Connections Between Inclusive and Exclusive Semileptonic $B$ Decay

Mark B. Wise

California Institute of Technology, Pasadena, CA 91125, USA

Abstract. $B$ decay sum rules relate exclusive $B$ semileptonic decay matrix elements to forward $B$-meson matrix elements of operators in HQET. At leading order the operators that occur are the $b$-quark kinetic energy $\lambda_1$ and chromo-magnetic energy $\lambda_2$. The latter is determined by the measured $B^* - B$ mass splitting. The derivation of these sum rules is reviewed and perturbative QCD corrections are discussed. A determination of $\lambda_1$ and the energy of the light degrees of freedom in a $B$-meson, $\Lambda$, from semileptonic $B$ decay data is presented. Future prospects for improving these sum rules are discussed.

I INTRODUCTION

In this lecture I review some connections between inclusive and exclusive semileptonic $B$-meson decays. These arise from sum rules that relate the form factors for exclusive semileptonic decays to nonperturbative QCD matrix elements that occur in the inclusive semileptonic decay rate. Sum rules that relate inclusive $B$ transitions to a sum over exclusive states were first derived by Bjorken [1,2] and Voloshin [3]. Then a general framework for $B$-decay sum rules was presented by Bigi, et al. in [4]. Since this very important work, there has been a considerable amount of theoretical activity in the area of $B$ decay sum rules [5–12].

For inclusive decays it is possible using the operator product expansion and a transition to the heavy quark effective theory (HQET) [13] to show that at leading order in $\Lambda_{\text{QCD}}/m_b$ the $B$ semileptonic decay rate is equal to the $b$-quark decay rate [14,15]. There are no nonperturbative corrections to this at order $\Lambda_{\text{QCD}}/m_b$ [14]. The first corrections arise at order $\Lambda_{\text{QCD}}^2/m_b^2$, and are
characterized by matrix elements [16] that are related to the $b$-quark kinetic energy

$$\lambda_1 = \frac{1}{2m_B} \langle B(v)|\bar{h}_v^{(b)}(iD)^2h_v^{(b)}|B(v)\rangle,$$

and the color magnetic energy

$$\lambda_2 = \frac{1}{6m_B} \langle B(v)|\bar{h}_v^{(b)}\frac{g}{2}\sigma_{\mu\nu}G^{\mu\nu}h_v^{(b)}|B(v)\rangle.$$  \hspace{1cm} (2)

The parameters $\lambda_1$ and $\lambda_2$ are independent of the heavy quark mass and occur in the formulas for the $B$, $B^*$, $D$, and $D^*$ meson masses:

$$m_B = m_b + \bar{\Lambda} - (\lambda_1 + 3\lambda_2)/2m_b,$$
$$m_{B^*} = m_b + \bar{\Lambda} - (\lambda_1 - \lambda_2)/2m_b,$$
$$m_D = m_c + \bar{\Lambda} - (\lambda_1 + 3\lambda_2)/2m_c,$$
$$m_{D^*} = m_c + \bar{\Lambda} - (\lambda_1 - \lambda_2)/2m_c.$$  \hspace{1cm} (3)

The measured $B^* - B$ mass splitting $(46 \pm 0.6)$ MeV implies that $\lambda_2 = 0.12$ GeV$^2$. The quantity $\bar{\Lambda}$ represents the energy of the light degrees of freedom for the ground state $s^e_\ell$ = $\frac{1}{2}^-$ multiplet in the $m_{b,c} \to \infty$ limit. Note that in the average masses $\bar{m}_B = (m_B + 3m_{B^*})/4$ and $\bar{m}_D = (m_D + 3m_{D^*})/4$ the parameter $\lambda_2$ cancels out.

The leading order prediction of the operator product expansion for the $B$ semileptonic decay rate involves quark masses, which are not known experimentally. What is measured are the hadron masses. It is possible using eq. (3) to express the quark masses, $m_b$ and $m_c$, in terms of the hadron masses, $\bar{m}_B$ and $\bar{m}_D$, and the parameters $\lambda_1$ and $\bar{\Lambda}$. When this is done the semileptonic $B$-meson decay rate depends on the unknown parameters $\lambda_1$ and $\bar{\Lambda}$ that are of order $\Lambda^2_{QCD}$ and $\Lambda_{QCD}$ respectively. In this way of looking at the predicted decay rate there are contributions of order $\Lambda_{QCD}/m_{c,b}$, but they are given in terms of the single parameter $\bar{\Lambda}$.

