AC

UGVA-DPNC 1997/2-168
February 1997

FERMION MASS SINGULARITIES IN QED AND QCD AND THE MEANING OF THE KINOSHITA-LEE-NAUENBERG THEOREM

J.H. Field

Département de Physique nucléaire et corpusculaire
Université de Genève, 24, quai Ernest-Ansermet
CH - 1211 Genève 4

SW 9713

ABSTRACT

An analysis is presented of radiative corrections due to fermion vacuum polarisation for the process $e^-\mu^- \rightarrow e^-\mu^-$ and the full $O(\alpha^2)$ final state QED radiative correction to $e^+e^- \rightarrow f\bar{f} (f \neq e)$. It is demonstrated that in both cases, the corrections contain next-to-leading logarithmic terms of the form $\alpha^2 \ln \frac{Q}{m}$ where Q is the large external scale and m the fermion mass. These radiative corrections are infinite in the massless limit for any non vanishing value of α . The rôle of Landau singularities and the relation of the above results to the KLN Theorem are discussed. Similar considerations apply also to the fermion sector of QCD where m is the quark mass. Thus contrary to many statements to be found in the literature no finite massless version of either QED or QCD can exist.

To be published in Modern Physics Letters A

It has become standard practice in perturbative QCD calculations to describe the theory as "massless" when, apparently, the massless limit is taken in the energy-momentum 4-vectors and propagators of all particles in the contributing Feynman diagrams. This is often justified by reference to the KLN (Kinoshita-Lee-Nauenberg) Theorem [1], [2]. The purpose of this letter is to point out, that for certain classes of diagrams occurring in next-to-leading order radiative corrections to a wide range of different physical quantities, both in QED and QCD, the truly "massless" theory is spoiled by logarithmic divergences and (in QED) by their associated Landau singularities [3]. It is in one sense correct to refer to the QCD calculations in question, which use the 't Hooft-Veltman dimensional regularisation scheme [4] as "massless" since no mass parameters appear explicitly in the equations of the theory. However, it will be shown that non-zero fermion masses are always implicit in other parameters of the theory when finite predictions are obtained. The discussion will be limited to fermion mass singularities where the structure of the mass singular diagrams is relatively simple and generalisation to the QCD case particularly straight forward. However, as discussed previously in the literature [5-8], similar considerations apply also to UV (Ultra-Violet) singular diagrams containing gluons.

The essential conclusion may be briefly stated at the outset. Diagrams with a tree structure and an arbitrary number of external photon or gluon lines, together with the associated (by Gauge Invariance) non-UV divergent virtual diagrams, give a finite contribution to the invariant amplitude in the limit $\lambda \rightarrow 0$, $m \rightarrow 0$. Here λ is a generic photon or gluon mass and m is a generic fermion mass. This is the "LN" [2] part of the KLN Theorem. On the other hand many diagrams containing an internal off shell photon or gluon line with a fermion vacuum polarisation insertion have logarithmic mass singularities in the associated invariant amplitude after renormalisation to remove the UV divergence associated with the fermion loop. This is the "K" [1] part of the KLN Theorem.

Two specific QED examples will now be discussed : one loop fermion vacuum polarisation insertion corrections to the amplitude for the process $e^- \mu^- \rightarrow e^- \mu^-$, and the complete $O(\alpha^2)$ final state corrections to the total cross section for the process $e^+ e^- \rightarrow \gamma^* \rightarrow f \bar{f}$ at energies $\sqrt{s}$ such that the $m_f/\sqrt{s} \ll 1$. In the first example only the "K" Theorem applies, in the second both the "K" and the "LN" Theorems are relevant.

