A New State Of Hadronic Matter At High Density

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Abstract

We propose in this article that if the chemical potential exceeds a critical value in dense hadronic medium, a first-order phase transition to a new state of matter with Lorentz symmetry \textit{spontaneously broken} (in addition to the explicit breaking) takes place. As a consequence, light vector mesons get excited as “almost” Goldstone bosons. Since the light vector mesons dominantly couple to photons, the presence of these new vector mesons could lead to an enhancement in the dilepton production from dense medium at an invariant mass lower than the free-space vector-meson mass. We provide a low-energy quark model which demonstrates that the above scenario is a generic case for quark theories with a strong interaction in the vector channel. We discuss possible relevance of this phase to the phenomenon of the enhanced dilepton production at low invariant masses in relativistic heavy-ion collisions.

1 Introduction

Recently, the CERES and HELIOS collaborations reported the exciting observation [1, 2] that the lepton pair production is enhanced in S–Au collisions in the invariant mass range of 300...600 MeV compared with the collisions p–Be and p–Au. This observation provides an important clue as to what happens to hadronic matter when it is compressed to a high density. Unlike the case of temperature, lattice calculations are not yet in a position to provide information on the effect of density on QCD vacuum and hence practically nothing is understood of density-driven QCD phase transitions. Random matrix studies show indeed that the effect of a chemical potential can be exceedingly subtle from the point of view of QCD [3]. In the paucity of any first-principle guidance, there is a wide range of theoretical ideas to explore. It will ultimately be up to experiments to weed out wrong ideas and to guide us towards a viable scenario.

From a detailed study of the collisions with the help of covariant transport equations, it became clear that the dilepton yield of the collisions p–A in the above mass range can be well understood by resorting only to the decays of the η, ρ, ω and φ-mesons [4]. The fact that a large pion density is produced by the collision and the experimental observation that the dilepton enhancement sets in at roughly twice the pion mass have led to the conjecture that the enhancement is due to π⁺π⁻–annihilation processes [1]. The quantitative analysis, however, showed that the experimental data cannot be explained using “free” meson masses and form factors [4, 6]. A change of the state of matter or at least a change of the meson properties in medium seem to be required. Several proposals [5, 6, 7, 8, 9, 10] have been made, most of which focusing on the role of the ρ vector meson. This meson is of particular importance for the dilepton enhancement effect, since the ρ-meson directly couples to the photons. The coupling of the ρ-meson to two pion states and the change of the pion propagation in medium generically results in a broadening of the ρ-meson peak [6, 7]. The observed increase in the dilepton production rate is compatible within the error bars of the present data. On the other hand, QCD sum rules [8] predict a decreasing ρ-meson mass for increasing matter density. Whereas the sum rule approach is restricted to small densities, a decrease of the ρ-mass as function of the matter density should hold due to the onset of the chiral phase transition [9]. These considerations supplemented with further support from the results from the Skyrme model can be summarized in the scaling in medium of the hadron “quasi-particle” masses known as BR-scaling [10]. Including a ρ-meson mass shift to smaller values in the calculation of the relativistic transport theory, a good agreement of the theoretical predictions with the observed dilepton spectra in the S–Au collision is achieved [11]. The two approaches, one based on many-body correlations starting from strongly coupled hadrons whose properties are defined
in matter-free space [6, 7] (with broadening widths) and the other based on the notion of both bosonic and fermionic quasi-particles with parameters defined in a medium background field [12], somewhat contradictory to each other though they may appear to be, are probably related to each other when applied to the dilepton phenomena in question. Whether or not the two ways of looking at dense matter can be mapped to each other for other physical observables is not clear.

In this paper, based on a rather generic argument, we propose a novel state of hadronic matter which is unstable in the (zero-density) vacuum, but which forms the state with the lowest energy density, if matter is present. In addition to the explicit breaking of Lorentz invariance due to a finite baryonic density, Lorentz symmetry is also spontaneously broken in this new matter state. This provides a mechanism for exciting low-mass vector mesons as “almost” Goldstone bosons. We will discuss the properties of such vector mesons in some detail. Our considerations are based solely on the realization of symmetries. This model-independent argument will be given a support by an effective low-energy quark model which will illustrate that the new matter state is generically present in theories with vector-type quark interactions as it is the case for QCD.

2 Induced Spontaneous Symmetry Breaking

In matter-free space, the QCD vacuum is characterized by a non-vanishing value of the quark condensate and zero baryonic density, i.e.

\[ \langle \bar{q}q \rangle \neq 0, \quad 3 \rho_B = \langle \bar{q}\gamma_0 q \rangle = 0 \quad \text{State (V).} \quad (1) \]

The non-zero condensate implies that chiral symmetry of QCD is spontaneously broken (apart from a small explicit breaking through current quark masses). This particular realization of chiral symmetry allows the interpretation of the light pseudo-scalar mesons as Goldstone bosons [13] and provides a model-independent explanation of the particular role of the pion in the meson mass spectrum.

The key observation in this paper is that there is a second state which is not realized in the vacuum but appears as a meta-stable state having a higher energy density than the vacuum. This additional state is characterized by a vanishing (scalar) quark condensate, and a vanishing baryon density (represented in terms of quark fields) \( \langle \bar{q}\gamma_0 q \rangle \), i.e.

\[ \rho_B, \langle \bar{q}q \rangle = 0, \quad \zeta^2 := \langle \bar{q}\gamma_\mu q \bar{q}\gamma^\mu q \rangle \neq 0 \quad \text{State (II),} \quad (2) \]

and the symmetry is in a Kosterlitz-Thouless type of realization [15].
Our crucial point is that when a small explicit breaking of Lorentz symmetry is introduced via the chemical potential, two important things happen. First, at certain chemical potential $\mu_c$, the state (II) becomes energetically favored and a phase transition takes place from the state (V) to the state (II). Second, the presence of matter selects a particular Lorentz frame, since it favors the zero component of the vector current. The matter state with the lowest energy density is then described by

$$\langle \bar{q}q \rangle = 0, \quad 3\rho_B = \langle \bar{q}\gamma_0 q \rangle \neq 0 \quad \text{State (M)}.$$  

(3)

In addition to the contribution due to the chemical potential, the baryonic density acquires a large contribution from the dynamics of the theory. Since the chemical potential $\mu$ must exceed a critical value $\mu_c$ to generate this dynamical contribution, we shall refer to this scenario as induced spontaneous symmetry breaking (ISSB).

