divisible by two for $i \neq j$, and $c_1(L_i)c_1(L_j)c_1(L_k)$ is integral for $i \neq j \neq k$. Taking combinatoric factors into account, this gives for the quantization condition

$$ c_{ijk} \in \mathbb{Z} \quad \text{for any } i,j,k. \quad (2.8) $$

For a $U(1)^r$ gauge theory with no matter, quantum effects do not change the prepotential, so $\mathcal{F} = \mathcal{F}_{\text{classical}}$.

Consider now a gauge theory coupled to matter fields and study a neighborhood in the Coulomb branch of its moduli space including the point where the matter fields are
massless. At a generic point in this neighborhood the prepotential is as in (2.1). The cubic terms in the prepotential are a sum of a term which exists at tree level and a quantum term which is generated by loops. The quantum contribution is not smooth in this neighborhood—it has singularities at points where new massless particles exist. However, the classical part must be smooth. Their sum must respect the quantization conditions discussed above.

For example, consider a $U(1)$ gauge theory with $n_f$ “electrons” which are massless at $\phi = 0$. The quantum contribution to the prepotential was computed in [16]

$$c_{\text{quantum}} = -\frac{n_f}{2} \text{sign}(\phi). \quad (2.9)$$

Combining this with a classical cubic term $c_{\text{classical}}$ we learn from (2.6) that for consistency

$$c_{\text{classical}} - \frac{n_f}{2} \in \mathbb{Z} \quad (2.10)$$

and in particular, for odd $n_f$ $c_{\text{classical}}$ cannot vanish\(^1\). Thus, for odd $n_f$, the theory necessarily has $C$ and $P$ broken, with $CP$ preserved. It is easy to generalize to the case of electrons with masses $m_f$

$$c_{\text{quantum}} = -\frac{1}{2} \sum_{f=1}^{n_f} \text{sign}(\phi + m_f). \quad (2.11)$$

Consider integrating out $n$ massive electrons with mass $m > 0$. They induce $c_{\text{induced}} = -\frac{n}{2}$, which appears as a classical term in the low-energy theory of the $n_f - n$ massless electrons.

Hence, one can interpret $c_{\text{classical}}$ as being induced by loops of massive electrons.

Another way of expressing this phenomenon is the following. The fermion determinant in a $U(1)$ gauge theory with odd number of flavors is not gauge invariant. It is multiplied by $-1$ under certain gauge transformations. The Chern–Simons term with half-integer $c_{\text{classical}}$ plays the role of a Wess–Zumino term. Its lack of gauge invariance compensates for the lack of gauge invariance of the fermion determinant.

This understanding is in accord with a phenomenon noticed in [15] when the five-dimensional theory was reduced on a circle to four dimensions. The moduli space of the four-dimensional theory is a cylinder. The monodromy around the point with $n_f$ massless electrons is $T^{n_f}$. The monodromies around the two circles of the cylinder are $T^{(n_f/2)+c}$ and

\(^1\) A similar phenomenon in three dimensions was observed in [17,18].
\(T^{(n_f/2) - c}\). For consistency, \((n_f/2) + c \in \mathbb{Z}\) [15]. This provides an independent derivation of our anomaly.

Note that in (2.9), because \(\phi\) is odd under both \(\mathcal{P}\) and \(\mathcal{C}\), the induced Chern–Simons term properly respects \(\mathcal{P}\) and \(\mathcal{C}\) if the classical theory does, \(c_{\text{classical}} = m = 0\). On the other hand, with \(m \neq 0\) only \(\mathcal{CP}\) is a symmetry, which is exhibited in the low-energy theory with the massive matter integrated out by the non-zero constant \(c_{\text{induced}}\). (Because there can be a \(c_{\text{classical}}\) which is tuned to cancel \(c_{\text{induced}}\), two classical violations of \(\mathcal{C}\) and \(\mathcal{P}\) can cancel in the infrared.)

For \(G = U(1)^r\), and matter fields of charges \((q_f)_i\) under the \(U(1)_i\), the above have obvious generalizations; for example, the generalization of (2.11) is

\[
(c_{\text{quantum}})_{ijk} = -\frac{1}{2} \sum_{f=1}^{n_f} (q_f)_i (q_f)_j (q_f)_k \text{sign}(m_f) \phi + m_f). 
\]

(2.12)

2.2. Non-Abelian gauge theory and the Wess–Zumino term

We now consider a non-Abelian simple gauge group \(G\) with matter, “quarks” in a representation \(\oplus_f r_f\) where \(r_f\) are irreducible representations of \(G\) with masses \(m_f\). The classical prepotential is

\[
\frac{m_0}{2} \text{Tr} \Phi^2 + \frac{c_{\text{classical}}}{6} \text{Tr} \Phi^3, 
\]

(2.13)

where \(m_0 = 1/g^2_{\text{cl}}\) is the classical gauge coupling; it has dimensions of mass in a five-dimensional theory. Take \(\Phi = \phi^a T_a\), with the index \(a\) running over the generators of \(G\) which are represented by the matrices \(T_a\) in the fundamental representation of \(G\). The cubic term in (2.13) leads to terms in the Lagrangian proportional to

\[
c_{\text{classical}} d_{abc} \phi^a \phi^b \phi^c + c_{\text{classical}} d_{abc} \phi^a F^b F^c,
\]

(2.14)

with the \(d\) symbol defined by

\[
d_{abc} = \frac{1}{2} \text{Tr} T_a (T_b T_c + T_c T_b),
\]

(2.15)

and the non-Abelian Chern–Simons term

\[
\frac{c_{\text{classical}}}{24\pi^2} \text{Tr} (A \wedge F \wedge F - \frac{1}{2} A \wedge A \wedge A \wedge F + \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A).
\]

(2.16)

Such a \(c_{\text{classical}}\) term is possible only for \(G = SU(N), N \geq 3\), as those are the only (simple) gauge groups with a nontrivial third-order Casimir.
As in the $U(1)$ example above, the cubic term in (2.13) can be interpreted as arising from integrating out massive quarks. More explicitly, for $G = SU(N)$ (with $N \geq 3$) integrating out $n_f$ fundamental matter fields of mass $m$ induces

$$c_{\text{induced}} = -\text{sign}(m)\frac{n_f}{2}.$$  \hspace{1cm} (2.17)

(The generalization for integrating out massive matter in other representations modifies (2.17) by a factor of the cubic index of the representation.) Consider now the $SU(N)$ theory with quarks, all taken to be massive, in the fundamental representation. If there are $n_f^+$ massive quark flavors with masses $m_f > 0$ and $n_f^-$ with masses $m_f < 0$, the low-energy effective $c$ is

$$c_{\text{eff}} = c_{\text{classical}} - \frac{n_f^+ - n_f^-}{2}.$$  \hspace{1cm} (2.18)

The Chern–Simons term (2.16) is not gauge invariant. For gauge transformations $g$ which are nontrivial in $\pi_5(SU(N)) = \mathbb{Z}$, it changes by

$$\frac{c_{\text{eff}}}{240\pi^2} \int \text{Tr} g^{-1}dg g^{-1}dgg^{-1}dg^{-1}dg = c_{\text{eff}} 2\pi i \ell,$$  \hspace{1cm} (2.19)

with $\ell \in \mathbb{Z}$. We see that for consistency we need

$$c_{\text{eff}} = c_{\text{classical}} - \frac{n_f^+ - n_f^-}{2} \in \mathbb{Z}.$$  \hspace{1cm} (2.20)

If we consider now taking the masses of the flavors to zero, the numbers $n_f^\pm$ become ambiguous but $n_f^+ - n_f^- = n_f \mod 2$. Therefore, in the theory with $n_f$ massless flavors

$$c_{\text{eff}} = c_{\text{classical}} - \frac{n_f}{2} \in \mathbb{Z}.$$  \hspace{1cm} (2.21)

In particular, for odd number of flavors we must add the cubic term in (2.13). Note that unlike the $U(1)$ case mentioned above, here we do not need the requirement of special five-manifolds to get the correct quantization condition.

For gauge groups other than $SU(N)$, as there is no nontrivial third-order Casimir, $c_{\text{classical}} = 0$ and lack of gauge invariance under a nontrivial element of $\pi_5(G)$ would render the theory inconsistent. The only other cases of nontrivial $\pi_5$ are $Sp(N)$, which have $\pi_5(\text{Sp}(N)) = \mathbb{Z}_2$. Correspondingly, there is a global anomaly, similar to that of [19], which implies that $Sp(N)$ theories must have an even number of half-hypermultiplets in the fundamental representation. One way to see that is to consider a $d = 5$ instanton associated with $\pi_4(\text{Sp}(N)) = \mathbb{Z}_2$. It has one zero-mode for each half-hypermultiplet in the fundamental representation; similar to the situation for the global anomaly in four dimensions, this implies that the five-dimensional theory with an odd number of half-hypermultiplets is inconsistent. The requirement that the number of half-hypermultiplets of $Sp(N)$ be even is also consistent with compactification to four dimensions, where the number of half-hypermultiplets has to be even because of $\pi_4(\text{Sp}(N)) = \mathbb{Z}_2$. 

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2.3. Low-energy effective theory

Along the Coulomb branch the non-Abelian gauge group \( G \) is broken to \( U(1)^r \), with \( r = \text{rank}(G) \), and the theory is described by an Abelian low-energy effective theory for the \( A^i \) in \( A = \sum_{i=1}^r A^i T_i \), with \( T_i \) the Cartan generators of \( G \). Five dimensional gauge invariance implies that the exact quantum prepotential \( F(A^i) \) is at most cubic in the \( A_i \).

Indeed, only if \( \partial_i \partial_j \partial_k F \equiv c_{ijk} \) is a constant, could the quantization condition (2.8) possibly be satisfied. Because it is at most cubic, the exact low-energy effective prepotential is determined by a one-loop calculation. For arbitrary gauge group \( G \) and matter multiplets in representations \( r_f \), with masses \( m_f \), the result for the exact quantum prepotential (written for the scalar components \( \phi^i \) of \( A_i \)) is

\[
F = \frac{1}{2} m_0 h_{ij} \phi^i \phi^j + \frac{c_{\text{classical}}}{6} d_{ijk} \phi^i \phi^j \phi^k + \frac{1}{12} \left( \sum_{R} |R \cdot \phi|^3 - \sum_{f} \sum_{w \in W_f} |w \cdot \phi + m_f|^3 \right).
\] (2.22)

Here \( h_{ij} = \text{Tr}(T_i T_j) \) and \( d_{ijk} \) is (2.15), both evaluated for the Cartan generators. \( R \) are the roots of \( G \), and \( W_f \) are the weights of \( G \) in the representation \( r_f \). The first two terms in (2.22) are the classical prepotential (2.13) along the Coulomb branch. The last two terms in (2.22) are the quantum contributions of the massive charged vector and matter multiplets, respectively, which contribute with opposite sign to the prepotential. Explicit expressions for (2.22) will be presented in later sections.

Note that (2.22) agrees with what we find when we first integrate out the massive matter to induce a cubic term in the non-Abelian theory (it is only nontrivial for \( G = SU(N) \) with \( N \geq 3 \)) and then go along the flat directions of the Coulomb branch. In addition, there are extra quadratic terms induced in the prepotential upon integrating out massive flavors; these terms can always be canceled by a shift of \( m_0 \) proportional to \( m_f \).

Within the Weyl chamber for \( \phi \), each term \( R \cdot \phi \) in (2.22) is either everywhere positive or everywhere negative, without changing sign; the possible zeros are all at the boundaries of the Weyl chamber. The absolute value for these terms in (2.22) thus simply amount to assigning each term the appropriate sign throughout the Weyl chamber. On the other hand, it is possible for some terms \( w \cdot \phi + m_f \) to change sign (even when \( m_f = 0 \)), within the Weyl chamber. When that happens, because of the absolute value in (2.22), there are different prepotentials in different sub-wedges of the Weyl chamber, within which the terms \( w \cdot \phi + m_f \) are of definite sign; the boundaries of the sub-wedges are where some \( w \cdot \phi + m_f \) is zero. At the boundaries of the sub-wedges, \( F \) is not smooth but, because
of the $|\mathbf{w} \cdot \phi + m_f|^3$ dependence, $\mathcal{F}$ and its Hessian, the metric $g^{-2}$, are continuous across the boundaries of the sub-wedges. Exactly as in (2.9), the discontinuity in $\partial_i \partial_j \partial_k \mathcal{F}$ is associated with charged matter fields which become massless on the boundaries of these sub-wedges.

In the explicit expressions for (2.22) which will be given in later sections, it can be seen that the low-energy effective Abelian theory has a $(c_{eff})_{ijk}$ which satisfies the Abelian quantization condition (2.8) in all cases as long as the non-Abelian theory does not suffer from a global anomaly associated with $\pi_5(G)$. Here it is necessary to use a basis and normalization for the Cartan torus where all Abelian charges, the roots and weights, are properly quantized to be integers. The contribution of the massive gauge fields, the sum over the roots in (2.22), always leads to an integer contribution to $c_{ijk}$ for any gauge group. For $G = SU(N)$ the appropriate basis for the Cartan sub-algebra is generated by $(T^{(i)})^j_k = (\delta^{i-j} - \delta^{i+1-j})\delta^j_k$, $i = 1, \ldots N - 1$, and the classical and matter multiplet contribution to $c_{ijk}$ is $c_{i,i+1,i} = -(c_{i+1,i+1,i} = (c_{\text{classical}} - \frac{1}{2}n_f)$. The $c_{ijk}$ thus satisfy the Abelian quantization condition (2.8) precisely when the non-Abelian theory satisfies the non-Abelian quantization condition (2.21).

Similarly, for $G = Sp(N)$ with $n_f$ hypermultiplets, i.e. $2n_f$ half-hypermultiplets, the contribution is $c_{iii} = -n_f$. The Abelian quantization condition is satisfied since, as seen above, $n_f$ must be an integer—the theories with an odd number of half-hypermultiplets have a global anomaly. For all other cases of gauge groups and matter the effective $c_{ijk}$ satisfy (2.8).

### 3. Nontrivial Renormalization Group Fixed Points

The metric $t(\phi)_{ij} = \partial_i \partial_j \mathcal{F}$ should be non-negative throughout the Weyl chamber of the Coulomb branch. As discussed in [4], when the quantum contribution to the metric, which is linear in $\phi$, is negative, at best the theory can be made sensible in a subspace near the origin by taking the classical part $t_{ij}^{(0)} = m_0 h_{ij} > 0$. Eventually, away from the origin, the negative quantum part becomes larger than $t_{ij}^{(0)}$, which is a reflection of the fact that the theory is non-renormalizable and eventually hits a Landau pole. On the other hand, when the quantum contribution to the metric is positive on the entire Coulomb branch (Weyl chamber), it is possible to have a scale invariant fixed point theory with $m_0 = g_{cl}^{-2} = 0$ [4]. The theory is then sensible on the entire moduli space of vacua. A necessary (though perhaps not sufficient) condition [4] for a nontrivial strong coupling fixed point is thus
that the quantum part of the metric, $(\partial_i \partial_j \mathcal{F})d\phi^i d\phi^j$ must be non-negative throughout the Weyl chamber. We will examine general conditions on the gauge theory for this to be the case.

The condition that the Hessian $\partial_i \partial_j \mathcal{F}$ be a positive matrix is equivalent to the condition that the prepotential $\mathcal{F}$ be a convex function throughout the Weyl chamber: for any two points $x$ and $y$ in the Weyl chamber, the prepotential must satisfy $\mathcal{F}(\lambda x + (1-\lambda)y) \leq \lambda \mathcal{F}(x) + (1-\lambda)\mathcal{F}(y)$ for $0 \leq \lambda \leq 1$. (A function which satisfies the opposite inequality is referred to as concave.)

We will investigate the conditions required on the gauge group and matter content for the prepotential (2.22) with $m_0 = 0$ to be a convex function on the entire Weyl chamber. Because $\mathcal{F}$ and $\partial_i \mathcal{F}$ vanish at the origin, a consequence of the convexity of $\mathcal{F}$ is that $\mathcal{F} \geq 0$ throughout the Weyl chamber Coulomb branch.