The form factors for semileptonic $B \to D^{(*)}\bar{e}\nu_e$ decay are defined by

$$\frac{\langle D(v')|V^\mu|B(v)\rangle}{\sqrt{m_Bm_D}} = h_+(w)(v + v')^\mu + h_-(w)(v - v')^\mu,$$
$$\frac{\langle D^*(v',\epsilon)|V^\mu|B(v)\rangle}{\sqrt{m_Bm_{D^*}}} = ih_V(w)\epsilon^{\mu\alpha\beta}v'_\alpha v_\beta,$$
$$\frac{\langle D^*(v',\epsilon)|A^\mu|B(v)\rangle}{\sqrt{m_Bm_{D^*}}} = h_{A_1}(w)(w + 1)\epsilon^{*\mu} - h_{A_2}(w)(\epsilon^* \cdot v)v^\mu - h_{A_3}(w)(\epsilon^* \cdot v)v'^\mu.$$  \hspace{1cm} (4)

Here $V^\mu = \bar{c}\gamma^\mu b$ and $A^\mu = \bar{c}\gamma^\mu\gamma_5 b$ are the vector and axial vector currents. The four-velocities of the initial and final states are denoted by $v$ and $v'$. 
respectively. The dot product of these four-velocities is \( w = v \cdot v' \) and at the zero recoil point, where \( v = v', \ w = 1 \). Up to corrections suppressed by powers of \( \alpha_s(m_{c,b}) \) and \( \Lambda_{QCD}/m_{c,b} \), \( h_-(w) = h_{A_2}(w) = 0 \) and \( h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w) \), where the Isgur–Wise function [17] \( \xi \) is evaluated at a subtraction point around \( m_{c,b} \). The differential decay rates are

\[
\frac{d\Gamma(B \to D^*(\bar{\nu}_e))}{dw} = \frac{G_F^2 m_B^5}{48\pi^3} r^{*3} (1 - r^*)^2 (w^2 - 1)^{1/2} (w + 1)^2 \times \left[ 1 + \frac{4w - 1 - 2wr^* + r^{*2}}{w + 1} (1 - r^*)^2 \right] |V_{cb}|^2 |\mathcal{F}_{B \to D^*}(w)|^2,
\]

\[
\frac{d\Gamma(B \to D(\bar{\nu}_e))}{dw} = \frac{G_F^2 m_B^5}{48\pi^3} r^{*3} (1 + r)^2 (w^2 - 1)^{3/2} |V_{cb}|^2 |\mathcal{F}_{B \to D}(w)|^2,
\]

where \( r^{(*)} = m_{D^{(*)}}/m_B \). The functions \( \mathcal{F}_{B \to D^*} \) and \( \mathcal{F}_{B \to D} \) are given in terms of the form factors of the vector and axial vector currents defined in eq. (4) as

\[
|\mathcal{F}_{B \to D^*}(w)|^2 = \left[ 1 + \frac{4w - 1 - 2wr^* + r^{*2}}{w + 1} (1 - r^*)^2 \right]^{-1} \times \left\{ \frac{1 - 2wr^* + r^{*2}}{(1 - r^*)^2} 2 \left[ h_{A_1}^2(w) + \frac{w - 1}{w + 1} h_{V}^2(w) \right] + \left[ h_{A_1}(w) + \frac{w - 1}{1 - r^*} \left( h_{A_1}(w) - h_{A_3}(w) - r^* h_{A_2}(w) \right) \right]^2 \right\},
\]

\[
\mathcal{F}_{B \to D}(w) = h_+(w) - \frac{1 - r}{1 + r} h_-(w).
\]

Note that \( \mathcal{F}_{B \to D^*}(1) = h_{A_1}(1) \) and due to Luke’s theorem [18]

\[
h_{A_1}(1) = \eta_A + \mathcal{O}(\Lambda_{QCD}^2/m_{c,b}^2).
\]

For the \( B \to D \) case \( \mathcal{F}_{B \to D}(1) = \eta_V + \mathcal{O}(\Lambda_{QCD}/m_{c,b}) \). The quantities \( \eta_A \) and \( \eta_V \) relate the axial and vector currents in the full theory of QCD to those in HQET at zero recoil. Including corrections of order \( \alpha_s^2/\beta_0 \) [19,20]

\[
\eta_A = 1 - \alpha_s(\sqrt{m_b m_c}) \left( \frac{1 + z}{1 - z} \ln z + \frac{8}{3} \right) - \alpha_s^2(\sqrt{m_b m_c}) \beta_0 \frac{5}{24} \left( \frac{1 + z}{1 - z} \ln z + \frac{44}{15} \right),
\]

where \( z = m_c/m_b \) and \( \beta_0 = 11 - 2N_f/3 \) is the 1-loop beta function. In eq. (8) and hereafter dimensional regularization with \( \overline{MS} \) subtraction is used. The full order \( \alpha_s^2 \) expression for \( \eta_A \) is known [21] and the \( \alpha_s^2/\beta_0 \) part presented in eq. (8) dominates it.
II SUM RULES

