AC

UNIFORM STABILITY OF MONOPOLES

David STUART

SW 9713

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Février 1997

IHES/M/97/16

Uniform Stability of Monopoles

D. Stuart*
IHES, Le Bois Marie
91440 Bures-sur-Yvette
France
email: stuart@math.ucdavis.edu

February 24, 1997

MSC numbers 35L70 81T13 58E15 53C07

*Supported by NSF. Permanent address: University of California, Davis.

Abstract

Two proofs of uniform stability in energy norm of BPS monopoles in the Yang-Mills-Higgs equations on Minkowski space-time are presented. BPS monopoles are minimisers of the static Yang-Mills-Higgs functional. The space of monopoles is an infinite dimensional submanifold whose quotient by the group of gauge transformations is finite dimensional. The problem of proving the existence and regularity of monopoles which are closest to the solution at each time is also discussed. The first proof establishes the existence of monopoles which are closest in L^2 ; this is achieved using Schoen-Uhlenbeck regularity theory for harmonic maps, and works for solutions of fairly high regularity (H^k_{loc} with $k \geq 4$). A uniform bound is then obtained for the distance function which gives the distance from the solution to the projected point in a certain norm, first defined by Taubes, which is related to the energy. The second proof involves the direct minimisation of the distance function with respect to which stability is proved. Although this function is minimised only subject to a closeness condition, this approach has the advantage of yielding a proof of uniform stability valid for solutions with only the regularity required for finite energy (H^1_{loc}). There are two principal sources of difficulty in the present problem: the fact that the continuous spectrum of the Hessian extends all the way to zero and the presence of the infinite dimensional group of gauge symmetries.

MSC numbers 35L70 81T13 58E15 53C07

Contents

1	Introduction	4
1.1	Background material and notation	4
1.2	Discussion and statement of results	6
2	L^2 Projection onto the space of monopoles	8
2.1	Gauge Invariance	8
2.2	Regularity	10
2.3	Solution space and moduli space	14
2.4	Proof of 2.1	16
3	Uniform Stability	16
3.1	Proof of theorem 1.2	16
3.2	Proof of theorem 1.3	18
A	Spaces of functions	22
A.1	Quaternion notation	22
A.2	Morrey and Campanato spaces	23
A.3	Covariant Sobolev completions	23
A.4	Dual norms	25
B	Operators	25
C	Proofs of some lemmas	27
D	Acknowledgements	29

1 Introduction

We consider Bogomol'nyi-Prasad-Sommerfeld (BPS) monopoles in the Yang-Mills-Higgs (YMH) equations on Minkowski space $\mathbf{R}^{1+3}$, on which we use standard coordinates (t, x) . Monopoles are minima of the static action

$$V(A, \Phi) = \int_{\mathbf{R}^3} (|F_A|^2 + |\nabla_A \Phi|^2) d^3x$$

with

$$(1.1) \quad \lim_{|x| \rightarrow \infty} |\Phi(x)| = 1$$

where $A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ is an $SU(2)$ connection, Φ is an $su(2)$ -valued function, $\nabla_A = d + [A, \cdot]$ and $F_A = dA + \frac{1}{2}A \wedge A$ are (respectively) the covariant derivative and the curvature (at each fixed time) of A . Using quaternionic notation (see appendix A) the YMH equations are a system of nonlinear wave equations for $\Psi = \Phi + \sum_{i=1}^3 \tau^i A_i : \mathbf{R}^{1+3} \rightarrow su(2) \otimes \mathbf{H}$ which take the following form (in temporal gauge):

$$(YMH) \quad \partial_t^2 \Psi + V'(\Psi) = 0.$$

These are supplemented by the following constraint equation (which is preserved by the time evolution):

$$(C) \quad \nabla_A \cdot \dot{A} + [\Phi, \dot{\Phi}] = 0$$

where $[\cdot, \cdot]$ is the commutator and $\dot{\cdot} \equiv \frac{d}{dt}$.

The aim of this paper is twofold. Firstly, in theorems 1.2 and 1.3, to prove the uniform stability of monopoles in the dynamical system (YMH, C) and secondly, in theorems 2.1 and 3.4 to prove the existence of projections onto the space of monopoles. The stability issue is dealt with most satisfactorily in theorem 1.3 which is based on projection with respect to Taubes' distance function (see (1.3) and (1.10)). Theorem 1.2 is based upon an L^2 projection which is technically more complicated due to regularity issues. This provides an approach to stability which works for higher regularity solutions whereas theorem 1.3 gives stability in an H_{loc}^1 setting. We pursue the L^2 projection method in detail because of its close relationship with the geodesic approximation of Manton (see [Man82, Stu94]) as is discussed in more detail below. The stability issue for monopoles is difficult because of the absence of a spectral gap in the linearised problem and the presence of gauge symmetry. The reader will find a general discussion of stability for solitary waves in the book [Str89] and an introduction to static and dynamic monopoles in [JT82, AH88].

1.1 Background material and notation

We will consider monopoles having fixed winding number

$$\#(A, \Phi) \equiv \frac{1}{4\pi} \int_{\mathbf{R}^3} \nabla_A \Phi \wedge F_A.$$

It is proved in [Gro84] that this number is an integer under the assumption $V(A, \Phi) < \infty$. Throughout this paper we will fix the winding number to be some positive integer, so that we may write ([Bog76])

$$V(A, \Phi) = \int |*F_A - \nabla_A \Phi|^2 + 8\pi\#.$$

We now consider the space of monopoles, which are pairs $(A, \Phi) \in H_{loc}^1$ with $V(A, \Phi) < +\infty$ and which satisfy the Bogomol'nyi equations:

$$(1.2) \quad *F_A = \nabla_A \Phi \quad (\text{denoted } \mathcal{B}(A, \Phi) = 0).$$

These are precisely the minimisers of V and we will call the minimum value $V_{min} = 8\pi\#$. The space of monopoles is nonempty and contains elements in the space $\cap_{k=1}^{\infty} W^{k,\infty}$ ([JT82]) of which we fix one, $\Psi_* = (a_*, \phi_*)$, throughout this paper. Any dependencies of constants in estimates on either the winding number or Ψ_* will be suppressed throughout this paper. Write ∇_* for the covariant derivative with respect to a_* . Generally we will consider functions taking values in $su(2) \otimes \mathbf{R}^4$, where $\mathbf{R}^4$ is identified with the space of quaternions $\mathbf{H}$. Additional definitions and conventions are given in appendix A.

For $k = 1, 2, \dots$ define the spaces $\mathcal{A}_k = \Psi_* + H^k$ where H^k is the standard Sobolev space on $\mathbf{R}^3$ with norm $\left(\sum_{|\alpha|=0}^k \|\partial^\alpha \psi\|_{L^2}^2 \right)^{\frac{1}{2}}$. These are affine spaces modelled on the Hilbert spaces H^k . The index k will be omitted if $k=1$. These will be referred to as the energy spaces. The winding number $\#(A, \Phi)$ is a continuous integer-valued function on $\mathcal{A}_k$ and hence constant.

Following Taubes define for $P = (a, \phi) \in \mathcal{A}$ the completion, H_P , of all $\psi \in C_0^\infty(\mathbf{R}^3; su(2) \otimes \mathbf{H})$ with respect to the norm

$$(1.3) \quad \|\psi\|_P^2 = \|\nabla_a \psi\|_{L^2}^2 + \|[\phi, \psi]\|_{L^2}^2.$$

We also use higher derivative norms $\|\cdot\|_{k,P}$ defined, for $P \in \mathcal{A}_{k-1}$, $k = 2, 3, \dots$ by

$$(1.4) \quad \|\psi\|_{k,P}^2 \equiv \sum_{|\beta|=2}^k \|\nabla_a^\beta \psi\|_{L^2}^2 + \|\psi\|_P^2.$$

with corresponding completions $H_{k,P}$. For the case $P = \Psi_*$ write these norms as $\|\cdot\|_{k,*}$ etc. It is proved in section A.3 that for all $P \in \mathcal{A}_{k-1}$ the norms $\|\cdot\|_{s,P}$ are equivalent on C_0^∞ for $s = 1, \dots, k$, and hence we shall write $H_{s,*}$ for all these spaces.

The gauge group $\mathcal{G}$ is the completion, with respect to the norm $\|\cdot\|_{2,*}$, of the space of smooth functions $h : \mathbf{R}^3 \rightarrow su(2)$ such that $h - 1$ is compactly supported. The action of $g \in \mathcal{G}$ on pairs $(A, \Phi) \in \mathcal{A}$ is given by

$$(1.5) \quad g \cdot (A, \Phi) = (g^{-1}dg + g^{-1}Ag, g^{-1}\Phi g).$$

Its Lie algebra $\text{Lie } \mathcal{G}$ is the completion of the space of compactly supported smooth functions $h : \mathbf{R}^3 \rightarrow su(2)$ with respect to the norm $\|\cdot\|_{2,*}$. We will use notation $\mathcal{G}_k$, $\text{Lie } \mathcal{G}_k$ for the corresponding spaces defined with the norm $\|\cdot\|_{k,*}$, with the index k omitted if $k = 2$.