The renormalised invariant amplitude for the process $e^- \mu^- \rightarrow e^- \mu^-$, given by the sum of the two diagrams shown in Fig. 1 may be written as :

$$M^{(1)} = \bar{u}_B \gamma^\rho u_A \left(\frac{g_{\rho\nu}}{q^2} \right) \bar{u}_D \gamma^\nu u_C \alpha_\mu^i d_i^{(1)}(q^2, \mu^2, m_\mu^2, m^2) \quad (1)$$

where

$$d_i^{(1)} = 1 + \Pi_i^{(1)}$$

α_μ^i is one loop running fine structure constant at scale μ , and $\Pi_i^{(1)}$ is the one loop photon proper self energy function (PPSEF) which is a function of: the 4-momentum transfer squared $q^2 = (p_B - p_A)^2$, ($q^2 < 0$), flowing in the photon propagators, the subtraction scale μ ($\mu^2 > 0$), the one loop running mass of the fermion f at scale μ , m_μ and the physical (pole) mass of the fermion defined as $m \equiv m_m$. The label i denotes the renormalisation scheme, as discussed in detail by Coqueroaux [9]. Two schemes are of particular interest. (i) The On-Shell (OS) scheme conventionally used in QED calculations. (ii) The $\overline{MS}$ scheme [10] as conventionally used in

QCD. The corresponding scheme for QED introduced in Ref. [9] was referred to there as the JB ("John Brown") scheme. Here the name " $\overline{\text{MS}}$ " is used for both QED and QCD. The following expressions are obtained for the PPSEF in the OS and $\overline{\text{MS}}$ schemes :

$$\Pi_{\text{OS}}^{(1)} = \frac{2\alpha_0^{\text{OS}}}{3\pi} L(\rho) \quad , \quad \rho = \frac{-q^2}{4m^2} \quad (2)$$

$$\Pi_{\overline{\text{MS}}}^{(1)} = \frac{2\alpha_\mu^{\overline{\text{MS}}}}{3\pi} \left[L(\rho_\mu) - \ln \frac{\mu}{m_\mu} \right] \quad , \quad \rho_\mu = \frac{-q^2}{4m_\mu^2} \quad (3)$$

Here $\alpha_0^{\text{OS}} = \alpha$ is the usual fine structure constant ($\alpha^{-1} = 137.036\dots$), while $\alpha_\mu^{\overline{\text{MS}}}$ is the $\overline{\text{MS}}$ scheme (running) fine structure constant at scale μ . The function L , determined by the integration over the momentum transfer flowing in the fermion loop is [11] :

$$L(\rho) = \frac{1}{2} \left(1 - \frac{1}{2\rho} \right) \sqrt{1 + \frac{1}{\rho}} \ln \left[\frac{\sqrt{1 + 1/\rho} + 1}{\sqrt{1 + 1/\rho} - 1} \right] + \frac{1}{2\rho} - \frac{5}{6} \quad (4)$$

In order to derive Eqn (2) it is assumed that $m \neq 0$, and that, at the subtraction point - $q^2 \ll m^2$. The OS scheme, although m dependent, is therefore not suitable for a rigorous discussion of the massless limit. Assuming for the $\overline{\text{MS}}$ case that $\mu^2, -q^2 \gg m_\mu^2$ and using the limiting form of Eqn (4) for $\rho \gg 1$:

$$L(\rho) \equiv \frac{1}{2} \ln(4\rho) - \frac{5}{6} \quad (\rho \gg 1) \quad (5)$$

gives the following "asymptotic" form for $\Pi_{\overline{\text{MS}}}^{(1)}$:

$$\Pi_{\overline{\text{MS}}}^{(1)\infty} = \frac{\alpha_\mu^{\overline{\text{MS}}}}{3\pi} \left[\ln \left(\frac{-q^2}{\mu^2} \right) - \frac{5}{3} \right] \quad (6)$$