To derive the sum rules, we consider the time-ordered product

\[ T_{\mu \nu} = \frac{i}{2m_B} \int d^4x \ e^{-iqx} \langle B \mid T\{J_{\mu}^+(x), J_{\nu}(0)\} \mid B \rangle, \tag{9} \]

where \( J_{\mu} \) is a \( b \to c \) axial or vector current, the \( B \) states are at rest, \( \vec{q} \) is fixed, and \( q_0 = m_B - E_{D^{(*)}} - \epsilon \). Here \( E_{D^{(*)}} = \sqrt{m_{D^{(*)}}^2 + |\vec{q}|^2} \) is the minimal possible energy of the hadronic final states that can be created by the current \( J_{\mu} \) at fixed \(|\vec{q}|\). (We deal with cases where the lowest energy state is either a \( D \) or a \( D^* \).) With this definition of \( \epsilon \), \( T_{\mu \nu} \) has a cut in in the complex \( \epsilon \) plane that lies along \( 0 < \epsilon < \infty \), corresponding to physical intermediate states with a charm quark. At the same value of \(|\vec{q}|\) the cut at the parton level lies in the smaller region \( \epsilon > \Lambda(w-1)/w + \mathcal{O}(\Lambda^2_{QCD}/m_{c,b}^2) \). \( T_{\mu \nu} \) has another cut corresponding to physical states with two \( b \)-quarks and a \( \bar{c} \) quark that lies along \( -\infty < \epsilon < -2E_{D^{(*)}} \). To separate out specific hadronic form factors, one contracts the currents in eq. (9) with a suitably chosen four-vector \( \vec{a} \). Inserting a complete set of states between the currents yields

\[ a^\mu T_{\mu \nu}(\epsilon) a^\nu = \frac{1}{2m_B} \sum_X (2\pi)^3 \delta^3(\vec{q} + \vec{p}_X) \frac{\langle B \mid J^+ \cdot a^* \mid X \rangle \langle X \mid J \cdot a \mid B \rangle}{E_X - E_{D^{(*)}} - \epsilon} + \ldots, \tag{10} \]

where the ellipses denote the contribution from the cut corresponding to two \( b \)-quarks and a \( \bar{c} \) quark. The sum over \( X \) includes the usual phase space factors, \( d^3p/(2\pi)^32E_X \), for each particle in the state \( X \).

While \( T_{\mu \nu}(\epsilon) \) cannot be computed for arbitrary values of \( \epsilon \), its integrals with appropriate weight functions are calculable using the operator product expansion and perturbative QCD. Consider integrating the product of a weight function \( W_\Delta(\epsilon) \) with \( T_{\mu \nu}(\epsilon) \) along the contour \( C \) surrounding the physical cut shown in Fig. 1. Assuming \( W \) is analytic in the shaded region enclosed by this contour, we get

\[ \frac{1}{2\pi i} \int_C d\epsilon W_\Delta(\epsilon) [a^\mu T_{\mu \nu}(\epsilon) a^\nu] = \sum_X W_\Delta(E_X - E_{D^{(*)}}) (2\pi)^3 \delta^3(\vec{q} + \vec{p}_X) \frac{\langle X \mid J \cdot a \mid B \rangle^2}{2m_B}. \tag{11} \]

The weight function is assumed to be positive along the cut and to satisfy the normalization condition \( W_\Delta(0) = 1 \). Then \( W_\Delta \cdot \langle X \mid J \cdot a \mid B \rangle^2 \) is positive for all states \( X \), and eq. (11) implies an upper bound on the magnitude of form factors for semileptonic \( B \) decays to the ground states \( D^{(*)} \).

\[ \frac{|\langle D^{(*)} \mid J \cdot a \mid B \rangle|^2}{4m_B E_{D^{(*)}}} < \frac{1}{2\pi i} \int_C d\epsilon W_\Delta(\epsilon) [a^\mu T_{\mu \nu} a^\nu]. \tag{12} \]
FIGURE 1. The integration contour $C$ in the complex $\epsilon$ plane. The cuts extend to $\text{Re} \epsilon \to \pm \infty$.

In eq. (12) a sum over $D^*$ polarizations is understood. It is also possible to derive a lower bound if some model dependent assumptions concerning the spectrum of final states $X$ are made.

A possible set of weight functions is [6],

$$W_{\Delta}^{(n)}(\epsilon) = \frac{\Delta^{2n}}{\epsilon^{2n} + \Delta^{2n}}, \quad (n = 2, 3, \ldots). \quad (13)$$