The spaces of minimisers, or equilibria, are given by

$$\mathcal{S} \equiv \{(A, \Phi) \in \mathcal{A} : *F_A = \nabla_A \Phi\} \quad \mathcal{S}_k \equiv \mathcal{S} \cap \mathcal{A}_k.$$

These will be referred to as solution spaces. Let $\mathcal{S}_k$ be given the topology induced from the H^k norm on $\mathcal{A}_k$. These are infinite dimensional spaces. After dividing out by gauge invariance one obtains finite dimensional manifolds (called moduli spaces) whose dimension is $4\# - 1$, where $\#$ is the winding number. These spaces are fairly well understood (see [AH88] and references therein.)

To state the local existence theorem from [BO86], which is based on [GV81, EM82], think of (YMH, C) as a system of equations for the variable $\mathcal{U} : \mathbf{R}^{1+3} \rightarrow (su(2))^{14}$ where

$$\mathcal{U} \equiv (\Psi - \Psi_*, F_A - F_{a_*}, \nabla_A \Phi - \nabla_* \phi_*, \partial_t \Psi).$$

Theorem 1.1 (Existence theorem) *Assume that the initial data satisfy $\mathcal{U}(0) \in H^k$ for $k = 2, 3, \dots$ then there exists a global unique solution to (YMH) , also satisfying (C) , with*

$$\mathcal{U} \in \cap_{r=0}^k C^r((-\infty, +\infty); H^{k-r}).$$

There is also a version of the existence theory which provides a global unique solution under the assumption of finite energy only ([KM95, Kee95]).

1.2 Discussion and statement of results

The dynamical role of monopoles. We can now seek to understand the significance of monopoles in the time-dependent equations (YMH, C). The following questions are some of the most natural to address:

(i) Given a solution $\Psi(t)$ to the Cauchy problem, is there a natural way to associate to this, at each time t , a monopole $P(t)$ which best approximates $\Psi(t)$? In other words, is there a natural projection operator onto the static solution space?

(ii) Is there a norm in which the monopoles are uniformly stable? By uniformly stable we mean that the solution remains uniformly close to the space of monopoles for all time. Here we remark that one can only expect a **modulational** stability statement since the spaces of minima of V are highly degenerate as remarked above.

(iii) Is there a simple description of the dynamics of monopoles? An answer to this was given by the geodesic approximation of Manton ([Man82]), which was subsequently justified mathematically in [Stu94]. This states that for appropriate initial data, in a slow motion (adiabatic) limit, the infinite dimensional dynamical system (YMH) projects (approximately) to geodesics on the moduli space with respect to the metric induced from L^2 .

(iv) Are monopoles asymptotically stable, i.e., do small perturbations of the monopoles vanish asymptotically in some norm as $t \rightarrow \infty$? A related problem is to extend (at least under some circumstances) the validity of the geodesic approximation of (iii) to infinite times. These problems are completely open at present.

In this paper we make progress in understanding questions (i) and (ii). In particular theorems 2.1 and 3.4 provide a (partial) answer to question (i) by proving that with respect to two natural distance functions, namely L^2 and that given by the norm 1.3, there exists a closest monopole to $\Psi(t)$. Our two main results, theorems 1.2 and 1.3, assert both that monopoles are uniformly stable with respect to the norm (1.3) of Taubes ([Tau83]) and the existence and optimal regularity of closest monopoles at each time. These two results are complementary in that the first holds for $\Psi(T)$ of fairly high Sobolev regularity while the second holds for H^1 regularity.

Theorem 1.2 (Uniform stability I) *Let $\Psi(t)$ be a solution to the Cauchy problem for (YMH) satisfying (C), with regularity as described in theorem 1.1 with $k \geq 4$. Assume in addition that at time $t = 0$ there exists $P(0) \in \mathcal{S}_k$ such that*

$$(1.6) \quad \|\partial_t \Psi(0)\|_{L^2} < \epsilon, \text{ and}$$

$$(1.7) \quad \|\Psi(0) - P(0)\|_{P(0)} < \epsilon.$$

Then there exists ϵ_ such that if $\epsilon < \epsilon_*$ there exists a $a > 0$, depending only upon ϵ_* , and for all $t > 0$ there exists $P(t) \in \mathcal{S}_k$ satisfying*

$$(1.8) \quad \|\Psi(t) - P(t)\|_{L^2}^2 = \inf_{Q \in \mathcal{S}_k} \|\Psi(t) - Q\|_{L^2}^2, \text{ and}$$

$$(1.9) \quad \|\Psi(t) - P(t)\|_{P(t)} < a\epsilon.$$

If we introduce the distance function

$$d(t) \equiv \|\Psi(t) - P(t)\|_{P(t)}$$

then the previous theorem states, in the case $\partial_t \Psi(0) = 0$, that if $d(0)$ is sufficiently small then $d(t) < ad(0)$. The approach to stability via (1.8) is very natural for the following reason: the Euler-Lagrange equation for minimisation in (1.8) implies that $\Psi(t) - P(t)$ is L^2 -orthogonal to the tangent space to $\mathcal{S}_k$, which is described in lemma 2.9. These are precisely the conditions used in [Stu94] in proving the validity of Manton's geodesic prescription for monopole dynamics. Thus the present approach to stability has the attractive property of picking out the physical dynamics of the monopoles. There are, however, other senses in which it is less satisfactory:

(i) Since the stability statement is for the distance function $d(t)$ it might be regarded as more natural to pick out the closest monopole with respect to this distance function, and then use this as a starting point to prove uniform stability. This is done below (see (1.10)).

(ii) There is a strong regularity restriction in theorem 1.2 (the condition $k \geq 4$). This is related to the fact that in theorem 2.1 it is non-trivial to prove that the closest monopole has the expected optimal regularity as it hinges upon the regularity of certain minimising gauge transformations which in turn depends on the Schoen-Uhlenbeck regularity theory for harmonic maps into spheres. Ideally one would like to consider the Yang-Mills equations under conditions of finite energy only, and indeed the equations are known to be well-posed in H^1 norm ([KM95, Kee95]).

Thus in section 3.2 we give an alternative approach to the problem using a method suggested by (i). Thus it is based upon defining the closest monopole $\tilde{P}(t)$ to be the minimiser of

$$(1.10) \quad \|\Psi(t) - \tilde{P}(t)\|_{\tilde{P}(t)}^2 = \inf_{Q \in \mathcal{S}} \|\Psi(t) - Q\|_Q^2.$$

This has the disadvantage that (at present) it is only proved that such a minimiser exists under a smallness assumption (see theorem 3.4). However it allows us to prove a uniform stability theorem for solutions of finite energy only. Thus consider solutions of (YMH, C) with regularity:

$$(1.11) \quad \Psi - \Psi_* \in C([0, \infty); H^1) \cap C^1((0, \infty); L^2).$$

We consider the distance function

$$\tilde{d}(t) = \|\Psi(t) - \tilde{P}(t)\|_{\tilde{P}(t)}$$

and prove stability with respect to this distance. Notice that if both d and $\tilde{d}$ are defined we have by definition $\tilde{d}(t) \leq d(t)$, but the advantage of using $\tilde{d}$ is that it still leads to a proof of stability for solutions satisfying only (1.11).

Theorem 1.3 (Uniform Stability II) *Let $\Psi(t)$ be a solution of (YMH, C) satisfying (1.6, 1.11) such that at $t = 0$ there exists a minimiser $\tilde{P}(0)$ for (1.10) with $d(0) < \epsilon$. Then there exists $\tilde{\epsilon}_*$ such that for $\epsilon < \tilde{\epsilon}_*$:*

- (i) *there exists at each time $t > 0$ a minimiser $\tilde{P}(t) \in \mathcal{S}$ for (1.10),*
- (ii) *$\tilde{d}(t)$ is a continuous function of time,*
- (iii) *there exists $b > 0$ such that for all $t > 0$*

$$\tilde{d}(t) < b\epsilon.$$

We should emphasize that these two theorems do not provide a completely satisfactory picture of monopoles in the system (YMH, C) . In particular one would like to know that the curves $t \rightarrow P(t), \tilde{P}(t)$ are C^1 curves for all times. Although we prove our stability theorems without knowing this to be true this is somewhat unsatisfactory from the phenomenological point of view, according to which one would like to view the dynamical system (YMH, C) as describing a system of interacting monopoles evolving smoothly in time. However, one can make assertions about the regularity of the projected monopole for finite times as we now explain. Since the space of monopoles is a smooth submanifold it follows from the tubular neighbourhood theorem that, as long as $\tilde{d}(t)$ is sufficiently small, $\tilde{P}(t)$ is unique and will be a continuous function of time. However, it is not clear that there is a fixed number δ such that if $\tilde{d}(0) < \delta$ this is so for all time. (We have to expect that this number may depend upon the region of $\mathcal{S}$ to which we are close, and since $\mathcal{S}$ is not compact it is possible that no such uniform lower bound exists). This is the reason that we have given a proof of stability which does not depend upon the smoothness of the projected point in time. However by choosing $\tilde{d}(0)$ sufficiently small we can ensure that, for an arbitrarily large time, $t \rightarrow \tilde{P}(t)$ is a C^1 curve. It is conjectured that it is a C^1 curve for all time. Similar remarks apply for $P(t)$, although in that case since we are using L^2 projection the standard tubular neighbourhood theorem does not apply. An analysis of $P(t)$ in a similar setting is given in [DS96].

