The interpretation of the solutions of the Wheeler De Witt equation

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Abstract
We extract transition amplitudes among matter constituents of the universe from the solutions of the Wheeler De Witt equation. The physical interpretation of these solutions is then reached by an analysis of the properties of the transition amplitudes. The interpretation so obtained is based on the current carried by these solutions and confirms ideas put forward by Vilenkin.
1 Introduction

In the Hamiltonian formulation of General Relativity coupled to matter fields, the Euler-Lagrange equations split into 6 dynamical equations plus the equations for the matter fields and 4 constraints which express the invariance of the theory under local changes of coordinates. Therefore, when quantizing this system in a gauge invariant manner, i.e. à la Dirac, the “wave function” must be annihilated by these four constraints that are now operator valued. This clear structure dictated by the reparametrization invariance turns out to lead to very complicated problems concerning the interpretation of this “wave function”. By interpretation, one designates the problem of extracting (probabilistic) predictions concerning the evolution of matter and gravity configurations. The origin of the difficulties can be blamed on the absence of time and the accompanying unitary evolution on which quantum mechanics is based [1][2][3][4][5].

In order to describe more precisely some of these difficulties, from now on we shall pursue the discussion in mini-superspace. This drastic restriction to homogeneous and isotropic three-geometries offers the double advantage of removing the U.V. problem that plagues the local theory while keeping the problem concerning the interpretation of the wave function. In this restricted configuration space, gravity is described by the scale factor $a$, matter by homogeneous fields that we shall denoted collectively by $\phi$ and the wave $\Psi(a, \phi)$ is constrained to satisfy a single global Wheeler De Witt (WDW) equation. The questions are the following: How to proceed to read from $\Psi(a, \phi)$ predictions concerning quantum events such as, for instance, transitions rates among the $\phi$ fields? and: Are those transitions described by a unitary evolution?

Both questions have received attention and many different schemes have been proposed [1]. Let us mention here only the scheme based on the hypothesis that $|\Psi(a, \phi)|^2$ does posses a probabilistic interpretation [2] (at least in terms of conditional probabilities [1]) and the scheme based on the conserved current [5] $\Psi^* i \partial_a \Psi$. The answers to the second question are partially related with this choice and range from, “Yes, the evolution is unitary [6][7]”, to “no, unitarity is violated[4]”, and includes the middle attitude: “unitarity is only approximatively conserved [5]”. The peculiar aspect of these widely distributed answers is that they arise from the same starting point: the WDW equation and its solutions. The disagreements build up with the choice of the treatment required to extract information from $\Psi(a, \phi)$.

In the present article, we clarify the mathematical aspects of these treatments by analysing matter interactions from the solutions of the WDW equation. More precisely, by studying transitions, we identify the coefficient $C_n(a)$ that weights the n-th state at $a$ and that replaces the amplitude $c_n(t)$ to be in the n-th matter state at time $t$ in conventional quantum mechanics. The unambiguous identification is based on two criteria: 1) In the absence of transitions, $C_n(a)$ must be constant. 2) When one simplifies the equation governing their dependence in $a$ by treating gravity in the background field approximation (BFA), the resulting equation must be the Schroedinger equation. This mathematical procedure of extracting the $C_n(a)$ is based on [8][9][10] and does not require an a priori interpretation of $\Psi(a, \phi)$. On the contrary, our program is first to determine the properties of the coefficients $C_n(a)$ and only then to examine the question of its
interpretation in the light of these properties.