One might object at this point that the ISSB scenario is in contradiction to the Vafa-Witten theorem [16], which states that vector symmetries cannot be spontaneously broken in QCD. In fact, what the theorem is telling us is that the correlation function in the vector channel has, for a given gluon configuration, an upper bound provided by an exponentially decreasing function of distance. Since the QCD weight as employed by averaging over all gluon configurations is positive, the full correlation function has the same upper bound. This would rule out a massless vector state. In our case, the state (M) becomes the ground state for $\mu > \mu_c$, with the vector particle becoming light, but not massless due to the additional explicit breaking of Lorentz symmetry. Therefore the correlation function will be decreasing with an exponential slope. Thus the ISSB scenario does not contradict the Vafa-Witten theorem

Let us discuss the consequences of the ISSB scenario. Assume that the matter phase is realized in the state (M) and that the small explicit breaking via the chemical potential $\mu > \mu_c$ can be taken into account perturbatively. Due to the presence of the condensate $\langle \bar{q}\gamma_0 q \rangle$, the symmetry with respect to Lorentz boost transformation, $\Lambda^\nu_\nu = \exp\{\omega^\nu_\nu\}$,

$$q(x) \rightarrow q'(x') = S(\Lambda)q(x), \quad S(\Lambda) = \exp\{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\},$$  

(4)

is spontaneously broken (in addition to the small explicit breaking induced by $\mu$). What is the corresponding Goldstone boson? In order to answer this question, we resort to standard techniques which were developed in the context of the spontaneous breakdown of chiral symmetry [17]. For this purpose, first note that the quark

\footnote{Furthermore, the Vafa-Witten theorem is proven in Euclidean space. In the presence of a chemical potential (and a gluonic background field), the determinant develops a phase, upsetting the positivity of the measure required for the proof. We would like to thank Maciek Nowak for reminding us of this caveat.}
propagator of the state \((M)\) satisfying the Dyson-Schwinger equation possesses the general structure

\[
s(k) = \frac{Z(k_0, \vec{k})}{\vec{k} - V_0(k_0, \vec{k})\gamma_0 - \Sigma(k_0, \vec{k})}.
\] (5)

In the low-energy regime (where the momentum transfer \(k\) is much smaller than the typical gluonic energy scale), one expects that the quark theory can be approximated by a Nambu-Jona-Lasinio type effective model in which case the momentum and energy dependence of the functions \(Z\), \(V_0\) and \(\Sigma\) in (5) can be neglected [18]. For simplicity, we will make this assumption in the following. Furthermore note that the transformed propagator, i.e.

\[
S(\Lambda)s(k')S^{-1}(\Lambda),
\] (6)

also satisfies the Dyson-Schwinger equation, since the latter equation is manifestly Lorentz covariant for \(\mu = 0\). Considering infinitesimal Lorentz boost transformations induced by \(\omega_{0i}\), one concludes that the vertex function

\[
P^V \propto V_0[\gamma_0, \sigma_{0i}] \propto V_0 \gamma_i
\] (7)

satisfies a Bethe-Salpeter equation with zero mass. The important finding is that, at least at sufficiently small energies, the corresponding particle is of pure vector type. The emergence of the light vector particle is a consequence of the spontaneous breaking of the Lorentz group down to the non-relativistic rotational group \(SO(3)\) and is analogous to the emergence of scalar Goldstone bosons for a spontaneously broken internal symmetry group. Since in the present case the broken symmetry is a space-time symmetry, the corresponding massless particles are (non-relativistic) spin one excitations (for more details see section 3).

If one wants to relax the restriction to the low-energy regime, one might resort to the full vertex function, which is provided by the Noether charge density generated by the Lorentz boost (4), i.e.

\[
Q_{i}^{(L)}(x) = (x^0 T^{0i}(x) - x^i T^{00}(x)) + \bar{q}(x) \gamma_i q(x),
\] (8)

where \(T^{\mu\nu}\) is the energy-momentum tensor written in terms of quark fields. Here, the first term represents the “orbital” part while the second term gives rise to the “spin” part of the vertex function.

The state \((M)\) in (3) exists only for finite values of the chemical potential \(\mu\), which explicitly breaks Lorentz invariance. This explicit breaking has the consequences that the vertex function \(P^V\) in (7) acquires additional parts and that the Goldstone vector particle gets a small mass. To demonstrate this, we use the variational approach of [19], which is a kind of relativistic RPA approach – and a convenient
one if the light mesons are treated as Goldstone bosons. As a trial state for the variational approach, we allow the vertex function $P^V$ to gain additional vector parts at finite values of $\mu$. The vector field operator is chosen in terms of the quark operators as

$$V_i(t) = \kappa_L \int d^3x \, Q_i^{(L)}(x) + \kappa_V \int d^3x \, \bar{q}(x)\gamma_i q(x),$$  \hspace{1cm} (9)$$

where $\kappa_{L/V}$ are variational constants. Using the techniques of [19], it is straightforward to relate the mass of the vector Goldstone boson $m_V$ to the explicit breaking via the chemical potential $\mu$. We find

$$m_V^2 f_V^2 = 2\mu \langle \bar{q}\gamma_0 q \rangle,$$  \hspace{1cm} (10)$$

where $f_V$ is a decay constant defined by

$$\langle \Omega | Q_i^{(L)}(x=0) | \mathcal{V}_k(m_V, \vec{p}=0) \rangle = i f_V m_V \delta_{ik}.$$  \hspace{1cm} (11)$$

Here $|\mathcal{V}(p)\rangle$ denotes the state of the Goldstone vector with four momentum $p$, and $|\Omega\rangle$ is the matter ground state. Equation (10) is nothing but the analog of the Gell-Mann-Oakes-Renner relation [20]. The decay constant $f_V$ describes the decay of the Goldstone vector into photons. A calculation of this quantity in a simple model will be presented later. Since the electro-magnetic U(1) gauge invariance is broken by the presence of matter, the coupling of quarks and photons is non-minimal, but acquires additional pieces proportional to the "angular momentum" density (the first term in eq. (8)).

The local minimum of the effective potential at $\langle \bar{q}q \rangle = 0$ and $\langle \bar{q}\gamma_0 q \rangle \neq 0$ (i.e. the state (M)) becomes the global minimum, i.e. the ground state, when $\mu$ exceeds a critical value. In this new state, the condensate $\langle \bar{q}\gamma_0 q \rangle$ does not scale with $\mu$ anymore, but acquires a strong contribution from the interaction. As shown above, this mechanism results in a light iso-scalar vector meson. The occurrence of the light vector meson is obviously accompanied by the spontaneous generation of baryon density $\langle \bar{q}\gamma_0 q \rangle$. On the other hand, the $U(1)_V$ vector symmetry is conserved in our approach implying that baryon number is conserved. Both statements above do not contradict each other, since we have assumed an infinite system at finite baryon density. In practical applications a finite baryon number localized in space is the interesting case. In this case, we cannot assume a homogeneous phase. Rather we expect that the gradients discarded in the above description will lead to a domain structure, where each domain is characterized by a different value of the order parameter $\langle \bar{q}\gamma_0 q \rangle$.