This paper is arranged as follows. In section 2 we prove existence and regularity of L^2 projections onto $\mathcal{S}_k$. The main result in that section is theorem 2.1, whose proof depends essentially upon lemma 2.3, which constructs global generalised Coulomb gauges, and lemma 2.10 which is a regularity theorem for monopoles satisfying the generalised Coulomb gauge condition. In section 3 we prove our main results on uniform stability stated above. In the appendices we have collected together various analytical facts from the papers [Tau82, Tau83, Stu94] which are needed, as well as proofs of some technical results of a standard type used in the paper.

2 L^2 Projection onto the space of monopoles

In this section we will prove the following result

Theorem 2.1 *For all $\Psi \in \mathcal{A}_k$, $k = 4, 5, \dots$, there exists $P = P(\Psi) \in \mathcal{S}_k$ such that*

$$\|\Psi - P\|_{L^2} = \inf_{Q \in \mathcal{S}_k} \|\Psi - Q\|_{L^2}.$$

The proof of this requires some subsidiary results which we give in the following sections. The most important of these are lemmas 2.3 and 2.10.

2.1 Gauge Invariance

In this section we present some results about how to treat gauge invariance. The first of these is a Helmholtz or Hodge type decomposition which explains how to divide out by gauge invariance at the infinitesimal level.

Lemma 2.2 *For $\Psi = (A, \Phi) \in \mathcal{A}_k$ there is an L^2 -orthogonal decomposition:*

$$T_\Psi \mathcal{A}_k = H^k = T(\mathcal{G}_{k+1} \cdot \Psi) \oplus \mathcal{SL}_{k,\Psi} \quad T(\mathcal{G}_k \cdot \Psi) \cap \mathcal{SL}_{k,\Psi} = \{0\}$$

where

$$(2.1) \quad T(\mathcal{G}_{k+1} \cdot \Psi) \equiv \{(\nabla_A h, [\Phi, h]) : h \in \text{Lie } \mathcal{G}_{k+1}\}$$

$$(2.2) \quad \mathcal{SL}_{k,\Psi} \equiv \{(\beta, \eta) \in H_k : \nabla_A \cdot \beta + [\Phi, \eta] = 0\}.$$

The corresponding projection operator

$$\Lambda_{k,\Psi} : H^k \rightarrow \mathcal{SL}_{k,\Psi}$$

is a continuous linear operator. The subindex k will often be omitted from $\mathcal{SL}_{k,\Psi}$ etc.

Proof. See appendix C. □

The following notation will be useful:

$$(2.3) \quad \mathbf{d}_\Psi h = \sum_{i=1}^3 \tau^i (\nabla_A)_i h + [\Phi, h] \quad \mathbf{d}_\Psi : \text{Lie } \mathcal{G}_{k+1} \rightarrow H^k$$

$$(2.4) \quad \delta_\Psi(\psi_0 + \sum_{i=1}^3 \tau^i \psi_i) = \sum_{i=1}^3 (\nabla_A)_i \psi_i + [\Phi, \psi_0] \quad \delta_\Psi : H_{k+1,*} \rightarrow H^k(\mathbb{R}^3; su(2)).$$

The space $T(\mathcal{G}_{k+1} \cdot \Psi)$ is the tangent space to the orbit of the gauge group through Ψ , and will be referred to as the space of **infinitesimal gauge transformations**. The condition that (β, η) lie on $\mathcal{SL}_\Psi$ is to be thought of as a generalised Coulomb condition. Application of this condition allows the construction of a slice for the group action; this is done in lemma C.1. The next lemma shows that this gauge condition can be applied even globally through minimisation. It relies on the regularity theory of [SU82, SU84].

Lemma 2.3 Given $P', \Psi \in \mathcal{A}_k, k \geq 4$ there exists $g \in \mathcal{G}_{k+1}$ such that if $P = g \cdot P'$

$$(2.5) \quad \|\Psi - P\|_{L^2} = \|\Psi - g \cdot P'\|_{L^2} = \inf_{h \in \mathcal{G}_{k+1}} \|\Psi - h \cdot P'\|_{L^2}$$

and $(\Psi - P) \in \mathcal{SL}_{k,\Psi}$. Also g satisfies an estimate of the form

$$\|(g - 1)\|_{k+1,*} \leq N \quad N = N(k, \|P' - \Psi_*\|_{H^k}, \|\Psi - P'\|_{H^k}).$$

We will write

$$\mathcal{SL}_{k,\Psi}^0 \equiv \{P \in \mathcal{SL}_{k,\Psi} : \|\Psi - P\|_{L^2} = \inf_{h \in \mathcal{G}_{k+1}} \|\Psi - h \cdot P\|_{L^2}\}.$$

Proof. Step one. In this proof we consider $SU(2)$ as the unit sphere $S^3 \subset \mathbf{R}^4$, so that the function g may be regarded as $\mathbf{R}^4$ valued. We will write $a : b$ for the standard inner product in $\mathbf{R}^4$ and as before $\Psi = (A, \Phi)$ and $P' = (a, \phi)$. We wish to minimise the function

$$I(g) = \|A - g^{-1}dg - g^{-1}ag\|_{L^2}^2 + \|\Phi - g^{-1}\phi g\|_{L^2}^2$$

over $g \in \mathcal{G}_{k+1}$.

Step two. We first notice that I is coercive on $\mathcal{G}_1$ since

$$(2.6) \quad \begin{aligned} I(g) &= \|\nabla_* g - a_* g + g a_* - g A + a g\|_{L^2}^2 + \|g \phi_* - \phi_* g - g(\phi_* - \Phi) + (\phi_* - \phi)g\|_{L^2}^2 \\ &= \|\nabla_* g + (a - a_*)g + g(a_* - A)\|_{L^2}^2 + \|[g, \phi_*] - g(\phi_* - \Phi) + (\phi_* - \phi)g\|_{L^2}^2 \\ &\geq \frac{1}{2}(\|\nabla_* g\|_{L^2}^2 + \|[\phi_*, g]\|_{L^2}^2) - c(\|\Psi - \Psi_*\|_{L^2}^2 + \|P' - \Psi\|_{L^2}^2). \end{aligned}$$

From this we obtain (by taking $g \equiv 1$ as a comparison map) the following estimate

$$(2.7) \quad \|\nabla_* g\|_{L^2} \leq c(\|\Psi - \Psi_*\|_{L^2}^2 + \|P' - \Psi\|_{L^2}^2).$$

Now let g_i be a minimising sequence then since $\mathcal{G}_1 \subset H^1(\mathbf{R}^3; \mathbf{R}^4)$ is a weakly closed subset we find there exists $g \in \mathcal{G}_1$ such that, after taking and relabelling a subsequence,

$$g_i \rightharpoonup g$$

weakly in the topology of $\|\cdot\|_*$ and strongly in $L^2(K)$ for compact K . But

$$I(g) \leq \liminf_{i \rightarrow \infty} I(g_i)$$

so that I attains its minimum at $g \in \mathcal{G}_1$. It remains to prove that in fact $g \in \mathcal{G}_{k+1}$.

Step three. We now apply the regularity results of [SU82, SU84] to obtain bounds for $g \in \mathcal{G}_{k+1}$. The bounds themselves, in particular (2.9), are derived in section 2.2. The functional $I(g)$ can be written in the form

$$(2.8) \quad I(g) = \int |dg|^2 + \sum_{i=1}^3 \gamma_i(x, g) \partial_i g + \Gamma(x, g)$$

with the constraint that $g(x) \in SU(2) = S^3$. Notice that γ, Γ are at most quadratic in the components of Ψ, P' . If we assume that Ψ, P are in $\mathcal{A}_k$ with $k = 4, 5, \dots$ then γ, Γ are C^2 and the functional is in the form required to apply the regularity theory of [SU82, SU84]. This tells us that the minimiser will be locally Hölder continuous. From this one can deduce regularity by Schauder theory ([Gia93]). To be precise it is proved in theorem 2.4 that there exists a number L such that

$$(2.9) \quad \|\nabla_* g\|_{L^\infty} \leq L \text{ where } L = L(\|\Psi\|_{W^{1,\infty}}, \|P'\|_{W^{1,\infty}}, \|\Psi - P'\|_{L^2}, \|P' - \Psi_*\|_{L^2}).$$

Given this we now show that $g \in \mathcal{G}_{k+1}$. The Euler-Lagrange equation for the minimiser is the generalised Coulomb gauge condition

$$-\nabla_A \cdot (a_g - A) - [\Phi, (\phi_g - \Phi)] = 0.$$

This can be written out fully as

$$-\Delta_* g = |\nabla_* g|^2 g + g \nabla_* \cdot (g^{-1}(a - a_*)g - (A - a_*)) + g[A - a_*, g^{-1} \nabla_* g + g^{-1}(a - a_*)g] + g[\Phi, g^{-1} \phi g - \Phi].$$

From this we deduce that $\Delta_* g \in L^2$, and hence by (A.6) we get $\nabla_* \nabla_* g \in L^2$. The process can be continued: since $\nabla_* g \in L^\infty$ one obtains from Moser's Sobolev chain and product rules ([Maj84]) the implication

$$g \in \mathcal{G}_s \cap W^{1,\infty} \text{ for } s = 1, \dots, k \Rightarrow |\nabla_* g|^2 g \in H_{s-1,*}.$$

This in turn implies by lemma B.2 that $\Delta_* g \in H_{k-1,*}$ and so $g \in \mathcal{G}_{k+1}$, with norm depending only on the stated quantities, which was to be proved. $\square$

2.2 Regularity

In this section we prove theorem 2.4 which implies the validity of the assertion (2.9). We use the same notation as in the proof of lemma 2.3 in addition to some definitions given in section A.2. The local Hölder continuity of g follows from theorem IV of [SU82] and the main result of [SU84] while theorem 2.4 provides estimates for the L^∞ bounds of ∇g .