Since in Eqn (6) all explicit m dependence has disappeared, it might be assumed that $\Pi_{\overline{\text{MS}}}^{(1)\infty}$ is finite in the massless limit. In fact a similar assumption was made in Ref. [12] leading to the erroneous conclusion that QED with massless electrons is, in general, a finite theory. In fact, as will be shown below, the m dependence in Eqn (6) is implicit in the quantity $\alpha_\mu^{\overline{\text{MS}}}$. To study the sensitivity of the theory to mass singularities it is evident that a mass dependent renormalisation scheme must be used. A suitable choice is the "Momentum Subtraction" [13-15] or "Canonical" [9] scheme, with subtraction at the point - $q^2 = m^2$. Denoting this scheme by MM, the PPSEF is found to be :

$$\Pi_{\text{MM}}^{(1)} = \frac{2\alpha_m^{\text{MM}}}{3\pi} \left[L(\rho) - L\left(\frac{1}{4}\right) \right] \quad (7)$$

where

$$L\left(\frac{1}{4}\right) = \frac{7}{6} - \frac{\sqrt{5}}{2} \ln \left(\frac{\sqrt{3} + 3}{2} \right) = 0.0906$$

As $m \rightarrow 0$ at fixed q^2 the approximate form Eqn (5) of L may be used. It is then clear that : $\Pi_{MM}^{(1)}$ is logarithmically divergent as $m \rightarrow 0$ at fixed q^2 for all non-vanishing values of α_m^{MM} . To see the physical consequences of this divergence it is necessary to perform the Dyson sum of diagrams containing one loop fermion vacuum polarisation insertions to all orders. Thus $M^{(1)} \rightarrow M^{(\infty)}$ where $M^{(\infty)}$ is defined by Eqn (1) with the replacement :

$$d_i^{(1)} \rightarrow d_i^{(\infty)} = 1 + \Pi_i^{(1)} + (\Pi_i^{(1)})^2 + \dots$$

Performing the geometric sum gives :

$$\alpha_{\text{eff}}(q^2) \equiv \alpha_\mu^i d_i^{(\infty)} = \frac{\alpha_\mu^i}{1 - \Pi_i^{(1)}} \quad (8)$$

The "effective charge" [9] α_{eff} is a Renormalisation Scheme, Renormalisation Scale and Gauge Invariant quantity. Choosing $i = MM$ it is then clear that the QED prediction for $M^{(\infty)}$ is ill-defined in the massless limit due to the logarithmic divergence in $\Pi_{MM}^{(1)}$. In fact the theory already breaks down at a *finite* value of m due to the Landau [3] singularity in Eqn (8) when $\Pi_i^{(1)} = 1$. This singularity occurs at fixed q^2 and for any non-vanishing value of $\alpha_m^{MM} \equiv \alpha_L^{MM}$ when $m = m_L^{MM}$ where :

$$(m_L^{MM})^2 = (-q^2) / \exp \left[\frac{3\pi}{\alpha_L^{MM}} + 2L\left(\frac{1}{4}\right) + \frac{5}{3} \right] \quad (9)$$

Comparing Eqns (2) and (7) it can be seen that the mass singular logarithms are the same in the OS and MM schemes, so that inspection of the result of an OS scheme calculation is sufficient to establish whether a given physical quantity is ill-defined in the massless limit or not. Using the renormalisation scheme and renormalisation scale invariance of the effective charge, the relation between the running fine structure constants α^i, α^j in two different schemes i, j is given by the relation [9] :

$$\alpha_\mu^i = \frac{\alpha_\nu^j}{1 - \alpha_\nu^j [\tilde{\Pi}_j(q^2, \nu^2) - \tilde{\Pi}_i(q^2, \mu^2)]} \quad (10)$$

where

$$\Pi_i(q^2, \mu^2) = \alpha_\mu^i \tilde{\Pi}_i(q^2, \mu^2)$$

For $q^2 \gg m^2$ Eqn (10) gives with $i = \overline{MS}$, $j = MM$.

$$\alpha_\mu^{\overline{MS}} = \frac{\alpha_m^{MM}}{1 - \frac{\alpha_m^{MM}}{3\pi} \left[\ln \left(\frac{\mu^2}{m^2} \right) - 2L\left(\frac{1}{4}\right) \right]} \quad (11)$$