It is easily seen that the term in (2.22) associated with the vector multiplets is purely convex: each term in the sum on $\mathbf{R}$ is obviously convex as a function of $\mathbf{R} \cdot \phi$, and the sum of convex functions is convex. The quantum terms in (2.22) associated with the hypermultiplets, because of the sign difference, similarly leads to a purely concave contribution to $\mathcal{F}$. Therefore, $\mathcal{F}$ will be convex provided there isn’t too much matter. This is analogous to the requirement of asymptotic freedom in four-dimensional gauge theories. Finally, the $c_{\text{classical}}$ term is neither purely convex nor purely concave on the Weyl chamber.

We can immediately make some general comments about when there can or cannot be a nontrivial fixed point:

1. There cannot be interesting fixed points associated with Abelian gauge groups; unless there are gauge fields which make a convex contribution to $\mathcal{F}$, the matter contribution would lead to a concave $\mathcal{F}$.

2. On the other hand, for any non-Abelian gauge theory without matter hypermultiplets (2.22) is convex (taking $c_{\text{classical}} = 0$) and, therefore, there could be a fixed point.

3. There can be no interesting fixed point with theories containing representations with weights $\mathbf{W}_f$ which are of equal or longer length than those of the adjoint, the roots $\mathbf{R}$.

4. Turning on a quark mass $m_f$ changes the prepotential by

$$
\delta \mathcal{F} = \frac{1}{12} \sum_{\mathbf{w} \in \mathbf{W}_f} \left( |\mathbf{w} \cdot \phi|^3 - |\mathbf{w} \cdot \phi + m_f|^3 \right).
$$

(3.1)
We can now take $m_f \to \infty$ and derive the effective prepotential of the problem with one fewer quark. The terms which are quadratic in $\phi$ can be canceled by adjusting the bare coupling $m_0$. The cubic terms \( \frac{1}{12} \sum_{\mathbf{w} \in \mathbf{W}_f} (|\mathbf{w} \cdot \phi|^3 - \text{sign}(m_f)(\mathbf{w} \cdot \phi)^3) \) are convex. (The last term in $\delta F$ is the $c_{\text{induced}}$ discussed in the last section—it vanishes for all cases except $SU(N), N \geq 3$.) Therefore, if the prepotential with some matter content is convex, the prepotential obtained by giving mass to some matter fields and integrating them out will also be convex—less matter means the prepotential is even more convex. This is consistent with flowing from a nontrivial fixed point to another nontrivial fixed point upon perturbing by mass terms.

5. We can immediately rule out new fixed points associated with product gauge groups. In order for such a theory to give something new, the gauge groups must be coupled via matter fields which transform nontrivially under more than one gauge group. Concretely, with gauge group $G_1 \times G_2$, there must be hypermultiplets in representations like $(\mathbf{r}_1, \mathbf{r}_2)$. Now go along the Coulomb branch, breaking $G_1$ without breaking $G_2$. These matter fields in (2.22) lead to a negative contribution for the effective coupling of $G_2$ which is proportional to the $G_1$ Coulomb modulus and can thus be made arbitrarily negative. Hence the effective gauge coupling is not non-negative on the entire Coulomb branch and there must be a finite bare coupling $g_2$, ruling out a fixed point. We thus only need to consider simple gauge groups to completely categorize the possible fixed points.

6. By the decoupling of vector multiplets and hypermultiplets, any theory obtained via Higgsing from a theory with convex $F$ will also have a convex $F$. The converse need not be true: a theory with convex $F$ can be obtained via Higgsing from one with a $F$ which is not convex—this just means that the photons associated with the dangerous eigenvalues got lifted by the Higgsing.

In what follows we will examine in the various possible cases the condition that the prepotential be convex, the necessary conditions for a nontrivial fixed point.

4. $Sp(N)$ Gauge Theories

The Coulomb branch of the moduli space is given by $\Phi = \text{diag}(a_1, \ldots, a_N, -a_1, \ldots, -a_N)$, modulo the Weyl group action, which permutes these elements. It can thus be taken to be the Weyl chamber $a_1 \geq a_2 \geq \ldots \geq a_N \geq 0$. There are various enhanced gauge
symmetries on the walls of the Weyl chamber: with \( k \) zero \( a_i \) and \( p \) equal but non-zero \( a_i \), the unbroken gauge group is \( U(p) \times Sp(k) \times U(1)^{N-k-p} \). We consider theories with \( n_A \) matter hypermultiplets in the antisymmetric and \( n_F \) matter flavors (that is, \( 2n_F \) half-hypermultiplets of matter) in the fundamental representation of \( Sp(N) \). The global symmetry of the classical theory is \( Sp(n_A) \times Spin(2n_F) \times U(1)_I \), where \( U(1)_I \) is the symmetry \([4]\) associated with the current \( * (F \wedge F) \).

The prepotential (2.22) with \( m_0 = 0 \) and masses \( m_f = 0 \) is given by

\[
F = \frac{1}{6} \left( \sum_{i<j} [(a_i - a_j)^3 + (a_i + a_j)^3](1 - n_A) + \sum_i a_i^3(8 - n_F) \right). \tag{4.1}
\]

The effective gauge coupling matrix is given by its Hessian, which is

\[
(g^{-2})_{ii} = 2[\sum_{k=1}^{i-1} a_k](1 - n_A) + a_i(8 - n_F),
\]
\[
(g^{-2})_{i<j} = 2(1 - n_A)a_j. \tag{4.2}
\]

Note that for \( n_A = 1 \) the matrix (4.2) is diagonal and the condition for an interacting fixed point, positivity of the diagonal elements, is satisfied for \( n_F \leq 7 \). For \( n_A = 1 \) and \( n_F = 8 \), (4.2) is identically zero and the Coulomb moduli space collapses to nothing.

For arbitrary \( n_A \) and \( n_F \), consider the direction of the Coulomb branch along which \( Sp(N) \) is broken to \( SU(k) \times Sp(N-k) \times U(1) \): \( a_1 = a_2 = \ldots = a_k \neq 0 \) with the remaining \( a_i = 0 \). In this limit, the eigenvalues of (4.2) are \( a_1[2(N+k-2)(1 - n_A) + 8 - n_F] \), \( k - 1 \) eigenvalues \( a_1[2(N-2)(1 - n_A) + 8 - n_F] \), and \( N - k \) eigenvalues \( 2a_1(1 - n_A) \). Therefore, we see that necessary conditions for \( F \) to be convex and the possibility of an interacting fixed point are: \( n_A \leq 1 \); for \( n_A = 0 \), \( n_F \leq 2N + 4 \). (An exception is the \( Sp(1) \cong SU(2) \) case in the original work of \([4]\), where \( N = k = 1 \) and \( g^{-2} \) is given by the first of these eigenvalues, which is positive for \( n_F \leq 7 \).)

The condition, for \( n_A = 0 \), that \( n_F \leq 2N + 4 \) is not only a necessary condition for convex \( F \), and hence an interesting fixed point, it is also sufficient. Indeed, for \( n_A = 0 \) and \( n_F = 2N + 4 \) the eigenvalues of (4.2) are especially simple and non-negative everywhere on the Weyl chamber; they are: \( 2(\sum_{i=1}^{k} a_i - ka_{k+1}) \), for \( k = 1, \ldots, N - 1 \), and \( 2 \sum_{i=1}^{N} a_i \).

As discussed in the previous section, because \( F \) is convex on the entire Coulomb branch for \( n_F = 2N + 4 \), it will be convex for any \( n_F \leq 2N + 4 \).
Bare masses can also be introduced. For example, giving the fundamentals masses $m_f$, $f = 1 \ldots n_F$, (4.2) becomes

$$
(g^{-2})_{ii} = 2[(N - i)a_i + \sum_{k=1}^{i-1} a_k](1 - n_A) + 8a_i - \frac{1}{2} \sum_{f=1}^{n_F} (|a_i + m_f| + |a_i - m_f|),
$$

(4.3)

$$
(g^{-2})_{i<j} = 2(1 - n_A)a_j.
$$

The theories with $n_A = 1$ and $n_F \leq 7$ are expected to have an interacting fixed point by the same argument as in [4] but with $N$ four-brane probes. And, as in [4], the global symmetry should be enhanced from $Sp(1) \times D_{n_F} \times U(1)_I$ to $Sp(1) \times E_{n_F+1}$. For $n_A = 1$ and any $n_F$, there is a Higgs branch which, for finite coupling, is the moduli space of $N$ $D_{n_F}$ instantons, exactly as in [20] in six dimensions. For the new fixed points with $g_{cl}^{-2} \to 0$, we expect from string theory that the Higgs branch should become the moduli space of $N$ $E_{n_F+1}$ instantons.

Starting from the theories with $n_A = 1$ and $n_F \leq 7$ and giving the antisymmetric tensor a large mass, we can flow to the theories with $n_A = 0$ and $n_F \leq 7$. The mass term breaks the $Sp(1) \times E_{n_F+1}$ global symmetry of the strong coupling fixed point to $U(1) \times E_{n_F+1}$, where the $U(1)$ acts only on the massive antisymmetric tensor and thus decouples from the low-energy theory. Therefore, at least for $n_A = 0$ and $n_F \leq 7$, we expect there to be a fixed point with $E_{n_F+1}$ global symmetry for any $Sp(N)$. For $n_A = 0$ with $7 < n_F \leq 2N + 4$ our consistency condition is satisfied but in order to construct examples we must use Calabi–Yau models. We shall do this in sect. 9.

As in [6,7], there are actually two fixed point theories associated with the theories with $n_A = 0, 1$ and $n_F = 0$, associated with $\pi_4(Sp(N)) = \mathbb{Z}_2$: the analogues of $E_1$ and $\tilde{E}_1$ in [6]. For $n_A = 0$ and $n_F = 0$, there is one fixed point with global symmetry $E_1 = SU(2)$ and another with global symmetry $\tilde{E}_1 = U(1)$. For $n_A = 1$ and $n_F = 0$, there is a similar situation, with an additional global $Sp(1)$. Also, exactly as in [6], there are $Sp(N)$ analogues of the $E_0$ theory, with no global symmetry for $n_A = 0$ and global symmetry $Sp(1)$ for $n_A = 1$.

Consider the behavior near the boundaries of the Weyl chamber, where $U(1)^N$ is enhanced to some bigger subgroup of $Sp(N)$. At the codimension $p + k - 1$ boundary where $k$ of the $a_i$ are zero and $p$ equal but non-zero, there is an enhanced $U(p) \times Sp(k) \times U(1)^{N-k-p}$ with $2N \to (p_{\pm 1}, 1)_{(0, \ldots, 0)} + (1_0, 2k)_{(0, \ldots, 0)} + ((1_0, 1)_{\pm 1, 0, \ldots, 0} + \text{permutations})$. Similarly decomposing the $Sp(N)$ vector multiplet and taking into account the masses that fields
charged under the $U(1)s$ get along this direction of the Coulomb branch, the prepotential (4.1) is, of course, properly reproduced in the low-energy theory.

For $n_F \geq 2$ there is a Higgs branch, which connects to the Coulomb branch along the boundaries of the Weyl chamber discussed above with $k > 0$. In particular, along $a_N = 0$, $Sp(N)$ with $n_F$ flavors can be Higgsed to $Sp(N - 1)$ with $n_F - 2$ flavors. Setting $a_N = 0$ in (4.1), the prepotential reduces to exactly that of $Sp(N - 1)$ with $n_F - 2$ flavors. This is in agreement with the expected decoupling of vector and hypermultiplets.

It is interesting to compare our condition for an interacting fixed point in five dimensions to the condition of asymptotic freedom in four dimensions. Reducing the five-dimensional supersymmetric theory to four dimensions yields a theory with $N = 2$ supersymmetry. There the condition for asymptotic freedom for $Sp(N)$ with $n_A$ hypermultiplets in the antisymmetric representation and $n_F$ in the fundamental is $2(N + 1) - n_F - 2n_A(N - 1) \geq 0$. For example, for $n_A = 0$ the condition is $n_F \leq 2(N + 1)$, which is stronger than our condition in five dimensions. On the other hand, for $N = 2$ and $n_F = 0$, there can be $n_A \leq 3$ for four-dimensional asymptotic freedom, which is weaker than our condition in five dimensions. So, in general, the condition for a fixed point in five dimensions can be either stronger or weaker than that of asymptotic freedom in four dimensions.

4.1. $Sp(2) \cong Spin(5)$ in more detail

The Weyl chamber is given by $a_1 \geq a_2 \geq 0$. On the boundary $a_2 = 0$ there is an unbroken $Sp(1) \times U(1)$ with the fundamental decomposing as $4 \to 2_0 + 1_{\pm 1}$, the antisymmetric tensor decomposing as $5 \to 2_{\pm 1} + 1_0$, and the adjoint decomposing as $10 \to 3_0 + 2_{\pm 1} + 1_{\pm 2} + 1_0$. Along this boundary, the $SU(2)$ has $n_F$ massless doublets, with those coming from the 5 for $n_A = 1$ getting a mass from their coupling to the $U(1)$. On the boundary $a_1 = a_2$ there is an unbroken $SU(2) \times U(1)$, with the representations decomposing as $4 \to 2_{\pm 1}$, $5 \to 3_0 + 1_{\pm 2}$, $10 \to 3_{\pm 2} + 3_0 + 1_0$. The unbroken $SU(2)$, with zero bare masses, has no massless doublets, as those coming from the 4’s get a mass via their coupling to the $U(1)$. For $n_A = 1$, however, there can be a massless 3 of $SU(2)$.

By introducing a bare mass $m$ for the matter fields in the 4, the unbroken $SU(2)$ will have $n_F$ massless doublets at $a_1 = a_2 = |m|$. On the boundary $a_1 = a_2$, on one side of this point, the $SU(2)$ will go to the $E_1$ fixed point, and on the other side it will flow to the $\tilde{E}_1$ fixed point discussed in [6].

For $n_F \geq 2$ and zero bare masses, there is a Higgs branch which meets the Coulomb branch at the boundary $a_2 = 0$.  

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Theories with \( n_A = 1 \) and \( n_F \leq 7 \) lead to new fixed points via a generalization of the argument in [4] to one involving two probes. By giving a mass to the 5 and integrating it out, the theories with \( n_A = 0 \) and \( n_F \leq 7 \) thus also lead to new fixed points. By the general analysis above, there is one more case where there could possibly be an interesting fixed point: \( n_A = 0, n_F = 8 \). The existence of this fixed point, and more generally all of the possible fixed points with \( n_F \leq 2N + 4 \), is demonstrated in sect. 9.

5. SU\((N)\) Gauge Theories

The Coulomb branch of the moduli space is given by \( \Phi = \text{diag}(a_1, \ldots a_N) \), with \( \sum_i a_i = 0 \), modulo the Weyl group action, which permutes the \( a_i \). It can thus be taken to be the Weyl chamber \( a_1 \geq a_2 \geq \ldots \geq a_N \). On the walls of the Weyl chamber there is enhanced gauge symmetry: with \( k \) equal \( a_i \), \( U(1)^{N-1} \) is enhanced to \( U(k) \times U(1)^{N-1-k} \).

We only need to consider theories with matter in the symmetric, antisymmetric, and fundamental representations; \( n_S, n_A \), and \( n_F \) are the number of hypermultiplets in these representations. For other representations, there cannot be an interesting fixed point. (We will shortly argue that \( n_S = 0 \) is also necessary.) There is a \( U(n_S) \times U(n_A) \times U(n_F) \times U(1)_I \) classical global symmetry. As discussed in sect. 2.2, it is possible and sometimes necessary to include a classical \( c_{cl} \) term for \( SU(N) \). The condition on \( c_{cl} \) is

\[
c_{cl} + \frac{1}{2} n_F + \frac{1}{2} N(n_A + n_S) \in \mathbb{Z},
\]

where we used the fact that the (anti)symmetric tensor representation has cubic index \( N + (-)4 \).