Theorem 2.4 *Let $\Psi = (A, \Phi)$, $P' = (a, \phi)$ lie in $\mathcal{A}_k$ for $k \geq 4$. Assume that g is a continuous weak solution of the equation*

$$(2.10) \quad \nabla \cdot (g^{-1} \nabla g + g^{-1} a g - A) + [A, g^{-1} \nabla g + g^{-1} a g] + [\Phi, g^{-1} \phi g] = 0.$$

and also a minimiser of $I(g)$ in (2.8), i.e. we assume that if Ω is an open set with smooth boundary and $g - h \in H_0^1(\Omega)$ then $I(g) \leq I(h)$. There exists a constant L , with dependency

$$L = L(\|\Psi\|_{W^{1,\infty}}, \|P'\|_{W^{1,\infty}}, \|\Psi - P'\|_{L^2}, \|P' - \Psi_*\|_{L^2})$$

such that

$$\|\nabla g\|_{L^\infty} \leq L.$$

This is proved below, after some lemmas. First of all recall the following ϵ -regularity theorem from [SU82].

Lemma 2.5 *Let g be as in theorem 2.4. For any $B > 0$ there exist $L_2 > 0$, $\epsilon_2 = \epsilon_2(B)$ and $r_2 = r_2(B, \|\Psi\|_{L^\infty}, \|P'\|_{L^\infty})$ such that if for $r \leq r_2$ there exists g_* such that*

$$\frac{1}{r} \int_{B(x,r)} |\nabla g|^2 \leq B \text{ and } \int_{B(x,r)} |g - g_*|^2 \leq r^3 \epsilon_2$$

then

$$|g(y) - g(z)| \leq \frac{L_2}{r^\alpha} |y - z|^\alpha \quad \forall y, z \in B(x, \frac{r}{2}).$$

Proof of lemma 2.5. This is obtained from 4.5 of [SU82] by rescaling. Rescale to a ball of radius one: let $y = \frac{x}{r}$ and $w(y) = g(ry)$. Then

$$\begin{aligned} \int_{B(y,1)} |\nabla w|^2 d^3 y &= (r)^{-1} \int_{B(x,r)} |\nabla g|^2 d^3 x \leq B \\ \int_{B(y,1)} |w - g_*|^2 d^3 y &= (r)^{-3} \int_{B(x,r)} |g - g_*|^2 d^3 x \leq \epsilon_2. \end{aligned}$$

The function w will now be a minimiser for the functional

$$I'(g) = \int |dg|^2 + r \sum_{i=1}^3 \gamma_i(x, g) \partial_i g + r^2 \Gamma(x, g).$$

By choosing r_2, ϵ_2 small enough we may ensure that the quantities

$$\int_{B(y,1)} |w - g_*|^2 d^3 y$$

and

$$r|\gamma(x, g)| + r|\partial_g \gamma(x, g)| + r|\Gamma(x, g)|^{\frac{1}{2}} + r|\partial_g \Gamma(x, g)|^{\frac{1}{2}}$$

are as small as necessary to apply the quoted theorems in [SU82] to deduce the result. $\square$

We now work under the assumption:

$$(2.11) \quad \max\{\|P'\|_{L^\infty}, \|\Psi\|_{L^\infty}, \sup_{p \in \mathbb{R}^3} \|\nabla g\|_{L^2(B(p,1))}\} \leq M_1.$$

Notice that by (2.7) we have

$$\sup_{p \in \mathbb{R}^3} \|\nabla g\|_{L^2(B(p,1))} \leq c.(\|\Psi - \Psi_*\|_{L^2}^2 + \|P' - \Psi\|_{L^2}^2).$$

Lemma 2.5 is useful on account of the monotonicity formula ([SU82, proposition 2.4]) which implies that under the assumption (2.11) there exists $B = B(M_1)$ such that

$$(2.12) \quad \sup_{p \in \mathbb{R}^3} \sup_{\sigma \leq 1} \sigma^{-1} \int_{B(p,\sigma)} |\nabla g|^2 \leq B.$$

Lemma 2.6 *Assume that $\max\{\|P'\|_{L^\infty}, \|\Psi\|_{L^\infty}, \|\nabla_* g\|_{L^2}\} \leq M_1$ then there exists $\Lambda = \Lambda(M_1)$ such that for all $p \in \mathbb{R}^3$ there exists $r_p > \Lambda$ and a constant g_* such that*

$$\int_{B(p,r_p)} |g - g_*|^2 < r_p^3 \epsilon_2$$

where ϵ_2 is as in lemma 2.5.

Proof. We will assume the contrary and derive a contradiction. Thus consider a sequence $\{\Psi_\mu, P_\mu\}_{\mu=1}^\infty$ in $\mathcal{A}_k \times \mathcal{S}_k$, with corresponding minimisers g_μ , obeying (2.11). If the result is false we may assume, possibly after applying translations, that the minimisers g_μ satisfy (2.11) and the following:

for a sequence of positive numbers r_μ such that $\lim_{\mu \rightarrow \infty} |r_\mu| = 0$ we have $\int_{B(r)} |g - \langle g \rangle_r|^2 \geq r^3 \epsilon_2$, for all $r \geq r_\mu$.

We now derive a contradiction. First by (2.12) and [SU82, proposition 4.6] we can extract a subsequence which converges weakly in $H^1(B(1))$ and strongly in $H^1(B(\frac{1}{2}))$ to a limit g_0 which is Hölder continuous outside a closed set of zero one dimensional Hausdorff measure. Furthermore the convergence is uniform away from this closed singular set. We claim that g_0 is Hölder continuous. To see this assume to the contrary that p is a singular point for g_0 . Then by [SU82, propositions 4.7] there exists a sequence of real numbers $\lambda(i) \rightarrow 0$ such that $g_0(\lambda(i)(x - p))$ converge in $H^1(B(1))$ to a harmonic map of the form $f(x/|x|)$. Furthermore the singular set of this map will have zero one dimensional Hausdorff measure, therefore since it is of the given form it can have at most a point singularity at the origin; it is thus what is known as a tangent map. Therefore by [SU82, proposition 5.2] it is a minimising tangent map and so constant by [SU84, proposition 1.2]. Therefore p is a regular point and so g_0 is indeed Hölder continuous on $B(\frac{1}{2})$, and the g_μ converge uniformly to it. Therefore by the Hölder continuity of g_0 (see (2.18))

$$\lim_{r \rightarrow 0} r^{-3} \int_{B(r)} |g_0(x) - \langle g_0 \rangle_r|^2 dx = 0.$$

Now let R be such that $r < R$ implies

$$\int_{B(r)} |g_0(x) - \langle g_0 \rangle_r|^2 dx < r^3 \epsilon_2,$$

then choosing μ sufficiently large we find

$$\int_{B(r)} |g_\mu(x) - \langle g_\mu \rangle_r|^2 \leq \int_{B(r)} |g_\mu(x) - \langle g_0 \rangle_r|^2 dx < r^3 \epsilon_2$$

which gives a contradiction to the assumption above since $\lim_{\mu \rightarrow \infty} (r - r_\mu) > 0$. $\square$

Corollary 2.7 *There exists a positive number $\rho = \rho(M_1)$ such that on any ball $B(p, \rho)$ g is Hölder continuous and satisfies*

$$|g(y) - g(z)| \leq \frac{L_2}{\rho^\alpha} |y - z|^\alpha,$$

where L_2, α are as in lemma 2.5.

We now prove that Hölder continuity of g implies that of ∇g . Although this fact is well known, we provide a proof as we need dependencies for the estimates which are hard to find in the literature. Thus we consider a small ball of radius $R \leq r_3$ (where $r_3 \leq \rho$ is to be determined) about an arbitrary point, which we take to be the origin without loss of generality. Inside this ball g is taken to be a Hölder continuous weak solution. The notation and background needed for the following proof is given in section A.2.