Hence, for non zero μ and α_m^{MM} , $\alpha_\mu^{\overline{MS}}$ is logarithmically divergent in the massless limit. The value of m at which the theory breaks down is actually finite, and corresponds to the Landau

singularity on the R.H.S. of Eqn (11). It occurs, for any non zero values of μ and, $\alpha_m^{MM} \equiv \alpha_L$, when $m = m_L^{\overline{MS}}$ where :

$$\left(m_L^{\overline{MS}}\right)^2 = \mu^2 / \exp \left[\frac{3\pi}{\alpha_L} + 2L \left(\frac{1}{4} \right) \right] \quad (12)$$

Thus, when the $\overline{MS}$ scheme is used the mass singular terms are implicit in $\alpha_\mu^{\overline{MS}}$. Since the effective charge is renormalisation scheme invariant the singular behaviour found in the massless limit using the MM scheme must of course be present for all other schemes in Eqn (8). This is shown explicitly for the $\overline{MS}$ scheme by Eqns (6) and (11).

The second example to be discussed, the $O(\alpha^2)$ correction to the total cross section for the process $e^+e^- \rightarrow \gamma^* \rightarrow f\bar{f}$ is particularly instructive since both aspects of the KLN Theorem, mentioned above are clearly demonstrated. It is convenient to classify the $O(\alpha^2)$ corrections in terms of different cuts of the forward virtual photon scattering amplitude for which some representative diagrams are shown in Fig. 2. There are two different types of contribution, those involving two photon lines, as illustrated on Fig. 2a), or two photon lines and a massive fermion (F) loop, as shown in Fig. 2b). Combining the recent calculation [16] of the fermion loop contribution with a previous calculation [7] of the complete $O(\alpha^2)$ correction, the separate contributions from diagrams of the type shown in Figs 2a), 2b) may be evaluated. Assuming that the fermions f, F have the electron charge, and neglecting terms of order m^2/s , M^2/s where m, M are the masses of the fermions f, F respectively and $\sqrt{s}$ is the e^+e^- collision energy, then the $O(\alpha^2)$ contributions to the total cross section are [7, 16] :

$$\frac{\sigma(f\bar{f}F\bar{F})}{\sigma(f\bar{f})} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{4} \ln \left(\frac{s}{M^2} \right) + \zeta(3) - \frac{11}{8} \right] \quad (13)$$

$$\frac{\sigma(f\bar{f}\gamma\gamma) + \sigma(f\bar{f}F\bar{F})}{\sigma(f\bar{f})} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{4} \ln \left(\frac{s}{M^2} \right) + \zeta(3) - \frac{47}{32} \right] \quad (14)$$

where $\xi(3) = 1.20205\dots$ and $\sigma(f\bar{f})$ is the total cross section for the Born term $e^+e^- \rightarrow f\bar{f}$. Subtracting Eqn (13) from Eqn (14) gives :

$$\frac{\sigma(f\bar{f}\gamma\gamma)}{\sigma(f\bar{f})} = - \left(\frac{\alpha}{\pi}\right)^2 \frac{3}{32} \quad (15)$$

which may be compared to the $O(\alpha)$ correction :

$$\frac{\sigma(f\bar{f}\gamma)}{\sigma(f\bar{f})} = \left(\frac{\alpha}{\pi}\right) \frac{3}{4} \quad (16)$$

$\sigma(f\bar{f}\gamma\gamma)$ contains three types of contributions, corresponding to different cuts of diagrams such as these shown in Fig. 2a) : (i) a positive definite contribution due to final states with two real photons (ii) a contribution from amplitudes with one real and one virtual photon interfering with the $f\bar{f}\gamma$ amplitude (iii) a contribution from $f\bar{f}$ amplitudes with two virtual pho-

tons in interfering with the Born amplitude. Eqn (15) shows that destructive interference effects predominate. The "LN" part of the KLN Theorem is exemplified by Eqns (15) and (16). The contributions of diagrams of the type shown in Fig. 2a) (and the similar diagrams with only one photon line that give the $0(\alpha)$ correction) are "infra-red safe". No logarithms in the masses λ , m of the photon and light fermion appear in Eqns (15) and (16). The contributions from these diagrams are finite as $\lambda \rightarrow 0$, $m \rightarrow 0$. In contrast the contribution from the "heavy fermion" loop diagrams as in Fig. 2b) are logarithmically divergent as $M \rightarrow 0$. This feature of QED was first pointed out by Kinoshita [1]. It is the "K" part of the "KLN" Theorem.