They reveal the existence of three regimes which are delineated by the values of the parameters describing matter transitions in quantum cosmology. In the case of weak interactions occurring close to equilibrium in a macroscopic\textsuperscript{5}\textsuperscript{11}, the coefficients $C_n(a)$ are equal to the amplitudes $c_n(t)$, solutions of the corresponding Schroedinger equation evaluated in the geometry described by $a(t)$. In the second regime, the departure from equilibrium and/or the importance of the interactions lead to $C_n(a)$ that no longer coincide with $c_n(t)$. However when gravity is still correctly described by WKB waves, the $C_n(a)$ still satisfy the “unitary” equation $\sum_n |C_n(a)|^2 = 1$, up to negligible corrections. In the third regime, the interactions are so violent that the propagation of $a$ is affected by the quantum transition acts. In that case, $\sum_n |C_n(a)|^2 \neq 1$. This “violation” is a direct manifestation of the modification of the propagation of $a$ by the transitions themselves. It is also kinematically related to the conservation of the current carried by $\Psi(a,\phi)$. Under these extreme conditions, there is no possibility of defining a background. Neither, therefore, should there be any possibility of interpreting $\Psi(a,\phi)$ using the conventional rules of quantum mechanics. This does not mean that no predictions can be made, it simply means that the conventional analysis cannot be performed \textit{in situ}. By propagating $\Psi(a,\phi)$ outside this regime, one can then perfectly determine its physical outcome.

In conclusion, in this article, we shall show that when one requires that the conventional description of matter transitions is recovered from quantum cosmology, the probabilistic interpretation of $\Psi(a,\phi)$ must be based on its current and not on its norm. We shall also show that the interpretation of $C_n(a)$ as the amplitude of probability to find the $n$-th matter state at $a$ is valid as long as the propagation of gravity is not significantly affected by the interactions. Both aspects have been put forward by Vilenkin in [5]. However, to the knowledge of the author, they have never been made as explicit as in the present paper. Furthermore, in contradistinction to [5], our small parameter is the coupling constant among the quantum systems and not their energy. This allows to reach more general conclusions.

2 The identification, the evolution and the meaning of the coefficients $C_n(a)$

As said in the Introduction, we shall use perturbation theory applied to matter interactions as a guide to identify the coefficients $C_n(a)$. Before accomplishing this program, we briefly present the kinematical properties at work in Quantum Cosmology, in the absence of these interactions, see [8] for more details.

The coefficients in absence of interactions

For simplicity, the matter system is chosen in such a way that the free hamiltonian $H_m^0(a)$

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does not lead to transitions. Thus, the general solution of the Schroedinger equation is
\[ \chi(t, \phi) = \sum_n c_n e^{-i \int^t dt' E_n(t')} \langle \phi | n \rangle \] (1)
where the eigenstates of the free hamiltonian \( H_0^m(a) \) satisfy
\[ H_0^m(a(t)) | n \rangle = E_n(a(t)) | n \rangle \] (2)
Their time dependence arises only through the equation governing the background propagation \( a = a(t) \).

The coefficients \( c_n \) are interpreted as the amplitudes of probability to find the matter system in the n-th state. Therefore, by convention, they are normalized so that \( \sum_n |c_n|^2 = 1 \). Furthermore, in the present case, they are constant. In this respect, notice that one must consider either interactions with the external world, or self interactions, or non adiabatic transitions[12] in order to give physical substance to the probabilistic interpretation of \( c_n \). Indeed, in the absence of interactions, no interference among the \( c_n \) will show up.

We shall now determine to what extend these properties are recovered in quantum cosmology from the solutions of the Wheeler De Witt equation.

In minisuperspace, when matter is characterized by an energy \( E_n(a) \), the gravitational part of the action satisfies the Hamilton-Jacobi constraint
\[ H_G(a) + E_n(a) = \frac{-G^2 (\partial_a S_G(a))^2 + \kappa a^2 + \Lambda a^4}{2Ga} + E_n(a) = 0 \] (3)
where \( G \) is Newton’s constant, \( \kappa \) is equal to \( \pm 1 \) or 0 for respectively open, closed and flat three surfaces and \( \Lambda \) is the cosmological constant. The solution of this equation is simply \( S_n(a) = \int^a da' p_n(a') \) where the momentum of \( a \) driven by the \( E_n(a) \)
\[ p_n(a) = \mp G^{-1} \sqrt{\kappa a^2 + \Lambda a^4 + 2GaE_n(a)} \] (4)
where \( \mp \) correspond respectively to expanding and contracting universes.