Lemma 2.8 *Assume that $g : \overline{B(R)} \rightarrow S^3$ is a continuous function satisfying the following two conditions: (i) for all $\phi \in C(\overline{B(R)}) \cap H_0^1(B(R))$ we have*

$$\sum_{i=1}^3 \int_{B(R)} \nabla_i \phi : (g^{-1} \nabla_i g) + \alpha_i \nabla_i g : \phi + F_i : \nabla_i \phi = 0,$$

$$(ii) [g]_{C^{0,\alpha}(\overline{B(R)})} + [F]_{C^{0,\alpha}(\overline{B(R)})} + \|\alpha\|_{L^\infty(B(R))} \leq M_2.$$

Then for $\beta \in (0, \frac{\alpha}{2})$ there exists $r_3 = r_3(\beta, M_2)$, and $L_3 = L_3(\beta, M_2)$ such that if $R \leq r_3$

$$(2.13) \quad y, z \in B(x, R) \quad \Rightarrow \quad |\nabla g(y) - \nabla g(z)| \leq L_3 |y - z|^\beta.$$

Proof. *Step one.* We may assume that $x = 0$ and $g(0) = 1$ without loss of generality. It is convenient to rewrite the weak form of the equation as follows: let $p \in B(R)$ and $r > 0$ be such that $B(p, r) \subset B(R)$ then for any $\phi \in C(\overline{B(p, r)}) \cap H_0^1(B(p, r))$

$$(2.14) \quad \int_{B(p, r)} \sum_{i=1}^3 \nabla_i g : \nabla_i \phi + \alpha_i \nabla_i g : \phi = - \int_{B(p, r)} \sum_{i=1}^3 \{ \nabla_i \phi : ((g^{-1} - 1) \nabla_i g) + \nabla_i \phi : F \}.$$

Let v be the solution of $\Delta v = 0$, $v = g$ on $\partial B(p, r)$. Then for all $\phi \in C(\overline{B(p, r)}) \cap H_0^1(B(p, r))$ we have

$$(2.15) \quad \int_{B(p, r)} \sum_{i=1}^3 \nabla_i v : \nabla_i \phi = 0.$$

We need the following facts about v :

- (i) v is Hölder continuous of exponent α with Hölder constant determined by that of g . Therefore since it agrees with g on $\partial B(p, r)$ we have, on $B(p, r)$, $|v - g| \leq cr^\alpha$, where $c = c(M)$.
- (ii) $\int_{B(p, r)} |\nabla v|^2 \leq \int_{B(p, r)} |\nabla g|^2$ by the minimising characterisation of the Dirichlet problem.

(iii) If $s < r$ we have, for some universal constant c ([Gia93, theorem 3.2]),

$$(2.16) \quad \int_{B(p,s)} |v|^2 \leq c \left(\frac{s}{r}\right)^3 \int_{B(p,r)} |v|^2$$

$$(2.17) \quad \int_{B(p,s)} |v - \langle v \rangle_s|^2 \leq c \left(\frac{s}{r}\right)^5 \int_{B(p,r)} |v - \langle v \rangle_r|^2$$

using the notation in section A.2. *Step two. Claim* There exists a constant c such that

$$\begin{aligned} \int_{B(p,r)} |\nabla(g-v)|^2 dx &\leq c R^{3/2} |\alpha|_{L^\infty} |g-v|_{L^\infty(B(p,r))} \left(\int_{B(p,r)} |\nabla g|^2 dx \right)^{\frac{1}{2}} \\ &\quad + c |g^{-1} - 1|_{L^\infty(B(p,r))}^2 \int_{B(p,r)} |\nabla g|^2 dx + c \int_{B(p,r)} |F - \langle F \rangle_R|^2 dx. \end{aligned}$$

This is proved by subtracting (2.15) from (2.14), substituting $\phi = g - v$, and then applying the Hölder and Cauchy-Schwarz inequalities.

Step three. As in [Gia93, p. 48] we have, using step two and (2.16)

$$\begin{aligned} \int_{B(p,s)} |\nabla g|^2 &\leq c \left(\frac{s}{r}\right)^3 \int_{B(p,r)} |\nabla v|^2 + \int_{B(p,r)} |\nabla(g-v)|^2 \\ &\leq c \left(\frac{s}{r}\right)^3 \int_{B(p,r)} |\nabla g|^2 + c R^3 + c R^{2\alpha} \int_{B(p,r)} |\nabla g|^2 + c \int_{B(p,r)} |F - \langle F \rangle_R|^2. \end{aligned}$$

The Hölder continuity of F implies that (in 3 space dimensions)

$$(2.18) \quad \int_{B(p,r)} |F - \langle F \rangle_r|^p \leq c r^{3+2p} \quad c = c(M_2).$$

Now let $\chi(s) = \int_{B(p,s)} |\nabla g|^2$ then choosing r_3 small enough we can apply lemma A.1 to deduce that if $\delta \in (0, 3)$ there exists $c(\delta, M_2)$ such that

$$\chi(s) \leq c \left(\left(\frac{s}{R}\right)^3 \chi(R) + s^{3-\delta}\right)$$

so that (using the notation in section A.2) $\|\nabla g\|_{3-\delta, B(R)} \leq c$ where $c = c(\delta, M_2)$.

Step four. We now make a slightly more refined estimate. Let $\delta \in (0, \alpha)$ and calculate, again as in [Gia93], using step two and (2.17):

$$\begin{aligned} \int_{B(p,s)} |\nabla g - \langle \nabla g \rangle_{p,s}|^2 &\leq c \left(\frac{s}{r}\right)^5 \int_{B(p,r)} |\nabla g - \langle \nabla g \rangle_{p,r}|^2 + c \int_{B(p,r)} |\nabla(g-v)|^2 \\ &\leq c \left(\frac{s}{r}\right)^5 \int_{B(p,r)} |\nabla g - \langle \nabla g \rangle_{p,r}|^2 + c r^{3+\alpha-\delta} \\ &\quad + c r^{3+\alpha} + c \int_{B(p,r)} |F - \langle F \rangle_{p,r}|^2 \\ &\leq c \left(\frac{s}{r}\right)^5 \int_{B(p,r)} |\nabla g - \langle \nabla g \rangle_{p,r}|^2 + c r^{3+\alpha-\delta}, \end{aligned}$$

where we used (2.18) in the last step. Again applying lemma A.1 we deduce that (using the notation in section A.2)

$$[\nabla g]_{3+\alpha-\delta, B(R)} \leq c(\alpha, \delta, M_2)$$

which implies (2.13) by (A.1). $\square$

Proof of theorem 2.4. First of all from (2.7) we have an estimate

$$(2.19) \quad \|\nabla_* g\|_{L^2} \leq c (\|\Psi - P'\|_{L^2} + \|P' - \Psi_*\|_{L^2}).$$

This allows us to apply corollary 2.7 to deduce the uniform continuity of g . We next wish to apply lemma 2.8 to deduce the uniform continuity of ∇g . Looking at the equation (2.10) we see that if it is put in the form required by lemma 2.8 we have

$$[F]_{C^0, \alpha \overline{B(R)}} + \|\alpha\|_{L^\infty(B(R))} \leq c.(\|P'\|_{W^{1,\infty}} + \|\Psi\|_{W^{1,\infty}}).$$

Thus we deduce that for any positive ϵ there exists

$$\delta = \delta(\epsilon, \|\Psi\|_{W^{1,\infty}}, \|P'\|_{W^{1,\infty}}, \|\Psi - P'\|_{L^2}, \|P' - \Psi_\star\|_{L^2})$$

such that

$$(2.20) \quad |y - z| < \delta \Rightarrow |g(y) - g(z)| + |\nabla g(y) - \nabla g(z)| < \epsilon.$$

The result of the theorem is implied by (2.19) and (2.20). $\square$

2.3 Solution space and moduli space

In this section we discuss the static solution spaces for the Bogomol'nyi equations (1.2). It is convenient to rewrite (1.2) in the following way; assume that $P = Q + \psi$ where $Q \in \mathcal{S}_k$ and $(P - \Psi) \in \mathcal{SL}_{k,P}$. Write $P = (a, \phi)$, $Q = (a_0, \phi_0)$ and $\Psi = (A, \Phi)$ then the condition that P satisfies (1.2) reduces to

$$(2.21) \quad \mathcal{D}_Q \psi = g(\psi, \psi) + h(\Psi, Q, P)$$

(See [JT82, IV.2] for a similar calculation). Here g is a bilinear function $\mathbf{R}^4 \times \mathbf{R}^4 \rightarrow \mathbf{R}^4$ given in quaternionic notation (see section A) by

$$g(\psi, \psi) = \sum_{i=1}^3 \tau_i \left(\frac{1}{2} \epsilon_{ijk} [\beta_i, \beta_j] - [\beta_i, \eta] \right) \text{ for } \psi = (\beta, \eta) = (a - a_0, \phi - \phi_0).$$

$\mathcal{D}_Q$ is a first order elliptic operator whose principal symbol is the (Euclidean) Dirac operator on $\mathbf{R}^3$ given (in quaternionic notation) by

$$(2.22) \quad \mathcal{D}_Q = -\tau^i (\nabla_{a_0})_i + [\phi_0, \cdot].$$

The three imaginary components of $\mathcal{D}_Q \psi$ form the linearised Bogomol'nyi equations while the real component is $\delta_Q \psi$, defined in (2.3). Finally h is given by

$$(2.23) \quad h = \nabla_{a_0} \cdot (A - a_0) + [\phi_0, \Phi - \phi_0] + [a - a_0, A - a_0] + [\phi - \phi_0, \Phi - \phi_0].$$

Lemma 2.9 (Submanifold) *For $k \geq 1$ the solution space $\mathcal{S}_k$ is a smooth infinite-dimensional submanifold of $\mathcal{A}_k$ with tangent and normal spaces given respectively by*

$$\begin{aligned} T_\Psi \mathcal{S}_k &= T(\mathcal{G}_k \cdot \Psi) \oplus \text{Ker} \mathcal{D}_\Psi \\ N_\Psi \mathcal{S}_k &= \mathcal{SL}_{k,\Psi} \cap (\text{Ker} \mathcal{D}_\Psi)^\perp \end{aligned}$$

where $(\text{Ker} \mathcal{D}_\Psi)^\perp$ is the L^2 orthogonal complement of $\text{Ker} \mathcal{D}_\Psi$, which is well defined by lemma (B.2).