They satisfy the following properties: (i) $W_{\Delta}$ is positive along the cut so that every term in the sum over $X$ on the hadron side of the sum rule is positive; (ii) $W_{\Delta}(0) = 1$; (iii) $W_{\Delta}$ is flat near $\epsilon = 0$; and (iv) $W_{\Delta}$ falls off rapidly for $\epsilon > \Delta$. For values of $n$ of order unity all the poles of $W_{\Delta}^{(n)}$ lie at a distance of order $\Delta$ away from the physical cut. As $n \to \infty$, $W_{\Delta}^{(n)}$ approaches $\theta(\Delta - \epsilon)$ for $\epsilon > 0$, which corresponds to summing over all final hadronic resonances up to excitation energy $\Delta$ with equal weight. In this limit the poles of $W_{\Delta}^{(n)}$ approach the cut, and the contour $C$ is forced to pinch the cut at $\epsilon = \Delta$. Then the evaluation of the contour integrals using perturbative QCD relies on local duality at the scale $\Delta$. In practice, for $n > 3$ the results obtained are very close to those for $n = \infty$ and for the remainder of this lecture I will only quote results obtained from the weight $W_{\Delta}^{(\infty)}(\epsilon) = \theta(\Delta - \epsilon)$. 
III APPLICATION OF SUM RULES AT ZERO RECOIL

The sum rule bound in eq. (12) is made explicit by using the operator product expansion and perturbative QCD to evaluate the right-hand side. The most important kinematic point is the zero recoil point where $\vec{q} = 0$. Choosing $a$ to be a spatial vector $a = (0, \hat{n})$ and averaging over directions of the unit vector $\hat{n}$, we obtain for the axial vector current

$$|\mathcal{F}_{B \to D^*}(1)|^2 \leq \eta_A^2 - \frac{\lambda_2}{m_c^2} + \left( \frac{\lambda_1 + 3\lambda_2}{4} \right) \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right)$$

$$+ \frac{\alpha_s(\Delta)}{\pi} X_{AA}(\Delta) + \frac{\alpha_s^2(\Delta)}{\pi^2} \beta_0 Y_{AA}(\Delta),$$

and for the vector current

$$0 \leq \frac{\lambda_2}{m_c^2} - \left( \frac{\lambda_1 + 3\lambda_2}{4} \right) \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} - \frac{2}{3m_c m_b} \right)$$

$$+ \frac{\alpha_s(\Delta)}{\pi} X_{VV}(\Delta) + \frac{\alpha_s^2(\Delta)}{\pi^2} \beta_0 Y_{VV}(\Delta).$$

Eqs. (14) and (15) include terms of order $\Lambda_{QCD}^2/m_c^2$ coming from dimension five operators in the operator product expansion for the time ordered product of currents. The coefficients of these operators are evaluated at tree level. Also included is the contribution from the dimension 3 operator $\bar{h}_v (b) \Gamma h_v (b)$ evaluated to order $\alpha_s^2 \beta_0$. There are two distinct sources of perturbative QCD corrections. Those in $\eta_{AA}$ correspond to a final state $X_c$ that at the parton level is a single charm quark. These terms are independent of $\Delta$ and come from matching of the axial vector current onto its HQET counterpart, i.e., $A^\nu = \eta_A \bar{h}_v (b) \gamma^\nu \gamma_5 h_v (c)$. The part of the QCD correction involving $X_{AA}, Y_{AA}, X_{VV}$ and $Y_{VV}$, comes at the parton level from states with a charm quark and a gluon or even more partons. These corrections depend on $\Delta$. Since $\Delta$ is the cut off on the invariant mass of the final hadronic states it seems most natural to write these terms as a power series in $\alpha_s(\Delta)$. If one used $\alpha_s(\mu)$ with $\mu$ much different from $\Delta$ the coefficients $Y_{AA}$ and $Y_{VV}$ would contain large logarithms of $\Delta/\mu$. Analytic expressions for the order $\alpha_s$ corrections are known [6]

$$X_{AA}(\Delta) = \frac{\Delta(\Delta + 2m_c)[2(\Delta + m_c)^2 - 2m_b^2 - (m_b + m_c)^2]}{18m_b^2(\Delta + m_c)^2}$$

$$+ \frac{3m_b^2 + 2m_b m_c - m_c^2}{9m_b^2} \ln \left( \frac{\Delta + m_c}{m_c} \right),$$

$$X_{VV}(\Delta) = \frac{\Delta(\Delta + 2m_c)[2(\Delta + m_c)^2 - 2m_b^2 - (m_b - m_c)^2]}{18m_b^2(\Delta + m_c)^2}$$

$$+ \frac{3m_b^2 - 2m_b m_c - m_c^2}{9m_b^2} \ln \left( \frac{\Delta + m_c}{m_c} \right).$$
FIGURE 2. $X(\Delta)$ and $Y(\Delta)$ for the a) axial, and b) vector coefficients. Thick solid lines are $X$ while thick dashed lines are $Y$. The thin solid and dashed lines are $X$ and $Y$ to order $\Delta^2/m_{c,b}^2$.

For small $\Delta$, $X_{AA}$ and $X_{VV}$ are of order $\Delta^2/m_{c,b}^2$; however, even when $\Delta = 1$ GeV, terms higher order in $\Delta/m_{c,b}$ are important (the small $\Delta$ approximation to $X_{AA}$ and $X_{VV}$ was calculated in Ref. [4]). The values of $Y_{AA}$ and $Y_{VV}$ have been determined numerically [6]. In Fig. 2, $X_{AA}$, $Y_{AA}$, $X_{VV}$, and $Y_{VV}$ are plotted as functions of $\Delta$ in the region $\Delta < 2$ GeV. The values of $Y$ are quite close to $X$ in this region.