The result of Eqn (13) may be used to give the prediction for the analogous QCD process $e^+e^- \rightarrow q\bar{q}Q\bar{Q}$ given by the replacements $f \rightarrow q$, $F \rightarrow Q$ and where the virtual photons inside the f loop in diagrams such as Fig. 2b) are replaced by gluons. Introducing the ratio r_{QED} defined by :

$$r_{\text{QED}} \equiv \frac{\sigma(\bar{f}\bar{f}) + \sigma(\bar{f}\bar{f}\gamma) + \sigma(\bar{f}\bar{f}\gamma\gamma) + \sigma(\bar{f}\bar{f}F\bar{F})}{\sigma(\bar{f}\bar{f})} \quad (17)$$

and using Eqns (14) and (16) :

$$r_{\text{QED}} = 1 + \left(\frac{\alpha}{\pi}\right) \frac{3}{4} + \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{4} \ln\left(\frac{s}{M^2}\right) + \zeta(3) - \frac{11}{8} \right] \quad (18)$$

Changing renormalisation schemes from OS to $\overline{\text{MS}}$ using the analog of Eqn (10) for time-like momentum transfers, and with the choice $\mu = \sqrt{s}$ of renormalisation scale, Eqn (18) may be written as :

$$r_{\text{QED}} = 1 + \frac{\alpha_{\overline{\text{MS}}}(s)}{\pi} \frac{3}{4} + \left(\frac{\alpha_{\overline{\text{MS}}}(s)}{\pi}\right)^2 \left[\zeta(3) - \frac{11}{8} \right] + 0 \left[\left(\alpha_{\overline{\text{MS}}}(s)\right)^3 \right] \quad (19)$$

The logarithmic mass singular term has been absorbed into $\alpha_{\overline{\text{MS}}}(s)$. However, as demonstrated above, this quantity has a Landau singularity as $M \rightarrow 0$. The analogous QCD quantity r_{QCD} is readily derived from Eqn (19) by inserting the appropriate colour factors and making the replacement $\alpha_{\overline{\text{MS}}}(s) \rightarrow \alpha_s(s)$ [16] :

$$r_{\text{QCD}} = 1 + \frac{\alpha_s(s)}{\pi} + \left(\frac{\alpha_s(s)}{\pi}\right)^2 \left[\frac{2}{3} \zeta(3) - \frac{11}{12} + \dots \right] \quad (20)$$

As in the QED case the mass singular logarithm is absorbed into $\alpha_s(s)$, which in consequence is divergent in the $M \rightarrow 0$ limit. Due however to contributions in α_s due to the non-abelian gluon self coupling, no simple conclusions may be drawn as to the precise pathology of α_s in the massless fermion limit. This question merits further study, but is beyond the scope of the present letter.

Some other examples are now given of fully inclusive physical quantities for which the QED prediction is divergent as the mass of the electron, m_e , tends to zero, for non zero values of α .

- a) The muon lifetime τ_μ (Fig. 3a) [17] :

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2 M^5}{192\pi^3} \left[1 - 8 \left(\frac{m_e}{M} \right)^2 \right] \left\{ 1 + \frac{\alpha}{2\pi} \left(\frac{25}{4} - \pi^2 \right) \left[1 + \frac{2\alpha}{3\pi} \ln \left(\frac{M}{m_e} \right) \right] \right\} \quad (21)$$

$M \equiv$ muon mass, $G_\mu \equiv$ Fermi constant.