Upon quantizing gravity, the Hamilton-Jacobi constraint becomes the operator valued equation (the WDW equation)
\[ [H_G(a) + H_0^m(a, \phi)] \Psi(a, \phi) = 0 \] (5)
When the quantum matter states are characterized by constants of motion, \( \Psi(a, \phi) \) can be decomposed as
\[ \Psi(a, \phi) = \sum_n C_n \Psi(a; n) \langle \phi | n \rangle \] (6)
where the n-th gravitational wave, entangled\(^1\) to the n-th matter state, is a solution of
\[ \left[ G^2 \partial_a^2 + \kappa a^2 + \Lambda a^4 + 2GaE_n(a) \right] \Psi(a; n) = 0 \] (7)
\(^1\)This is the main reason for which we have chosen to work with matter states characterized by constants of motion. It allows an unambiguous decomposition of the total wave in terms of products. This is to be compared with the difficulties to perform this decomposition in the general case wherein no clear principle seems to exist, see e.g. [4] after eq. (2.35) “We choose D in such a way that the equations become simple”. Very important also is the fact that our decomposition keeps the linearity of the WDW equation when used in a perturbative treatment.
This equation is second order in $\partial_a$ and has therefore two independent solutions. This has to be the case since classically we can work either with expanding or contracting universes. Indeed when using the WKB approximation

$$\Psi(a; n) = e^{\int p_n(a')da'} \frac{1}{\sqrt{2|p_n(a)|}}$$

one verifies that solutions with positive (negative) wronskian

$$W_n = \Psi^*(a; n) \leftrightarrow \Psi'(a; n)$$

Correspond to expanding (contracting) universes.

At this stage, several remarks should be made. Firstly, the decomposition of the general solution of the WDW equation is performed by using the same set of quantum numbers $n$ that the one used in eq. (2) wherein gravity was treated in the background field approximation (BFA). The enlargement of the dynamics to $a$ is compensated by the WDW constraint. Secondly, it is now through the sign of the Wronskian rather than at the classical level that one now chooses expanding or contracting universes. Thirdly, with our definition of $\Psi(a; n)$, the coefficients $C_n$ are constant.

Having recall these kinematical properties, we can now formulate precisely our mathematical claim[8]: in order for $C_n$ to satisfy the Schrödinger equation when gravity is treated in the BFA, all Wronskians $W_n$ must be equal to the same constant. This can be already guessed by considering the conserved current[5] carried by $\Psi(a, \phi)$

$$J = \int d\phi \left( i \partial_a \Psi^*(a, \phi) \rightarrow i \partial_a \Psi(a, \phi) = \sum_n |C_n|^2 W_n \right)$$

To work with both $J = 1$ and unit Wronskians suggests an identification of $C_n$ with $c_n$. However, at this moment, no physically relevant conclusion$^2$ can be made concerning a probabilistic interpretation of the $C_n$. Indeed it is mandatory to consider self interactions leading to transitions in order for the $C_n$ to vary and interfere, see the remark made after eq. (2).

The $a$-dependence of the $C_n$

We return for a moment to quantum mechanics and consider the time dependent perturbation theory for allowing a comparison with the corresponding equations derived in quantum cosmology. Upon introducing an interacting hamiltonian $H_{int}$ possessing non