In order to roughly estimate the order of magnitude of the parameters involved, we assume that the phase transition of the vacuum (1) to the state (M) (3) occurs at
a Fermi momentum \( k_f \approx M_c \), where \( M_c \approx 300 \) MeV is the constituent quark mass. This is the standard value for \( k_f \), where one expects the chiral phase transition to occur. The estimate (based on a constituent quark model) of the corresponding chemical potential is therefore \( \mu \approx \sqrt{M_c^2 + k_f^2} \approx \sqrt{2} M_c \). Note that this value for \( \mu \) is of the same order of magnitude as the generic energy scale of the phase transition implying that corrections to (10) might become important.

For a first guess, we further use the relation \( \rho_B \approx 2 k_f^3 / 3 \pi^2 \) of the constituent quark model. Combining these rough estimates, we find \( m_V f_V \sim (260 \) MeV \( )^2 \) from (10). If the mass of the “almost” Goldstone vector bosons is small, the resonance will be broad, since the coupling to the photons \( f_V \) becomes large.

### 3 Model Calculation

The results of the previous section are model-independent, relying solely on a specific phase structure of the field theory. In this section, we illustrate with the help of a simple model that the particular phase structure required by the ISSB scenario is actually a generic case for effective quark models with a strong vector-current interaction.

For this purpose, we study an effective low energy quark model [13] defined at finite chemical potential \( \mu \), by the generating functional for Euclidean Green’s functions (see e.g. [14])

\[
Z[s, j_\mu] = \int Dq D\bar{q} D\sigma D\pi DV_\mu e^{\int d^4x \left[ L + s(x)\sigma(x) + j_\mu(x)V_\mu(x) \right]}, \tag{12}
\]

\[
L = \bar{q}(x) \left( i\partial - \sigma(x) + i\gamma_5\pi(x) + iV_\mu(x)\gamma^\mu \right)q(x) \tag{13}
\]

\[
- \frac{N}{2} g_s \left[ (\sigma(x) - m)^2 + \pi^2(x) \right]
- \frac{N}{2} \left\{ V_\mu(x) \left( -\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu \right) V_\nu(x) + m^2_v \left[ (V_0(x) - \mu)^2 + V_\mu^2(x) \right] \right\},
\]

where \( N \) is the number of colors (the color index of the quark fields is not shown) and \( m \) is the current quark mass. Our definitions of the Euclidean space can be found in appendix A. In particular, we have defined the square of an Euclidean vector by \( \partial^2 := \partial_\mu \partial^\mu = -\partial_\mu \partial^\mu \). For simplicity we consider the case of one light-quark flavor. Generalizations to the iso-spin \( I = 1/2 \) will be discussed in the next section.

Let us look at the model (12) in Minkowski space. For this purpose, we integrate out the meson fields \( \sigma, \pi \) and \( V_\mu \) and perform the analytic continuation of the Euclidean quark theory to Minkowski space. Our conventions can be found in appendix A. Integrating out the vector fields \( V_\mu \) yields a non-local current-current interaction. If
we neglect the momentum transfer in this interaction, the low energy quark theory is of NJL-type, i.e.

\[ L_M = \bar{q}_M(x)(i\partial - m - \mu \gamma_0^{(M)}) q_M(x) \]  

(14)

\[ + \frac{1}{2Ng_s} \left[ (\bar{q}_M(x)q_M(x))^2 - (\bar{q}_M(x)\gamma_5 q_M(x))^2 \right] \]

\[ + \frac{1}{2Nm_0^2} q_M(x)\gamma_\mu^{(M)} q_M(x) \bar{q}_M(x)\gamma_\mu^{(M)} q_M(x). \]

Note that the vector current interaction contributes with a plus sign to the Lagrangian \( L_M \). We call this an attractive interaction in the vector channel. It will turn out that this sign is crucial for the ISSB scenario. We should stress that there is nothing to indicate that such an interaction is incompatible with light-quark hadron phenomenology. Now if we compare (14) with (13), we find that the parameter \( \mu \) in (13) can be interpreted as the chemical potential. Any non-vanishing expectation value of \( V_0 \) obviously acts as a chemical potential.

Despite the kinetic term of the vector field (we will discuss its role below), the model (12) is designed as a low-energy effective quark theory, with the quark loop momenta cut off at an energy scale \( \Lambda \) [18]. We stress that the regularization procedure must not spoil the Euclidean O(4) invariance, that is, the Lorentz symmetry in Minkowski space. The magnitude of the cut-off \( \Lambda \) is of the order of the gluonic energy scale.

In the chiral limit \( m \to 0 \), the Lagrangian \( L \), (13), is invariant under chiral transformations. If the scalar field acquires a non-vanishing vacuum expectation, i.e. \( \langle \sigma \rangle \neq 0 \), the chiral symmetry is spontaneously broken (SSB). The beautiful concept of the spontaneous breakdown of chiral symmetry allows to interpret the light pseudo-scalar mesons as Goldstone bosons, and hence provides a natural and model-independent explanation of the particular role of the pions in the meson mass spectrum.