The vector current sum rule bound in eq. (15) implies a bound on $\lambda_1$. This bound is strongest for $m_c \gg m_b \gg \Delta$. In that limit it becomes [4,6]

$$\lambda_1 \leq -3\lambda_2 + \frac{\alpha_s(\Delta)}{\pi} \Delta^2 \left(\frac{4}{3}\right) + \frac{\alpha_s^2(\Delta)}{\pi^2} \beta_0 \Delta^2 \left(\frac{13}{9} - \frac{2\ln 2}{3}\right).$$

(18)

The parameter $\Delta$ must be chosen large enough that perturbative QCD is meaningful. However the bounds on $\lambda_1$ and $|F_{B\to D^*}|^2$ become stronger the smaller the value of $\Delta$. The smallest value of $\Delta$ for which one can imagine using perturbative QCD is 1 GeV. Using $\Delta = 1$ GeV, $\alpha_s(1\text{GeV}) = 0.45$, $\lambda_2 = 0.12\text{GeV}^2$, eq. (18) implies

$$\lambda_1 \leq (-0.36 + 0.19 + 0.20) \text{GeV}^2.$$

(19)

The three terms on the right-hand side of eq. (19) correspond respectively to the contribution of $\lambda_2$, the perturbative part of order $\alpha_s(\Delta)\Delta^2/\pi$, and the perturbative part of order $[\alpha_s(\Delta)/\pi]^2\Delta^2$. Notice that with $\Delta = 1$ GeV the $\alpha_s^2$ term is as large as the order $\alpha_s$ term. It may be a mistake to conclude from this that $\Delta = 1$ GeV is too low for QCD perturbation theory to be meaningful. It has been conjectured that $\lambda_1$ has a renormalon ambiguity of order $\Lambda_{QCD}^2$. 
(one does not see this from the usual sum of bubble graphs) [22]. Even though the renormalon ambiguity arises from large orders of perturbation theory, it is possible that the bad behavior of the first few terms in the perturbative series presented in eq. (18) is a reflection of this uncertainty.

In this lecture the matrix element $\lambda_1$ is defined using dimensional regularization and \( \overline{\text{MS}} \) subtraction. If $\lambda_1$ has a renormalon ambiguity (of order $\Lambda_{\text{QCD}}^2$), the perturbative QCD series that relates it to a physical quantity, for example computed in lattice QCD, is not Borel summable. However, there is no evidence that this is a serious problem. Whenever $\lambda_1$ occurs in an expression for some measurable quantity, e.g., the bound on $|F_{B \to D^*}(1)|^2$, there is another perturbative series that when combined with the series in $\lambda_1$ (e.g., from matching onto lattice QCD) probably has no renormalon ambiguity (of order $\Lambda_{\text{QCD}}^2$) [23].

Next consider the bound on $|F_{B \to D^*}(1)|^2$ in eq. (14). We can eliminate $\lambda_1$ from it by combining (14) and (15). This gives

\[
|F_{B \to D^*}(1)|^2 \leq \eta_A^2 - \frac{\lambda_2}{m_c^2} + \frac{\alpha_s(\Delta)}{\pi} \left[ X_{AA}(\Delta) + \frac{1}{3} \left( \frac{\Delta^2}{m_c^2} + \frac{\Delta^2}{m_b^2} + \frac{2\Delta^2}{3m_c m_b} \right) \right] + \frac{\alpha_s^2(\Delta)}{\pi^2} \beta_0 \left[ Y_{AA}(\Delta) + \left( \frac{13}{36} - \frac{\ln 2}{6} \right) \left( \frac{\Delta^2}{m_c^2} + \frac{\Delta^2}{m_b^2} + \frac{2\Delta^2}{3m_c m_b} \right) \right].
\]

Neglecting the nonperturbative correction factor of $-\frac{\lambda_2}{m_c^2}$ and again using $\Delta = 1$ GeV, the above bound is

\[
|F_{B \to D^*}(1)|^2 \leq 1 - 0.074 - 0.020 + 0.044 + 0.046 = 1 - 0.030 + 0.026.
\]