- b) The total cross section σ_{4e}^{TOT} for the process $e^+e^- \rightarrow e^+e^-e^+e^-$ via two-photon exchange (Fig. 3b)) [18] :

$$\sigma_{4e}^{\text{TOT}} = \frac{\alpha^4}{27\pi m_e^2} \left[28L^3 - 178L^2 - (82\pi^2 - 490)L + 1203\zeta(3) + 78\pi^2 \ln 2 + \frac{458\pi^2}{3} - 676 \right] \quad (22)$$

$$L \equiv \ln \left(\frac{s}{m_e^2} \right)$$

- c) The anomalous magnetic moment of the muon. The diagram shown in Fig. 3c) gives the following contribution [19] to $a_\mu \equiv (g_\mu - 2)/2$:

$$a_\mu^{\text{Vac.Pol.}} = \left(\frac{\alpha}{\pi} \right)^2 \left[\frac{1}{2} \ln \left(\frac{M}{m_e} \right) - \frac{25}{36} + 0 \left(\frac{m_e}{M} \right) \right] \quad (23)$$

The following comments may be made on these results : For a) and c) the logarithmic term may be derived using the Renormalisation Group. It is given by substituting $\alpha \rightarrow \alpha(M)$ in the

$O(\alpha)$ correction term [20]. That the $O(\alpha^2)$ radiative correction to τ_μ contains a mass singular term was already pointed out by Kinoshita [1] before the explicit calculation was done. For b)

the leading logarithmic term in $\ln^3 \left(\frac{s}{m_e^2} \right)$ was derived long ago by Landau and Lifshitz [21].

In this case there are no Renormalisation Group arguments to derive the logarithmic terms. Note that the cross section also has a $(m_e)^{-2}$ power-law divergence.

If the muon in Fig. 3c) is replaced by an electron, the logarithmic term in Eqn (23) vanishes. Hence the electron anomalous magnetic moment a_e is free of logarithmic mass singularities and, in contrast to a_μ , is finite as $m_e \rightarrow 0$. Indeed a_e may well be the unique physical quantity that has UV divergent contributions before renormalisation and yet is finite as $m_e \rightarrow 0$ after renormalisation, so providing a counter example to the "K" part of the KLN Theorem. This is a consequence of naive dimensional analysis. The large external scale and the infra-red cut-off scale are, for a_e , one and the same quantity m_e . The dimensionless quantity a_e must depend on the ratio of these scales, which is unity, so all logarithmic terms must vanish. Other mass dependent terms proportional to positive powers of m_e/M (from muon loops) vanish in the $m_e \rightarrow 0$ limit.

Very frequently, in the literature on QCD the massless theory is discussed and it is claimed that many fully inclusive quantities are completely "infra-red safe", that is finite, in the massless fermion limit. For example, in a recent review article on pQCD [22], it is stated that there e^+e^- total cross section "possesses a perturbative expansion in the running coupling constant that is free of logarithms or other sensitive functions that depend on large ratios such as Q/m with m a parton mass and Q the overall momentum scale". Although this is true of the *coefficients* of the expansion, if this is expressed in terms of $\alpha_s(Q)$ it is not so far the quantity itself since, as explained above, terms $\approx \ln Q/m$ are contained in $\alpha_s(Q)$. The above statement of finiteness is justified by reference to several papers [23-25] in which the results found by Kinoshita for QED [1] were generalised to non-abelian gauge theories. These papers demonstrated that the *unrenormalised* amplitudes for fully inclusive physical quantities are infrared finite in the massless fermion limit. In every case UV divergent terms were left aside and "All renormalisation is assumed to have been done "off-shell" in such away as to introduce no new mass divergences" [25]. But as pointed out in Section 12 of Kinoshita's paper [1], this cannot be done. In fact the renormalisation constants contain mass singularities, that cancel against *similar divergences in the renormalised amplitude* in such a way that the unrenormalised amplitude is singularly free [26]. It is easily demonstrated that $\alpha_s(Q)$ does contain singular terms $\approx \ln Q/m$ by including heavy quark threshold factors (Eqn (4) above) either in the β -function coefficients [27] or directly in the Renormalisation Group Equation for the running coupling constant [28-30]. In the limit $Q/m \gg 1$ these threshold factors reduce to terms $\approx \log Q/m$ [Eqn (5) above]. The mass singularities which appear in the renormalised amplitudes are literally "hidden under" the UV divergences in the unrenormalised one. The unrenormalised amplitude contains UV divergent terms of the form $\ln \Lambda/m$ where Λ is the UV cut-off, that, after renormalisation, give the mass singular $\ln Q/m$ terms. It is reiterated that the divergences *cannot* be simply removed by off-shell renormalisation. $\alpha_s(Q)$ is renormalisation scale invariant quantity, so if it is demonstrably divergent for any choice of scale it must be so for all scales.