Proof. See appendix C. $\square$

Remark. The quotient space $M = \mathcal{S}_k / \mathcal{G}_{k+1}$ is independent of k and is known as the extended moduli space of monopoles. It is a circle bundle over the regular moduli space (which is defined as the space of smooth solutions of the Bogomol'nyi equations divided by the group of smooth gauge transformations); these spaces are well understood (see [AH88] and references therein). In particular the extended moduli space is a connected smooth 4#-manifold and can be identified with the space of based rational maps ([Don84]). M inherits a natural metric from L^2 . Let dist be the corresponding Riemannian distance. This is given in the following way. Let $\pi(Q), \pi(P)$ be two points in M and let $q : [0, T] \rightarrow M$ be a minimal geodesic joining them. Then a curve

$\tilde{P} : [0, T] \rightarrow \mathcal{S}_k$ is called a horizontal lift of the curve q starting at Q if it satisfies the following conditions:

- (i) $\tilde{P}(0) = Q$,
- (ii) $\frac{d\tilde{P}}{dt} \in \mathcal{SL}_{\tilde{P}}$,
- (iii) $\pi(\tilde{P}(T)) = \pi(P)$.

It is a direct consequence of lemma B.3 that horizontal lifts always exist ([Stu94]). In the case that $\tilde{P}$ is the horizontal lift of a minimal geodesic from $\pi(P)$ to $\pi(Q)$ the distance between these two points is given by

$$(2.24) \quad \text{dist}(\pi(P), \pi(Q)) \equiv \int_0^T \left\| \frac{d\tilde{P}}{dt} \right\|_{L^2} dt.$$

Lemma 2.10 *Given $k \geq 4$, $\Psi \in \mathcal{A}_k$ and $P \in \mathcal{S}_k$ such that $(\Psi - P) \in \mathcal{SL}_{k, \Psi}^0$ there exist positive numbers $c_1, c_2 > 0$ such that*

$$(2.25) \quad \|P - \Psi_*\|_{k, *} \leq c_1(\text{dist}(\pi(\Psi_*), \pi(P))) + c_2$$

Here $c_1 = c_1(k)$ while $c_2 = c_2(k, \|\Psi - \Psi_*\|_{H^k})$. Furthermore if $P, Q \in \mathcal{S}_k$ and $\|P - Q\|_{L^2} \leq M$ then $\text{dist}(\pi(P), \pi(Q)) \leq c = c(M)$.

Proof. *Step one.* Let $\tilde{P} : [0, T] \rightarrow \mathcal{S}_k$ be the horizontal lift of a minimal geodesic joining $\pi(\Psi_*)$ to $\pi(P)$. Write $\tilde{P} = (\tilde{a}, \tilde{\phi})$ and consider $P' = \tilde{P}(T)$. We may write

$$P' - \Psi_* = \int_0^T \frac{d\tilde{P}}{dt} dt.$$

Now at each time we have

$$\mathcal{D}_{\tilde{P}(t)} \frac{d\tilde{P}}{dt} = 0$$

and the L^2 estimates for this equation imply

$$\left\| \frac{d\tilde{P}}{dt} \right\|_{\tilde{P}} \leq c \cdot \left\| \frac{d\tilde{P}}{dt} \right\|_{L^2}.$$

But also $\mathcal{D}_{\tilde{P}}^* \mathcal{D}_{\tilde{P}} \frac{d\tilde{P}}{dt} = 0$ from which we find an estimate for $\|\Delta_{\tilde{a}} \frac{d\tilde{P}}{dt}\|_{L^2}$ and hence

$$\left\| \frac{d\tilde{P}}{dt} \right\|_{L^\infty} \leq c \cdot \left\| \frac{d\tilde{P}}{dt} \right\|_{L^2}.$$

From this we now obtain, by differentiating, the estimate

$$\left\| \frac{d\tilde{P}}{dt} \right\|_{k, *} \leq c \cdot \left\| \frac{d\tilde{P}}{dt} \right\|_{L^2},$$

again with c depending only on k and the winding number. Now reparametrise the curve so that $T = \text{dist}(\pi(\Psi_*), \pi(P))$ and $\left\| \frac{d\tilde{P}}{dt} \right\|_{L^2} = 1$. We then find that

$$(2.26) \quad \|P' - \Psi_*\|_{k, *} \leq c \cdot \text{dist}(\pi(\Psi_*), \pi(P)).$$

Step two. P' will not generally satisfy $(\Psi - P') \in \mathcal{SL}_{k, \Psi}^0$; to obtain the required estimate we must apply lemma 2.3. If $P' = g \cdot P$ then g will satisfy the estimates of lemma 2.3 leading to $\|g - 1\|_{k, *} \leq N$ where $N = N(k, \|P' - \Psi_*\|_{H^k}, \|\Psi - P'\|_{H^k})$. This combines with (2.26) to imply (2.25).

Step three. The final statement follows from [AH88, page 24]. $\square$

2.4 Proof of 2.1

Step one. Fix $Q \in \mathcal{S}_k$. By lemma 2.3 we may restrict ourselves to $P \in \mathcal{S}_k$ such that $\Psi - P \in \mathcal{SL}_{k,\Psi}^0$. Let P_i be a minimising sequence so that $\|P_i - Q\|_{L^2}$ is bounded and so by lemma 2.10

$$\|P_i - Q\|_{H^k} \leq c$$

so that $\psi_i = P_i - Q$ is bounded in H^k , and we obtain a weakly convergent subsequence; call the limit ψ and define $P = \Psi - \psi$. Weak lower semi-continuity ensures that P is minimising as required, and it lies in $\mathcal{A}_k$. Since $\psi \in H^k$ with $k \geq 2$ we have

$$\lim_{|x| \rightarrow \infty} \psi(x) = 0$$

and hence P and Ψ have the same winding number.

Step two. It remains to show that $P(\Psi) \in \mathcal{S}_k$; this follows from the version (2.21) of the Bogomol'nyi equations as follows. For each i

$$\mathcal{D}_Q(P_i - Q) = g(P_i - Q, P_i - Q) + h(\Psi, Q, P_i).$$

The distributional form of this equation is preserved under the weak limit above since by the Rellich theorem all the nonlinear terms (which are quadratic) converge since P_i converge strongly in $L^2(K)$ for compact K . Therefore the weak limit indeed lies in $\mathcal{S}_k$.

3 Uniform Stability

3.1 Proof of theorem 1.2

In this section we prove theorem 1.2. Given $\Psi(t)$ we obtain at each time an L^2 closest point $P(t) \in \mathcal{S}_k$ by application of theorem 2.1. Since we do not generally have uniqueness for $P(t)$ we cannot expect that $P(t)$ is a continuous function of time. Nevertheless the associated distance functions are continuous.

Lemma 3.1 (Continuity) *The functions of time*

$$\begin{aligned} d_1(t) &\equiv \|\Psi(t) - P(t)\|_{L^2} \\ d(t) &\equiv \|\Psi(t) - P(t)\|_{P(t)} \end{aligned}$$

are continuous.

This can be proved more directly; we prefer the proof given here for the reason explained in the remark following the proof. We will require the following compactness result which follows from the fact that the static equations are elliptic (modulo gauge invariance.)

Lemma 3.2 (Compactness) *Let $k \geq 2$ and assume that for $i = 1, 2, \dots$ $P_i \in \mathcal{S}_k, \Psi_i \in \mathcal{A}_k, \Psi \in \mathcal{A}_k$,*

$$P_i - \Psi_i \in \mathcal{SL}_{\Psi_i}^0,$$

and

$$\|P_i - \Psi_i\|_{L^2} \leq c$$

while

$$\lim_{i \rightarrow \infty} \|\Psi_i - \Psi\|_{H^k} = 0.$$

Then there exists a subsequence, which we relabel i , and a point $P \in \mathcal{S}_k$ such that

$$P_i - P \rightharpoonup 0$$

weakly in H^k .

