Asymptotic Expansions of Two-Loop Feynman Diagrams
in the Sudakov Limit

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Abstract

Recently presented explicit formulae for asymptotic expansions of Feynman diagrams in the Sudakov limit [1] are applied to typical two-loop diagrams. For a diagram with one non-zero mass these formulae provide an algorithm for analytical calculation of all powers and logarithms, i.e. coefficients in the corresponding expansion $(Q^2)^{-2} \sum_{n,j=0} c_{nj} t^{-n} \ln^j t$, with $t = Q^2/m^2$ and $j \leq 4$. Results for the coefficients at several first powers are presented. For a diagram with two non-zero masses, results for all the logarithms and the leading power, i.e. the coefficients $c_{nj}$ for $n = 0$ and $j = 4, 3, 2, 1, 0$ are obtained. A typical feature of these explicit formulae (written through a sum over a specific family of subgraphs of a given graph, similar to asymptotic expansions for off-shell limits of momenta and masses) is an interplay between ultraviolet, collinear and infrared divergences which represent themselves as poles in the parameter $\epsilon = (4 - d)/2$ of dimensional regularization. In particular, in the case of the second diagram, that is free from the divergences, individual terms of the asymptotic expansion involve all the three kinds of divergences resulting in poles, up to $1/\epsilon^4$, which are successfully canceled in the sum.

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1 Introduction

The simplest explicit formulae for asymptotic expansions of off-shell Feynman diagrams in various limits of momenta and masses\(^2\) [2, 3, 4] (see [5] for a brief review) have been recently generalized [1, 6] for two on-shell limits: the limit of the large momenta on the mass shell with the large mass and the Sudakov limit, with the large momenta on the massless mass shell.

To derive these formulae a method based on the construction of a remainder of the expansion and using then diagrammatic Zimmermann identities [7] and applied in refs. [8] for operator product expansions within momentum subtractions and later for diagrammatic and operator expansions within dimensional regularization and renormalization [4] has been used. These general prescriptions were illustrated in ref. [1] through one-loop examples (the triangle diagram of Fig. 1a in the case of the Sudakov limit) and, in ref. [6], illustrated and applied for calculation of a typical two-loop diagram in the case of the first of two limits.

The purpose of this paper is to present similar results in the latter case, i.e. for the Sudakov limit. We shall consider two typical vertex two-loop diagrams. The former is the Feynman integral \(F_1(p_1, p_2, m)\) corresponding to Fig. 2a with \(m_1 = \ldots = m_5 = 0, m_6 = m\), and the latter is \(F_2(p_1, p_2, m)\) with \(m_1 = \ldots = m_4 = 0, m_5 = m_6 = m\). The limit under consideration is \(q^2 = (p_1 - p_2)^2 \equiv -Q^2 \to -\infty\), with \(p_1^2 = p_2^2 = 0\). In the case of \(F_1\), the general prescription will be used to obtain an algorithm for analytical calculation of arbitrary coefficients \(c_{nj}^{(1)}\) in the corresponding expansion

\[
F_i(p_1, p_2, m) \quad q^2 \to -\infty \quad (Q^2)^{-2} \sum_{n=0}^{\infty} \sum_{j=0}^{4} c_{nj}^{(i)} (m^2/Q^2)^n \ln^j(Q^2/m^2).
\]  

(1)

Results for the coefficients at several first powers will be presented. For \(F_2\), the coefficients for all the logarithms and the leading power, i.e. \(c_{nj}^{(2)}\) for \(n = 0\) and \(j = 4, 3, 2, 1, 0\) will be analytically evaluated.

As in the case of the off-shell explicit formulae [2, 3, 4] the prescriptions of ref. [1] are written through a sum over a specific family of subgraphs of a given graph. A typical feature of the off-shell formulae is an interplay between ultraviolet and infrared divergences which appear in individual terms of that sum but are mutually canceled, provided the initial Feynman integral is finite. It turns out that, for the Sudakov limit, one meets a more general interplay between ultraviolet, collinear and infrared divergences which, at first sight, seem to be of very different nature. In particular, in the case of \(F_2\), which does not have divergences from the very beginning, individual terms of the corresponding asymptotic expansion involve all three kinds of divergences resulting in poles up to \(1/\epsilon^4\), with \(\epsilon = (4 - d)/2\) parameter of dimensional regularization [9]. However these poles are successfully canceled in the sum and one obtains expansion (1) in the limit \(\epsilon \to 0\).