$^2$It is nevertheless interesting to compare the present treatment based on the current $J$ to the one in which it is the norm that determines probabilities. In that latter case, by definition, the probability to be in the $n$-th state is given by $|\langle n|\Psi(a, \phi)\rangle|^2 / \langle \Psi|\Psi \rangle = |C_n|^2 p_n^{-1}(a) / \sum_m |C_m|^2 / p_m^{-1}(a)$, where we have used the WKB form for the waves $\Psi(a; n)$. This “probability” depends on $a$ through the $a$-dependent norm of each $\Psi(a; n)$, a feature that I find unattractive. Notice however, that in the doubled limit of well grouped states living in a macroscopic universe, this dependence in $a$ vanishes, see latter in the text for more precision concerning these limits.
vanishing matrix elements $\langle n|H_{int}|m\rangle$, the time dependence of the coefficients $c_n$ is given by

$$i\partial_t c_n(t) = \sum_m \langle n|H_{int}|m\rangle \ c_m(t) \ e^{-i\int^t dt'[E_m(t')-E_n(t')]},$$

(11)

Since this equation is linear in $c_n$ and first order in $i\partial_t$, when the Hamiltonian $H_{int}$ is hermitian, one obtains $\Sigma_n |c_n(t)|^2 = 1$, a necessary condition to keep the probabilistic interpretation of $|c_n(t)|^2$.

In quantum cosmology, the fact of taking into account the Hamiltonian $H_{int}$ is quite different from what we just did in usual quantum mechanics wherein one has an external time parameter at our disposal. Indeed, $H_{int}$ modifies the propagation of both gravity and matter through the modified WDW equation given by

$$\left[H_G(a) + H^0_m(a,\phi) + H_{int}(a,\phi)\right] \Xi(a,\phi) = 0$$

(12)

By decomposing the interacting wave $\Xi(a,\phi)$ in terms of the free components $\Psi(a;n)$

$$\Xi(a,\phi) = \sum_m C_m(a) \ \Psi(a;m) \langle \phi | m \rangle$$

(13)

and by projecting eq. (12) into the bra $\langle n |$, we obtain

$$\frac{\partial_a \Psi(a;n)}{\Psi(a;n)} \partial_a C_n(a) + \frac{1}{2} \partial^2_a C_n(a) + \frac{a}{G} \sum_m C_m(a) \ \langle n | H_{int} | m \rangle \ \frac{\Psi(a;m)}{\Psi(a;n)} = 0$$

(14)

Since this equation is second order in $\partial_a$, some analysis is required in order to reveal the properties of the evolution it encodes. To this end, we shall first simplify it by making use of three approximations. In the next subsection, we shall evaluate the errors they induce on the basis of the analysis of [8][9].

The first approximation consists in using the WKB approximation for $\Psi(a;n)$. Its validity requires $\partial_a p_n(a)/p^2_n(a) \ll 1$, an inequality which is satisfied when the second condition is met. This second condition requires that the universe be macroscopic [5][11], i.e. that the matter sources driving gravity must be macroscopic. By denoting $M$ the rest mass of the atoms and $\Delta m$ the energy change induced by the transitions engendered by $H_{int}$, the macroscopic limit guarantees that $\Delta m/E_n \simeq \Delta m/N_M M \ll 1$ since $N_M$, the total number of atoms, satisfies $N_M \gg 1$. Thirdly, we require that the dimensionless constant $g^2$ that characterizes the transition rates be smaller or comparable to unity ($g$ is related to $H_{int}$ by $\langle n | H_{int} | m \rangle \approx g \Delta m$).

By applying these three approximations to eq. (14), one can drop the second term and write the two other terms as

$$i\partial_a C_n(a) \approx \sum_m \frac{a}{G} \frac{\langle n | H_{int} | m \rangle}{\sqrt{p_n(a) p_m(a)}} \ C_m(a) \ e^{i\int^a da'[p_m(a')-p_n(a')]} \ \sqrt{\frac{W_m}{W_n}}$$

(15)

$^3$Notice that this development does not coincide with the Born-Oppenheimer treatment. What would be closer to that treatment, would consist in working with states which diagonalise the total Hamiltonian $H^0_m(a,\phi) + H_{int}(a,\phi)$ at fixed $a$. Together with S. Massar, we shall present this adiabatic treatment applied to quantum cosmology in [12].
This equation deserves a few comments.