### 3.1 Ground state properties

We will now show that the simple model (12) exhibits a vacuum phase \( (\mu = 0) \) where chiral symmetry is spontaneously broken (SSB) and a \( \mu \)-induced transition to a phase where the (scalar) quark condensate vanishes and a spontaneous breakdown of Lorentz symmetry occurs (ISSB) on top of the explicit breaking. The convenient quantity by means of which this mechanism can be illustrated is the effective potential as function of the scalar and vector fields, i.e. \( U(\sigma, V_\mu) \). The effective action \( \Gamma \) is defined by a Legendre transform of the generating functional \( \ln Z[s,j_\mu] \) with respect to the external sources \( s(x) \) and \( j_\mu(x) \), respectively, i.e.

\[ \Gamma(\sigma, V_\mu) = -\ln Z[s,j_\mu] + \int d^4x \left[ s(x)\sigma(x) + j_\mu(x)V_\mu(x) \right], \]  

(15)
Figure 1: Lines of constant effective potential $U$ in the plane of $\sigma$ and $\zeta^2 = V_\mu V_\mu$ for $\mu = 0$ (left picture) and in the plane of $\sigma$ and $V_0$ for $\mu = 0.2\sigma_V$ (right picture). $\sigma_V$ is the vacuum expectation value of $\sigma$. The numbers provide the effective potential $U$ in units of $|U_V|$, where $U_V$ is the vacuum value of the potential $U$. $(V)$ and $(M)$ indicate the position of the vacuum state and of the ground state in matter, respectively.

\[ V_\mu(x) = \frac{\delta \ln Z[s, j_\mu]}{\delta j_\mu(x)} , \quad \sigma(x) = \frac{\delta \ln Z[s, j_\mu]}{\delta s(x)} . \quad (16) \]

To leading order in the large $N$ expansion, it is sufficient to evaluate the functional integral (12) in a mean-field (stationary phase) approximation, since fluctuations around the mean-fields are suppressed by a factor $1/N$. A straightforward calculation yields

\[
\frac{1}{N} \Gamma(\sigma, V_\mu) = -\frac{1}{N} \text{Tr} \ln \{i\partial - \sigma(x) + i\pi(x)\gamma_5 + i\gamma^\mu V_\mu(x)\} \\
+ \int d^4x \left[ \frac{g_s}{2} \left[(\sigma - m)^2 + \pi^2\right] \right] \\
+ \frac{1}{2} \left\{ V_\mu(x) \left(-\partial^2 \delta_\mu\nu + \partial_\mu \partial_\nu\right) V_\nu(x) + m_v^2 \left[(V_0 - \mu)^2 + V_k^2\right] \right\} + O\left(\frac{1}{N}\right)
\]

where the trace extends over Lorentz indices as well as over the Euclidean spacetime. Note that a regularization which preserves the O(4) invariance is understood in (17) in order to define the trace term. We then obtain the potential $U(\sigma, V_\mu)$ from the effective action by confining ourselves to constant classical fields. Assuming a vanishing mean field for the pionic field $\pi(x)$, we find for a sharp momentum cutoff
\[ \frac{1}{N} U(\sigma, V_\mu) = -\frac{1}{8\pi^3} \int_{-1}^{1} dx \sqrt{1-x^2} \int_0^{\Lambda^2} du \ln \left[ (u - \zeta^2 + \sigma^2)^2 + 4\zeta^2 x^2 u \right] \]
\[ + \frac{g_s}{2} [\sigma - m]^2 + \frac{m_v^2}{2} [(V_0 - \mu)^2 + V_k^2] + \mathcal{O}\left(\frac{1}{N}\right) \]

where \( \zeta^2 := V_\mu V_\mu \). Due to O(4)–invariance, the effective potential \( U \) depends only on the O(4)–invariant field combination \( \zeta^2 \) in the case \( \mu = 0 \). Minima of the effective potential serve as possible candidates for the ground state. The global minimum, i.e. the state with the lowest vacuum energy density, represents the vacuum. The left-hand picture of Figure 1 shows the effective potential \( U \) for \( g_s = m_v^2 = \Lambda^2/8\pi^3 \). At zero chemical potential, the global minimum of \( U \) is located at \((V)\) in Figure 1. The corresponding vacuum properties are precisely characterized by (1). Chiral symmetry is spontaneously broken. In addition, the effective potential possesses a local minimum (as indicated by \((II)\) in Figure 1). This minimum corresponds to a meta-stable state with the properties (2).

The picture changes drastically, if the chemical potential is increased. If the chemical potential exceeds a critical strength \( \mu_c \), the global minimum flips from the state \((V)\) to the state \((II)\) (see right hand side of figure 1). Since the chemical potential selects the zeroth component of the O(4)–invariant combination \( V_\mu V_\mu \), the ground state in matter will be characterized by (3). This means that in addition to a small explicit breaking, the O(4) (Lorentz) symmetry is spontaneously broken. Thus the model (12) exhibits the ISSB-mechanism discussed above. Finally, let us mention that a phase structure similar to the one of our toy model has been also found in phenomenologically successful effective nucleon-meson theories [21]. An analogous phenomenon occurs in the random-matrix study of the QCD phase transition in the presence of chemical potential[3].

### 3.2 The “almost” Goldstone vector boson

In this subsection, we study small fluctuations \( v_\mu(x) \) of the vector field \( V_\mu(x) \) around its mean-field value \( V_\mu^B \), i.e. \( V_\mu(x) = V_\mu^B + v_\mu(x) \). We assume that a small chemical potential is sufficient to induce the transition from the state \( V \) to the state \( M \) and treat the influence of the small explicit breaking of the O(4) symmetry by the chemical potential \( \mu \) as a perturbation. We will find that in this case the vector fields \( v_{k=1,...,3}(x) \) emerge as massless excitations from the Bethe-Salpeter equation, if \( \mu \) goes to zero. For this purpose, we expand the effective action (17) up to second order in the fields \( v_\mu \), i.e.

\[ \frac{1}{N} \Gamma^{(2)} = \frac{1}{2N} \text{Tr} \left\{ \frac{i}{i\bar{\phi} + iV_B^B - \sigma} \psi \frac{i}{i\bar{\phi} + iV_B^B - \sigma} \psi \right\} \]
where \((p)\) is the shorthand for \(d^4p/(2\pi)^4\). The explicit calculation of the trace term in (19) is left to appendix C. The final result can be written as

\[
\frac{1}{N} \Gamma^{(2)} = \frac{1}{2} \int (p) v_k(p) \Pi_{kl}(p) v_l(-p) .
\] (20)

Mass eigenstates appear as solutions of the Bethe-Salpeter equation

\[
\Pi_{kl}(p^2_0, \vec{p} = 0) v_l(-p) = 0 ,
\] (21)

where \(m_{V}\) is the mass of the excitation. For simplicity, we here consider the Bethe-Salpeter equation for vanishing spatial momentum. It is also sufficient for our purposes to study this equation in a derivative expansion with respect to the meson momentum \(p^2\). Exploiting the gap equation for the mean field \(V_\mu^B = (V_0, 0, 0, 0)\), one finds (see appendix C)

\[
\Pi_{kl}(p^2_0, \vec{p} = 0) = [1 - f(\sigma, V_0^B)] p^2 \delta_{kl} + \frac{m_v^2}{V_0} \mu \delta_{ik} .
\] (22)