\(^2\)In the large mass limit, one can also apply the same ‘off-shell’ formulae for any values of the external momenta.
In the next section we shall briefly recall the general prescriptions and one-loop example of ref. [1]. In the following sections these prescriptions will be applied to the Feynman integrals $F_1$ and $F_2$.

2 General prescriptions and one-loop example

The asymptotic expansion of a Feynman integral $F_\Gamma(p_1, p_2, m)$, corresponding to a vertex graph $\Gamma$, in the Sudakov limit takes the following explicit form [1]:

$$F_\Gamma(p_1, p_2, m, \epsilon) \overset{q^2 \to -\infty}{\sim} \sum_\gamma M_\gamma F_\Gamma(p_1, p_2, m, \epsilon).$$  \hspace{1cm} (2)

Here the sum runs over subgraphs $\gamma$ of $\Gamma$ for which at least one of the following conditions holds:

(i) In $\gamma$ there is a path between the end-points 1 and 3. (Three end-points of the diagram are numerated according to the following order: $p_1, p_2, q = p_1 - p_2$.) The graph $\gamma$ obtained from $\gamma$ by identifying the vertices 1 and 3 is 1PI.

(ii) Similar condition with $1 \leftrightarrow 2$.

The pre-subtraction operator $M_\gamma$ is defined as a product $\prod_j M_{\gamma_j}$ of operators of Taylor expansion acting on 1PI components and cut lines of the subgraph $\gamma$. Suppose that the above condition (i) holds and (ii) does not hold. Let $\gamma_j$ be a 1PI component of $\gamma$ and let $p_1 + k$ be one of its external momenta, where $k$ is a linear combination of the loop momenta. (We imply that the loop momenta are chosen in such a way that $p_1$ flows through all $\gamma_j$ and the corresponding cut lines). Let now $q_j$ be other independent external momenta of $\gamma_j$. Then the operator $M$ for this component is defined as

$$T_{\kappa - \left(\frac{p_1 k}{(p_1 p_2)}\right)\kappa_j, m_j}, \text{ where } m_j \text{ are the masses of } \gamma_j.$$  \hspace{1cm} (3)

In other words, it is the operator of Taylor expansion in $q_j$ and $m_j$ at the origin and in $k$ at the point $\tilde{k} = \frac{(p_1 k)}{(p_1 p_2)} p_2$ (which depends on $k$ itself).

For the cut lines the same prescription is adopted. If $p_1 + k$ is the momentum of the cut line, then the corresponding operator acts as $T_{\kappa - \left(\frac{1}{\kappa \kappa_j - m_j^2} + 2p_1 k\right)\kappa_j - 1} \bigg|_{\kappa = 1}$. If both conditions (i) and (ii) hold the corresponding operator performs Taylor expansion in the mass and the external momenta of subgraphs (apart from $p_1$ and $p_2$).

For example, in the case of the one-loop triangle diagram of Fig. 1a,

$$F_\Gamma(p_1, p_2, m, \epsilon) = \int \frac{d^d k}{(k^2 - 2p_1 k)(k^2 - 2p_2 k)(k^2 - m^2)},$$  \hspace{1cm} (3)

this family of subgraphs is shown in Fig. 1. Besides the graph itself (a), these are two subgraphs consisting, respectively, of the left and the right line (b and c). The subtraction operator for (a) expands the propagator with the mass $m$ in Taylor series in
\( m \), and the subtraction operators \( \mathcal{M} \) for (b) and (c) act at the first and, respectively, the second factor in the integrand in (3) as follows:

\[
\mathcal{M}_i^a \left( \frac{1}{k^2 - 2p_i k} \right) = \sum_{j=0}^{a} \frac{(k^2)^j}{(-2p_i k)^j+1}, \quad i = 1, 2.
\]  

(4)