The effective prepotential on the Coulomb branch, taking \( m_0 = 0 \), is given by (the case with no matter and \( c_{cl} = 0 \) has already appeared in [21])

\[
\mathcal{F} = \frac{1}{12} \left( 2 \sum_{i<j} (a_i - a_j)^3 + 2c_{cl} \sum a_i^3 - (n_A + n_S) \sum_{i<j} |a_i + a_j|^3 - (n_F + 8n_S) \sum a_i^3 \right).
\]

The prepotential (5.2) should be written in terms of \( N - 1 \) independent variables, solving for the constraint \( \sum_{i=1}^N a_i = 0 \). Because a symmetric tensor makes the same contribution to \( \mathcal{F} \) as an antisymmetric tensor and 8 fundamentals, we will no longer need to write the dependence on \( n_S \) explicitly.
With $n_F, n_A, n_S \neq 0$, there is not a single prepotential but, as discussed in sect. 2.3, prepotentials defined in sub-wedges of the Weyl chamber corresponding to different values of the signs of those terms in (5.2) which have absolute values. There are massless charged matter fields at the boundaries of these sub-wedges. For example, for $n_A + n_S = 0$ there are $N - 1$ sub-wedges in the Weyl chamber corresponding to the $N - 1$ choices of where zero appears in $a_1 \geq a_2 \geq \ldots \geq a_N$; in the $k$-th sub-wedge, $k$ of the $a_i$ are negative. (For $n_F \geq 2$ there is a Higgs branch which connects to the Coulomb branch along the boundaries of these sub-wedges.) The prepotential in the $p$-th sub-wedge, for $p = 2 \ldots N - 1$, is given by $F_p = F_1 + (n_F/6) \sum_{i=1}^{N-1} a_{N-i}^3$, where $F_1$ is the prepotential for the first sub-wedge, where $a_{N-1} \geq 0$.

We now find the conditions on the number of flavors and $c_{cl}$ for the prepotential (5.2) to be convex throughout the Weyl chamber. Because the charge conjugation operation, which acts on the Weyl chamber as $a_i \rightarrow -a_{N+1-i}$, $i = 1 \ldots N$, takes $c_{cl} \rightarrow -c_{cl}$, any condition on $c_{cl}$ will imply a similar condition on $-c_{cl}$. Therefore, the convexity condition on $c_{cl}$ will be in terms of $|c_{cl}|$.

In the sub-wedge with $a_{N-1} \geq 0$, the metric $g^{-2}$ in terms of the Hessian of $F$ with respect to the $a_i$, $i = 1 \ldots N - 1$, with $a_N = -a_T \equiv -\sum_{i=1}^{N-1} a_i$, is given by:

\[
(g^{-2})_{ii} = (N + 4 + c_{cl} - 2i)a_i + 2 \sum_{k=1}^{i-1} a_k \\
+ (N + 2 - c_{cl})a_T - \frac{1}{2}((N - 2)n_A + n_F)(a_i + a_T),
\]

\[
(g^{-2})_{i<j} = (2 - n_A)a_j + \frac{1}{2}(2N + 4 - n_A(N - 4) - n_F - 2c_{cl})a_T - n_Aa_i.
\]

The metric $g^{-2}$ in the $p$-th sub-wedge differs from (5.3) by $\Delta^{(p)}_{ij} = \sum_{k=1}^{p-1} \delta_{N-k,i}\delta_{ij}$.

Along the direction where $a_i = a$, $i = 1 \ldots N - 1$, where $SU(N)$ is broken to $SU(N-1) \times U(1)$, (5.3) has $N - 2$ eigenvalues $a[2N - n_A(3N - 8) - n_F + 2c_{cl}] / 2$ and one eigenvalue $a[2N^3 - n_A(N - 2)(N^2 - 4N + 8) - n_F(N^2 - 2N + 2) - 2c_{cl}N(N - 2)] / 2$. For $N > 2$ a necessary condition for convex prepotential is thus $2N - n_A(3N - 8) - n_F - 2|c_{cl}| \geq 0$, where the absolute value follows from the charge conjugation operation mentioned above. When this condition is satisfied, both of the above eigenvalues are non-negative. Putting $n_A = n_S$ and $n_F = 8n_S$, we see that $n_S = 0$ is required for the first eigenvalue to be non-negative. Similarly, $n_A = 0$ is required for $N > 8$. For $N \leq 8$, these eigenvalues are also non-negative with $n_A = 1$ antisymmetric tensor and $n_F \leq 8 - N - 2|c_{cl}|$ fundamentals.
For \( N = 4 \) they are also non-negative for \( n_A = 2, n_F = c_{cl} = 0 \). For all \( N \), with \( n_A = 0 \), we get

\[
n_F + 2|c_{cl}| \leq 2N
\]  
(5.4)

as a necessary condition for a nontrivial fixed point. Thus \( n_F \leq 2N \) and \(|c_{cl}| \leq N\).

For \( n_F = 2N \), (5.4) requires \( c_{cl} = 0 \), which is compatible with (5.1). For \( n_F = 2N \) and \( c_{cl} = 0 \) the gauge coupling matrix (5.3) has especially simple eigenvalues: \( 2(\sum_{i=1}^{k} a_i - ka_{k+1}) \), for \( i = 1, \ldots, N - 2 \), and \( 2N \sum_{i=1}^{N-1} a_i \). Because these are all non-negative in the Weyl chamber, \( n_F \leq 2N \) is, in fact, necessary and sufficient for having \( F \) be convex in the entire first sub-wedge. The prepotential in the \( p \)-th sub-wedge, for \( p = 2 \ldots N - 1 \), is given by \( F(p) = F(1) + (n_F / 6) \sum_{i=1}^{p-1} a_{N-i}^3 \), where \( F(1) \) is the prepotential for the first sub-wedge, which we just found to be convex throughout the entire Weyl chamber. The additions in the \( (p) \) sub-wedge, on the other hand, are purely concave since the \( a_{N-i} < 0 \). Therefore, one might worry that \( F(p) \) could fail to be convex. But even in the \( N - 1 \)-th sub-wedge, which is the worst case in terms of possibly not being convex, \( F \) is, in fact, convex. Indeed, the \( N - 1 \)-th sub-wedge is related to the first sub-wedge by taking \( \Phi \to -\Phi \) (for \( c_{cl} = 0 \) this reflects the underlying charge conjugation symmetry), from which it follows that \( F(N-1) \) will be convex since \( F(1) \) is.

Since the prepotential is convex for zero quark masses for \( n_F = 2N \) and \( c_{cl} = 0 \), it follows from comment 4 in sect. 3 that it is also convex when the masses are turned on. It also follows that the inequality (5.4) is necessary and sufficient for all \( n_F \) and \( c_{cl} \). Any theory satisfying (5.4) can be obtained from \( n_F = 2N \) and \( c_{cl} = 0 \) by adding mass terms. Different values of \( c_{cl} \) are induced by choosing appropriate signs for the mass terms (see equation (2.18)). By comment 4, they will all have convex prepotentials. Therefore, there can be interesting fixed points whenever (5.1) and (5.4) are satisfied. An exception is \( SU(2) \), where \( c_{cl} \) doesn’t arise and where the fixed points extend to \( n_F \leq 7 \) [4].

For \( n_F \geq 2 \) there is a Higgs branch associated with the Higgsing: \( N \to N - 1, n_F \to n_F - 2 + n_A, n_A \to n_A \). This Higgs branch connects to the Coulomb branch along the \( N - 2 \) boundaries of the sub-wedges, where \( a_l = 0 \) for \( l = 2 \ldots N - 2 \). The above expressions are compatible with this Higgsing. In particular, setting \( a_l = 0 \) in (5.2), the prepotential exactly reproduces that of the low-energy \( SU(N - 1) \) theory obtained along the Higgs branch.

Bare masses can be added for the matter fields, with the prepotential modified as in (2.22). Giving the fundamentals masses \( m_f \), this modifies the metric (5.3) by replacing
\[ n_Fa_i \text{ with } \sum_{f=1}^{n_F} |a_i + m_f| \text{ and } n_Fa_T \text{ with } \sum_{f=1}^{n_F} |a_T - m_f|. \]

The mass dependence is compatible with flowing between fixed points upon adding masses for the matter fields, with an addition of \( c_{induced} \) as in (2.17) and a shift of \( m_0 \).

5.1. \( SU(3) \) in more detail

We now consider \( SU(3) \) in a bit more detail, explicitly verifying that the gauge coupling eigenvalues are indeed non-negative for all \( n_F \leq 6 \). Taking \( a_3 = -a_1 - a_2 \), the Weyl chamber is \( a_1 \geq a_2 \geq -\frac{1}{2}a_1 \). At this boundary of the Weyl chamber there is an unbroken \( SU(2) \times U(1) \) with the fundamental decomposing as \( 3 \rightarrow 2 + 1 \) and the adjoint as \( 8 \rightarrow 3 + 1 + 2 \). Because of the non-zero \( U(1) \) charge of the matter in the \( 2 + 1 \), the \( SU(2) \) theory has no massless matter fields for zero bare mass. By introducing a bare mass \( m \) for the matter fields, the point \( a_1 = a_2 = |m| \) has an \( SU(2) \) with \( n_F \) massless doublets. On the rest of the Weyl boundary, the doublets get a mass of one sign on one side of this point and another sign on the other side of this point.

For \( n_F > 0 \) the prepotential (5.2) differs in the two sub-wedges, \( a_2 \geq 0 \) and \( a_2 \leq 0 \), of the Weyl chamber. Explicitly, the metric in the sub-wedge \( a_1 \geq a_2 \geq 0 \) is

\[
t(a) = \begin{pmatrix}
(10 - n_f)a_1 + (5 - c_{cl} - \frac{1}{2}n_f)a_2 & 2a_2 + (5 - c_{cl} - \frac{1}{2}n_f)(a_1 + a_2) \\
2a_2 + (5 - c_{cl} - \frac{1}{2}n_f)(a_1 + a_2) & (7 - c_{cl} - \frac{1}{2}n_f)a_1 + (8 - n_f)a_2
\end{pmatrix}.
\] (5.5)

In the sub-wedge \( 0 \geq a_2 \geq -\frac{1}{2}a_1 \) the metric is given by

\[
t(a) = \begin{pmatrix}
(10 - n_f)a_1 + (5 - c_{cl} - \frac{1}{2}n_f)a_2 & 2a_2 + (5 - c_{cl} - \frac{1}{2}n_f)(a_1 + a_2) \\
2a_2 + (5 - c_{cl} - \frac{1}{2}n_f)(a_1 + a_2) & (7 - c_{cl} - \frac{1}{2}n_f)a_1 + 8a_2
\end{pmatrix}.
\] (5.6)

The eigenvalues of these are complicated in general but are non-negative when (5.4) is satisfied.

6. \( Spin(M) \) Gauge Theories

For \( Spin(2N) \) or \( Spin(2N+1) \) the Coulomb branch moduli space is given by the Weyl chamber \( a_1 \geq a_2 \geq \cdots \geq a_N \geq 0 \), corresponding to the block diagonal elements of \( \Phi \). On the boundary of the Weyl chamber with \( k \) zero \( a_i \)'s and \( p \) equal but non-zero \( a_i \)'s, there is an unbroken \( Spin(2k + \epsilon) \times U(p) \times U(1)^{N-k-p} \) gauge group, where \( \epsilon = 0 \) (1) for \( M \) even (odd).

Because the antisymmetric tensor representation is the adjoint, there cannot be any matter fields in the symmetric or antisymmetric tensor representations. The theory with
$n_V$ hypermultiplets in the vector representation has a classical global $Sp(n_V) \times U(1)_I$ symmetry. Our discussion will be for $Spin(M)$ with $M \geq 5$; $Spin(3) \cong SU(2)$ is covered by the original work of [4] and, as discussed in sect. 3, $Spin(4) \cong SU(2) \times SU(2)$ is not interesting because it is not simple.

The prepotential for $Spin(2N)$ with $n_V$ hypermultiplets in the vector representation is

$$F = \frac{1}{6} \left( \sum_{i<j} [(a_i + a_j)^3 + (a_i - a_j)^3] - n_V \sum a_i^3 \right). \quad (6.1)$$

The corresponding effective gauge coupling matrix is

$$(g^{-2})_{ii} = 2[(N - i)a_i + \sum_{k=1}^{i-1} a_k] - n_V a_i,$$

$$(g^{-2})_{i<j} = 2a_j. \quad (6.2)$$

The prepotential for $Spin(2N + 1)$ is

$$F = \frac{1}{6} \left( \sum_{i<j} [(a_i + a_j)^3 + (a_i - a_j)^3] + (1 - n_V) \sum a_i^3 \right). \quad (6.3)$$

The corresponding matrix is

$$(g^{-2})_{ii} = 2[(N - i)a_i + \sum_{k=1}^{i-1} a_k] + (1 - n_V)a_i,$$

$$(g^{-2})_{i<j} = 2a_j. \quad (6.4)$$

Consider the direction in the Coulomb branch where $Spin(M)$ is broken to $U(k) \times Spin(M - 2k)$: $a_1 = \ldots = a_k$, with the remaining $a_i = 0$. In this limit, (6.2) or (6.4) has an eigenvalue $a_1 [M + 2k - 4 - n_V]$, $k - 1$ eigenvalues $a_1 (M - 4 - n_V)$, and $N - k$ eigenvalues $2a_1 k$. Therefore, a necessary condition for non-negative eigenvalues is $n_V \leq M - 4$. For $n_V = M - 4$ the eigenvalues of (6.2) and (6.4) simplify; they are $2(\sum_{i=1}^{p} a_i - pa_{p+1})$, $p = 1 \ldots N - 1$, and $2 \sum_{i=1}^{N} a_i$. These eigenvalues are indeed all non-negative throughout the Weyl chamber, showing that $n_V \leq M - 4$ is a necessary and sufficient condition for everywhere convex prepotential $F$. (The condition for asymptotic freedom in the four-dimensional, $N = 2$ version of this is $n_V \leq M - 2$.)

In the direction on the Coulomb branch where there are $k$ equal eigenvalues $a_i = a$, $Spin(M)$ is broken to $SU(k) \times U(1)^{N-k+1}$. If the $n_V$ matter fields originally had masses $m_i$, the $SU(k)$ theory will have $n_F = 2n_V$ matter fields with pairs having masses $m_i \pm a$. 20
For \( n_V \geq 1 \) there is a Higgs branch corresponding to the Higgsing: \( M \rightarrow M - 1 \) with \( n_V \rightarrow n_V - 1 \). For \( M = 2N \), this Higgs branch connects to the Coulomb branch along the boundary \( a_N = 0 \) of the Weyl Chamber. For \( M = 2N + 1 \) this Higgs branch connects to the entire Coulomb branch. The above expressions are compatible with this Higgsing. In particular, starting from \( \text{Spin}(2N) \), setting \( a_N = 0 \) in (6.1) the prepotential exactly reduces to that of \( \text{Spin}(2N - 1) \) with \( n_v - 1 \) vectors. Similarly, starting from \( \text{Spin}(2N + 1) \), (6.3) is, over the entire Coulomb branch, exactly the same as the prepotential (6.1) for the \( \text{Spin}(2N) \) theory with \( n_V - 1 \) vectors obtained along the Higgs branch.

The dimension of the full Higgs branch for \( n_V \leq M \) is \( \frac{1}{2} n_V (n_V + 1) \), with the gauge group generically broken to \( \text{Spin}(M - n_V) \) with no matter. In particular, for our critical case of \( n_V = M - 4 \), the gauge group can be broken to \( \text{Spin}(4) \cong SU(2) \times SU(2) \) with no matter on the Higgs branch. A simple way to see that \( n_V \leq M - 4 \) is necessary for a new fixed point is to note that for larger \( n_V \) it is possible to Higgs to \( \text{Spin}(4) \) with matter fields in the \((2, 2)\) which, as discussed in sect. 3, necessarily leads to negative eigenvalues.

Note that for \( n_V = M - 4 \) or \( n_V = M - 3 \) the Higgs branch connects to \( \text{Spin}(5) \cong Sp(2) \) with \( n_A = 0 \) or \( n_A = 1 \) antisymmetric tensors which, as discussed in sect. 2, we know have nontrivial fixed points in analogy with \( E_1, \tilde{E}_1, \) and \( E_0 \) of [6]. For \( n_V = 0 \), for \( M > 5 \), there is expected to be only one fixed point because \( \pi_4(\text{Spin}(M)) = 0 \).