The main observation is that for \(\mu = 0\) and \(V_0 \neq 0\) the mass term drops out and that a massless excitation \((p^2 = 0)\) with the quantum numbers of a (non-relativistic) vector field occurs. For a small explicit breaking \(\mu\) of the Lorentz symmetry, one can cast the Bethe-Salpeter equation (21) into \((p^2 = -m_{V}^2)\)

\[
[1 - f(\sigma, V_0^B)] m_{V}^2 = \frac{m_v^2}{V_0} \mu.
\] (23)

Using the gap equation (55) (in appendix C), we find

\[
m_{V}^2 V_0 = \langle \bar{q} \gamma_0 q \rangle + O(\mu) .
\] (24)

With this result, eq.(23) can be cast into the form of eq.(10), i.e.

\[
f_V^2 m_{V}^2 = 2 \mu \langle \bar{q} \gamma_0 q \rangle + O(\mu^2) , \quad f_V = \frac{\sqrt{2}}{\sqrt{1 - f(\sigma, V_0^B)}} V_0 .
\] (25)

If the first-order phase transition from the vacuum state \((V)\) to the matter state \((M)\) takes place, the constituent quark mass \(\sigma\) drops to the value of the current mass. Therefore, we expect \(\sigma \ll V_0\). Let us study the function \(f(\sigma, V_0^B)\) for the case \(\sigma = 0\). The explicit calculation of this function is left to appendix C. The function \(f\) is shown in Figure 2. It diverges logarithmically at the origin and decreases rapidly for large values of \(V_0\).

\textsuperscript{2}Note that the derivative expansion becomes exact for (massless) Goldstone bosons.
It turns out that the function $f(\sigma, V_0^B)$ in (22) is always positive. The “1” in the square bracket stems from the bare kinetic term of the vector fields in (13), whereas $f$ is the contribution of the quark loop to the vector kinetic term. One observes that the quark-loop-induced kinetic term favors gradients in the vector field $v_k(x)$. If the parameter set is chosen such that $f(\sigma, V_0^B) < 1$, we can interpret the small amplitude fluctuations $v_k(x)$ as particle excitations in the usual way. In the case $f(\sigma, V_0^B) \geq 1$, the small amplitude fluctuations exponentially grow, and the matter state would favor large gradients. This would probably lead to the formation of domain walls. In the present model, $f$ is always much less than 1 in the parameter range of interest. We believe that $f > 0$ is a generic feature, implying that quark loop contributions support instabilities. In contrast, we expect that the parameters $(\sigma, V_0)$ that would produce instability (i.e., leading to $f = 1$) are highly model-dependent.

4 The Particle Spectrum of the ISSB Scenario

The toy model considered above with one flavor of quarks gives “almost” Goldstone vectors of the $\omega$ meson quantum number. To be realistic, we need at least two light flavors. Let us consider this case. For vanishing current mass and chemical potential, the quark sector exhibits chiral symmetry, i.e.

$$SU_V(2) \times SU_A(2),$$

and Lorentz invariance. In particular, the $SU_A(2)$ transformations relate scalar particles with pseudo-scalar particles, and transform vector current into axial-vector currents. The vacuum state is characterized by a spontaneous breakdown of the axial part of (26). According to Goldstone’s theorem, each generator of the spontaneously broken symmetry gives rise to a massless particle. In the case of QCD, the iso-triplet pions can be identified with the Goldstone bosons. However, the iso-triplet pions are
not massless, but possess a mass which is small compared with the hadronic energy scale, since the chiral symmetry is also explicitly broken by small current masses.

In the case of the ISSB scenario, the light particle content of the spectrum changes drastically. The crucial fact is the occurrence of the condensate (that is, in Minkowski space)

\[ \langle \bar{q} \gamma_0 1 q \rangle , \]  

(27)

where the unit operator in (27) indicates that the condensate is iso-scalar. Lorentz symmetry is spontaneously broken over and above the explicit breaking via the chemical potential. On the other hand, the quark condensate \( \langle \bar{q} q \rangle \) is proportional to the current quark mass and vanishes in the chiral limit. Since the condensate (27) is invariant under a chiral rotation of the quark fields, chiral symmetry is not spontaneously broken and the mass of the pions is not necessarily small.

The quantum numbers of the Goldstone vector bosons can be most easily seen by going to Euclidean space where the Lorentz group becomes the SO(4) group, which is equivalent to SU(2) \( \times \) SU(2). Expanding the anti-symmetric matrices \( \omega_{\mu\nu} \) of the Euclidean (Lorentz) transformation (44) in terms of the ’t Hooft symbols [22] \( \eta_{\mu\nu}, \eta_{\mu\nu}^\dagger \), which form a complete basis of self-dual and anti-self-dual matrices,

\[ \omega_{\mu\nu} = \theta_k \eta_{\mu\nu}^k + \bar{\theta}_k \bar{\eta}_{\mu\nu}^k , \]  

(28)

the generators of Euclidean (Lorentz) transformations can be written as

\[ S(\Lambda) = \exp \left[ -i \left( \theta_k \Sigma_R^k + \bar{\theta}_k \Sigma_L^k \right) \right] , \]  

(29)

where

\[ \Sigma_R^k = \frac{1}{4} \eta_{\mu\nu}^k \sigma_{\mu\nu} , \quad \Sigma_L^k = \frac{1}{4} \bar{\eta}_{\mu\nu}^k \sigma_{\mu\nu} . \]  

(30)

In the direct product representation of the \( \gamma \)-matrices [23], one finds [24]

\[ \Sigma_{R/L}^k = P_{R/L} \times \sigma^k , \quad P_{R/L} = \frac{1}{2} (1 \pm \gamma_5) . \]  

(31)

The matrices \( P_{R/L} \) are the right and left handed projectors and \( \sigma^k \) are the familiar Pauli spin matrices. With (31) the Euclidean (Lorentz) transformation (29) has precisely the form of a chiral transformation with the iso-spin (or in general flavor) matrices replaced by the spin matrices. Defining \( \theta_{V/A}^k = \frac{1}{2} (\theta^k \pm \bar{\theta}^k) \), eq. (29) becomes

\[ S(\Lambda) = \exp \left[ -i \theta_V^k (1 \times \sigma^k) - i \bar{\theta}_A^k (\gamma_5 \times \sigma^k) \right] . \]  

(32)

This representation of the Euclidean (Lorentz) group corresponds to the coset decomposition

\[ SU_L(2) \times SU_R(2) = SU_V(2) \times [SU_L(2) \times SU_R(2)/SU_V(2)] . \]  

(33)
It is the coset symmetry $SU_L(2) \times SU_R(2)/SU_V(2)$ which is spontaneously broken in the state (M) (3). Using the analogy between the Lorentz transformations (32) and a chiral transformation, it becomes clear that in the same way as the Goldstone bosons of spontaneous broken chiral symmetry carry iso-spin, the massless particles of spontaneously broken Lorentz symmetry carry spin and are hence vector particles. They are given by the spatial components of iso-scalar vectors $\omega_i := \bar{q} \gamma_i q$.

