Inverse Amplitude Method and Chiral Zeros

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The inverse amplitude method has previously been successfully applied to ππ scattering in order to extend the range of applicability of chiral perturbation theory. However, in order to take the chiral zeros into account systematically, the previous derivation of the inverse amplitude method has to be modified. It is shown how this can be done to both one and two loops in the chiral expansion. In the physical region, the inclusion of these chiral zeros has very little significance, whereas they become essential in the sub-threshold region. Finally, the crossing properties of the inverse amplitude method are investigated in the sub-threshold region.

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Chiral perturbation theory (ChPT) [1,2] has become a very successful methodology for low-energy hadron physics. Within this methodology one obtains a systematic expansion in powers of external momenta and light quark masses, or equivalently in the number of loops. However, unitarity is only satisfied perturbatively in the chiral expansion, which gives a severe restriction on the applicability of ChPT. Therefore, in order to extend the validity of ChPT, several methods have been proposed to combine exact unitarity and the chiral expansion. One such method is the inverse amplitude method (IAM), which has been analyzed to both one-loop [3] and two-loop [4] order in the chiral expansion. This has shown that the IAM is indeed a successful and systematic method to extend the range of applicability of ChPT.

However, the IAM has to be somewhat modified when there are zeros in the amplitudes [5]. This is the case for ππ scattering where chiral dynamics demand that the S waves have zeros below threshold, whereas P and higher partial waves have fixed kinematical zeros at threshold. The previous derivation of the IAM [3,4] has been based upon the assumption that these zeros occur at the same energy as for the lowest order ChPT result. This assumption is in fact true for the P and higher partial waves, whereas the same is not necessarily the case for the S waves. Thus, the derivation of the IAM should be somewhat modified in the case of the S waves in order to systematically account for the occurrence of the chiral zeros. In this brief report it is shown how this is possible to both one and two-loop order in the chiral expansion. Since the main interest will be in the sub-threshold behavior, the following derivation of this generalized IAM will be restricted to the elastic approximation.

The elastic unitarity relation for the ππ partial waves t is given by

\[ \text{Im} t^I_I(s) = \sigma(s) |t^I_I(s)|^2, \]

where s is the square of the c.m. energy, I and l are isospin and angular momentum indices, and σ is the phase-space factor. In ChPT the elastic ππ partial waves are now known to two loops in the chiral expansion [6,7]. These partial waves are given by the following expansion

\[ t^I_I(s) = t^{(0)}_I(s) + t^{(1)}_I(s) + t^{(2)}_I(s), \]

where \( t^{(0)}_I \) is the lowest order result, \( t^{(1)}_I \) is the one-loop correction, and \( t^{(2)}_I \) the additional two-loop correction. They satisfy the elastic unitarity relation (1) perturbatively

\[ \begin{align*}
\text{Im} t^{(0)}_I(s) &= 0, \\
\text{Im} t^{(1)}_I(s) &= \sigma(s) t^{(0)}_I(s), \\
\text{Im} t^{(2)}_I(s) &= \sigma(s) 2t^{(1)}_I(s) \text{Re} t^{(1)}_I(s).
\end{align*} \]

Since perturbative unitarity only works well rather close to threshold, this will give a severe restriction on the applicability of ChPT. However, with the use of the IAM, the range of applicability of ChPT can be substantially extended. The previous starting point for this method [3,4] was to write down a dispersion relation for the function \( \Gamma = t^{(2)}_I / t \). However, in order to systematically account for the occurrence of the chiral zeros, one has to modify the function \( \Gamma \) and now write it as \( \Gamma = (t^{(0)} + \epsilon)^2 / t \). The \( \epsilon \) parameter parameterizes the change in the position of the zero in the partial wave \( t \) away from the lowest order result, i.e., \( \epsilon \) is determined by the equation

\[ t^{(0)}_I(s_z) + \epsilon I = 0, \]

where \( s_z \) is the position of the zero in the partial wave \( t \). Of course, the precise value of \( s_z \) and therefore also the value of \( \epsilon \) is unknown, but they may both be approximated by the use of the chiral expansion, which should apply unambiguously in the sub-threshold region. Unfortunately, the zero in the chiral expansion can only be found analytically to lowest order. However, the zero may be determined very accurately to a given order in the chiral expansion by solving the equation