Proof. First of all recall from lemma 2.10 that if $P_i - \Psi_*$ is bounded in L^2 then $\text{dist}(P_i, \Psi_*)$ is bounded. Now apply lemma 2.10 to deduce that $\|P_i - \Psi_*\|_{H^k}$ is bounded. By the theorem of Banach-Alaoglu $P_i - \Psi_*$ has a weak limit, ψ say. Now define $P = \Psi_* + \psi$ so that $P_i - P \rightarrow 0$ in H^k . The same remarks about winding number apply as in the proof of theorem 2.1. $\square$

Proof of lemma 3.1 It follows from the existence theorem that both of these are finite at each time. Consider $d_1(t)$ first. General results ([Die76, Section XII.7]) imply it is upper semi-continuous, i.e. for any sequence $t_i \rightarrow T$

$$d_1(T) \geq \limsup_{i \rightarrow \infty} d_1(t_i).$$

We assume that d_1 is not continuous at T so that there is a sequence $t_i \rightarrow T$ along which

$$(3.1) \quad \liminf_{i \rightarrow \infty} d_1(t_i) < \limsup_{i \rightarrow \infty} d_1(t_i).$$

Choose a subsequence $\{t_j\} \subset \{t_i\}$ such that

$$\lim_{j \rightarrow \infty} d_1(t_j) = \liminf_{i \rightarrow \infty} d_1(t_i) < d_1(T).$$

But in this circumstance the compactness lemma 3.2 implies that for a further subsequence $\{t_k\} \subset \{t_j\}$ there exists $Q \in \mathcal{S}_k$ such that

$$P(t_k) - Q \rightarrow 0$$

weakly in H^k . By the existence theory we know that $(\Psi(T) - \Psi(t_k)) \rightarrow 0$ strongly in H^k which gives

$$\Psi(t_k) - P(t_k) \rightarrow \Psi(T) - Q$$

and so

$$\begin{aligned} \limsup_{k \rightarrow \infty} d_1(t_k) &\leq \limsup_{i \rightarrow \infty} d_1(t_i) \leq d_1(T) \leq \|\Psi(T) - Q\|_{L^2} \leq \liminf_{k \rightarrow \infty} \|\Psi(t_k) - P(t_k)\|_{L^2} \\ &\leq \lim_{k \rightarrow \infty} d_1(t_k) = \liminf_{i \rightarrow \infty} d_1(t_i) \end{aligned}$$

which contradicts (3.1) and implies the continuity of d_1 immediately. Weak convergence combined with norm convergence implies strong convergence so that we get

$$\lim_{i \rightarrow \infty} \|P(t_i) - Q\|_{L^2} = 0.$$

This also implies that we may take $P(T) = Q$, remembering however that in the absence of a uniqueness statement for $P(T)$ other choices may be possible. If we apply lemma 2.10 we deduce immediately the boundedness of $P(t_i) - P(T)$ in H^k . Now $\psi_i = P(t_i) - P(T)$ satisfies the equation

$$\mathcal{D}_{P(T)} \psi_i = g(\psi_i, \psi_i) + h(\Psi(t_i), P(T), P_i)$$

as in (2.21). Using the fact that $P(T) - \Psi(T) \in \mathcal{S}_{k, \Psi(T)}^0$ we find that

$$\|h\|_{L^2} + \|g\|_{L^2} \leq c.(\|\psi_i\|_{L^2} + \|\Psi(t_i) - \Psi(T)\|_Q).$$

From this the elliptic estimate (see section B) gives $\psi \rightarrow 0$ strongly in H_Q also, from which follows the continuity of $d(t)$. $\square$

Remark. The method of proof shows that *subsequentially* $P(t_i) \rightarrow P(T)$ as $t_i \rightarrow T$. One would really like to know that $t \rightarrow P(t)$ is a C^1 curve.

Theorem 3.3 Assume $\Psi = P + \psi$ with $\Psi \in \mathcal{A}_k, P \in \mathcal{S}_k$ and, using the notation of lemma 2.9,

$$(3.2) \quad \psi \in N_P \mathcal{S}_k.$$

Then there exist positive numbers L_1, δ_1 such that if $\|\psi\|_P < \delta_1$ then

$$V(\Psi) \geq V_{\min} + L_1 \|\psi\|_P^2.$$

Proof. *Step one.* Use notation $P = (a, \phi)$, $\Psi = (A, \Phi)$ Let g be as in 2.21 then by (A.5) we have

$$(3.3) \quad \|g(\psi, \psi)\|_{L^2} \leq \kappa \|\psi\|_P^2$$

where κ depends on the measure of the set on which $|\phi| < \frac{1}{2}$.

Step two. The condition $\psi \in N_P \mathcal{S}_k$ implies that ψ is L^2 orthogonal to $\text{Ker } \mathcal{D}_P$ and that $\delta_P \psi = 0$. Therefore if we use (2.21) then $h = 0$.

Step three. We now apply the Bogomol'nyi formula

$$\begin{aligned} V(\Psi) &= V_{min} + \int |*F_A - \nabla_A \phi|^2 \\ (3.4) \quad &= V_{min} + \int |\mathcal{D}_P \psi + g(\psi, \psi)|^2 \\ &\geq V_{min} + \frac{1}{2} \int |\mathcal{D}_P \psi|^2 - 2 \|g(\psi, \psi)\|_{L^2}^2 \\ &\geq V_{min} + \frac{\gamma^2}{2} \|\psi\|_P^2 - 2\kappa^2 \|\psi\|_P^4 \quad \text{by lemma B.1(ii)} \\ &\geq V_{min} + \frac{1}{2} \|\psi\|_P^2 (\gamma^2 - 4\kappa^2 \|\psi\|_P^2) \\ (3.5) \quad &\geq V_{min} + \frac{1}{4} \gamma^2 \|\psi\|_P^2, \end{aligned}$$

as long as $\delta_1 < \frac{\gamma^2}{8\kappa^2}$. □

Proof of theorem 1.2 The energy at time t

$$\mathcal{E}(\Psi(t)) = \|\dot{\Psi}\|_{L^2}^2 + V(\Psi(t))$$

is independent of time; call this constant value $\mathcal{E}$. Therefore we obtain the estimate

$$V(\Psi(t)) \leq \mathcal{E}$$

Now by theorem 2.1 at each time we may decompose

$$\Psi(t) = P(t) + \psi(t)$$

and the Euler-Lagrange equation for this minimisation implies that $\psi(t)$ satisfies (3.2) relative to $P(t)$. Therefore as long as $d(t) = \|\psi\|_{P(t)} < \delta_1$ the following estimate holds

$$\|\psi\|_{P(t)}^2 \leq \frac{\mathcal{E} - V_{min}}{L}.$$

Now (1.6, 3.3-3.4) imply that

$$\mathcal{E} < V_{min} + c.(d(0)^2 + d(0)^4) + \epsilon^2$$

so we may choose initial data such that

$$\frac{\mathcal{E} - V_{min}}{L} < \delta_1^2.$$

The result then follows since $d(t) = \|\psi\|_{P(t)}^2$ is a continuous function of time. □

3.2 Proof of theorem 1.3

In this section we explain the alternative approach to uniform stability discussed in the introduction. Thus we first try to define the closest monopole with respect to the norm $\|\cdot\|_P$ directly. Following this line of thought we will obtain a proof of uniform stability holding for solutions with regularity as in (1.11). The first stage is to define the projected point with respect to the norm (1.3). This requires a smallness condition.

Theorem 3.4 Given $\Psi = (A, \Phi) \in \mathcal{A}$ there exists a positive number δ_2 , depending only upon the Lebesgue measure of the set on which $|\Phi| < \frac{1}{2}$, such that if there exists $Q \in \mathcal{S}$ with

$$\|\Psi - Q\|_Q < \delta_2$$

then there exists $\tilde{P} = \tilde{P}(\Psi) \in \mathcal{S}$ such that

$$(3.6) \quad \|\Psi - \tilde{P}\|_{\tilde{P}}^2 = \inf_{Q \in \mathcal{S}} \|\Psi - Q\|_Q^2.$$

First of all we record the following technical lemma which is a generalisation of lemma 6.6 in [Tau82] (see also (A.5)) and is proved in exactly the same way.

Lemma 3.5 Let $\Psi = (A, \Phi), P = (a, \phi)$ lie in $\mathcal{A}$ and $u, v \in H_*$ then there exists a number c , depending only upon the Lebesgue measure of the set on which either $|\Phi| < \frac{1}{2}$ or $|\phi| < \frac{1}{2}$, such that

$$(3.7) \quad \|[u^i, v^j]\|_{L^2} \leq c.(\|u\|_{\Psi}\|v\|_P + \|u\|_P\|v\|_{\Psi}).$$

Here u^i, v^j are the $su(2)$ -valued components of u, v .