Here we used $m_c = 1.4$ GeV, $m_b = 4.8$ GeV, $\alpha_s(\sqrt{m_c m_b}) = 0.28$, $\alpha_s(1 \text{ GeV}) = 0.45$, and $\beta_0 = 9$. The first row is the perturbative expansion of $\eta_A^2$, the second row is the order $\alpha_s(\Delta)$ and order $\alpha_s(\Delta)^2$ terms, and the third row sums the columns. There is a renormalon ambiguity of order $\Lambda_{\text{QCD}}^2/m_c^2 m_b$ that cancels between the perturbative series for $\eta_A^2$ and the series in $\alpha_s(\Delta)$. This bound on the physical quantity $|F_{B \to D^*}(1)|^2$ is not very strong (even when the factor of $-\frac{\lambda_2}{m_c^2}$ is included), and furthermore the third row of eq. (21) seems to indicate that with $\Delta = 1$ GeV QCD perturbation theory is not very well behaved. However, this does not mean that the sum rule for $|F_{B \to D^*}(1)|^2$ in eq. (14) is not useful. Consider the perturbative part of eq. (14), neglecting for now both the terms of order $\frac{\lambda_1}{m_c^2}$ and $\frac{\lambda_2}{m_c^2}$. Numerically, with $\Delta = 1$ GeV, this gives

\[
|F_{B \to D^*}(1)|^2 \leq 1 - 0.074 - 0.020 + 0.013 + 0.017 = 1 - 0.061 - 0.003.
\]
Again, the first row is the perturbative expansion of $\eta_2^A$ and the second row are the terms of order $\alpha_s(\Delta)$ and $\alpha_s^2(\Delta)$. The third row of eq. (22) does not indicate that there is any breakdown of QCD perturbation theory. If $\lambda_1$ can be determined experimentally from, for example, the electron spectrum in inclusive semileptonic $B$-decay then the sum rule in eq. (14) may lead to an important constraint. For example, with $\Delta = 1$ GeV and $\lambda_1 = -0.2$ GeV$^2$, eq. (14) implies the bound

$$|F_{B\to D^*}(1)|^2 \leq 0.90,$$

which is smaller than $\eta_2^A = 0.91$.

**IV THE LEPTON ENERGY SPECTRUM AND THE PARAMETERS $\lambda_1$, $\bar{\Lambda}$**

The CLEO collaboration has measured the lepton energy spectrum for inclusive $B \to X \ell \bar{\nu}_\ell$ decay, both demanding only one charged lepton (single tagged data) and two charged leptons (double tagged sample) [24,25]. In the double tagged sample the charge of the high momentum lepton determines whether the other lepton comes from a semileptonic $B$ decay (primary lepton) or the semileptonic decay of a $D$-meson (secondary lepton). The single tagged sample is presented in 50 MeV bins while the double tagged data is presented in 100 MeV bins. The single tagged sample has much higher statistics, but is significantly contaminated by secondaries below $E_\ell = 1.5$ GeV.

The operator product expansion for semileptonic $B$ decay does not reproduce the physical lepton spectrum point by point near the maximal electron energy. Near the endpoint, comparison of theory with data can only be made after smearing or integrating over a large enough region. The minimal size of this region has been estimated to be about 500 MeV. As was mentioned in the introduction, the theoretical prediction for the lepton energy spectrum depends on $\lambda_1$ and $\bar{\Lambda}$, so we can try to use the data to determine these quantities. We want to consider observables sensitive to $\bar{\Lambda}$ and $\lambda_1$, but we also want deviations from the $b$-quark decay rate to be small enough so that contributions from even higher dimension operators in the operator product expansion are small. Ref. [26] uses $R_1$ and $R_2$, where

$$R_1 = \frac{\int_{1.5\text{GeV}}^{1.7\text{GeV}} (d\Gamma/dE_\ell) E_\ell \, dE_\ell}{\int_{1.5\text{GeV}}^{1.7\text{GeV}} (d\Gamma/dE_\ell) \, dE_\ell},$$

and

$$R_2 = \frac{\int_{1.5\text{GeV}}^{1.7\text{GeV}} (d\Gamma/dE_\ell) E_\ell \, dE_\ell}{\int_{1.5\text{GeV}}^{1.7\text{GeV}} (d\Gamma/dE_\ell) \, dE_\ell}.$$
Here $E_\ell$ denotes the lepton energy. The variable $R_1$ has dimensions of mass and values for it will be given in GeV. Ratios are considered so that $|V_{ub}|$ cancels out. Before comparing the experimental data with theoretical predictions derived from the operator product expansion and QCD perturbation theory, it is necessary to include electromagnetic corrections and effects of the boost to the laboratory frame. This gives

$$R_1 = 1.8059 - 0.309 \left( \frac{\bar{\Lambda}}{m_B} \right) - 0.35 \left( \frac{\bar{\Lambda}}{m_B} \right)^2 - 2.32 \left( \frac{\lambda_1}{m_B^2} \right) - 3.96 \left( \frac{\lambda_2}{m_B^2} \right)$$

$$- \frac{\alpha_s}{\pi} \left( 0.035 + 0.07 \frac{\bar{\Lambda}}{m_B} \right) + |V_{ub}|^2 \left( 1.33 - 10.3 \frac{\bar{\Lambda}}{m_B} \right)$$

$$- \left( 0.0041 - 0.004 \frac{\bar{\Lambda}}{m_B} \right) + \left( 0.0062 + 0.002 \frac{\bar{\Lambda}}{m_B} \right),$$