\[ t^{(0)}_I(s) + t^{(1)}_I(s^{(0)}_z) + t^{(2)}_I(s^{(0)}_z) + \cdots = 0, \]

where \( s^{(0)}_z \) is the position of the zero in the lowest order result \( t^{(0)}_I \). This has been checked to work very well indeed for both one and two-loop ChPT. Therefore, comparing with Eq. (4) one finds that \( \epsilon \) may be expanded in ChPT as
\[ \epsilon_i^f = \epsilon_i^{(1)} + \epsilon_i^{(2)} + \cdots, \]  
\[ \epsilon_i^{(1)} = t_i^{(1)}(s_z^{(0)}), \quad \epsilon_i^{(2)} = t_i^{(2)}(s_z^{(0)}), \quad \cdots. \]  

Having expanded \( \epsilon \) in ChPT it is now possible to obtain the function \( \Gamma \) to a given order in the chiral expansion. To two loops one finds that the result \( \Gamma^{(2)} \) is given by

\[ \Gamma_i^{(2)}(s) = t_i^{(0)}(s) - t_i^{(1)}(s) + t_i^{(2)}(s) - t_i^{(0)}(s) + 2\epsilon_i^{(1)} \left( 1 - \frac{t_i^{(1)}(s)}{t_i^{(0)}(s)} \right) + 2\epsilon_i^{(2)} + \epsilon_i^{(1)^2}. \]  

This expression for \( \Gamma^{(2)} \) is well-defined at \( s_z^{(0)} \) where it coincides with the truncation of the chiral expansion (2). In fact, it is analytic in the whole complex \( s \) plane with cuts for \( -\infty < s < 0 \) and \( 4M_*^2 < s < \infty \). Therefore, it is possible to write down a dispersion relation for \( \Gamma^{(2)} \) with four subtractions in order to ensure the convergences of the integrals. For the right cut perturbative unitarity (3) gives \( \text{Im}\Gamma^{(2)} = -\sigma(t^{(0)^2} + 2\epsilon^{(1)^2}(t^{(0)}) \), whereas the left cut is simply given by \( \text{Im}\Gamma^{(2)} \).

Turning to the original function \( \Gamma = (t^{(0)} + \epsilon)^2/t \), it is known from the fundamental principles of S matrix theory that the partial wave \( t \) is analytic in the complex \( s \) plane with a right cut for \( 4M_0^2 < s < \infty \) and a left cut for \( -\infty < s < 0 \). Assuming that the only zero in \( t \) is the one below threshold demanded by chiral dynamics, the function \( \Gamma \) is also analytic in the whole complex \( s \) plane with the same cut structure as \( t \). It is therefore possible to write down a dispersion relation for \( \Gamma \) in the same way as for \( \Gamma^{(2)} \). In this case, however, exact unitarity is used in order to compute the right cut, i.e., Eq. (1) gives \( \text{Im}\Gamma = -\sigma(t^{(0)} + \epsilon)^2 \). Since the precise value of \( \epsilon \) is unknown, the terms involving \( \epsilon \) have to be expanded in ChPT using Eq. (5). This gives \( \text{Im}\Gamma = -\sigma(t^{(0)^2} + 2\epsilon^{(1)^2}(t^{(0)}) \) to two loops in the chiral expansion, which is used for the right cut in the dispersion relation for \( \Gamma \).

The left cut and the subtraction constants cannot be computed in the same way, but it is possible to evaluate them to a given order in the chiral expansion. Evaluating both the left cut and the subtraction constants to two loops in the chiral expansion, one finds that the dispersion relation for \( \Gamma \) is exactly the same as the dispersion relation for \( \Gamma^{(2)} \). Therefore, one has the following relation

\[ \left( \frac{t_i^{(0)}(s) + \epsilon_i^f}{t_i^f(s)} \right)^2 = \Gamma_i^{(2)}(s), \]  

from where it is possible to determine the position of the zero in \( t \) by setting the right hand side equal to zero and solving for \( s \). One can then determine \( \epsilon \) from Eq. (4) with a result that should of course be close to the value obtained from the chiral expansion to two loops (5). From Eq. (8) one finds that the final result can be written in the form

\[ t_i^f(s) = \frac{(t_i^{(0)}(s) + \epsilon_i^f)^2}{\Gamma_i^{(2)}(s)}. \]  