Firstly, as eqs. (12, 14), it is linear in $C_n(a)$. We point out this fact since many approximation schemes[4][6][7][11], as the semi-classical treatment, destroy the linearity of the WDW equation and therefore the superposition principle. The loss of linearity in these treatments results from a quantum averaging performed at an earlier stage.

Secondly, in order for eq. (15) to coincide with eq. (11) in the background field approximation, the Wronskians must satisfy $W_m/W_n = 1$ for all $m, n$. The validity of the further simplification which consists in treating eq. (15) in the BFA, requires that the $C_n(a)$ be grouped together such that their spread around the mean energy $E_\bar{n}$ satisfies $(E_n - E_\bar{n})/E_\bar{n} \ll 1[5][11]$. Only then can one correctly describe the evolution in terms of a single time parameter[13] defined, from the propagation of $a$, by

$$t_\bar{n}(a) = \int_{a_0}^{a} da' \frac{a'}{G p_n(a')} \quad (16)$$

and develop the $a$-dependent phase of eq. (15) to first order in $E_m - E_n$ around $E_\bar{n}$, see [9]. Using eq. (4) and $t_\bar{n}(a)$, one finds, for an expanding universe,

$$- \int_{a_0}^{a} da' (p_m(a') - p_n(a')) = \int_0^{t_\bar{n}(a)} dt' (E_m(t') - E_n(t')) + O((E_m - E_n)/E_\bar{n}) \quad (17)$$

Thirdly, when $W_m/W_n = 1$, eq. (15) leads to “unitary” evolution in the sense that $\Sigma_n|C_n(a)|^2 = 1$. We emphasize that this does not imply that, starting with $C_n$ that coincide with $c_n$ at $a_0 (t = 0), C_n(a)$ will evolve like $c_n(t(a))$. Indeed, as shown in eq. (17), the phases of eq. (15) equal the corresponding phases of eq. (11) only when developed to first order in $\Delta E$ and evaluated for the mean energy $E_\bar{n}$. Therefore, the non-linear terms will induce increasing additional phase shifts. (This is not particular to quantum cosmology. Indeed, whenever the quantum dynamics is enlarged to a variable formerly treated classically, non-linear phase shifts appear, see [15].) Then, after a certain lapse of $t(a)$, the interferences amongst the $C_n(a)$ will posess no relation to those amongst the $c_n(t(a))$. Moreover, remote configurations evolve with their own time[5], see eq. (16) for the dependence on $E_n$ in $t_n$.

Fourthly, eq. (15) might have been obtained by “first quantizing” the wave $\Xi(a, \phi)$. In that framework, one postulates that the fundamental equation is

$$-i\partial_a \Xi(a, \phi) = \sqrt{H_0(a, \phi) + H_{int}(a, \phi)} \Xi(a, \phi) \quad (18)$$

instead of eq. (12), see [1]. Then, by developing the square root to first order in $H_{int}$ one obtains eq. (15) exactly like eq. (11) is obtained[9]. The main weakness of this ad hoc approach is that there is no a priori justification to eliminate half of the solutions of the Hamilton Jacobi equation before quantization. Furthermore, whether or not it offers a good approximation, the importance of the neglected terms cannot be estimated without considering the solutions to eq. (12).

Finally, the interpretation of the wave $\Xi(a, \phi)$ as the conditional amplitude of probability to find $\phi$ at $a$ is rejected by the present analysis of the solutions of eq. (12). Indeed, by adopting this interpretation, one would obtain a non-linear equation for the
conditional amplitudes since these are non linearly related to $C_n(a)$—recall the presence of the normalisation factor $p_n^{-1/2}(a)$ stemming from current conservation, see the second footnote—. Notice that this would not have been the case if one would have worked with the solutions of eq. (18). Notice also that, in contrast to what is presented in [3], our derivation of eqs. (14, 15) which encode the correlations between matter and gravity in no way requires that $\Xi(a, \phi)$ be peaked around certain configurations. But it does require that gravity be modified by the transitions, otherwise $a$ could not be used to parametrize their amplitudes.