All the resulting one-loop integrals are easily evaluated for any order of the expansion. The operator \( \mathcal{M}_0 \) gives the following contribution at \( \epsilon \neq 0 \):

\[
\mathcal{M}_0 F_\Gamma = -i\pi^{d/2} \frac{1}{(-q^2)^{1+\epsilon}} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + \epsilon) \Gamma(-n - \epsilon)^2}{\Gamma(1 - n - 2\epsilon)} \left( \frac{m^2}{q^2} \right)^n.
\]  

(5)

The terms \( \mathcal{M}_1 F_\Gamma \) and \( \mathcal{M}_2 F_\Gamma \) are not individually regulated by dimensional regularization but their sum exists for general \( \epsilon \neq 0 \):

\[
(\mathcal{M}_1 + \mathcal{M}_2) F_\Gamma = i\pi^{d/2} \frac{1}{q^2(m^2)^\epsilon} \Gamma(1 - \epsilon) \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 - n - \epsilon)}
\times \left[ \ln(Q^2/m^2) + \psi(\epsilon) + \psi(n + 1) - \psi(1 - \epsilon) - \psi(1 - n - \epsilon) \right] \left( \frac{m^2}{q^2} \right)^n.
\]  

(6)

Here \( \psi \) is the logarithmic derivative of the gamma function. In the limit \( \epsilon \to 0 \), one arrives at the known result:

\[
F_\Gamma \mid_{\epsilon=0} \sim q^2 \to -\infty = -i\pi^2 \frac{1}{q^2} \left[ \text{Li}_2 \left( \frac{1}{t} \right) + \frac{1}{2} \ln^2 t - \ln t \ln(t - 1) - \frac{1}{3} \pi^2 \right],
\]  

(7)

where \( \text{Li}_2 \) is the dilogarithm and \( t = Q^2/m^2 \).

## 3 Diagram with one non-zero mass

The two-loop Feynman integrals under consideration can be written as

\[
F_i(p_1, p_2, m, \epsilon) = \int \int \frac{d^dk d^dl}{(l^2 - 2p_1 l)(l^2 - 2p_2 l)(k^2 - 2p_1 k)(k^2 - 2p_2 k)(k^2 - m_5^2)((k - l)^2 - m^2)},
\]  

(8)

where \( m_{51} = 0 \) and \( m_{52} = m \). Let us apply the above prescriptions to them. The family of subgraphs that give non-zero contributions is shown in Fig. 2(a–f). Let us begin with the first case, when subgraphs (c) and (d) also happen to give zero contributions.

The contribution of Fig. 2a is obtained by expanding the massive propagator in the geometric series in \( m \). In the leading order, this is Fig. 2a with pure zero masses which was calculated in [10, 11, 12]:

\[
\left( \frac{i\pi^{d/2}}{(Q^2)^{2+2\epsilon}} \right) \epsilon \left\{ \frac{1}{2\epsilon} G_2(2, 2) G_3(2 + \epsilon, 1, 1) - G_2(2, 1) \left[ \frac{1}{\epsilon} G_3(2, 1, 1 + \epsilon) + G_3(1, 1, 1) \right] \right\},
\]  

(9)
where \( G_2(a,b) \) and \( G_3(a,b,c) \) are expressed through gamma functions and give, respectively, the value of the general one-loop propagator-type integral and triangle integral with the given kinematics, in the case of arbitrary indices of lines.

All the resulting Feynman integrals in arbitrary order of the expansion are evaluated by integration by parts [13] (as in [12]) and are expressed, for general \( \epsilon \), in terms of gamma functions.

To write down the contribution of Fig. 2b it is necessary to expand the initial diagram, with the 6th line omitted, in Taylor series (under the sign of the integral over the loop momenta) in the loop momentum flowing through this line in the whole diagram and then insert the result into the reduced diagram which is the tadpole generated by the 6th line. This subgraph Fig. 2b contributes to the expansion starting from the next-to-leading order, \( m^2/(Q^2)^3 \). The calculation of this contribution is rather simple and reduces to successive application of the one-loop integration formulae, with the \( G \)-functions \( G_2 \) and \( G_3 \) in the result.