An important property of this generalized IAM is that it coincides with the chiral expansion (2) up to two-loop order. This is due to the fact that the left cut, the subtraction constants, and \( \epsilon \) in the dispersion relation for the IAM have been evaluated to two-loop order in ChPT. If the left cut had been approximated by something else [5], the IAM would not have been consistent with the chiral expansion (2) up to two-loop order. Furthermore, the generalized IAM will satisfy unitarity exactly up to two loops in \( (t^{(0)} + \epsilon)^2 \), which should work extremely well due to the suppression of the higher order contributions. Finally, with the assumption that \( \epsilon \) to all orders in ChPT is exactly zero, one finds that Eq. (9) coincides with the [0,2] Padé approximant previously derived in Ref. [4]. Since this assumption is in fact true for the \( F \) partial wave, one finds that the generalized IAM in this case coincides with the previously obtained result.

The generalized IAM can in general be applied to any given order in the chiral expansion and therefore also to the one-loop approximation. In this case one finds the result

\[ t_i^f(s) = \frac{(t_i^{(0)}(s) + \epsilon_i^f)^2}{\Gamma_i^{(1)}(s)}, \]  

where \( \Gamma^{(1)} \) is given by

\[ \Gamma_i^{(1)}(s) = t_i^{(0)}(s) - t_i^{(1)}(s) + 2\epsilon_i^{(1)} \].

Expanding this result to one loop in the chiral expansion, one finds that it coincides with one-loop ChPT. Furthermore, neglecting the \( \epsilon \) terms one finds the [0,1] Padé approximant, which has previously been extensively analyzed [3].

In the following the pion mass \( M_* \) and the pion decay constant \( F_\pi \) are set equal to \( M_* = 139.6 \text{ MeV} \) and \( F_\pi = 92.4 \text{ MeV} \), respectively. In addition, the chiral expansion is given in terms of a number of low-energy constants, which have to be determined before it is possible to obtain numerical results for the IAM. The low-energy constants occurring in the IAM to one and two loops have previously been determined without taking the chiral zeros into account [4]. Thus, in order to show how important the inclusion of the chiral zeros is, the same values of the low-energy constants are also used in the generalized IAM to one and two loops.

With the low-energy constants fixed at the values given in Ref. [4], the parameter \( \epsilon \) in the generalized IAM to one and two loops can be determined with results that
are indeed close to the chiral expansion of $\epsilon$ to one and two loops, respectively. Thus, with all the parameters in the generalized IAM fixed, one can compare with the results obtained previously without taking the occurrence of the chiral zeros into account [4]. In the physical region one finds that the two approaches are practical identical, indicating that the inclusion of the chiral zeros in the IAM has very little significance in the physical region. However, in the sub-threshold region, the inclusion of these zeros becomes essential as shown in Figs. 1 and 2. Indeed, the generalized IAM generates the chiral zeros, whereas the IAM without taking the chiral zeros into account clearly fails in the displayed energy regions.

An important constraint on the $\pi\pi$ scattering amplitude is due to crossing symmetry. Of course ChPT satisfies this symmetry exactly, whereas the same is not necessarily true in the case of the IAM. Therefore, it would be interesting to investigate how well the IAM actually satisfies crossing. In order to do this one needs to express the consequences of crossing in terms of a finite set of conditions for crossing. There are five axiomatic constraints are also satisfied in this case.

Since the crossing sum rules involve the partial waves in the sub-threshold region, where the inclusion of the chiral zeros in the IAM becomes essential, the measure (13) is only evaluated for the generalized IAM with the results given in Table I. From this table it is observed that the IAM to two loops significantly improves the crossing symmetry of the one-loop approximation. In fact, crossing is satisfied very well indeed in the two-loop approximation, the violation being less than 0.2% in all cases. For the one-loop approximation the violation is larger but still less than 1.3% in all cases.