The iso-triplet $\rho$-mesons with a reduced mass in matter play a central role for the explanation of the dilepton enhancement in the CERES and HELIOS experiments in the approach of [11]. It is therefore interesting to look at the properties of the $\rho$-mesons in the ISSB scenario. First note that in the vacuum the $\omega$-meson is somewhat heavier than the $\rho$-meson and that the overlap of the $\rho$-meson and $\omega$-meson wave-functions is non-zero [25]. At some large density, a light meson with the quantum numbers of the $\omega$-meson appears as an “almost” Goldstone boson. As density decreases from the critical density as the system expands, it can happen that the levels cross with the $\omega$ and $\rho$ becoming degenerate at the crossing point. In this case, the $\rho\omega$-mixing will become 50% independently of the strength of the overlap matrix elements.

5 Discussions and Conclusions

We have shown that a meta-stable state with the properties (2) exists at vanishing chemical potential, if the effective low-energy quark interaction in the time component of the vector channel is attractive and strong enough. Using a simple effective quark model, we argued that this situation is generic for a wide range of parameter choices. We have suggested that the presence of this meta-stable state has important consequences at finite baryon density. If the chemical potential exceeds a critical value, a first-order phase transition from the vacuum phase to the former meta-stable state can take place. The scalar quark condensate vanishes and chiral symmetry is restored. In this phase, pions are no longer Goldstone bosons. In this new state of matter (state (M)), Lorentz symmetry is spontaneously broken (over and above the explicit breaking via the chemical potential), and light iso-scalar vector particles are excited. The light iso-scalar vector particles will dominantly couple to photons.

If the density is further increased until it becomes large compared with the fundamental gluonic energy scale, one expects that the interaction between the quarks becomes weak due to asymptotic freedom. In this case, the iso-scalar condensate $\langle \bar{q} \gamma_0 q \rangle$ loses the strong contribution from the interaction and will scale proportional to the chemical potential. This implies that the Goldstone mechanism no
longer applies and that the iso-scalar vector mesons become heavy again before they dissolve in $q$-$\bar{q}$ pairs at very high density, where one expects a phase transition to the quark gluon plasma. The content of light particles of hadronic matter for several values of the density is illustrated in Figure 3.

What are possible implications of the new state of matter at medium densities in heavy-ion collisions? Since the phase transition from the vacuum state to the new matter state is first order, the matter state (M) will appear in bubbles in the standard matter phase where the mesons experience a small change of the vacuum properties. The experimental observation could be a superposition of the results of the standard “hadronic cocktail” and of the ISSB scenario. The quantitative outcome will depend on how much of the bubbles are nucleated. It is obvious that the contribution of the new matter state to the observables will be more pronounced in Pb-Au than in p-Au collisions. It would be interesting to see whether this scenario has any role in the dilepton enhancement actually observed in the CERES and HELIOS experiments.

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A Notation and conventions

The metric tensor in Minkowski space is

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

(34)
We define Euclidean tensors $T_{(E)}$ from the tensors in Minkowski space $T_{(M)}$ by

$$T_{(E)}^{\mu_1 \ldots \mu_N}{}_{\nu_1 \ldots \nu_n} = (i)^r (-i)^s \ T_{(M)}^{\mu_1 \ldots \mu_N}{}_{\nu_1 \ldots \nu_n},$$

where $r$ and $s$ are the numbers of zeros within $\{\mu_1 \ldots \mu_N\}$ and $\{\nu_1 \ldots \nu_n\}$, respectively. In particular, we have for the Euclidean time and the Euclidean metric

$$x^0_{(E)} = i x^0_{(M)}, \quad g_{(E)}^{\mu\nu} = \text{diag}(-1,-1,-1,-1). \quad (36)$$

Covariant and contra-variant vectors in Euclidean space differ by an overall sign. For a consistent treatment of the symmetries, one is forced to consider the $\gamma^\mu$ matrices as vectors. Therefore, one is naturally led to anti-hermitian Euclidean matrices via

$$\gamma^0_{(E)} = i \gamma^0_{(M)}, \quad \gamma^k_{(E)} = \gamma^k_{(M)}. \quad (37)$$

In particular, one finds

$$\left(\gamma^\mu_{(E)}\right)^\dagger = -\gamma^\mu_{(E)}, \quad \{\gamma^\mu_{(E)},\gamma^\nu_{(E)}\} = 2g_{(E)}^{\mu\nu} = -2\delta_{\mu\nu}. \quad (38)$$

The so-called Wick rotation is performed by considering the Euclidean tensors (35) as real fields.

In addition, we define the square of an Euclidean vector field, e.g. $V_\mu$, by

$$V^2 := V_\mu V^\mu = -V_\mu V^\mu. \quad (39)$$

This implies that $V^2$ is always a positive quantity (after the wick rotation to Euclidean space).

The Euclidean action $S_E$ is defined from the action $S_M$ in Minkowski space by

$$\exp\{iS_M\} = \exp\{S_E\}. \quad (40)$$

Using (35), it is obvious that the Euclidean Lagrangian $L_E$ is obtained from the Lagrangian $L_M$ in Minkowski space by replacing the fields in Minkowski space by Euclidean fields, i.e. $L_E = L_M$.

Let the tensor $\Lambda^\mu_{\nu}$ denote a Lorentz transformation in Minkowski space, i.e.

$$\Lambda^\mu_{\alpha} \Lambda^\nu_{\beta} g^{\alpha\beta} = g^{\mu\nu}. \quad (41)$$

Using the definition (35), one easily verifies that $\Lambda^\mu_{(E)\nu}$ are elements of an O(4) group, i.e. $\Lambda^T_{(E)} \Lambda_{(E)} = 1$, which is the counterpart of the Lorentz group in Euclidean space.
In order to define the Euclidean quark fields, we exploit the spinor transformation of the Euclidean quark field

\[ q(E)(x_E) \rightarrow q'(E)(x'_{E}) = S(\Lambda(E))q(E)(x_E), \quad x_E \rightarrow x'_{E} = \Lambda x_E, \quad (42) \]

\[ S(\Lambda(E))\gamma^\mu(E)S^\dagger(\Lambda(E)) = (\Lambda^{-1}(E))^\mu_{\nu}\gamma^\nu(E), \quad (43) \]

where the matrices

\[ S(\Lambda(E)) = \exp\left\{-\frac{i}{4}\omega_{\mu\nu}\sigma_{(E)}^{\mu\nu}\right\} \quad (44) \]

are unitary. It is obvious that one must interpret

\[ \bar{q}(E) = q^\dagger(E) \quad (45) \]

in order to ensure that e.g. the quantity \( \bar{q}(E)\gamma^\mu(E)q(E) \) transform as an Euclidean vector. We suppress the index \( E \) throughout the paper and mark tensors with an index \( M \), if they are Minkowskian.