To repeat again the essential conclusions : UV finite but potentially IR divergent terms related to real or virtual photons or gluons radiated from external lines, having a leading-log structure :

$$\alpha_s^n \ln^n \left(\frac{Q}{m} \right) , \quad \alpha_s^n \ln^n \left(\frac{Q}{\lambda} \right)$$

cancel completely for final state radiation in fully inclusive quantities. This was shown using a general quantum mechanical argument, by Lee and Nauenberg [2], and by consideration of sets of gauge invariant Feynman diagrams by Kinoshita [1]. The literature confirming this "LN"* part of the KLN theorem for QCD is extensive [31-36]. On the other hand next-to-leading-log terms

$$\alpha^n \ln^{n-1} \left(\frac{s}{m^2} \right)$$

are UV divergent before renormalisation, and remain after as manifestly mass singular terms in the predictions of even fully inclusive physical quantities. Four specific examples are given above. Because of their one-to-one relationship with UV divergent terms in the unrenormalised amplitude these mass singular terms are, by the Renormalisation Group, all absorbed into a process-independent "running coupling constant". This is the content of the "K" part of

the KLN Theorem. As a final summary of its physical meaning one can do no better than to quote directly from Kinoshita's paper of 1962 ([1]; P673).

"The discussion above reveals a rather puzzling feature in that while the unrenormalised theory suffers from ultraviolet divergences, but behaves in a reasonable way at m singularities, the situation is reversed in the renormalised theory. Of course, this is no problem from a practical point of view since the m divergences do not occur for the observed values of the masses",

and later, on a slightly more hopeful note for the existence of a massless theory :

"Thus we still do not have a consistent formulation of quantum electrodynamics for zero-mass electron. It should be pointed out however that this is based on the assumption that the renormalisation method used in the case $m \neq 0$ can also be applied to the case $m = 0$. This may not be correct, since, as mentioned above, one particle state cannot be defined for $m = 0$ because of the degeneracy of the eigenstates of the total energy-momentum vector of the interacting fields. It would be extremely interesting if this point can be clarified".

It has however, been demonstrated above that the theory breaks down already for small, but non-vanishing, masses due to the presence of the Landau singularities.

Acknowledgments

I should like to thank W.L. van Neerven for first pointing out to me that Kinoshita's theorem on the cancellation of mass singularities applies only to unrenormalised amplitudes, and B. Lautrup and O. Piguet for discussions of an earlier version of this letter.