We could include \( n_S \) spinors by including a term in the prepotential (6.1) for \( M = 2N \):

\[
-\frac{1}{12} n_S \sum_{\{\epsilon_i\}} \frac{1}{2} \sum_i \epsilon_i a_i |^3, 
\]

where \( \{\epsilon_i\} \) runs over all \( 2^{N-1} \) sign choices for \( \epsilon_i = \pm 1 \) with an even (odd) number of \( - \) signs for the spinor (conjugate spinor) representation. In the limit where only \( a_1 \neq 0 \), this would lead to an eigenvalue \( a_1 [2(N - 1) - n_V - n_S 2^{N - 5}] \) and \( N - 1 \) eigenvalues \( 2a_1 (1 - n_S 2^{N - 6}) \), showing that we need, for any \( n_V, n_S \leq 2^{6-N} \) and thus \( n_S = 0 \) for \( M = 2N > 12 \). Considering also the limit where only \( a_1 = a_2 \neq 0 \) again yields \( n_V \leq M - 4 \) as a necessary condition on \( n_V \) for any \( n_S \). Note that when \( n_V = M - 4 \) there is a Higgs branch where \( \text{Spin}(2N) \) is broken to \( \text{Spin}(4) \cong SU(2) \times SU(2) \) with \( 2^{N-3} n_S \) hypermultiplets in the \((2, 1) + (1, 2)\). The condition that \( n_S \leq 2^{6-N} \) then corresponds to the condition that each \( SU(2) \) can have at most 8 flavors before leading to negative eigenvalues. Similar considerations apply for \( M = 2N + 1 \), showing that \( n_S \leq 2^{5-N} \).
7. Exceptional Gauge Groups

7.1. \(G_2\)

\(G_2\) with \(n_7\) matter fields in the 7 has a \(Sp(n_7) \times U(1)_I\) global symmetry. We can obtain the conditions for \(G_2\) simply by decomposing it in terms of \(SU(3)\): \(7 \rightarrow 3 + \bar{3} + 1, 14 \rightarrow 8 + 3 + \bar{3}\). Therefore, the effective gauge coupling for \(G_2\) can be obtained by substituting \(n_3 = 2(n_7 - 1)\) in the \(SU(3)\) answer. Corresponding to the condition \(n_3 \leq 6\) found in sect. 3, the necessary condition for convex \(\mathcal{F}\) for \(G_2\) is thus \(n_7 \leq 4\).

In terms of the variables \(a_i\) used in sect. 3 for \(SU(3)\), the Weyl chamber for \(G_2\) is \(a_1 \geq a_2 \geq 0\). At the boundary \(a_1 = a_2\), \(G_2\) is broken to \(SU(2) \times U(1)\), with the representations decomposing as \(7 \rightarrow 2_{\pm 1} + 1_{\pm 2} + 1_0\) and \(14 \rightarrow 2_{\pm 1} + 1_{\pm 2} + 3_0 + 2_{\pm 3} + 1_0\). At the other boundary, \(a_2 = 0\), \(G_2\) is broken to \(SU(2)' \times U(1)\), with the representations decomposing as \(7 \rightarrow 3_0 + 2_{\pm 1}\) and \(14 \rightarrow 3_0 + 4_{\pm 1} + 1_{\pm 2} + 1_0\). For \(n_7 > 0\) there is everywhere a Higgs branch with \(G_2\) broken to \(SU(3)\) according to the above decomposition.

7.2. \(F_4\)

The classical theory with \(n_{26}\) matter fields in the fundamental representation has an \(Sp(n_{26}) \times U(1)_I\) global symmetry. To find the condition for a nontrivial fixed point, we can decompose in terms of \(Spin(9)\): \(26 \rightarrow 9 + 16 + 1\) and \(52 \rightarrow 36 + 16\). Therefore, the effective gauge coupling for \(F_4\) can be obtained by substituting \(n_V = n_{26}\) and \(n_S = n_{26} - 1\) in the \(Spin(9)\) answer. This gives \(n_{26} \leq 3\) as a necessary (though perhaps not sufficient) condition for convex \(\mathcal{F}\). For example, with \(n_{26} = 3\) there is a Higgs branch to \(Spin(6) \cong SU(4)\) with 8 matter fields in the 4, just making the condition for non-negative eigenvalues. On the other hand, for \(n_{26} = 4\) there is a Higgs branch to \(Spin(5) \cong Sp(2)\) with 12 matter fields in the 4, which yields a negative eigenvalue.

7.3. \(E_6\), \(E_7\), and \(E_8\)

For \(E_6\) there could be a nontrivial fixed point associated with theories with \(n_{27}\) matter fields in the 27. The classical global symmetry is \(SU(n_{27}) \times U(1)_I\). For \(n_{27} > 0\), there is a Higgs branch with \(E_6\) broken to \(F_4\) with \(n_{26} = n_{27} - 1\). Corresponding to the necessary (though perhaps not sufficient) condition found for \(F_4\), a necessary (though perhaps not sufficient) condition for convex \(\mathcal{F}\) is \(n_{27} \leq 4\).

For \(E_7\) there could be nontrivial fixed points with \(n_{56}\) matter fields in the 56. Because the 56 is pseudo-real, \(n_{56}\) can be half-integral. Unlike the \(Sp(N)\) case, where an anomaly
required an even number of half-hypermultiplets, here there is no such anomaly and the number \( n_{56} \) can be half-integral\(^2\). The classical global symmetry is \( \text{Spin}(2n_{56}) \). For \( n_{56} \geq 0 \), there is a Higgs branch with \( E_7 \) broken to \( E_6 \) with \( n_{27} = 2(n_{56} - 1) \). Corresponding to the condition found above for \( E_6 \), \( n_{56} \leq 3 \) is thus a necessary (though perhaps not sufficient) condition for convex \( \mathcal{F} \).

For \( E_8 \), because the fundamental representation is the adjoint, there can only be a new fixed point for the theory without matter. Because \( \pi_4(E_8) = 0 \), there should only be one such theory.

### 8. Calabi–Yau Interpretation

We now turn to the Calabi–Yau interpretation of these theories. Non-Abelian gauge symmetry in stringy models compactified on a Calabi–Yau threefold arises from singularities along holomorphic curves in the threefold [22,23]. This happens in type IIA compactifications, M-theory compactifications, and F-theory compactifications (to four, five and six dimensions, respectively), with the same geometry governing all cases [24,25,16,26]. The easiest way to describe the geometry is by considering a desingularization \( \pi : X \to \overline{X} \) of the original Calabi–Yau space \( X \), by a Calabi–Yau manifold \( X \) which contains a collection of complex surfaces \( S_j \) that shrink to a holomorphic curve \( \overline{C} \) on \( \overline{X} \). (This “shrinking” is accomplished by sending the Kähler class of \( X \) to a point on an appropriate boundary wall of the Kähler cone.) There are holomorphic curves \( \sigma_\alpha \) within those surfaces which shrink to zero size in this limit, and map to points on \( \overline{X} \); among these are the generic fiber \( \varepsilon_j \) of the map \( S_j \to \overline{C} \), for each \( j \).

The gauge group for M-theory compactified on \( X \) is\(^3\) \( H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) \). On \( \overline{X} \), this is enhanced to a non-Abelian group whose simple roots correspond to the cohomology classes \( [S_j] \in H^2(X, \mathbb{Z}) \). The charged matter which becomes massless when the symmetry is enhanced arises from 2-branes wrapping connected holomorphic curves \( \sigma \) on \( X \) which

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\(^2\) Indeed, compactification of the heterotic string on \( K3 \times S^1 \) leads to \( E_7 \) in five dimensions with \( n_{56} = \frac{1}{2}k - 2 \) for an \( E_8 \) with \( k \) instantons.

\(^3\) For the type IIA compactification on \( X \), the gauge group is \( (H^0(X, \mathbb{R}) \oplus H^2(X, \mathbb{R}))/ (H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})) \). If \( X \) is elliptically fibered, then the Cartan subgroup of the gauge group for the \( F \)-theory compactification on the base of the fibration is \( H^2(X, \mathbb{R})_0/H^2(X, \mathbb{Z})_0 \), where \( H^2(X)_0 \) is the subspace of \( H^2(X) \) orthogonal to the class of the elliptic curve. The non-Abelian enhancement in these cases is similar to that of the M-theory compactification.
shrink to points in $\overline{X}$. (More precisely, each such curve determines a hypermultiplet which contains 2-branes wrapping both the holomorphic curve $\sigma$ and the anti-holomorphic curve $\overline{\sigma}$.) The homology classes $[\sigma] \in H_2(X, \mathbb{Z})$ and $[\overline{\sigma}] = -[\sigma]$ determine the charges, so $H_2(X, \mathbb{Z})$ should be identified with the weight lattice. The homology classes of the generic fibers $\varepsilon_j$ are the adjoint weights $[\varepsilon_j]$ dual to the simple roots $[S_j]$.

The geometric configurations of surfaces $S_j$ which give rise to all possible non-Abelian gauge groups are known. Given such a collection of surfaces, the charged matter spectrum is determined by the parameter curves for the fibers $\varepsilon_j$ and by the possible singular fibers [27,28]. In many cases, it is known how to relate specific types of singular fibers to specific matter representations [29–33]. Here, we run the arguments in reverse, and determine the geometric structure when the matter representation is given, using the fact that for each class $[\sigma]$ in the matter representation, $\pm [\sigma]$ must be represented by a connected holomorphic curve.

For the compactification of M-theory on a Calabi–Yau threefold $X$, the scalars in the vector multiplets parameterize the Kähler classes of unit volume, and the prepotential is determined by the intersection numbers among those classes [34,35,36]. Concretely, if we choose a basis $J_i$ for the Kähler classes and write an arbitrary class in the form $J = \sum \phi^i J_i$, then these scalars can be identified with the $\phi^i$’s (subject to the constraint of the volume being one), and the coefficients $c_{ijk}$ in $F = \frac{1}{6} \sum_{i,j,k} c_{ijk} \phi^i \phi^j \phi^k$ are the topological intersection numbers $c_{ijk} = \int J_i \wedge J_j \wedge J_k$.

The Kähler classes on $X$ are related to the positive classes on $\overline{X}$ as follows. Given a positive class $\kappa$ on $\overline{X}$, consider the relative Kähler cone

\[ K(X/\overline{X}) = \{ S = \sum \psi_j S_j \mid \kappa + S \text{ defines a Kähler class on } X \}. \quad (8.1) \]

According to the Kleiman criterion for ampleness [37], this cone can also be described as

\[ K(X/\overline{X}) = \{ S = \sum \psi_j S_j \mid S \cdot \sigma > 0 \text{ for all } \sigma \text{ mapping to points on } \overline{X} \}, \quad (8.2) \]

which shows that the definition is independent of $\kappa$. This description can be used in two ways: the holomorphic classes $\sigma$ can be used to characterize the cone, or the cone can be used to determine which of $\pm \sigma$ is holomorphic.

The mathematical structure of the relative Kähler cone is known in considerable detail [38]. For many purposes, it is better to work with the negative of this cone, defined by

\[ -K(X/\overline{X}) = \{ S = \sum \psi_j S_j \mid -S \cdot \sigma > 0 \text{ for all } \sigma \text{ mapping to points on } \overline{X} \}; \quad (8.3) \]
this has the advantage that the coefficients $\psi_j$ will be positive. We identify the set of
volume one elements in this cone as being the portion of the vector multiplet moduli space
which is relevant for discussing the contraction $X \to \overline{X}$.

It is possible for $\overline{X}$ to have more than one desingularization which is a Calabi–Yau
manifold. That is, there could be distinct Calabi–Yau manifolds $X_1, X_2, \ldots, X_k$ all having
maps $\pi_\alpha : X_\alpha \to \overline{X}$ which resolve the singularities. In this situation, the various Calabi–
Yau manifolds must differ by flops [38,39]. There are canonical identifications among the
spaces $H^2(X_\alpha, \mathbb{R})$, in which the Kähler cones of the various $X_\alpha$’s share common boundary
walls. The vector multiplet moduli space should include the set of volume one elements
within the union of all of these cones [40,16].

The flops will affect the relative Kähler cones in a similar way: the cones $-K(X_\alpha/\overline{X})$
will share common boundary walls, and the relevant part of the vector multiplet moduli
space should include the set of volume one elements within the union of all of these cones.

The prepotential in these theories is determined by the intersection properties of
divisors on $X_\alpha$. Specifically, if we write a general element in the negative of the relative
Kähler cone $-K(X_\alpha/\overline{X})$ in the form $S = \sum \psi_j S_j$, then the coefficients $\psi_j$ are scalars which
parameterize the moduli space, in terms of which the prepotential can be written

$$\mathcal{F} = \frac{1}{6} \sum_{ijk} c_{ijk} \psi_i \psi_j \psi_k,$$

(8.4)

with the coefficients $c_{ijk}$ coinciding with the intersection numbers $S_i \cdot S_j \cdot S_k$, calculated on
the threefold $X_\alpha$. (We can express this rather compactly by writing $\mathcal{F} = \frac{1}{6} S^3$.) Since we are
using a basis of Kähler classes that came from integral cohomology classes $S_j \in H^2(X_\alpha, \mathbb{Z})$,
the coefficients $c_{ijk}$ are integers, as expected from the quantization condition (2.8). Notice
that when we move from $X_\alpha$ to $X_\beta$, these coefficients will change.

In the remainder of this section, we discuss the detailed geometry on the Calabi–
Yau space corresponding to various specific groups, with specified matter content. We
include adjoint matter in each of the cases we discuss (even though it is not needed for
strong coupling limits), since the geometry associated to the adjoint matter is somewhat
intertwined with that of the other matter representations. The number of hypermultiplets
in the adjoint representation is determined by the genus of the parameter curves for the
rulings on the ruled surfaces [25].

Our primary goal will be to make the geometry sufficiently explicit that the prepo-
tential $\mathcal{F} = \frac{1}{6} S^3$ can be calculated in detail. We will find that in each case we consider,
there is a remarkable correspondence with the group-theoretic calculations made earlier. The combinatorial manipulations which are needed to fully check this are quite involved, and the agreement we obtain looks like another “string miracle,” as anticipated by Harvey and Moore [12].

We limit our calculations to the classical groups, with some restriction on the matter content beyond that suggested by the strong-coupling limit problem. (These restrictions simplify the geometry substantially.) We are confident that similar calculations with more general matter content or for the exceptional groups would be equally successful (as was suggested explicitly for $E_8$ in [12]). We will turn to the consideration of strong coupling limits in the next section.

8.1. $Sp(N)$

We begin with the case of gauge group $Sp(N)$, with hypermultiplets in $g$ copies of the adjoint representation, $n_A$ copies of the antisymmetric representation, and $n_F$ copies of the fundamental representation. (More precisely, we have $2n_F$ half-hypermultiplets in the fundamental representation.) In order to produce $Sp(N)$ gauge symmetry when the ruled surfaces $S_j$ shrink to zero size, one of them—labeled $S_N$ below—must be ruled over a holomorphic curve of genus $g$, with the others being ruled over a holomorphic curve of genus $g'$ which has a two-to-one map to the curve of genus $g$. (This corresponds to the fact that the first $N - 1$ roots of $Sp(N)$ have length half that of the $N$-th root.) The surfaces $S_j$ and $S_{j+1}$ must meet along a holomorphic curve $\gamma_j$. When $j < N - 1$ this curve is a section for the ruling on each surface, while $\gamma_{N-1}$ is a section on $S_{N-1}$ and a 2-section on $S_N$. All of these curves of intersection have genus $g'$.

If we let $\varepsilon_j$ denote the class of a fiber of the ruling on $S_j$, it follows that

\[
S_j \cdot \varepsilon_k = \begin{cases} 
-2 & \text{if } k = j \\
1 & \text{if } |k - j| = 1, \ k \neq N \\
2 & \text{if } k = j + 1 = N \\
0 & \text{otherwise}, 
\end{cases}
\]  

which reproduces the Cartan matrix for $Sp(N)$ (with the usual sign change needed when passing between the conventions of Lie group theory and algebraic geometry). The configuration of surfaces is illustrated in figure 1. (Dotted lines in the figures are intended as an aid to visualization, and have no particular meaning; thick lines indicate the curves
Figure 1. The configuration of surfaces yielding $Sp(N)$ with $n_F$ fundamentals.

along which pairs of surfaces meet; thin lines indicate curves which are contained in only one surface.

Following [28], we note that each of the adjoint weights (which can all be written as combinations of the $\varepsilon_k$'s) is responsible for either $g$ or $g'$ hypermultiplets of charged matter, depending on whether the corresponding parameter curve has genus $g$ or $g'$. These hypermultiplets can be collected into $g$ copies of the adjoint representation and $g' - g$ copies of the antisymmetric representation. In fact, this is the only way that the antisymmetric representation can arise, so we find that $n_A = g' - g$.