What is more non-trivial is the evaluation of the contributions of the subgraphs of Fig. 2e and f. The contribution of Fig. 2e is obtained by expansion of the propagators \( 1/(l^2 - 2p_1l) \) and \( 1/(k^2 - 2p_1k) \) in geometrical series in \( l^2 \) and \( k^2 \), respectively:

\[
\sum_{N_1,N_3} \int \int \frac{d^d k d^d l (-1)^{N_1 + N_3} (l^2)^{N_1}}{(-2p_1 l)^{1 + N_1} (l^2 - 2p_2 l) (-2p_1 k)^{1 + N_3} (k^2 - 2p_2 k) (k^2)^{1 - N_3} ((k - l)^2 - m^2)} \quad (10)
\]

However these contributions are not individually regularized by dimensional regularization so that let us introduce, temporarily, analytic regularization into the lines number 3 and 4, with \( \lambda_3 \neq \lambda_4 \), then calculate the integrals involved and switch off the regularization in the sum.

Thus the problem reduces to calculation of the following family of integrals:

\[
J(N_1, \ldots, N_7) = \int \int \frac{d^d k d^d l (l^2)^{N_7}}{(-2p_1 l)^{1 + N_1} (l^2 - 2p_2 l)^{1 + N_2} (-2p_1 k)^{1 + N_3} (k^2 - 2p_2 k)^{1 + N_4} \frac{1}{((k - l)^2 - m^2)^{N_6}}} \quad , (11)
\]

with integer \( N_6 \) and \( N_7 \). Using integration by parts [13] it is possible to express any integral with \( N_7 > 0 \) through a linear combination of integrals with \( N_7 = 0 \).

For integral (11) with \( N_7 = 0 \), let us use the \( \alpha \)-parametric representation. Then, in the resulting 6-fold integral, it is possible to apply the Mellin-Barnes representation in such a way that the internal integration over \( \alpha \)-parameters can be performed explicitly and we are left with the following representation:

\[
J(N_1, N_2, N_3 + \lambda_3, N_4 + \lambda_4, N_5, N_6, 0) = \frac{(i \pi^{d/2})^2}{(Q^2)^{2 + N_1 + N_3 + \lambda_3} (m^2)^{2 \epsilon + N_2 + N_4 + N_5 + N_6 + \lambda_4}} \times \frac{\Gamma(-\epsilon + N_5 - N_6 - \lambda_3) \Gamma(-\epsilon + N_2 + N_4 + N_5 + N_6 + \lambda_4) \Gamma(2\epsilon + N_2 + N_4 + N_5 + N_6 + \lambda_4)}{\Gamma(1 + N_2) \Gamma(1 + N_4 + \lambda_4) \Gamma(1 + N_5) \Gamma(1 - \epsilon - N_5) \Gamma(-N_1 - N_3 - \epsilon - \lambda_3)}
\]

4
The contour of integration is chosen in the standard way: the IR poles are to the left and the UV poles are to the right. The singularities at $\lambda_3 \to \lambda_4$ are due to the fact that some UV poles become identical with IR poles. By shifting the contour of integration these pole contributions are explicitly picked up. The rest of the integral is finite in the limit $\lambda_3, \lambda_4 \to 0$, and the resulting integrals are easily evaluated by recurrence relations and finally using well-known integrals with four gamma functions.

Following this procedure, any integral that is present in the contribution of Fig. 2e and $f$ can be analytically evaluated, for arbitrary $\epsilon$. For example, the leading contribution takes the form

$$ -\left(\frac{i\pi^{d/2}}{(Q^2)^{2e}m^2}\right)^2 \Gamma(-\epsilon) \Gamma(\epsilon) \Gamma(2\epsilon) \left[ L + \psi(2\epsilon) + \psi(1 + \epsilon) - 2\psi(1) \right], $$

where $L = \ln Q^2/m^2$. To avoid the Euler constant in the results we present several first terms of the expansion of the function $F_1 \cdot (Q^2)^2/(i\pi^{d/2}\Gamma(1 + \epsilon))^2 = \sum_{n=0}^{\infty} C_n t^{-n}$:

$$ C_0 = \left(\frac{1}{2}L^2 + \frac{1}{3}\pi^2\right) \frac{1}{\epsilon^2} - \left(\frac{1}{3}L^3 + \frac{1}{6}\pi^2L - 3\zeta(3)\right) \frac{1}{\epsilon}, $$
$$ + \frac{L^4}{6} + \frac{1}{3}\pi^2L^2 - 11\zeta(3)L + \frac{61}{360}\pi^4, $$

$$ C_1 = -(L + 1) \frac{1}{\epsilon^2} + \left(\frac{1}{2}L^2 - L + \frac{1}{6}\pi^2\right) \frac{1}{\epsilon}, $$
$$ - \frac{2}{3}L^3 + 3L^2 - 6L - \frac{2}{5}\pi^2L - 26 + \frac{1}{2}\pi^2 + 11\zeta(3), $$

$$ C_2 = -\left(\frac{1}{3}L + \frac{1}{3}\right) \frac{1}{\epsilon^2} + \left(\frac{1}{2}L^2 - \frac{3}{4}L - \frac{5}{4} + \frac{1}{15}\pi^2\right) \frac{1}{\epsilon}, $$
$$ - \frac{1}{3}L^3 + \frac{13}{4}L^2 - 5L - \frac{1}{3}\pi^2L - \frac{17}{8} + \frac{17}{24}\pi^2 + \frac{11}{2}\zeta(3), $$

$$ C_3 = -\left(\frac{1}{3}L + \frac{1}{9}\right) \frac{1}{\epsilon^2} + \left(\frac{1}{3}L^2 - \frac{29}{18}L - \frac{3}{4} + \frac{1}{18}\pi^2\right) \frac{1}{\epsilon}, $$
$$ - \frac{2}{9}L^3 + \frac{17}{6}L^2 - \frac{35}{9}L - \frac{2}{5}\pi^2L - \frac{890}{81} + \frac{33}{36}\pi^2 + \frac{11}{3}\zeta(3). $$

These results have been obtained with the help of the Mathematica system [15].

## 4 Diagram with two non-zero masses

In the case of the second Feynman integral that we consider there are contributions from all the subgraphs of Fig. 2. The contribution of Fig. 2a is obtained by expanding the propagators of the 5th and the 6th lines in the geometric series in $m$. As in the previous case the resulting integrals are calculated by recurrence relations based on integration by parts. The leading contribution has the same form (9). The contribution of Fig. 2b in arbitrary order also poses no difficulties, with a result expressed through the $G$-functions $G_2$ and $G_3$. 

$$ \times \frac{1}{2\pi i} \int_C ds \frac{\Gamma(s - \epsilon - N_2 + 1)}{\Gamma(s - \epsilon - N_3 - \lambda_3 + 1)} \times \Gamma(s - N_1 - N_3 - \epsilon - \lambda_3) \Gamma(\epsilon + N_2 + N_4 + \lambda_4 - s) \Gamma(-s). $$

(12)
Consider now Fig. 2c and d. According to the general prescription, the contribution of Fig. 2c is written through the product of the expansion of the propagator $1/(k^2 - 2p_1 k)$ in geometrical series with respect to $k^2$ and an expansion of the lower triangle subdiagram which is a function $f(q_1^2, q_2^2, q_3^2)$ of the following external momenta squared: $q_1^2 = k^2 - 2p_1 k$, $q_2^2 = k^2 - 2p_2 k$ and $q_3^2 = q^2$. So, this second expansion of $f$ is in Taylor series (again in the sense of the expansion under the sign of integral) in $q_2^2$, with the subsequent expansion of the result, as a function of $k^2 - 2p_1 k$ in $k^2$. All the derivatives of that triangle in $q_2^2$ at $q_2^2 = 0$ are calculated by recurrence relations based on integration by parts (see, e.g., results for the first two derivatives in [14]).

This product of the expansion of the 4th line and the lower triangle is then integrated with the product of other propagators (number 4 and 6), using simple one-loop integration formulae. However this contribution of Fig. 2c alone is not regularized dimensionally so that it is natural to consider the sum of both contributions of Fig. 2c and d. Still to handle these terms individually, one can introduce a temporal analytic regularization, into the lines number 3 and 4, calculate them and switch off the analytic regularization in the sum.