References

- [1] T. Kinoshita, J. Math. Phys. **3** (1962) 650.
- [2] T.D. Lee and M. Nauenberg, Phys. Rev. **B153** (1964) 1549.
- [3] A.A. Akrikosov, I.M. Khatatinkov and D. Landau, Dokl. Akad. Nauk. SSSR **95** (1954) 1177. English translation, paper 80 in "Collected Papers" of L.D. Landau. Ed. D ter Haar Pergamon Press 1965.
- [4] G. 'tHooft and M. Veltman, Nucl. Phys. **B44** (1972) 189.
- [5] C.P. Korthales Altes and E. de Rafael, Phys. Lett. **B62** (1976) 320, Nucl. Phys. **B125** (1977) 275.
- [6] P. Cvitanovic, Phys. Lett. **B62** (1976) 72.
- [7] J.H. Field, Int. Journ. Mod. Phys. A, Vol. 9, No 18 (1994) 3283.
- [8] J.H. Field, University of Geneva Pre-print, UGVA-DPNC 1993/7-159, July 1993.
- [9] R. Coqueroaux, Ann. Phys. (N.Y.) **125** (1980) 401.
- [10] J.W. Bardeen, A. Buras, D. Duke and T. Muta, Phys. Rev. **D18** (1978) 3998.
- [11] J.M. Jauch and F. Röhrlich, "The Theory of Photons and Electrons", P194, Springer Verlag, New York/Berlin 1976.
- [12] T. Appelquist et al., Phys. Rev. Lett. **36** (1976) 768.
- [13] W. Celmaster and D. Sivers, Phys. Rev. **23D** (1981) 227.
- [14] W. Celmaster and R.J. Gonsalves, Phys. Rev. Lett. **42** (1979) 1435, Phys. Rev. **D20**, (1979) 1420.
- [15] R. Barbieri et al., Phys. Lett. **B81** (1979) 207.
- [16] A.H. Hoang et al., Phys. Lett. **B338** (1994) 330.
- [17] A. Sirlin, Phys. Rev. **D29** (1984) 89.
- [18] V.N. Baier et al., Physics Reports **78** (1981) 293.
- [19] H. Suura and E. Wichmann, Phys. Rev. **105** (1957) 1930. A. Petermann, Phys. Rev. **105** (1957) 1931.
- [20] R. Barbieri and E. Remiddi, Phys. Lett. **B57** (1975) 273.
- [21] L.D. Landau and E.M. Lifshitz, Sov. Phys. **6** (1934) 244.
- [22] G. Sterman et al. Review of Modern Physics **67** (1995) 57.
- [23] E.C. Poggio and H.R. Quinn, Phys. Rev. **D14** (1976) 578.
- [24] G. Sterman, Phys. Rev. **D14** (1976) 2123.
- [25] G. Sterman, Phys. Rev. **D17** (1978) 2773, 2789.
- [26] M. Gell-Mann and F.E. Low, Phys. Rev. **95** (1956) 1300.
- [27] H. Georgi and H.D. Politzer, Phys. Rev. **D14** (1976) 1829.
- [28] A. Petermann, CERN-TH.6487/92.
- [29] J.H. Field, Ann. Phys. **226** (1993) 209.
- [30] Yu L. Dokshitzer and D.V. Shirkov, Z. Phys. C **67** (1995) 449.
- [31] Y.P. Yao, Phys. Rev. Lett. **36** (1976) 653.

- [32] L. Tyburski, Phys. Rev. Lett. **37** (1976) 319.
- [33] J. Frenkel and J.C. Taylor, Nucl. Phys. **B116** (1976) 185.
- [34] J.M. Cornwall and G. Tiktopoulos, Phys. Rev. **D13** (1976) 3370, **D15** (1977) 2937.
- [35] G. Sterman and S. Weinberg, Phys. Rev. Lett. **39** (1977) 1436.

Figure Captions

Figure 1. Diagrams contributing to the invariant amplitude $M^{(1)}$ of the process $e\mu \rightarrow e\mu$.

Figure 2. Typical diagrams contributing to the $O(\alpha^2)$ radiative correction to the process $e^+e^- \rightarrow f\bar{f}$. a) UV finite diagrams containing only photon lines. b) Diagrams which are UV divergent before renormalisation containing a fermion loop.

Figure 3. Some examples of diagrams that contribute mass singular logarithms to fully inclusive physical quantities. a) The muon lifetime. b) The process $e^+e^- \rightarrow e^+e^-e^+e^-$. c) the anomalous magnetic moment of the muon. The QED predictions for all these processes are divergent as $m_e \rightarrow 0$ for any finite value of α .

Fig. 1

a)

b)

Fig. 2

Fig. 3