For each copy of the fundamental representation in the charged matter spectrum (each one filling out a half-hypermultiplet), let $\sigma_N^{(\alpha)}$ be the lowest weight in the representation, so that the other weights are given by

$$\sigma_k^{(\alpha)} = \varepsilon_k + \varepsilon_{k+1} + \ldots + \varepsilon_{N-1} + \sigma_N^{(\alpha)}.$$  \hfill (8.6)

These classes intersect the surfaces $S_j$ according to

$$S_j \cdot \sigma_k^{(\alpha)} = \begin{cases} -1 & \text{if } k = j \\ 1 & \text{if } k = j + 1 \\ 0 & \text{if } k \neq j, j + 1. \end{cases} \quad (8.7)$$

(As a partial converse, we can write

$$\varepsilon_k = \sigma_k^{(\alpha)} - \sigma_{k+1}^{(\alpha)}$$  \hfill (8.8)

for $k \leq N - 1$.)

Thus, if we let $S := \sum_{j=1}^N \varphi_j S_j$ be an arbitrary divisor supported on the exceptional locus, and introduce coordinates $a_k = -S \cdot \sigma_k^{(\alpha)} = \varphi_k - \varphi_{k-1}$ on the space of such divisors.
(setting \( \varphi_0 = 0 \) for convenience), we can describe the negative of the relative Kähler cone as being contained in another cone:

\[
-\mathcal{K}(X/X) \subseteq \{ S = \sum \varphi_j S_j \mid -S \cdot \varepsilon_k > 0 \} = \{ S \mid a_1 > a_2 > \cdots > a_N > 0 \}, \tag{8.9}
\]

where in the last equality we used (8.8), the definition of \( a_k \), and the fact that \(-S \cdot \varepsilon_N = 2a_N\). In other words, \(-\mathcal{K}(X/X)\) is contained in the usual Weyl chamber for \( Sp(N) \).

The weights \( \sigma_k^{(\alpha)} \) all define positive functions on the Weyl chamber, so we will have \(-S \cdot \sigma_k^{(\alpha)} > 0\) whenever \( S \in -\mathcal{K}(X/X) \). This tells us that the negative of the relative Kähler cone is the Weyl chamber, and also that all of the classes \( \sigma_k^{(\alpha)} \) are represented by unions of holomorphic curves. In fact, \( \sigma_N^{(\alpha)} \) will be represented by an irreducible holomorphic curve, while \( \sigma_k^{(\alpha)} \) for \( k < N \) will be represented by a reducible curve as specified in (8.6), with the fibers being chosen so that the resulting curve is connected. (Such fibers appear in figure 1 as thin lines.)

Note that \( \sigma_N^{(\alpha)} \) is an exceptional curve of the first kind on \( S_N \). The fiber which contains it must be reducible, with another component \( \sigma_N^{(\alpha')} := \varepsilon_N - \sigma_N^{(\alpha)} \) which is also an exceptional curve of the first kind, and which also serves as the lowest weight for a fundamental half-hypermultiplet. (One such pair is illustrated in figure 1.) Thus, we see that the fundamental representations must occur in pairs, i.e., the number of half-hypermultiplets is even, confirming one of our predictions from the field theory analysis. There are \( 2n_F \) exceptional curves of the first kind contained in fibers, combining to give \( n_F \) reducible fibers on the ruled surface \( S_N \).

We can compute the cubic intersection form as follows. First, on a minimal ruled surface \( S \) over a holomorphic curve of genus \( g \), any 2-section \( \gamma \) of genus \( g' \) must satisfy \( \gamma^2 = 4(g' - 2g + 1) \). To apply this in our situation, we must modify the formula by the \( n_F \) blowups to which \( S_N \) has been subjected; the result is

\[
S_{N-1}^2 S_N = (\gamma_{N-1})^2_{S_N} = 4(g' - 2g + 1) - n_F = (8 - 8g - n_F) + (4g' - 4). \tag{8.10}
\]

Second, the holomorphic curve \( \gamma_j = S_j \cap S_{j+1} \) has genus \( g' \), so its normal bundle must have degree \( 2g' - 2 \); this implies that

\[
S_j^2 S_{j+1} + S_j S_{j+1}^2 = (\gamma_j)^2_{S_{j+1}} + (\gamma_j)^2_{S_j} = 2g' - 2. \tag{8.11}
\]

Third, since each \( S_j \) for \( j < N \) is a minimal ruled surface with disjoint sections we find

\[
S_{j-1}^2 S_j + S_j S_{j+1}^2 = (\gamma_{j-1})^2_{S_j} + (\gamma_j)^2_{S_j} = 0 \quad \text{for all } j \leq N-1. \tag{8.12}
\]
Now an easy induction based on (8.10), (8.11), and (8.12) yields the formula
\[ S_j^2 S_{j+1} = (j - N + 3)(2g' - 2) + (8 - 8g - n_F) \]
\[ S_j S_{j+1}^2 = (N - j - 2)(2g' - 2) - (8 - 8g - n_F). \]
(8.13)

The only other intersection numbers which are non-zero are determined by the fact that the complex surfaces \( S_j \) for \( j < N \) are minimal ruled surfaces over a holomorphic curve of genus \( g' \), whereas \( S_N \) is ruled over a holomorphic curve of genus \( g \) with \( n_F \) reducible fibers. This implies that
\[ S_j^3 = (K_S_j)^2 S_j = \begin{cases} 8 - 8g' & \text{if } j < N \\ 8 - 8g - n_F & \text{if } j = N. \end{cases} \]
(8.14)

We can assemble (8.14) and (8.13) into a single formula for the cubic form:
\[ S^3 = \left( \sum_{j=1}^{N} \varphi_j S_j \right)^3 \]
\[ = (8 - 8g - n_F)\varphi_N^3 + (8 - 8g') \sum_{j=1}^{N-1} \varphi_j^3 + 3(g' - 1) \sum_{j=1}^{N-1} (\varphi_j^2 \varphi_{j+1} + \varphi_j \varphi_{j+1}^2) \]
\[ + 3 \sum_{j=1}^{N-1} ((8 - 8g - n_F) + (2j + 5 - 2N)(g' - 1))(\varphi_j^2 \varphi_{j+1} - \varphi_j \varphi_{j+1}^2). \]
(8.15)

To put this into a more convenient form, we need two algebraic identities, which can be established easily by induction from our basic relations \( a_j = \varphi_j - \varphi_{j-1}, \varphi_0 = 0 \).
\[ \sum_{j=k+1}^{N} a_j^3 = -\varphi_k^3 + \varphi_N^3 + 3 \sum_{j=k}^{N-1} (\varphi_j^2 \varphi_{j+1} - \varphi_j \varphi_{j+1}^2) \]
(8.16a)
\[ \sum_{1 \leq i < j \leq N} ((a_i - a_j)^3 + (a_i + a_j)^3) = 8 \sum_{j=1}^{N-1} \varphi_j^3 - 3 \sum_{j=1}^{N-1} (\varphi_j^2 \varphi_{j+1} + \varphi_j \varphi_{j+1}^2) \]
\[ + 3 \sum_{j=1}^{N-1} (2N - 2j - 5)(\varphi_j^2 \varphi_{j+1} - \varphi_j \varphi_{j+1}^2). \]
(8.16b)

Then (8.15) can be rewritten as
\[ S^3 = (8 - 8g - n_F) \sum_{j=1}^{N} a_j^3 + (1 - g') \sum_{1 \leq i < j \leq N} ((a_i - a_j)^3 + (a_i + a_j)^3) \]
\[ = (1 - g) \left( 8 \sum_{j=1}^{N} a_j^3 + \sum_{1 \leq i < j \leq N} ((a_i - a_j)^3 + (a_i + a_j)^3) \right) \]
\[ - n_F \left( \sum_{j=1}^{N} a_j^3 \right) - n_A \left( \sum_{1 \leq i < j \leq N} ((a_i - a_j)^3 + (a_i + a_j)^3) \right), \]
(8.17)
where in the last line we used the fact that $n_A = g' - g$. This formula is in perfect agreement with (4.1), since the prepotential in the Calabi–Yau theories is given by $F = \frac{1}{6} S^3$.

8.2. $SU(N)$

We next turn to the case of gauge group $SU(N)$, with hypermultiplets in $g$ copies of the adjoint representation and $n_F$ copies of the fundamental representation. (We omit the antisymmetric representation in order to keep the geometry simple.) Producing $SU(N)$ gauge symmetry is relatively straightforward: we need $N - 1$ complex surfaces $S_j$ which shrink to zero size, with each $S_j$ being ruled over a holomorphic curve of genus $g$. The surfaces $S_j$ and $S_{j+1}$ must meet along a holomorphic curve $\gamma_j$ which is a section for the ruling on each of them.

If we let $\varepsilon_j$ denote the class of a fiber of the ruling on $S_j$, it follows that

$$S_j \cdot \varepsilon_k = \begin{cases} -2 & \text{if } k = j \\ 1 & \text{if } |k - j| = 1 \\ 0 & \text{if } |k - j| > 1, \end{cases}$$  
(8.18)

which reproduces the negative of the Cartan matrix for $SU(N)$. The configuration of surfaces is illustrated in figure 2.

![Figure 2. The configuration of surfaces yielding $SU(N)$ with $n_F$ fundamentals.](image)

For each copy of the fundamental representation in the charged matter spectrum, let $\sigma_N^{(\alpha)}$ be the lowest weight in the representation, so that the complete set of weights in the representation are given by

$$\sigma_k^{(\alpha)} = \varepsilon_k + \varepsilon_{k+1} + \ldots + \varepsilon_{N-1} + \sigma_N^{(\alpha)}. \quad (8.19)$$
(Note that the weights occurring in the charged matter spectrum are then \( \pm \sigma_k^{(\alpha)} \), since the spectrum contains both the fundamental representation and its complex conjugate, pairing up to form hypermultiplets.) These classes intersect the surfaces \( S_j \) according to

\[
S_j \cdot \sigma_k = \begin{cases} 
-1 & \text{if } k = j \\
1 & \text{if } k = j + 1 \\
0 & \text{if } k \neq j, j + 1.
\end{cases} \tag{8.20}
\]

(Conversely, we can write

\[
\varepsilon_k = \sigma_k^{(\alpha)} - \sigma_{k+1}^{(\alpha)} \tag{8.21}
\]

for \( k \leq N - 1 \).

Thus, if we let \( S := \sum_{j=1}^{N-1} \varphi_j S_j \) be an arbitrary divisor supported on the exceptional locus, and introduce coordinates \( a_k = -S \cdot \sigma_k^{(\alpha)} = \varphi_k - \varphi_{k-1} \) on the space of such divisors (setting \( \varphi_0 = \varphi_N = 0 \) for convenience), we can describe the negative of the relative Kähler cone as being contained in another cone:

\[
-\mathcal{K}(X/\mathcal{X}) \subseteq \{ S = \sum \varphi_j S_j \mid -S \cdot \varepsilon_k > 0 \} = \{ S \mid a_1 > a_2 > \cdots > a_N \}, \tag{8.22}
\]

where in the last equality we used (8.21) and the definition of \( a_k \). In other words, \(-\mathcal{K}(X/\mathcal{X})\) is contained in the usual Weyl chamber for \( SU(N) \).

At any given point in the Weyl chamber, the functions defined by the weights \( \sigma_k^{(\alpha)} \) are positive for small values of \( k \), and negative for large values of \( k \). So on the cone \(-\mathcal{K}(X/\mathcal{X})\), there must be some index \( \ell \) such that \( a_\ell > 0 > a_{\ell+1} \) on that cone. That condition identifies \(-\mathcal{K}(X/\mathcal{X})\) as one of the sub-wedges we encountered when studying the gauge theory.

The classes represented by connected holomorphic curves on \( X \) are then \( \sigma_1, \ldots, \sigma_\ell \) and \(-\sigma_{\ell+1}, \ldots, -\sigma_N \). These can all be written in the form

\[
\sigma_k^{(\alpha)} = \varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_{\ell-1} + \sigma_\ell^{(\alpha)} \quad \text{for } k \leq \ell
\]

\[
-\sigma_k^{(\alpha)} = -\sigma_{\ell+1}^{(\alpha)} + \varepsilon_{\ell+1} + \varepsilon_{\ell+2} + \cdots + \varepsilon_{k-1} \quad \text{for } k \geq \ell + 1,
\tag{8.23}
\]

so we see that \( \sigma_\ell^{(\alpha)} \) and \(-\sigma_{\ell+1}^{(\alpha)} \) are represented by irreducible holomorphic curves. Moreover, the remaining fiber class can be written as \( \varepsilon_\ell = \sigma_\ell^{(\alpha)} + (-\sigma_{\ell+1}^{(\alpha)}) \), so we see that there are \( n_F \) reducible fibers in the ruling on \( S_\ell \), while the other \( S_j \)'s are minimal ruled surfaces. The fibers and reducible fibers which account for the matter representation are illustrated in figure 2 with thin lines.

The other sub-wedges within the Kähler cone must be related to the given one by performing flops. Explicitly in this case, if we flop the curves \( \sigma_\ell^{(\alpha)} \), we move the reducible

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fibers from $S_\ell$ to $S_{\ell-1}$, whereas if we flop the curves $-\sigma^{(\ell)}\omega_{\ell+1}$ we move the reducible fibers from $S_\ell$ to $S_{\ell+1}$.

We can compute the cubic intersection form and complete the specification of the geometric data as follows. First, the holomorphic curve $\gamma_j = S_j \cap S_{j+1}$ has genus $g$, so its normal bundle must have degree $2g - 2$; this implies that

$$S_j^2 S_{j+1} + S_j S_{j+1}^2 = (\gamma_j)^2 S_{j+1} + (\gamma_j)^2 S_j = 2g - 2. \quad (8.24)$$

Second, since each $S_j$ for $j \neq \ell$ is a minimal ruled surface with disjoint sections while $S_\ell$ has been blown up $n_F$ times from a minimal ruled surface, we find

$$S_{j-1}^2 S_j + S_j S_{j+1}^2 = (\gamma_{j-1})^2 S_j + (\gamma_j)^2 S_j = \begin{cases} 0 & \text{if } j \neq \ell \\ -n_F & \text{if } j = \ell. \end{cases} \quad (8.25)$$

These two relations together determine these intersection numbers up to one undetermined constant—there is no analogue of (8.10) available to us in this case. That is, although the relative values of the self-intersections of the holomorphic curves $\gamma_j$ have been determined, the actual values have not been. To specify these, we define

$$c' := \begin{cases} (N - 2)(g - 1) - S_1 S_2^2 & \text{if } \ell > 1 \\ (N - 2)(g - 1) - n_F - S_1 S_2^2 & \text{if } \ell = 1. \end{cases} (8.26)$$

Then an easy induction based on (8.24), (8.25), and (8.26) shows that

$$S_j^2 S_{j+1} = \begin{cases} (2j + 2 - N)(g - 1) + c' & \text{if } j < \ell \\ (2j + 2 - N)(g - 1) + n_F + c' & \text{if } j \geq \ell, \end{cases} \quad (8.27)$$

$$S_j S_{j+1}^2 = \begin{cases} (N - 2j)(g - 1) - c' & \text{if } j < \ell \\ (N - 2j)(g - 1) - n_F - c' & \text{if } j \geq \ell, \end{cases}$$

so that all values are determined by this one choice.