Following this procedure, the contribution of Fig. 2c and d can be evaluated in terms of gamma functions in every order of the expansion, for arbitrary $\epsilon$. For example, the leading, $1/(Q^2)^2$, contribution takes the form

$$
\frac{(i\pi^{d/2})^2}{(Q^2)^{2+\epsilon}(m^2)^\epsilon} \frac{\Gamma(\epsilon)^2 \Gamma(1-\epsilon)^2}{\epsilon \Gamma(1-2\epsilon)} \left\{ -2 \frac{\Gamma(-\epsilon)^2}{\Gamma(-2\epsilon)} + \ln t + \psi(\epsilon) - 2\psi(-\epsilon) + \psi(1) \right\}. \quad (15)
$$

To calculate the contribution of Fig. 2e and f it is reasonable to use the Mellin-Barnes representation for the 5th propagator and reduce the problem to calculation of the corresponding contribution for the first of our integrals, in the case when this line is analytically regularized. In particular, the leading, $1/(Q^2)^2$, contribution is obtained as the following integral of the analytically regularized version of eq. (13): \begin{align*}
- \frac{(i\pi^{d/2})^2}{(Q^2)^{2+\epsilon}(m^2)^\epsilon} \frac{1}{2\pi i} \int_C ds \Gamma(2\epsilon + s) \Gamma(\epsilon + s) \Gamma(-\epsilon - s) \Gamma(-s) \\
\times [\ln t + \psi(2\epsilon + s) + \psi(1 + \epsilon + s) - 2\psi(1 + s)]. \quad (16)
\end{align*}

The contour $C$ is chosen in the standard way, with the qualification that the pole $s = -\epsilon$ (which is both UV and IR) is to the right of it. The pole and the finite part of eq. (16) are evaluated separately and give the following result written up to $\epsilon^0$:

$$
- \frac{(i\pi^{d/2})^2}{(Q^2)^{2+\epsilon}(m^2)^\epsilon} \left\{ 2\zeta(3) \ln t + \frac{\pi^4}{36} \right. \\
+ \Gamma(-\epsilon) \Gamma(\epsilon) [\ln t + \psi(1 - \epsilon) - 2\psi(1 - 2\epsilon) - \psi(-\epsilon) + \psi(\epsilon) + \psi(2\epsilon)] \right\}. \quad (17)
$$

Now, the leading order of the asymptotic expansion of the second of our Feynman integrals is given explicitly by the sum of three terms corresponding to Fig. 2a,
Fig. 2(c&d) and Fig. 2(e&f), respectively, (9), (15) and (17). Note that the first term involves infrared and collinear divergences, the third term involves ultraviolet and collinear divergences, and the second term possesses all the three kinds of the divergences. However all the poles, up to \(1/\epsilon^4\), are canceled in the sum and we get

\[
F_2(Q^2, m^2) \quad Q^2 \to \infty \sim -\pi^4 \left( \frac{1}{24} \ln^4 t + \frac{\pi^2}{3} \ln^2 t - 6\zeta(3) \ln t + \frac{31\pi^4}{180} \right).
\]  

These results are in a good agreement with numerical MC calculations: e.g., there is 2% accuracy at \(t = 50\). However this accuracy is achieved if all the terms in the parenthesis are taken into account. In other words, it is necessary to exhaust all the powers of the logarithm until the remainder is determined by the next power, \(m^2/(Q^2)^3\).

The calculation of the contribution of Fig. 2e and f in arbitrary order of the expansion happens to be more elaborated and will be reported in future publications as well as details of the present calculations, similar results for non-planar diagrams, with subsequent application of the new information obtained in two loops to extend well-known results on asymptotic behaviour in the Sudakov limit in QED and QCD [16, 17, 18, 19].

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References


Figure 1: (a) One-loop vertex diagram. (a)–(c) Subgraphs contributing to the asymptotic expansion of the diagram (a) in the Sudakov limit.

Figure 2: (a) A typical two-loop vertex diagram. (a)–(f) Subgraphs contributing to the asymptotic expansion of the diagram (a) in the Sudakov limit.