(26)

and

$$R_2 = 0.6581 - 0.315 \left( \frac{\bar{\Lambda}}{m_B} \right) - 0.68 \left( \frac{\bar{\Lambda}}{m_B} \right)^2 - 1.65 \left( \frac{\lambda_1}{m_B^2} \right) - 4.94 \left( \frac{\lambda_2}{m_B^2} \right)$$

$$- \frac{\alpha_s}{\pi} \left( 0.039 + 0.18 \frac{\bar{\Lambda}}{m_B} \right) + \frac{V_{ub}}{V_{cb}} \left( 0.87 - 3.8 \frac{\bar{\Lambda}}{m_B} \right)$$

$$- \left( 0.0073 + 0.005 \frac{\bar{\Lambda}}{m_B} \right) + \left( 0.0021 + 0.003 \frac{\bar{\Lambda}}{m_B} \right).$$

(27)

In eqs. (26) and (27) the charm and bottom quark masses have been expressed in terms of $\bar{m}_B$, $\bar{m}_D$, $\bar{\Lambda}$, $\lambda_1$, and $\lambda_2$ using eq. (3), which is why $\bar{\Lambda}$ occurs in these formulas. The last two terms in eqs. (26) and (27) are from electromagnetic radiative corrections and from the boost to the laboratory frame respectively. The experimental values for $R_1$ and $R_2$ are $R_1 = 1.7831$ GeV and $R_2 = 0.6159$. These were obtained from the single tagged data with a correction for the secondary leptons obtained from the double tagged sample. For $R_1$ this correction is 0.0001 GeV and for $R_2$ it is 0.0051. Comparing experiment with eqs. (26) and (27) gives the central values $\bar{\Lambda} = 0.39 \pm 0.11$ GeV and $\lambda_1 = -0.19 \pm 0.10$ GeV$^2$. Fig. 3 shows the one sigma bands on the allowed values of $\bar{\Lambda}$ and $\lambda_1$ from $R_1$ and $R_2$. The narrower band corresponds to the $R_1$ constraint. The shaded ellipse is the one sigma allowed region for $\bar{\Lambda}$ and $\lambda_1$ including correlations between $R_1$ and $R_2$. The errors included in this analysis are just the statistical ones. An analysis of systematic errors has not been performed. However, they are only weakly energy dependent for $E_\ell \geq 1.5$ GeV and it is hoped that for $R_{1,2}$ systematic errors are smaller than the statistical ones. Note that the bands from $R_1$ and $R_2$ are almost parallel, so even small corrections can significantly change the central values for $\bar{\Lambda}$ and $\lambda_1$ obtained from Fig. 3.
One such set of corrections comes from higher dimension operator in the operator product expansion. At order $\Lambda_{\text{QCD}}^3/\bar{m}_B^3$ new terms occur characterized by two matrix elements $\rho_1$ and $\rho_2$ and two time ordered products. $\rho_1$ can be estimated by factorization, $\rho_1 = (2\pi\alpha_s/9)m_Bf_B^2 \approx (300 \text{ MeV})^3$, and $\rho_2$ is expected to be small [4]. Neglecting $\rho_2$ and the time ordered products gives the following order $\Lambda_{\text{QCD}}^3/\bar{m}_B^3$ corrections to $R_1$ and $R_2$,

$$
\delta R_1 = -(0.4\bar{\Lambda}^3 + 5.7\bar{\Lambda}\lambda_1 + 6.8\bar{\Lambda}\lambda_2 + 7.7\rho_1)/\bar{m}_B^3,
$$

$$
\delta R_2 = -(1.5\bar{\Lambda}^3 + 7.1\bar{\Lambda}\lambda_1 + 17.5\bar{\Lambda}\lambda_2 + 1.8\rho_1)/\bar{m}_B^3.
$$

Including these corrections shifts the central values for $\bar{\Lambda}$ and $\lambda_1$ to the location $\bar{\Lambda} = 0.35 \text{ GeV}$ and $\lambda_1 = -0.15 \text{ GeV}^2$ marked by the star in Fig. 3. For a more complete discussion of the order $\Lambda_{\text{QCD}}^3/\bar{m}_B^3$ corrections, see Ref. [27].

The bands in Fig. 3 were determined using $|V_{ub}/V_{cb}| = 0.08$. This value is model dependent. If $|V_{ub}/V_{cb}| = 0.10$ is used then the central values shift to $\bar{\Lambda} = 0.42 \text{ GeV}$ and $\lambda_1 = -0.19 \text{ GeV}^2$. In Fig. 3, $\alpha_s = 0.22$ was used corresponding to a subtraction point near $m_b$. With $\alpha_s = 0.35$ the central values shift to $\bar{\Lambda} = 0.36 \text{ GeV}$ and $\lambda_1 = -0.18 \text{ GeV}^2$.