Finally, since the complex surfaces $S_j$ are ruled over a holomorphic curve of genus $g$, and all of them except for $S_\ell$ are minimal ruled surfaces, whereas $S_\ell$ has $n_F$ reducible fibers, it follows that

$$S_j^3 = (K_{S_j})^2 = \begin{cases} 8 - 8g & \text{if } j \neq \ell \\ 8 - 8g - n_F & \text{if } j = \ell. \end{cases} (8.28)$$

We can assemble (8.27) and (8.28) into a single formula for the cubic form:

$$S^3 = \left( \sum_{j=1}^{N-1} \varphi_j S_j \right)^3 = -n_F \varphi_\ell^3 + (8 - 8g) \sum_{j=1}^{N-1} \varphi_j^3 + 3(g - 1) \sum_{j=1}^{N-2} (\varphi_j^2 \varphi_{j+1} + \varphi_j \varphi_{j+1}^2) \\
+ 3 \sum_{j=1}^{N-2} (c' + (2j + 1 - N)(g - 1))(\varphi_j^2 \varphi_{j+1} - \varphi_j \varphi_{j+1}^2) \\
+ 3n_F \sum_{j=\ell}^{N-2} (\varphi_j^2 \varphi_{j+1} - \varphi_j \varphi_{j+1}^2). \quad (8.29)$$

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To convert this expression into the $a_j$ coordinates, we need one additional algebraic identity, valid when $\varphi_N = 0$:

$$
\sum_{1 \leq i < j \leq N} (a_i - a_j)^3 = 8 \sum_{j=1}^{N-1} \varphi_j^3 - 3 \sum_{j=1}^{N-1} (\varphi_j^2 \varphi_{j+1} + \varphi_j \varphi_{j+1}^2) \\
+ 3 \sum_{j=1}^{N-1} (N - 2j - 1)(\varphi_j^2 \varphi_{j+1} - \varphi_j \varphi_{j+1}^2).
$$

(8.30)

Using this together with (8.16a,b) and exploiting the fact that $\varphi_N = 0$, we can rewrite (8.29) as

$$
S^3 = -n_F \sum_{j=\ell+1}^{N} a_j^3 + (1 - g) \sum_{1 \leq i < j \leq N} (a_i - a_j)^3 + c' \sum_{j=1}^{N} a_j^3
$$

(8.31)

or, using the fact that $a_\ell > 0 > a_{\ell+1}$ and so $\sum_{j=\ell+1}^{N} a_j^3 = -\frac{1}{2} \left( \sum_{j=1}^{N} a_j^3 - \sum_{j=1}^{N} |a_j|^3 \right)$, as

$$
S^3 = (1 - g) \sum_{i<j} (a_i - a_j)^3 + (c' + \frac{n_F}{2}) \sum_{j=1}^{N} a_j^3 - \frac{n_F}{2} \sum_{j=1}^{N} |a_j|^3.
$$

(8.32)

In this latter form, we have a single formula, valid throughout the Weyl chamber.

Comparing with (5.2), we see that $c = c' + \frac{n_F}{2}$ must be the coefficient of the Chern–Simons term in the field theory. (In fact, it was the presence of the additional degree of freedom $c'$ in specifying these $SU(N)$ Calabi–Yau theories which led us to discover the possibility of such a Chern–Simons term.) We immediately recover the condition that $c + \frac{n_F}{2}$ must be an integer.

### 8.3. Spin($2N + 1$)

We now consider the gauge group $Spin(2N + 1)$, with hypermultiplets in $g$ copies of the adjoint representation, and $n_{V'}$ copies of the vector representation. (To keep the geometry relatively simple, we do not allow matter in spinor representations, in either this case or the next one.) In order to produce $Spin(2N + 1)$ gauge symmetry when the ruled surfaces $S_j$ shrink to zero size, all but one of them must be ruled over a holomorphic curve of genus $g$, while the last surface—labeled $S_N$ below—must be ruled over a holomorphic curve of genus $g'$ which has a two-to-one map to the curve of genus $g$. (This is the opposite

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4 There is another way to get $Spin(2N + 1)$ gauge symmetry [28], but it leads to charged matter in the antisymmetric rather than the vector representation.
of what happened with $Sp(N)$, corresponding to the exchange of long and short roots.)

The surfaces $S_j$ and $S_{j+1}$ must meet along a holomorphic curve $\gamma_j$. When $j < N - 1$ this curve is a section for the ruling on each surface, while $\gamma_{N-1}$ is a 2-section on $S_{N-1}$ and a section on $S_N$. The curve of intersection $\gamma_j$ has genus $g$ for $j < N - 1$, while $\gamma_{N-1}$ has genus $g'$.

If we let $\varepsilon_j$ denote the class of a fiber of the ruling on $S_j$, it follows that

$$S_j \cdot \varepsilon_k = \begin{cases} 
-2 & \text{if } k = j \\
1 & \text{if } |k - j| = 1, j \neq N \\
2 & \text{if } k + 1 = j = N \\
0 & \text{otherwise.}
\end{cases} \quad (8.33)$$

which reproduces the negative of the Cartan matrix for $Spin(2N + 1)$. The configuration of surfaces is illustrated in figure 3.

![Figure 3. The configuration of surfaces yielding $Spin(2N + 1)$ with $n_V$ vectors.](image)

Following [28], we note that each of the adjoint weights (which can all be written as combinations of the $\varepsilon_k$’s) is responsible for either $g$ or $g'$ hypermultiplets of charged matter, depending on whether the corresponding parameter curve has genus $g$ or $g'$. These hypermultiplets can be collected into $g$ copies of the adjoint representation and $g' - g$ copies of the vector representation. In fact, this is the only way that the vector representation can arise, so we find that $n_V = g' - g$. Since this is the only matter content we have allowed, there will be no singular fibers in any of the rulings.

If we define

$$\sigma_k = \varepsilon_k + \ldots + \varepsilon_N, \quad (8.34)$$

then the classes in the vector representation consist of $\pm \sigma_1, \ldots, \pm \sigma_N$ and a neutral class. (These can be realized geometrically if we choose the fiber $\varepsilon_{N-1}$ to pass through one of
the ramification points of the two-to-one map $\gamma_{N-1} \to \gamma_{N-2}$, as illustrated in figure 3.) Conversely, we can write

$$\varepsilon_k = \sigma_k - \sigma_{k+1}$$

for $k \leq N$, setting $\varepsilon_{N+1} = 0$.

For ease of computation, we introduce $\tilde{S}_j$, defined as

$$\tilde{S}_j := \begin{cases} S_j & \text{if } j < N \\ \frac{1}{2} S_N & \text{if } j = N. \end{cases}$$

The intersection properties of these classes are

$$\tilde{S}_j \cdot \sigma_k = \begin{cases} -1 & \text{if } k = j \\ 1 & \text{if } k = j + 1 \\ 0 & \text{if } k \neq j, j + 1. \end{cases}$$

Thus, if we let let $S := \sum_{j=1}^{N} \varphi_j \tilde{S}_j$ be an arbitrary divisor supported on the exceptional locus, and introduce coordinates $a_k = -S \cdot \sigma_k = \varphi_k - \varphi_{k-1}$ on the space of such divisors (setting $\varphi_0 = 0$ for convenience), we can describe the negative of the relative Kähler cone as being contained in another cone:

$$-\mathcal{K}(X/X) \subseteq \{ S = \sum \varphi_j S_j \mid -S \cdot \varepsilon_k > 0 \} = \{ S \mid a_1 > a_2 > \cdots > a_N > 0 \},$$

where in the last equality we used (8.35) and the definition of $a_k$. In other words, $-\mathcal{K}(X/X)$ is contained in the usual Weyl chamber for $\text{Spin}(2N+1)$. In fact, since we have no singular fibers, we have accounted for all of the conditions and these two cones must coincide.

We can compute the cubic intersection form as follows. First, since $S_{N-1}$ is a minimal ruled surface over a holomorphic curve of genus $g$, and $\gamma_{N-1}$ is a 2-section on $S_{N-1}$, we have

$$S_{N-1} S_N^2 = (\gamma_{N-1})^2 S_{N-1} = 4(g' - 2g + 1) = (4g' - 4) - (8g - 8).$$

Moreover, since $\gamma_{N-1}$ has genus $g'$, its normal bundle has degree $2g' - 2$, from which it follows that

$$S_{N-1}^2 S_N + S_{N-1} S_N^2 = (\gamma_{N-1})^2 S_N + (\gamma_{N-1})^2 S_{N-1} = 2g' - 2.$$

Putting these together, we infer

$$S_{N-1}^2 S_N = (8g - 8) - (2g' - 2).$$
Calculating in terms of the $\tilde{S}_j$'s, we find
\[
\tilde{S}_{N-1}^2 \tilde{S}_N = \frac{1}{2} S_{N-1}^2 S_N = (4g - 4) + (1 - g') \tag{8.42}
\]
and
\[
\tilde{S}_{N-1}^2 \tilde{S}_N + \tilde{S}_{N-1}^2 \tilde{S}_N = \frac{1}{2} S_{N-1}^2 S_N + \frac{1}{4} S_{N-1}^2 S_N^2 = 2g - 2. \tag{8.43}
\]

Second, the holomorphic curve $\gamma_j = S_j \cap S_{j+1}$ has genus $g$ for $j \leq N - 2$, so its normal bundle must have degree $2g - 2$; this implies that
\[
S_j^2 S_{j+1}^2 + S_j S_{j+1}^2 = (\gamma_j^2 S_{j+1}^2 + (\gamma_j^2 S_j) = 2g - 2, \quad \text{for } j \leq N - 2. \tag{8.44}
\]

Combined with (8.43), we find
\[
\tilde{S}_j^2 \tilde{S}_{j+1}^2 + \tilde{S}_j \tilde{S}_{j+1}^2 = 2g - 2, \quad \text{for } j \leq N - 1. \tag{8.45}
\]

Third, the ruled surface $S_{N-1}$ has a 2-section $\gamma_{N-1}$ and a section $\gamma_{N-2}$ which are disjoint. Writing the 2-section in the form $\gamma_{N-1} = 2\gamma_{N-2} + k\varepsilon_{N-1}$ (in $H^2(S_{N-1})$), we find the relation $2\gamma_{N-2} + k = 0$, so that $\gamma_{N-1}^2 = 4\gamma_{N-2}^2 + 4k = -4\gamma_{N-2}^2$. In other words,
\[
4S_{N-2}^2 S_{N-1} + S_{N-1} S_N^2 = 4(\gamma_{N-2})_S_{N-1}^2 + (\gamma_{N-1})_S_{N-1}^2 = 0, \tag{8.46}
\]
from which it follows that
\[
\tilde{S}_{N-2}^2 \tilde{S}_{N-1} + \tilde{S}_{N-1} \tilde{S}_N^2 = 0. \tag{8.47}
\]

In addition, since $S_j$ is a minimal ruled surface with disjoint sections for $j \leq N - 2$, we find
\[
S_{j-1}^2 S_j + S_j S_{j+1}^2 = (\gamma_{j-1})_{S_j}^2 + (\gamma_j)_{S_j}^2 = 0, \quad \text{for } j \leq N - 2. \tag{8.48}
\]

Combined with (8.47), this implies
\[
\tilde{S}_{j-1}^2 \tilde{S}_j + \tilde{S}_j \tilde{S}_{j+1}^2 = 0, \quad \text{for } j \leq N - 1. \tag{8.49}
\]

Finally, all of the $S_j$'s being minimal ruled surfaces implies that
\[
S_j^3 = (K_{S_j})_{S_j}^2 = \begin{cases} 8 - 8g & \text{if } j < N \\ 8 - 8g' & \text{if } j = N, \end{cases} \tag{8.50}
\]
or in other words,
\[
\tilde{S}_j^3 = \begin{cases} 8 - 8g & \text{if } j < N \\ 1 - g' & \text{if } j = N. \end{cases} \tag{8.51}
\]
The relations (8.42), (8.45), (8.49), and (8.51) are precisely parallel to the relations (8.10), (8.11), (8.12), and (8.14), with the substitution of $1 - g'$ for $8 - 8g - n_F$ and $g$ for $g'$. We can thus immediately apply the results of our computation from the $Sp(N)$ case, and conclude that

$$\tilde{S}_j^{2} \tilde{S}_{j+1} = (j - N + 3)(2g - 2) + (1 - g')$$

$$\tilde{S}_j \tilde{S}_{j+1}^2 = (N - j - 2)(2g - 2) - (1 - g'),$$

as well as the final result

$$S^3 = (1 - g') \sum_{j=1}^{N} a_j^3 + (1 - g) \sum_{1 \leq i < j \leq N} ((a_i - a_j)^3 + (a_i + a_j)^3)$$

$$= (1 - g) \left( \sum_{j=1}^{N} a_j^3 + \sum_{1 \leq i < j \leq N} ((a_i - a_j)^3 + (a_i + a_j)^3) \right)$$

$$- n_V \left( \sum_{j=1}^{N} a_j^3 \right)$$

(8.53)

where in the last line we used the fact that $n_V = g' - g$. Since the prepotential is given by $F = \frac{1}{6} S^3$, we again find perfect agreement with the field theory formula (6.3).

8.4. $Spin(2N)$

Finally, we consider the gauge group $Spin(2N)$, with hypermultiplets in $g$ copies of the adjoint representation, and $n_V$ copies of the vector representation. As in the $SU(N)$ case, producing $Spin(2N)$ gauge symmetry is relatively straightforward: all of the complex surfaces $S_j$ must be ruled over a holomorphic curve of genus $g$, with $S_j$ and $S_{j+1}$ meeting along a holomorphic curve $\gamma_j$ which is a section of the rulings on both $(j \leq N - 2)$, and similarly for $S_{N-2}$ and $S_N$ meeting along $\gamma_{N-1}$.

If we let $\varepsilon_j$ denote the class of a fiber of the ruling on $S_j$, it follows that

$$S_j \cdot \varepsilon_k = \begin{cases} -2 & \text{if } k = j \\ 1 & \text{if } |k - j| = 1, j \neq N, k \neq N \\ 1 & \text{if } (j, k) = (N - 2, N) \text{ or } (N, N - 2) \\ 0 & \text{otherwise.} \end{cases}$$

(8.54)

which reproduces the negative of the Cartan matrix for $Spin(2N)$.

For each copy of the vector representation in the matter spectrum, there are weights $\sigma_1^{(\alpha)}, \ldots, \sigma_N^{(\alpha)}$, related to the adjoint weights by

$$\sigma_k^{(\alpha)} = \varepsilon_k + \cdots + \varepsilon_{N-1} + \sigma_N^{(\alpha)},$$

(8.55)
\[ \sigma^{(\alpha)}_{N-1} + \sigma^{(\alpha)}_N = \varepsilon_N, \quad (8.56) \]

such that \( \pm \sigma^{(\alpha)}_1, \ldots, \pm \sigma^{(\alpha)}_N \) are the weights occurring in the representation. Conversely, the \( \varepsilon_k \)'s can be determined from the \( \sigma^{(\alpha)}_k \)'s by (8.56) and

\[ \varepsilon_k = \sigma^{(\alpha)}_k - \sigma^{(\alpha)}_{k+1} \quad (8.57) \]

for \( k \leq N - 1 \).

Either \( \sigma^{(\alpha)}_N \) or \( -\sigma^{(\alpha)}_N \) must have positive intersection number with classes \( S \in -\mathcal{K}(X/X) \). In fact, since \( \sigma^{(\alpha)}_N = \frac{1}{2}(\varepsilon_N - \varepsilon_{N-1}) \), by exchanging labels on \( S_{N-1} \) and \( S_N \) if necessary, we can assume that

\[ -S \cdot \sigma^{(\alpha)}_N = -\frac{1}{2} S \cdot (\varepsilon_N - \varepsilon_{N-1}) > 0 \quad \text{for} \ S \in -\mathcal{K}(X/X). \quad (8.58) \]

(Such an exchange of labels is permissible due to the symmetry of the Dynkin diagram for \( Spin(2N) \).)

Since the classes \( \sigma^{(\alpha)}_k \) are all represented by connected holomorphic curves—and according to (8.55) all can be so represented in terms of \( \sigma^{(\alpha)}_N \) and various fibers of rulings—we see that \( \sigma^{(\alpha)}_N \) should actually be an irreducible holomorphic curve. Since \( S_N \cdot \sigma^{(\alpha)}_N = -1 \), this curve must be contained in \( S_N \) and be an exceptional curve of the first kind there. Moreover, since \( S_{N-1} \cdot \sigma^{(\alpha)}_N = 1 \) we see that \( S_N \) and \( S_{N-1} \) must meet each other.