The crossing symmetry of the IAM can also be investigated with the use of Martin inequalities [9,10]. These inequalities are exact consequences of the general principles of quantum field theory, which mainly consist in the constraints coming from analyticity, crossing symmetry, and positivity of absorptive parts. A very large number of inequalities has been derived, but the most interesting are for the $n^0 n^0$ $S$ partial wave, which is given by

$$L^{(0)}_0(s) = \frac{1}{2} (t^{(0)}_0(s) + 2t^{(0)}_3(s)) \quad \text{(14)}$$

With $s$ given in units of $M^2$, one has the following inequalities on the derivatives of $L^{(0)}_0$ [9]

$$\frac{dt^{(0)}_0(s)}{ds} < 0 \quad \text{for} \quad 0 < s < 1.219,$$

$$\frac{dt^{(0)}_0(s)}{ds} > 0 \quad \text{for} \quad 1.697 < s < 4,$$

$$\frac{d^2L^{(0)}_0(s)}{ds^2} > 0 \quad \text{for} \quad 0 < s < 1.7, \quad \text{(15)}$$

which implies that $L^{(0)}_0$ has a minimum in the interval $(1.219,1.697)$. In addition, the following inequalities have also been obtained [9]

$$L^{(0)}_0(1.20) > L^{(0)}_0(1.20) \quad \text{(16)}$$

One finds that the axiomatic constraints (15) and (16) are satisfied for the generalized IAM to both one and two loops. In the two-loop approach the minimum occurs at $s/M^2 = 1.652$, whereas in the one-loop approximation one obtains the minimum at $s/M^2 = 1.643$. If the neutral pion mass $M_\pi = 135.0$ MeV had been used instead of the charge pion mass $M_\pi = 139.6$ MeV, the result would have been slightly different. However, the axiomatic constraints are also satisfied in this case.

Inequalities involving the definite isospin $S$ and $P$ partial waves have also been obtained. For instance, one has the following axiomatic constraints [10]

$$at^{(0)}_0(0) + bt^{(2)}_3(0) + ct^{(0)}_1(0) < At^{(0)}_0(4) + Bt^{(0)}_3(4),$$

$$L^{(0)}_0(s) - L^{(0)}_3(s) + 9 \cos \theta L^{(0)}_1(s) > 0 \quad \text{(17)}$$

where the specified values of $s$ and $t$ together with the values of the coefficients are given in Ref. [10]. The inequalities (17) give altogether 27 different constraints on the partial waves, which have been tested in the case of the generalized IAM. For the two-loop approach, one finds that all the above constraints are satisfied, whereas the one-loop approximation slightly violates two of the constraints. This agrees nicely with the previous analysis based on the crossing sum rules, where it was shown that the generalized IAM to two loops improves the crossing symmetry of the one-loop approximation. This conclusion is not likely to change if even other axiomatic constraints had been tested.

To summarize, it has been shown how the IAM can be generalized in order to systematically take the chiral zeros into account. In the physical region one finds that the inclusion of these zeros has very little significance, which justify the previous neglect of these zeros.
However, below threshold it becomes essential to include the chiral zeros in the derivation of the IAM. Furthermore, the crossing symmetry of the IAM is investigated by the use of sub-threshold crossing sum rules and Martin inequalities. It is found that the generalized IAM to two loops satisfies crossing symmetry very well indeed, whereas the violation of crossing is somewhat larger for the one-loop approximation.

Recently, the IAM has been applied to coupled channels \cite{oller1} in order to extend the applicability of the single channel IAM to even higher energies. However, this coupled channel IAM generates poles in the sub-threshold region just as the previous derived single channel IAM did. Thus, the derivation of the coupled channel IAM should in principle be somewhat modified in order to remove these poles. However, since the poles in the single channel IAM can be removed without any significant influence in the physical region, the same in also likely to be true in the coupled channel case.

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TABLE I. The measure Eq. (13) for the generalized IAM to one loop (GIAM1) and for the generalized IAM to two loops (GIAM2). The numbers refer to the corresponding crossing sum rule as given in Ref. [5].

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FIG. 1. The partial wave $f_0^0$ in the region $0 \leq s \leq M_c^2$. The solid line is the generalized IAM to two loops, the dashed line the generalized IAM to one loop, the dashed-dotted line the IAM to two loops without taking the chiral zeros into account, and the dotted line the IAM to one loop without taking the chiral zeros into account.

FIG. 2. The partial wave $f_2^2$ in the region $1.9M_c^2 \leq s \leq 2.1M_c^2$. The curves are as in Fig. 1.