### B The gap equation

Let us calculate the trace term in the effective potential (17) for constant entries and for \( \pi(x) = 0 \). For this purpose, we write the trace as a sum over all eigenvalues \( \lambda \) of the Euclidean Dirac operator. For constant fields \( \sigma \) and \( V_{\mu} \), it is convenient to calculate the eigenvalues in momentum space. For a fixed momentum, one finds

\[ (\slashed{k} - \sigma + iV_{\mu}\gamma^\mu) \psi = \lambda(k) \psi. \quad (46) \]

A direct calculation yields

\[ \lambda_{\pm}(k) = -\sigma \pm i \sqrt{(k + iV)^2}, \quad (47) \]

where each eigenvalue is \( 2N \)-fold degenerated. The trace term is therefore given by

\[ -2\mathcal{V} \int_{k \leq \Lambda} (k) \ln (\lambda_{+}(k)\lambda_{-}(k)) = -2\mathcal{V} \int_{k \leq \Lambda} (k) \ln \left[ \sigma^2 + (k + iV)^2 \right], \quad (48) \]

where \( \mathcal{V} \) is the Euclidean space-time volume, and \( \Lambda \) is the sharp O(4) invariant cutoff. \( (k) \) is the shorthand for \( d^4k/(2\pi)^4 \). The integrand in (48) is complex. We will, however, see that the imaginary part drops out, if we perform the integration over the momentum. For this purpose, we write (48) as

\[ -\mathcal{V} \int_{k \leq \Lambda} (k) \ln \left[ \sigma^2 + (k + iV)^2 \right] - \mathcal{V} \int_{q \leq \Lambda} (q) \ln \left[ \sigma^2 + (q - iV)^2 \right] = \]

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\[-V \int_{k \leq \Lambda} (k) \ln \left[ (k^2 + \sigma^2 - V^2)^2 + 4 (k \cdot V)^2 \right],\]

where we have performed a change of integration variables \( q = -k \). We finally obtain

\[-\frac{V}{4\pi^2} \int_0^\pi \frac{d\alpha}{\pi} \sin^2 \alpha \int_0^\Lambda dk \: k^3 \ln \left[ \left( k^2 + \sigma^2 - V^2 \right)^2 + 4k^2V^2 \cos^2 \alpha \right].\] (49)

This expression directly enters the effective potential \( U(\sigma, V, \mu) \) in (18). Note that for \( V^2 > \sigma^2 \) the integrand in (49) becomes singular. It is, however, easy to show that this singularity is integrable and that no imaginary part is present. The singularity occurs for \( k^2 = V^2 - \sigma^2 \) in the angle integral. Using the principal-value prescription, this integral yields

\[2 \lim_{\epsilon \to 0} \int_\epsilon^{+1} dx \sqrt{1 - x^2} \log x^2 = -\pi \left( \ln 2 + \frac{1}{2} \right).\] (50)

An extremum of the effective potential occurs, if the (constant) vector fields satisfy the gap equation

\[-\frac{1}{N V} \text{Tr} \left\{ i \frac{i}{i\partial - \sigma + iV^\mu} \gamma^\mu \right\} + m_\nu^2 (V^\mu - \mu \delta^\mu_0) = 0.\] (51)

Let us study the case without an explicit breaking of the O(4) symmetry, i.e. \( \mu = 0 \). Since we use an O(4)-invariant regularization of the space-time trace in (51), the measure of the momentum integration is O(4) invariant. Let \( V^B_\mu \) denote a solution of the gap equation (51) (with \( \mu = 0 \)). Using the property (43), one easily shows that the rotated field \( \Lambda_{\mu\nu}V^B_\nu \) is also a solution. In this case, we multiply eq.(51) with \( V^B_\mu \) and obtain a single equation to determine the length \( V^B_\mu V^B_\mu \) of the vector field. For \( \mu \neq 0 \), it is easy to show that a solution of (51) is provided by \( V^B = (V_0,0,0,0) \).

Without loss of information, we calculate

\[-\frac{1}{N} \text{Tr} \left\{ i \frac{i}{i\partial - \sigma + iV} \gamma^\mu \right\} V^\mu\] (52)

for constant fields \( V^\mu \). Introducing momentum eigenstates and performing the trace over Dirac indices yield

\[-4iV \int_{k \leq \Lambda} (k) \frac{k \cdot V + iV^2}{(k + iV)^2 + \sigma^2},\] (53)

where \( k \cdot V := k_\mu V^\mu = -k_\mu V^\mu \). Introducing polar coordinates where \( \alpha \) denotes the angle between \( k \) and \( V \), one obtains

\[-\frac{iV}{\pi^2} \int_0^\pi d\alpha \sin^2 \alpha \int_0^\Lambda dk \: k^3 \frac{kV \cos \alpha + iV^2}{k^2 + \sigma^2 - V^2 + 2ikV \cos \alpha} = \]

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\[-\frac{V^2}{\pi^3} \int_0^\Lambda dk \int_{-1}^{+1} dx \frac{2k^2 x^2 - k^2 - \sigma^2 + V^2}{(k^2 + \sigma^2 - V^2)^2 + 4k^2 V^2 x^2}. \] 

(54)

This expression of course agrees with the result which is obtained by taking the derivative of (49) with respect to \( V_\mu \) and multiplying with \( V_\mu \).

C The Bethe-Salpeter equation

In order to observe the cancelation of the mass term for the “almost” Goldstone vectors, we need a certain relation which is satisfied for any solution \( V^B_\mu \) of the gap equation (51) with \( \mu = 0 \). Our first task in this section is to derive this relation.