From (8.55) and (8.56) we conclude that

\[ \varepsilon_N = \sigma^{(\alpha)}_{N-1} + \sigma^{(\alpha)}_N = \varepsilon_{N-1} + 2\sigma^{(\alpha)}_N. \]

Thus, we see that there is a particular fiber in the ruling on \( \varepsilon_N \) which can be written as \( \delta^{(\alpha)} + 2\sigma^{(\alpha)}_N \) for some irreducible holomorphic curve \( \delta^{(\alpha)} \), and \( \delta^{(\alpha)} \) also serves as a fiber in the ruling on \( S_{N-1} \). It is the curves \( \delta^{(\alpha)} \) along which \( S_N \) and \( S_{N-1} \) meet. This is illustrated in figure 4, where we show the configuration of surfaces, explicitly showing only one curve \( \delta^{(\alpha)} \) of intersection. The curve \( \sigma^{(\alpha)}_N \) (which lies on \( S_N \)) is indicated with a thin line.

There is one additional detail about the configuration which we need. Since \( S_N \cdot \delta^{(\alpha)} = S_N \cdot \varepsilon_{N-1} = 0 \), we see that \( \delta^{(\alpha)} \) is a holomorphic curve whose self-intersection on \( S_N \) is \(-2\). Under those circumstances, the only way that \( \delta^{(\alpha)} + 2\sigma^{(\alpha)}_N \) can be a fiber in the ruling is if the exceptional curve \( \sigma^{(\alpha)}_N \) passes through an ordinary double point, and indeed a local
computation shows that this happens [41]. This implies that the exceptional curve $\sigma_N^{(\alpha)}$ does not arise from an ordinary blowup, but rather from one of the “directed blowups of weight 1” introduced in [42]. The singular point is indicated in the figure with a dot.

For ease of computation, we introduce $\tilde{S}_j$, defined as

$$
\tilde{S}_j := \begin{cases} 
S_j & \text{if } j < N \\
\frac{1}{2}(S_N - S_{N-1}) & \text{if } j = N.
\end{cases}
$$

(8.59)

The intersection properties of the classes $\sigma_k^{(\alpha)}$ are

$$
\tilde{S}_j \cdot \sigma_k^{(\alpha)} = \begin{cases} 
-1 & \text{if } k = j \\
1 & \text{if } k = j + 1 \\
0 & \text{if } k \neq j, j + 1.
\end{cases}
$$

(8.60)

Thus, if we let $S := \sum_{j=1}^N \varphi_j \tilde{S}_j$ be an arbitrary divisor supported on the exceptional locus, and introduce coordinates $a_k = -S \cdot \sigma_k^{(\alpha)} = \varphi_k - \varphi_{k-1}$ on the space of such divisors (setting $\varphi_0 = 0$ for convenience), we can describe the negative of the relative Kähler cone as being contained in another cone:

$$
-K(X/X) \subseteq \{ S = \sum_{j=1}^N \varphi_j S_j \mid -S \cdot \varepsilon_k > 0, -S \cdot \sigma_N^{(\alpha)} > 0 \} = \{ S \mid a_1 > a_2 > \cdots > a_N > 0 \},
$$

(8.61)

where in the last equality we used (8.57), (8.58), and the definition of $a_k$. The cone on the right in (8.61) is the usual Weyl chamber for $Spin(2N)$, and all of the weights $\sigma_k^{(\alpha)}$ define positive functions on it; it follows that the two cones in (8.61) are equal. Note that in this case, in order to obtain the Weyl chamber we had to use more conditions than simply $-S \cdot \varepsilon_k > 0$.

We can compute the cubic intersection form as follows. First, all of the $S_j$’s are ruled surfaces over a holomorphic curve of genus $g$. These ruled surfaces are minimal if $j < N$,
whereas $S_N$ is obtained from a minimal ruled surface by making $n_V$ directed blowups of weight 1. Since each of these directed blowups decreases the value of $K_S^2$ by 2, we conclude that

$$S_j^3 = (K_{S_j})^2 S_j = \begin{cases} 8 - 8g & \text{if } j < N \\ 8 - 8g - 2n_V & \text{if } j = N. \end{cases} \quad (8.62)$$

Second, since there are $n_V$ of the holomorphic curves $\delta^{(\alpha)}$ which constitute the intersection of $S_N - 1$ and $S_N$, the numerical properties

$$S_{N-1} \cdot \delta^{(\alpha)} = -2, \quad S_N \cdot \delta^{(\alpha)} = 0 \quad (8.63)$$

immediately imply

$$S_{N-1}^2 S_N = -2n_V, \quad S_{N-1}^2 S_N^2 = 0. \quad (8.64)$$

Among the conclusions we can draw from (8.62) and (8.64) are

$$\widetilde{S}_{N-1} S_N = \frac{1}{2}(S_{N-1}^2 S_N - S_{N-1}^3) = 4g - 4 - n_V, \quad (8.65)$$

$$\widetilde{S}_{N-1}^2 S_N = \frac{1}{4}(S_{N-1}^2 S_N - 2S_{N-1}^2 S_N + S_{N-1}^3) = 2 - 2g + n_V, \quad (8.66)$$

and

$$\widetilde{S}_N^3 = \frac{1}{8}(S_N^3 - 3S_{N-1} S_N^2 + 3S_{N-1}^2 S_N - S_{N-1}^3) = -n_V. \quad (8.67)$$

The last one implies

$$\widetilde{S}_j^3 = \begin{cases} 8 - 8g & \text{if } j < N \\ -n_V & \text{if } j = N. \end{cases} \quad (8.68)$$

Third, the holomorphic curve $\gamma_j = S_j \cap S_{j+1}$ has genus $g$ for $j \leq N - 2$, so its normal bundle must have degree $2g - 2$; this implies that

$$S_j^2 S_{j+1} + S_j S_{j+1}^2 = (\gamma_j)_j^2 + (\gamma_j)_{j+1}^2 = 2g - 2, \quad \text{for } j \leq N - 2. \quad (8.69)$$

Similarly, $\gamma_{N-1} = S_{N-2} \cap S_N$ has genus $g$, so

$$S_{N-2} S_N^2 + S_{N-2} S_N = 2g - 2. \quad (8.70)$$

Adding (8.65) and (8.66) also yields $2g - 2$, so we find

$$\widetilde{S}_j^2 S_{j+1} + \widetilde{S}_j S_{j+1}^2 = 2g - 2, \quad \text{for } j \leq N - 1. \quad (8.71)$$
Finally, the minimal ruled surface $S_{N-2}$ has three sections $\gamma_{N-3}$, $\gamma_{N-2}$, and $\gamma_{N-1}$, which satisfy
\[(\gamma_{N-2} \cdot \gamma_{N-1})_{S_{N-2}} = n_V; \quad (\gamma_{N-3} \cdot \gamma_{N-2})_{S_{N-2}} = (\gamma_{N-3} \cdot \gamma_{N-1})_{S_{N-2}} = 0. \quad (8.72)\]
From these, it easily follows that $(\gamma_{N-3})_{S_{N-2}}^2 = -n_V$, $(\gamma_{N-2})_{S_{N-2}}^2 = (\gamma_{N-1})_{S_{N-2}}^2 = n_V$, or in other words,
\[S_{N-3}^2 S_{N-2} = -n_V; \quad S_{N-2} S_{N-1}^2 = S_{N-2} S_N^2 = n_V. \quad (8.73)\]
Combining this with (8.65) and (8.69) we find
\[\tilde{S}_{N-2}^2 \tilde{S}_{N-1} + \tilde{S}_{N-1} \tilde{S}_N^2 = S_{N-2}^2 S_{N-1} + 2 - 2g + n_V = -S_{N-2} S_{N-1}^2 + n_V = 0. \quad (8.74)\]
On the other hand, since $S_j$ is a minimal ruled surface with disjoint sections $\gamma_{j-1}$ and $\gamma_j$ for $j \leq N - 2$, we find
\[S_{j-1}^2 S_j + S_j S_{j+1}^2 = (\gamma_{j-1})_{S_j}^2 + (\gamma_j)_{S_j}^2 = 0, \quad \text{for } j \leq N - 2. \quad (8.75)\]
Combined with (8.74), this implies
\[\tilde{S}_{j-1}^2 \tilde{S}_j + \tilde{S}_j \tilde{S}_{j+1}^2 = 0, \quad \text{for } j \leq N - 1. \quad (8.76)\]
Before exploiting the parallel with the $Sp(N)$ computation, we must check that there are no other non-zero intersection numbers among the $\tilde{S}_j$'s. This is less obvious than in previous cases, since the $S_j$'s do have some other non-zero intersection numbers. We already saw non-zero values of $S_{N-2} S_N^2$ and $S_N^2 S_{N-2}$ occurring in (8.70) and (8.73); the only other such non-zero intersection number is
\[S_{N-2} \cdot S_{N-1} \cdot S_N = n_V. \quad (8.77)\]
To verify that the corresponding intersection numbers of the $\tilde{S}_j$'s vanish, we compute using the definition of $\tilde{S}_j$. On the one hand, from (8.69), (8.70), and (8.73) we find
\[\tilde{S}_{N-2}^2 \tilde{S}_N = \frac{1}{2} (S_{N-2}^2 S_N - S_{N-2}^2 S_{N-1}) = \frac{1}{2} ((2g - 2 - S_{N-2} S_N^2) - (2g - 2 - S_{N-2} S_{N-1}^2) = 0. \quad (8.78)\]
On the other hand, from (8.73) and (8.77) we find
\[\tilde{S}_{N-2} \tilde{S}_N^2 = \frac{1}{4} (S_{N-2}^2 S_N^2 - 2 S_{N-2} S_{N-1} S_N + S_{N-2}^2 S_{N-1}^2) = \frac{1}{4} (n_V - 2n_V + n_V) = 0, \quad (8.79)\]
and also
\[ \tilde{S}_{N-2} \cdot \tilde{S}_{N-1} \cdot \tilde{S}_N = \frac{1}{2} (S_{N-2} S_{N-1} S_N - S_{N-2} S_N^2) = \frac{1}{2} (n_V - n_V) = 0. \] (8.80)

Having checked that all other relations vanish, we now note that the relations (8.65), (8.71), (8.76), and (8.68) are precisely parallel to the relations (8.10), (8.11), (8.12), and (8.14), with the substitution of \(-n_V\) for \(8 - 8g - n_F\) and \(g\) for \(g'\). We can thus again apply the results of our computation from the \(Sp(N)\) case, and conclude that
\[ \tilde{S}_j \tilde{S}_{j+1}^2 = (j - N + 3)(2g - 2) - n_V \] \[ \tilde{S}_j \tilde{S}_{j+1}^2 = (N - j - 2)(2g - 2) + n_V, \] (8.81)
as well as the final result
\[ S^3 = (1 - g) \left( \sum_{1 \leq i < j \leq N} \left( (a_i - a_j)^3 + (a_i + a_j)^3 \right) \right) - n_V \left( \sum_{j=1}^{N} a_j^3 \right). \] (8.82)

Once again we find perfect agreement with the field theory formula (6.1).

9. Examples of the New Fixed Points from Calabi–Yau Degenerations

Having verified that our detailed descriptions of the geometry match the gauge theory in every detail, it remains to consider the question of when a strong coupling limit will exist.

The configurations of surfaces \(S_j\) which we have considered have the property that they can be contracted to a curve \(\Omega\) of singularities on a Calabi–Yau space \(\overline{X}\). The area of that curve (calculated with respect to some volume-one Kähler metric) corresponds to the classical gauge coupling \(1/g_0^2\) [6], so to find a strong coupling limit we need to be able to shrink the curve \(\Omega\) to zero size. Expressed a bit more directly, we need to find a Calabi–Yau space \(\hat{X}\) with a map \(X \to \hat{X}\) under which the surfaces \(S_j\) shrink to a single singular point \(\hat{P}\). In the case that there are several \(X_\alpha\)'s resolving \(X\) (differing by flops), all of them must map to \(\hat{X}\), giving different resolutions of the singular point \(\hat{P}\).

We have seen in our field theory analysis that the convexity of the prepotential \(F\) is a necessary condition for a strong coupling limit to exist. That can be seen directly in the geometry as well. For if we take any \(S = \sum \psi_j S_j \in -\mathcal{K}(X/\overline{X})\) with the coefficients \(\psi_j\) being rational numbers, then for a sufficiently large positive integer \(m\) there will be a
nonsingular surface $L$ on $X$ which is an ample divisor, such that $L$ and $-mS$ have the same intersection numbers with all curves $\sigma$ contracted by $\pi : X \to \hat{X}$. If the surfaces $S_j$ can be contracted to a point by some map $X \to \hat{X}$, then the curves $C_j = L \cap S_j$ can be contracted to a point by a map $L \to \hat{L}$ (where $\hat{L}$ is the image of $L$ in $\hat{X}$). However, a necessary [43] and sufficient [44] condition for such a contraction of $L$ to exist is that the intersection matrix $(C_i \cdot C_j)$ should be negative definite. Computing these intersection numbers on $X$, we see that $(L \cdot S \cdot S_j) = (\partial^2 F/\partial \psi_i \partial \psi_j | S) = (S \cdot S_i \cdot S_j)$ should be positive definite. If $(\partial^2 F/\partial \psi_i \partial \psi_j | S)$ is positive definite for all divisors $S = \sum \psi_j S_j$ in the cone for which the $\psi_j$’s are rational, it must be positive definite throughout the cone. But this was one of the characterizations of convexity of the prepotential.

A sufficient condition for the existence of such a contraction mapping is also known in algebraic geometry [45,46]: the classes in $\mathcal{K}(X/\hat{X})$ must restrict to positive classes (sometimes called “ample $\mathbb{R}$-divisors”) on each of the surfaces $S_j$. However, if $S_j$ is the blowup of $\mathbb{P}^2$ at more than 8 points (as happens in some of our configurations), it contains an infinite number of exceptional curves of the first kind. This fact makes the “sufficient condition” quite difficult to check, in general, so we shall have to be content with verifying it in a few examples. The field theory analysis, by renormalization group flow upon adding mass terms for the matter, implies that the condition will also be satisfied for examples with less matter. These examples are enough to conclude that most of the strong-coupling limit points which we have predicted based on the convexity analysis do in fact occur.

Our strategy for verifying the ampleness of these classes is as follows: we will analyze the divisors whose classes lie along the edges of the relative Kähler cone $\mathcal{K}(X/\hat{X})$, and show that each of them contracts most of the $S_j$’s to curves or to points, while serving as an ample divisor on some birational model of each $S_j$. From this analysis, it is easy to conclude that the classes in the interior of the cone correspond to ample divisors.

9.1. $Sp(N)$ with $n_A = 1$

In our first example, we consider $Sp(N)$ with $g = 0$ adjoints, $n_A = 1$ antisymmetric tensors, and $n_F \leq 7$ fundamentals. The geometry is represented by Figure 1, with $S_1, \ldots, S_{N-1}$ being elliptic ruled surfaces meeting along elliptic curves $\gamma_j$, and $S_N$ being a del Pezzo surface of degree $8 - n_F$. From (8.13), we see that

$$(\gamma_j^2)_{S_j} = S_j S_{j+1}^2 = -(8 - n_F),$$

(9.1)
for \( j < N \). Since \( n_F < 8 \), we can contract \( \gamma_j \) on \( S_j \) to obtain an elliptic cone \( \overline{S}_j \) for \( j < N \).

The divisor \( L_k := \sum_{j=1}^{k-1} jS_j + \sum_{j=k}^{N} kS_j \) lies on that edge of the cone \( -\mathcal{K}(X/X) \) for which \( a_1 = \ldots = a_k > a_{k+1} = \ldots = a_N = 0 \). We let \( \gamma_0 \) be a section of the ruling on \( S_1 \) disjoint from \( \gamma_1 \), and note that \( S_j|_{S_j} = K_{S_j} = -\gamma_j - 1 - \gamma_j \) for \( j < N \) and \( S_N|_{S_N} = K_{S_N} = -\gamma_{N-1} \). The restriction \( -L_k|_{S_j} \) can be computed as follows:

\[
-L_k|_{S_j} = \begin{cases} 
\gamma_j - 1 - \gamma_j & k > j \\
\gamma_j - 1 & k = j \\
0 & k < j.
\end{cases}
\] (9.2)

It is then easy to see that \( -L_k|_{S_j} \) is a nef divisor which contracts \( S_j \) along its ruling (mapping to a curve) for \( j < k \leq N \), maps \( S_j \) to a point for \( j > k \), is ample on the elliptic cone \( \overline{S}_k \) when \( j = k < N \). The remaining case, \( -L_N|_{S_N} \), is an anticanonical divisor on the del Pezzo surface \( S_N \); this is ample as well.