Theoretical uncertainty in this determination of $\bar{\Lambda}$ and $\lambda_1$ originate from the reliability of quark hadron duality at the limits of integration defining $R_{1,2}$. 

**FIGURE 3.** Allowed regions in the $\bar{\Lambda} - \lambda_1$ plane for $R_1$ and $R_2$. The bands represent the 1σ statistical errors, while the ellipse is the allowed region taking correlations into account. The star denotes where the order $\Lambda_{\text{QCD}}^3/\bar{m}_B^3$ corrections discussed in the text would shift the center of the ellipse.
order $\Lambda^3_{\text{QCD}}/\bar{m}_b^3$ corrections, and higher order perturbative QCD corrections. Recently the order $\alpha_s^2\beta_0$ terms in $R_1$ and $R_2$ have been computed [28]. They give corrections $\delta R_1 = -0.082\alpha_s^2\beta_0/\pi^2$ and $\delta R_2 = -0.098\alpha_s^2\beta_0/\pi^2$, moving the central values to $\bar{\Lambda} = 0.33\text{GeV}$ and $\lambda_1 = -0.17\text{GeV}^2$. Concerning duality, note that the lower limits on the lepton energy $E_\ell \geq 1.5\text{GeV}$ and $E_\ell \geq 1.7\text{GeV}$ used in $R_{1,2}$ correspond to summing over hadronic states $X$ with masses less than 3.6 GeV and 3.3 GeV respectively. Changing the lower limit in the numerator of $R_2$ to 1.8 GeV leads to central values $\bar{\Lambda} = 0.47\text{GeV}$ and $\lambda_1 = -0.26\text{GeV}^2$. The plot in Fig. 3 uses electron data only. Using the muon sample instead gives compatible central values of $\bar{\Lambda} = 0.43\text{GeV}$ and $\lambda_1 = -0.21\text{GeV}^2$.

It would be nice to have another constraint on $\lambda_1$ and $\bar{\Lambda}$ that would be less parallel than $R_1$ and $R_2$ are. This can be provided by the photon spectrum in inclusive $B \to X_s\gamma$ decay [29] which, when the data improves, will give a band almost parallel to the $\lambda_1$ axis of Fig. 3. Lattice QCD can also be used to determine $\lambda_1$ and $\bar{\Lambda}$, although for $\lambda_1$ there are serious difficulties coming from mixing with the lower dimension operator $\bar{h}^{(b)} h^{(b)}$ [30]. This mixing does not occur in the continuum if dimensional regularization with $\overline{\text{MS}}$ subtraction is used.

V CONCLUDING REMARKS

In this lecture I have reviewed the derivation of $B$-decay sum rules, discussed the perturbative QCD corrections, and reviewed the status of the determination of the nonperturbative QCD matrix element $\lambda_1$ that occurs in the sum rules.

If the contribution of the lowest lying excited states $X$ on the right-hand side of eq. (11) were known then this would imply a better bound on the ground state matrix elements. The lowest lying excited states are nonresonant $D^{(*)}\pi$. Their contribution, for low $D^{(*)}\pi$ invariant mass, is calculable [4] in terms of the one coupling constant, $g$, of heavy hadron chiral perturbation theory [31]. This coupling also determines the $D^*$ width, $\Gamma(D^{*+} \to D^{0}\pi^+)$ = $(g^2/6\pi f_\pi^2)|\vec{p}_\pi|^3$ (for the neutral pion mode there is an additional factor of 1/2). Unfortunately at the present time there is only a limit on the $D^*$ width and hence an upper bound on $g$. A measurement of the $D^*$ width would give a direct determination of this coupling. Then we would know the contribution of these nonresonant states to the sum rules. (Determining $g$ from various $D^*$ and $D^{(*)}_s$ branching ratios is discussed in Ref. [32]). Higher in mass is the $s^{\pi\ell}_t = \frac{3}{2}^+$ doublet of excited charmed mesons $D_1(2420)$ and $D^{*}_s(2460)$. These states are narrow with widths around 20 MeV. A doublet with $s^{\pi\ell}_t = \frac{1}{2}^+$ quantum numbers is expected to also exist, but these states are thought to be quite broad (i.e., widths greater than 100 MeV). Very recently the contribution of these excited charmed mesons to the sum rules has been discussed [33].
Perturbative QCD corrections to the sum rules have been calculated to order $\alpha_s^2\beta_0$. It is important to improve this to a full order $\alpha_s^2$ calculation. Finally, it is interesting to note that $B$-decay sum rules may also be important for the $b \to u$ transition, giving valuable information on exclusive $B \to \pi\ell\bar{\nu}_\ell$ or $B \to \rho\ell\bar{\nu}_\ell$ form factors [8]. At this time perturbative corrections to these $b \to u$ transition sum rules have not been computed.

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