We now address the problem of the corrections to eq. (15).

The $a$-dependence of $\Sigma_n|C_n(a)|^2$

Our aim is to determine how the terms neglected in passing from eq. (12) to eq. (15) affect $\Sigma_n|C_n(a)|^2$. This is most easily achieved by using the exact conservation law of the current carried by $\Xi(a, \phi)$. Indeed the current carried by $\Xi(a, \phi)$ is

$$ J = \int d\phi \left( \Xi^*(a, \phi) \frac{\partial}{\partial a} \Xi(a, \phi) \right) $$

$$ = \sum_n |C_n|^2 + \sum_n \left( C_n^*(a) \frac{\partial}{\partial a} C_n(a) \right) |\Psi(a; n)|^2 = 1 $$ (19)

We first notice that there is no systematic departure from unity. Indeed, when the interactions cease to act, due for instance to some cooling in an expanding universe, the final value of $\Sigma_n|C_n(a)|^2$ coincides with the initial one.

Before discussing the reasons that lead to the local departure from unity, we estimate the importance of this departure. To this end, it is sufficient to use eq. (15) and the WKB expression for the free waves $\Psi(a, \phi)$. One should also further specify the characteristics of the solution under examination. As an example, in the case of a matter dominated universe composed of $N_M$ “heavy” atoms of mass $M$ having transitions between inner states characterized by energy gaps $\Delta m$, this correction is

$$ \sum_n \left( C_n^* \frac{\partial}{\partial a} C_n \right) |\Psi(a; n)|^2 \simeq 2 \sum_n \sum_m C_m^* a \langle n | H_{int} | m \rangle G_{n}(a) G_{m}(a) \simeq \frac{g a N_M^{1/2} \Delta m}{G^2(a)} \lesssim 1 $$ (20)

far from a turning point[16] and at equilibrium. (The incoherence of the transitions close or at equilibrium brings the factor $N_M^{1/2}$ in the second approximation.) The departure from unity is therefore small for three independent reasons: the weakness of the transitions rates controlled by $g$, the smallness of the $\Delta m/M$ which allows to neglect the recoil of the atoms induced by the transitions and the macroscopic character of the universe $N_M \gg 1$.

The mechanism that leads to the departure from unity is clear: when the $C_n$ depend on $a$, they carry some current and therefore their norm should vary accordingly. Indeed, when the “potential term” of a second order differential equation varies, the induced modifications of the norm and the phase of the solutions are correlated since the Wronskian is conserved. The physical origin of these correlated modifications comes from the fact that the WDW imposes that the kinetic energy of gravity be modified
by the presence of $H_{int}$, see eq. (12). This in turn modifies the norm of the solution. Therefore, $\Sigma_n|C_n(a)|^2 \neq 1$ is a manifestation of the modification of the propagation of gravity induced by the interactions themselves.

To obtain explicitly the relation between the changes in phase and in amplitude, it is instructive to consider the simple case wherein the interactions induce a diagonal level shift $E_n(a) \rightarrow E_n(a) + \delta_n(a)$. Using eq. (4) in eq. (8), one immediately gets the desired relation through the change of the $p_n(a)$. Upon considering matter transitions, this modification of $p_n(a)$ has the following physical meaning. The transition probability is given by an integral over $a$ whose norm is $ada/p_n(a) = dt_n$, see [9]. Therefore, the modification of $p_n(a)$ is necessary in that it guarantees that the Golden Rule is obtained, i.e. that the transition probability increases linearly with the proper time lapse, eq. (16), evaluated in the universe wherein the transition occurs.