Generalizing the codimension 1 result of [6], there is a string theory argument that the deformations of the singularity in these cases, corresponding to the Higgs branch of the gauge theory in the strong coupling limit, must be the same as the moduli space of \( N, E_{n_F+1} \) instantons.

9.2. \( Sp(N) \) with \( n_A = 0 \)

In our next example, we consider \( Sp(N) \) with \( g = 0 \) adjoints, \( n_A = 0 \) antisymmetric tensors, and \( n_F = 2N + 4 \) fundamentals. The geometry is again represented by Figure 1, but this time \( S_1, \ldots, S_{N-1} \) are minimal rational ruled surfaces meeting along rational curves \( \gamma_j \), and \( S_N \) is the blowup of a minimal rational ruled surface at \( 2N + 4 \) points lying on a 2-section of the ruling. From (8.13), we see that

\[
(\gamma_{j-1}^2)_{S_j} = S_{j-1}^2 S_j = -2j,
\] (9.3)

for \( 1 < j \leq N \). We can thus contract \( \gamma_{j-1} \) on \( S_j \) to obtain a rational cone \( \overline{S}_j \) for \( 1 < j < N \). We can also contract \( \gamma_{N-1} \) (the proper transform of the 2-section) on \( S_N \) to obtain a surface \( \overline{S}_N \) which is somewhat more complicated than a cone. Note that \( S_1 \) can be either of the Hirzebruch surfaces \( \mathbb{F}_0 \) or \( \mathbb{F}_2 \); we let \( \overline{S}_1 \) denote either \( \mathbb{F}_0 \) or the contraction of the negative section on \( \mathbb{F}_2 \)—a quadric in either case.

The divisor \( L_k := \sum_{j=1}^{k-1} jS_j + \sum_{j=k}^{N} kS_j \) lies on that edge of the cone \( -\mathcal{K}(X/X) \) for which \( a_1 = \ldots = a_k > a_{k+1} = \ldots = a_N = 0 \). We define \( \gamma_0 = \gamma_1 - 2\epsilon_1 \) as a class on \( S_1 \), and note that \( S_j \cong \mathbb{F}_{2j} \) for \( j > 1 \), and hence that \( \gamma_{j-1} + 2j\epsilon_j \sim \gamma_j \) on \( S_j \). The normal
bundles are $S_j|_{S_j} = K_{S_j} = -\gamma_j - 2\varepsilon_j$ for $j < N$ and $S_N|_{S_N} = K_{S_N} = -\gamma_N + \varepsilon_N$, and the restriction $-L_k|_{S_j}$ can be computed as follows:

$$-L_k|_{S_j} = \begin{cases} 0 & k > j \\ \gamma_j & k = j < N \\ 2k\varepsilon_j & k < j \neq N, \end{cases}$$

$$-L_k|_{S_N} = \begin{cases} \gamma_{N-1} + N\varepsilon_N & k = N \\ k\varepsilon_N & k < N. \end{cases} \tag{9.4}$$

It is then easy to see that $-L_k|_{S_j}$ is a nef divisor which maps $S_j$ to a point for $j < k$, contracts $S_j$ along its ruling (mapping to a curve) for $j > k$, is ample on the quadric or rational cone $\mathcal{S}_k$ when $j = k < N$, and corresponds to the divisor $\gamma_{N-1} + N\varepsilon_N = -K_{S_N} + (N-1)\varepsilon_N$ on the surface $\mathcal{S}_N$ when $j = k = N$.

The key question now is: is the divisor $-K_{S_N} + (N-1)\varepsilon_N$ ample on the surface $\mathcal{S}_N$? We have not succeeded in settling this question in general; however, we can construct an example of such a surface on which the divisor is ample, as follows. Start with the Hirzebruch surface $\mathbb{F}_N$, and let $\sigma_\infty$ be the section of self-intersection $-N$, and $f$ be the class of the fiber. The divisor class $2\sigma_\infty + (2N + 2)f$ contains a smooth curve $C$ of genus $N+1$ which meets $\sigma_\infty$ in two points, and meets each fiber $f$ in two points. Let $\pi : S \to \mathbb{F}_N$ be the double cover of $\mathbb{F}_N$ branched along $C$, let $\varepsilon = \pi^{-1}(f)$ and let $\gamma = \pi^{-1}(\sigma_\infty)$. The surface $S$ is ruled by the rational curves $\varepsilon$, and $\gamma$ is a 2-section for this ruling which has self-intersection $-2N$. Moreover, the curves $\varepsilon$ are generically irreducible but become reducible precisely when the corresponding fiber $f$ is tangent to $C$, i.e., precisely where the two-to-one map $C \to \sigma_\infty$ has branch points. Since $C$ has genus $N+1$, there are precisely $2N + 4$ such branch points, and so precisely $2N + 4$ line pairs as singular fibers of the ruling on $S$. Thus, $S$ satisfies all of the properties needed for it to be used as $S_N$ in our description of the configuration leading to $Sp(N)$ gauge theory with $n_F = 2N + 4$.

It remains to verify that the divisor $-K_S + (N-1)\varepsilon$ is ample on the surface $\mathcal{S}$ obtained from $S$ by contracting $\gamma$. By construction,

$$-K_S + (N-1)\varepsilon = \pi^*(-K_{\mathbb{F}_N} - \frac{1}{2}C) + (N-1)\pi^*(f) = \pi^*(\sigma_\infty + Nf). \tag{9.5}$$

The divisor class $\sigma_\infty + Nf$ on $\mathbb{F}_N$ contracts $\sigma_\infty$ to a point, and embeds the resulting surface $\mathbb{F}_N$ into $\mathbb{P}^N$; it follows that its pullback to $\mathcal{S}$ must be ample on that surface.

Since there exist examples of the surfaces $\mathcal{S}_N$ with the required property, a strong coupling limit of the gauge theory will exist. This strongly suggests that any configuration of surfaces on a Calabi–Yau manifold giving rise to that gauge theory should have a strong coupling limit, and hence should have the property that $-K_{S_N} + (N-1)\varepsilon_N$ is in fact ample on $\mathcal{S}_N$.  

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As our next example, we consider $SU(N)$ with $g = 0$ adjoints and $n_F = 2N - 2$ fundamentals, with $c_{\text{classical}} = 0$. Although this is not quite the maximum number of fundamentals which we expect to be possible based on our convexity analysis, analyzing the ampleness condition is quite complicated in the case of $n_F = 2N$—even more complicated than the $Sp(N)$ case with $n_A = 0$ which we discussed in the previous subsection. However, it turns out that concrete examples with $n_F = 2N - 2$ can be given using toric geometry, so we shall present those instead.

The toric data needed for these examples is illustrated in figure 5, which shows a slice of a cone which could be used to describe an affine toric variety. The solid circles represent the divisors $S_1, \ldots, S_{N-1}$ which should be contracted, the open circles represent other toric divisors in a toric ambient space. To perform the contraction of all of these divisors to a point, one simply omits the solid circles and all associated edges.

The configuration of the divisors is determined by the connections between the dots, and there will be $N - 1$ different triangulations which must be considered, corresponding to the possibility of flopping among different resolutions. (Both relevant triangulations in the $SU(3)$ case are depicted in figure 5.) An easy exercise in toric geometry verifies that these configurations reproduce the data contained in figure 2 and the surround description, albeit in a slightly degenerate form in which the blowups from a minimal ruled surface have all been concentrated at two points (which are each blown up several times). Also from this data, one checks easily that $c' = 1 - N$, and hence that $c_{\text{classical}} = c' + \frac{n_F}{2} = 0$.

![Figure 5. Two triangulations of toric data for $SU(3)$ with $n_F = 4$.](image)

In our next example, we consider $Spin(2N + 1)$ with $g = 0$ adjoints and $n_V = 2N - 3$ vectors. The geometry is represented by Figure 3, with $S_1, \ldots, S_{N-1}$ being minimal
rational ruled surfaces meeting along rational curves $\gamma_j$ $(j \leq N - 2)$, and $S_N$ being a minimal ruled surface over a curve $\gamma_{N-1}$ of genus $g' = 2N - 3$. From (8.52), we see that

$$(\gamma_j^2)_{s_j} = S_{j-1}^2 s_j = \tilde{S}_{j-1}^2 \tilde{s}_j = -2j,$$  \hspace{1cm} (9.6)$$

for $1 < j < N$, and

$$(\gamma_{N-1}^2)_{s_N} = S_{N-1}^2 s_N = 2\tilde{S}_{N-1}^2 \tilde{s}_N = -4N.$$  \hspace{1cm} (9.7)$$

We can thus contract $\gamma_{j-1}$ on $S_j$ to obtain a rational cone $\overline{S}_j$ for $1 < j < N$, and contract $\gamma_{N-1}$ on $S_N$ to obtain a cone $\overline{S}_N$ over a curve of genus $2N - 3$. As before, $S_1$ can be either $F_0$ or $F_2$; we let $\overline{S}_1$ denote either $F_0$ or the contraction of the negative section on $F_2$.

The divisor $L_k := \sum_{j=1}^{k-1} jS_j + \sum_{j=k}^{N-1} kS_j + \frac{1}{2}kS_N$ lies on that edge of the cone $-K(X/\overline{X})$ for which $a_1 = \ldots = a_k > a_{k+1} = \ldots = a_N = 0$. We define $\gamma_0 = \gamma_1 - 2\varepsilon_1$ as a class on $S_1$, and note that $S_j \cong \overline{F}_2j$ for $1 < j < N$, and hence that $\gamma_{j-1} + 2j\varepsilon_j \sim \gamma_j$ on $S_j$. The normal bundles are $S_j|_{S_j} = K_{S_j} = -\gamma_{j-1} - \gamma_j - 2\varepsilon_j$ for $j < N$ and $S_N|_{S_N} = K_{S_N} = -\gamma_{N-1} + 4\varepsilon_N$, and the restriction $-L_k|_{S_j}$ can be computed as follows:

$$-L_k|_{S_j} = \begin{cases} 0 & k > j \\ \gamma_j & k = j < N \\ 2k\varepsilon_j & k < j \neq N, \end{cases} \hspace{1cm} -L_k|_{S_N} = \begin{cases} \gamma_{N-1} + 4N\varepsilon_N & k = N \\ 4k\varepsilon_N & k < N. \end{cases} \hspace{1cm} (9.8)$$

It is then easy to see that $-L_k|_{S_j}$ is a nef divisor which maps $S_j$ to a point for $j < k$, contracts $S_j$ along its ruling (mapping to a curve) for $j > k$, is ample on the quadric or rational cone $\overline{S}_k$ when $j = k < N$, and is ample on the irrational cone $\overline{S}_N$ when $j = k = N$.

9.5. Spin(2N)

As our final example, we consider Spin(2N) with $g = 0$ adjoints and $n_V = 2N - 4$ vectors. The geometry is represented by Figure 4, with $S_1, \ldots, S_{N-1}$ being minimal rational ruled surfaces meeting along rational curves $\gamma_j$ $(j \leq N - 2)$, and $S_N$ being a non-minimal rational ruled surface with double points, which meets both $S_{N-1}$ and $S_{N-2}$ along rational curves. More precisely, $S_N$ is obtained from a minimal ruled surface by performing $n_V$ directed blowups of weight 1, yielding exceptional divisors $\sigma_N^{(\alpha)}$ passing through double points, and the proper transforms $\delta^{(\alpha)}$ of the fibers passing through the points that were blown up.

From (8.81), we see that

$$(\gamma_{j-1}^2)_{s_j} = S_{j-1}^2 s_j = \tilde{S}_{j-1}^2 \tilde{s}_j = -2j,$$  \hspace{1cm} (9.9)$$

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for \( j \leq N - 1 \). We can thus contract \( \gamma_{j-1} \) on \( S_j \) to obtain a rational cone \( \overline{S}_j \) for \( j \leq N - 1 \).

As in some previous examples, \( S_1 \) can be either \( \mathbb{F}_0 \) or \( \mathbb{F}_2 \); we let \( \mathcal{F}_1 \) denote either \( \mathbb{F}_0 \) or the contraction of the negative section on \( \mathbb{F}_2 \).

From (8.70) and (8.73) we see that

\[
(\gamma_{N-2}^2)_{S_N} = S_{N-2}^2 S_N = -2 - n_V = 2 - 2N. \tag{9.10}
\]

We can thus contract \( \gamma_{N-1} \) and all \( \sigma_N^{(a)} \)'s (which are disjoint from it) on \( S_N \) to produce a rational cone \( \overline{S}_N \). Alternatively, since \( \gamma_{N-1} \) and the \( \delta^{(a)} \)'s have an intersection matrix

\[
\begin{pmatrix}
-2 - n_V & 1 & \cdots & 1 \\
1 & -2 & & \\
\vdots & & \ddots & \\
1 & & & -2
\end{pmatrix}
\tag{9.11}
\]

which is negative definite, we can contract all of them to produce a surface \( \overline{S} \), which is a sort of “cone with double points.”

The divisor \( L_k := \sum_{j=1}^{k-1} j S_j + \sum_{j=k}^{N-1} k S_j + \frac{1}{2} k (S_N - S_{N-1}) \) lies on that edge of the cone \(-K(X/\overline{X})\) for which \( a_1 = \ldots = a_k > a_{k+1} = \ldots = a_N = 0 \). We define \( \gamma_0 = \gamma_1 - 2\varepsilon_1 \) as a class on \( S_1 \), and note that \( S_j \cong \mathbb{F}_2 \) for \( 1 < j < N \), and hence that \( \gamma_j - 2j \varepsilon_j \sim \gamma_j \) on \( S_j \); also, \( \gamma_{N-1} + (2N - 4) \varepsilon_{N-2} \sim \frac{1}{2} \gamma_{N-2} + \frac{1}{2} \gamma_{N-1} \) on \( S_{N-2} \). The normal bundles are \( S_j|S_j = K_{S_j} = -\gamma_j - \gamma_j - 2\varepsilon_j \) for \( j < N \) and \( S_N|S_N = K_{S_N} = -\gamma_{N-1} - \delta + 2\varepsilon_N \), and the restriction \(-L_k|S_j \) can be computed as follows:

\[
-L_k|S_j = \begin{cases}
0 & k > j \\
\gamma_j & k = j < N \\
2k \varepsilon_j & k < j \neq N,
\end{cases}
\]

\[
-L_k|S_N = \begin{cases}
2\gamma_{N-1} + \delta + 2N \varepsilon_N & k = N \\
\gamma_{N-1} + (2N - 2) \varepsilon_N & k = N - 1 \\
2k \varepsilon_N & k < N - 1,
\end{cases}
\tag{9.12}
\]

where \( \delta = \sum \delta^{(a)} \).

It follows that \(-L_k|S_j \) is a nef divisor which maps \( S_j \) to a point for \( j < k \), contracts \( S_j \) along its ruling (mapping to a curve) for \( j > k \) (except \( (j,k) = (N,N-1) \)), is ample on the quadric or rational cone \( \overline{S}_k \) when \( j = k < N \) or \( (j,k) = (N,N-1) \), and is ample on the “cone with double points” \( \overline{S}_N \) when \( j = k = N \).

Note that \(-L_{N-1} \) might have been expected to leave only \( S_{N-1} \) uncontracted, it in fact leaves both of the divisors \( S_N \) and \( S_{N-1} \) uncontracted, mapping them to cones which meet along a common fiber. The reader may wonder if there is some divisor which contracts
all of the $S_j$'s other than $S_{N-1}$. Indeed there is, but the flop along the $\sigma_N^{(\alpha)}$'s must be performed before that divisor belongs to the relative Kähler cone.

Acknowledgments

We would like to thank Paul Aspinwall, Tom Banks, Ron Donagi, Antonella Grassi, Jeff Harvey, Sheldon Katz, Greg Moore, Miles Reid, Steve Shenker, and Edward Witten for discussions. D.R.M. would like to acknowledge the hospitality of the Research Institute for Mathematical Sciences, Kyoto University, where part of this work was done. The work of K.I. is supported by NSF PHY-9513835, the W.M. Keck Foundation, an Alfred Sloan Foundation Fellowship, and the generosity of Martin and Helen Chooljian. The work of D.R.M. is supported in part by the Harmon Duncombe Foundation and by NSF grants DMS-9401447 and DMS-9627351. The work of N.S. is supported in part by DOE grant #DE-FG02-96ER40559.

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