For this purpose, we note that the trace term in (51) transforms as an O(4) vector, i.e.

\[ B_\mu [V^B] = -4i \int (k) \frac{k_\mu + iV^B_\mu}{(k + iV^B)^2 + \sigma^2}. \] 

(55)

This is true, because we use an O(4) invariant regularization of the momentum integration. It is obvious that

\[ B_\mu [\Lambda V^B] = \Lambda_{\mu\nu} B_\nu [V^B], \quad \Lambda_{\mu\nu} = \exp \{\theta^a \eta^a\}_{\mu\nu}. \] 

(56)

The matrices \( \eta^a_{\mu\nu} \) are three of ’t Hooft’s antisymmetric matrices, which serve as three out of six generators of the O(4) transformation. Note that eq.(56) is satisfied for any choice of the angles \( \theta^a \). Taking the derivative of this equation with respect to \( \theta^a \) yields the identity

\[ \eta^a_{\mu\nu} B_\nu [V^B] = -4i \int (k) \frac{i\eta^a_{\mu\nu} V^B_\nu}{(k + iV^B)^2 + \sigma^2} + 4i \int (k) \frac{(k + iV^B)_\mu 2i (k_\alpha \eta^a_{\alpha\beta} V^B_\beta)}{((k + iV^B)^2 + \sigma^2)^2}. \] 

(57)

If we specialize to \( V^B_\mu = (V_0, 0, 0, 0) \), use \( \eta^a_{00} = \delta^{a1} \) and employ the gap equation, i.e. \( B_\mu = -m^2_0 (V_\mu - \mu \delta_{\mu3}) \), we finally obtain the desired equation \( (V_0 \neq 0) \), i.e.

\[ 4 \int (k) \left[ \frac{\delta^{a1}}{(k + iV^B)^2 + \sigma^2} \right] - \frac{2k_1 k_3}{(k + iV^B)^2 + \sigma^2} \right] + m^2_0 \delta^{a1} = \frac{m^2_0}{V_0} \mu \delta_{a1}. \] 

(58)

In the following, we evaluate the trace term in (19), which gives rise to the Bethe-Salpeter equation. Rewriting the trace as a sum over momentum eigenstates and inserting a complete set of these eigenstates, the trace term can be written as

\[ -\frac{1}{2} \int (p) v_\mu(p)v_\nu(-p) \int (k) \text{tr}_D \left\{ \frac{k_\mu + \frac{p_1}{2} + iV^B_\mu + \sigma}{(k + \frac{p}{2} + iV^B)^2 + \sigma^2} \gamma_\mu \frac{k_\nu - \frac{p_1}{2} + iV^B_\nu + \sigma}{(k - \frac{p}{2} + iV^B)^2 + \sigma^2} \gamma_\nu \right\}. \] 

(59)
where the trace $\text{tr}_D$ extends over Dirac indices only. Performing the Dirac trace and introducing a Feynman integral, the polarization tensor in (20) is given by

$$
\Pi_{ik}(p) = (p^2 \delta_{ik} - p_ip_k) + m_v^2 \delta_{ik}
$$

(60)

$$
- 4 \int_0^1 dx \int(q) \frac{2q_i q_k - 2x(1-x)p_i p_k - \delta_{ik} \left[ (q + \frac{1-2x}{2} p + iV^B)^2 - \frac{p^2}{4} + \sigma^2 \right]}{[(q + iV^B)^2 + x(1-x)p^2 + \sigma^2]^2} .
$$

In order to show that the vector fields are massless excitations for $\mu = 0$, we study the polarization tensor (60) at zero momentum, i.e.

$$
\Pi_{ik}(0) = m_v^2 \delta_{ik} - 4 \int(q) \frac{2q_i q_k - \delta_{ik} \left( (q + iV^B)^2 + \sigma^2 \right)}{[(q + iV^B)^2 + \sigma^2]^2}
$$

(61)

$$
= \frac{m_v^2}{V_0} \mu \delta_{ik} ,
$$

(62)

where we have used eq.(58). The crucial observation is that the expression (61) vanishes for $\mu = 0$ (and $V_0 \neq 0$).

If we are interested in the polarization tensor close to the mass shell, it is sufficient to study the Bethe-Salpeter equation in a derivative expansion. It is straightforward to extract the order $O(p^2)$ from (60). One finally finds

$$
\Pi_{ik}(p_0, \vec{p} = 0) = \delta_{ik} \left[ 1 - f(\sigma, V^B) \right] p^2 + \frac{m_v^2}{V_0} \mu \delta_{ik} + O(p^4)
$$

(63)

where

$$
f(\sigma, V^B) = \frac{2}{3} \int(q) \left\{ \frac{3}{[(q + iV^B)^2 + \sigma^2]^2} - \frac{4}{3} \frac{\vec{q}^2}{[(q + iV^B)^2 + \sigma^2]^3} \right\} .
$$

(64)

Let us study the interesting case $\sigma = 0$, $V_0 \neq 0$. For this purpose, we introduce polar-coordinates for the momentum integration in (64). The angle integration is tedious due to the complex functions. One must distinguish the cases $q > V_0$ and $q < V_0$. One finally obtains

$$
f(\sigma = 0, V^B) = \frac{1}{12\pi^2} \left\{ \int_0^{V_0} dq \ q^3 \frac{q^2 + 3V_0^2}{V_0(q^2 + V_0^2)} + \int_{V_0}^{\Lambda} dq \ q \frac{2}{q^2 + V_0^2} \right\}
$$

(65)

for $V_0 \leq \Lambda$, and

$$
f(\sigma = 0, V^B) = \frac{1}{12\pi^2} \int_0^{V_0} dq \ q^3 \frac{q^2 + 3V_0^2}{V_0(q^2 + V_0^2)}
$$

(66)
for $V_0 > \Lambda$. The final momentum integration is straightforward. We obtain

$$f(\sigma = 0, V^B) = \frac{1}{24\pi^2} \left\{ \frac{5}{2} - 4 \ln 2 + 2 \ln \left( \frac{\Lambda^2}{V_0^2} + 1 \right) \right\}, \quad \text{for } V_0 < \Lambda \quad (67)$$

$$f(\sigma = 0, V^B) = \frac{1}{24\pi^2} \left\{ \frac{2\Lambda^2}{V_0^2} + \frac{\Lambda^4}{2V_0^4} - 2 \ln \left( \frac{\Lambda^2}{V_0^2} + 1 \right) \right\}, \quad \text{for } V_0 > \Lambda \quad (68)$$

The final result depends logarithmically on the cutoff $\Lambda$ as expected from a naive power counting. We have numerically investigated the function $f(\sigma, V^B)$ for several values of $\sigma$. We find that generically $f(\sigma, V^B) > 0$.

References