In view of this, it is inviting to work with eigenstates of the total matter hamiltonian $H_0 + H_{int}$ rather than to work in perturbation theory with the free states. Indeed, when working with these eigenstates, one automatically includes the backreaction of the (adiabatic part of the) interaction hamiltonian in the definition of the WKB momentum $p_n(a)$. Since the induced modification of $p_n(a)$ will no longer be found, its contribution to $\Sigma_n|C_n(a)|^2 \neq 1$ will be also absent. This strongly suggests not to interpret $\Sigma_n|C_n(a)|^2 \neq 1$ as a violation of unitarity since the magnitude of the departure from unity depends on the perturbative scheme adopted. Together with S. Massar, we shall return to these aspects in [12].

Conclusions and additional remarks

In resume, our analysis of $C_n(a)$ shows that there are three regimes delineated by the values of the coupling constant $g$, the relative transition energy $\Delta m/M$ and the macroscopic character of the universe, here controlled by $N_M$.

In order to be in the first regime, the initial values $C_n(a_0) \neq 0$ must be grouped together so that the “mean” time parameter, eq. (16), correctly parametrizes the evolution of the whole system. One must also requires that the universe be macroscopic and that the transitions be not too violent. In that regime, $C_n(a) = c_n(t_n(a))$. Therefore, $C_n(a)$ is the amplitude of probability to find the matter system in the $n$-th state at $a$.

In the second regime, the spread of $C_n(a_0) \neq 0$ and/or the violence of the interactions and/or the “smallness” of the universe leads to an evolution that cannot be obtained from a Schrödinger equation based on a single time parameter. Nevertheless, when the WKB approximation for describing the free evolution of gravity is still valid, one still obtains a linear equation for the propagation of the $C_n(a)$ that guarantees that $\Sigma_n|C_n(a)|^2 = 1$. Moreover, upon considering a set of neighbouring $C_n(a)$ for a sufficiently small amount of time, the evolution of this restricted set still obeys a Schrödinger equation. Finally, as usually in quantum mechanics, remote configurations do not interfere. Therefore, it is still perfectly correct to interpret $C_n(a)$ as the amplitude of probability to find the $n$-th state at $a$. Notice however that the decoherence of the background time, that is the fact that remote configurations evolve with their own time, would leave no physical meaning to mean values of matter operators in which non-interfering configurations would contribute. Only summations over restricted sets of interfering configurations make physical
sense. In this we differ from [2][3] since, in these works, physical predictions are based on the peaks of $\Xi(a,\phi)$ which include summation over all states.

In the third regime, the importance of the interactions leads to backreaction effects on the propagation of gravity such that $\Sigma_n|C_n(a)|^2$ depends on $a$. A part of this dependence comes from the (adiabatic[12]) dressing energy brought in by the interactions. It reflects the fact that the relationship between proper time lapse and the momentum $p(a)$ has been modified by this dressing energy. Therefore the deviation from unity it engenders should certainly not be interpreted as a violation of unitarity. This deviation can be also understood from the necessity of applying a “reduction formula” to the universe’s wave function in order to obtain transition amplitudes amongst its constituents, see [9].

Far more difficult to interpret are the consequences of the corrections to the WKB approximation. Indeed upon dealing with exact solutions for the propagation of $a$, initially expanding (forward) solutions contain, at other radii, backward waves corresponding to contracting universes. Moreover, the conservation of the Wronskian tells us that the current of the forward part has increased. The only way to confront this “Klein paradox” seems to proceed to the so called third quantization[1].

However, we want to emphasize that by adopting this framework, one has not solve the question of extracting transition amplitudes occurring within expanding universes. Indeed, additional choices must be specified in order to know how to extract from superpositions of expanding and contracting universes predictions concerning transitions in the expanding sector. Without having further specified these choices and having considered combinations of forward and backward solutions, it is overhasty to conclude that unitarity will (not) be violated in Quantum Cosmology.

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