Proof of theorem 3.4. Let $Q = (a_0, \phi_0)$ and $P_\mu = (a_\mu, \phi_\mu)$ be a minimising sequence. In order to obtain bounds on this sequence we need coercivity. This is obtained using the smallness assumption. Indeed notice that if $\psi_\mu = \Psi - P_\mu$ then we may assume that $\|\psi_\mu\|_{P_\mu} < \delta_2$ for all μ and therefore

$$\begin{aligned} \|\Psi - P_\mu\|_{P_\mu}^2 &= \|\nabla_A \psi_\mu + [a_\mu - A, \psi_\mu]\|_{L^2}^2 + \|[\Phi, \psi_\mu] + [\phi_\mu - \Phi, \psi_\mu]\|_{L^2}^2 \\ &\geq \frac{1}{2}\|\psi_\mu\|_{\Psi}^2 - c.\|\psi_\mu\|_{P_\mu}^2 \|\psi_\mu\|_{\Psi}^2 \text{ by (3.7)} \\ &\geq \frac{1}{4}\|\psi_\mu\|_{\Psi}^2 \text{ for small enough } \delta_2. \end{aligned}$$

Since $P_\mu \in \mathcal{S}_k$ the measures of the sets $\{|\phi_\mu| < \frac{1}{2}\}$ are bounded solely in terms of the winding number (see [JT82]) which is fixed throughout this paper. Thus the requisite smallness of delta depends only on the quantities stated. This bound allows us to extract a subsequence converging weakly in the topology of H_Ψ to ψ . Let $P = \Psi - \psi = (a, \phi)$. By Rellich's theorem and a diagonal argument we have a subsequence converging strongly on compact sets, so that we can deduce that the limit P solves the Bogomol'nyi equations. Furthermore since P solves the Bogomol'nyi equations we know that $\lim_{|x| \rightarrow \infty} (1 - |\phi(x)|) = 0$ and by [Gro84, lemma 3] ϕ and ϕ_0 are uniformly continuous. But (A.4) implies $\phi - \phi_0 \in L^6$ so that $\lim_{|x| \rightarrow \infty} (|\phi(x) - \phi_0(x)|) = 0$ and therefore P has the same winding number as Q . Finally, P is indeed a minimiser by weak lower semicontinuity of $\|\Psi - Q\|_Q$ as a function of Q . $\square$

We also need a version of lemma 2.2 adapted to the inner product $(\cdot, \cdot)_P$. For $P \in \mathcal{S}$ this inner product can be written ([Tau83])

$$(f, g)_P = (\mathcal{D}_P^* f, \mathcal{D}_P^* g)_{L^2}.$$

We use notation $M_P = \mathcal{D}_P \mathcal{D}_P^*$. In this setting the appropriate space of infinitesimal gauge transformations is

$$\tilde{T}(\mathcal{G} \cdot P) = \{\mathbf{d}_P h : h \in \Gamma_P\}$$

where

$$\Gamma_P \equiv \{\text{measurable functions } h : \mathbf{R}^3 \rightarrow su(2) \text{ such that } \mathbf{d}_P h \in H_P\},$$

endowed with the norm

$$\|\tilde{h}\|_P \equiv \|\mathbf{d}_P h\|_P.$$

This is a norm since $\|\tilde{h}\|_P = 0$ implies that $\mathbf{d}_P h = 0$ which implies that $|h|$ is constant and $[\phi, h] = 0$. These together imply $h \equiv 0$ since the winding number is assumed nonzero. With respect to the inner product $(\cdot, \cdot)_P$ the orthogonal subspace to $\tilde{T}(\mathcal{G} \cdot P)$ is

$$\tilde{\mathcal{S}}\mathcal{L}_P \equiv \{\psi \in H_P : \delta_P(M_P \psi) = 0\}$$

in the weak sense that $(\psi, d_P h)_P = 0$ for all $h \in \Gamma_P$. It is useful to define the operator

$$\tilde{\delta}_P \psi = \delta_P(M_P \psi).$$

The condition $\tilde{\delta}_P \psi = 0$ can now be used as our gauge fixing condition.

Lemma 3.6 *For $P \in \mathcal{S}$ there exists a $(\cdot, \cdot)_P$ -orthogonal decomposition*

$$H_P = \Gamma_P \oplus \tilde{\mathcal{S}}\mathcal{L}_P,$$

with a corresponding continuous projection operator

$$\tilde{\Lambda}_P : H_P \rightarrow \tilde{\mathcal{S}}\mathcal{L}_P.$$

Proof. See appendix C. □

We also need an operator which is a slight modification of (2.22). Recall that the three imaginary components of $\mathcal{D}_P \psi$ are the linearisation of the Bogomol'nyi equations at P , while the real component is the slice condition operator $\delta_P \psi$. We define a new differential operator whose three imaginary components are exactly those of $\mathcal{D}_P \psi$ whereas the real component is zero:

$$DB_P \psi = \sum_{i=1}^3 \tau^i (\mathcal{D}_P \psi)_i.$$

The notation is chosen because this operator defines the linearisation of the Bogomol'nyi equations at P .

Lemma 3.7 *The operators Λ_P and $\tilde{\Lambda}_P$ restrict to isomorphisms between $\text{Ker} \mathcal{D}_P \cap H_P$ and $\text{Ker} DB_P \cap \tilde{\mathcal{S}}\mathcal{L}_P \cap H_P$.*

Proof. This follows immediately from the fact that the kernels of these operators consist entirely of infinitesimal gauge transformations, so the restrictions to $\mathcal{S}\mathcal{L}_P, \tilde{\mathcal{S}}\mathcal{L}_P$ define injections. But $\text{Ker} \mathcal{D}_P$ is known to be a finite dimensional space ([Tau83]), therefore so is $\text{Ker} DB_P \cap \tilde{\mathcal{S}}\mathcal{L}_P$ and the result follows. □

Now introduce the functional

$$(3.8) \quad Q_P(\psi) \equiv \|DB_P \psi\|_{L^2}^2 + (W_P(\psi))^2,$$

where

$$W_P(\psi) = \sup_{\{h: \|h\|_P=1\}} |(\psi, d_P h)_P|.$$

Lemma 3.8 *There exists a constant $\tilde{\gamma}$ such that*

$$Q_P(\psi) \geq \tilde{\gamma} \|\psi\|_P^2$$

for all $\psi \in H_P$ such that $(\psi, n)_P = 0$ if $n \in \text{Ker} DB_P \cap \tilde{\mathcal{S}}\mathcal{L}_P \cap H_P$.

Proof. *Step one.* This is proved by contradiction. Thus we assume that ψ_μ is a C_0^∞ minimising sequence with

- (i) $\|\psi_\mu\|_P = 1$.
- (ii) $(\psi_\mu, n)_P = 0$ for all $n \in \text{Ker} DB_P \cap \tilde{\mathcal{S}}\mathcal{L}_P \cap H_P$.
- (iii) $Q_P(\psi_\mu) \rightarrow 0$.

Now consider $\Lambda_P(\psi_\mu)$, where Λ_P was defined in lemma 2.2. There are two possibilities:

- (a) either $\limsup_{\mu \rightarrow \infty} \|\Lambda_P(\psi_\mu)\|_P = 0$, or
- (b) by rescaling and taking a subsequence we obtain a new sequence ψ'_ν which satisfies
 - (i') $\|\psi'_\nu\|_P = 1$.
 - (ii') $(\psi'_\nu, n)_P = 0$ for all $n \in \text{Ker} DB_P \cap \tilde{\mathcal{S}}\mathcal{L}_P \cap H_P$.
 - (iii') $\|\mathcal{D}_P \psi'_\nu\|_{L^2} \rightarrow 0$.

Step two. In case (a) we obtain a contradiction by noticing that if $\Lambda_P \psi_\mu = \psi_\mu + \mathbf{d}_P h_\mu$ then (i) and (a) imply $\|\mathbf{d}_P h_\mu\|_P \leq c$, while (iii) implies $\lim_{\mu \rightarrow \infty} W_P(\psi_\mu) = 0$, so that

$$(\psi_\mu, \mathbf{d}_P h_\mu)_P \rightarrow 0.$$

This implies that $h_\mu \rightarrow 0$ in Γ_P by the assumption (a) and so $\psi_\mu \rightarrow 0$ in H_P which contradicts (i).

Step three. In case (b) we extract a weak limit $\psi'_\nu \rightharpoonup \psi_\infty$ in H_P then $\mathcal{D}_P \psi_\infty = 0$. But combining this with (ii') implies that $\psi_\infty = 0$, and so (see [Tau83, proof of lemma 7.6]) we obtain a contradiction to (i'). $\square$

Proof of theorem 1.3. *Step zero.* We use the same notation as in the proof of theorem 3.4. The strategy of the proof is similar to the proof of theorem 1.2. Thus we apply theorem 3.4 to decompose

$$\Psi(t) = \tilde{P}(t) + \psi(t)$$

and proceed to estimate $\psi(t)$. In the first two steps we will prove the continuity of $\tilde{d}(t) = \|\psi(t)\|_{\tilde{P}(t)}$.

Step one. Assume that

$$(3.9) \quad \|\Psi_1 - Q\|_Q + \|\Psi_2 - Q\|_Q + \|Q - \Psi_*\|_{L^6} \leq c_1.$$

Then there exists $c_2 = c_2(c_1)$ such that

$$|\|\Psi_1 - Q\|_Q^2 - \|\Psi_2 - Q\|_Q^2| \leq c_2 \cdot \|\Psi_1 - \Psi_2\|_{H^1}.$$

To prove this we write out in detail:

$$\begin{aligned} \|\Psi - Q\|_Q^2 &= \|\nabla(\Psi - Q) + [a_0, \Psi - Q]\|_{L^2}^2 + \|[\phi_0, \Psi - Q]\|_{L^2}^2 \\ &= \|\nabla_*(\Psi - Q) + [a_0 - a_*, \Psi - Q]\|_{L^2}^2 + \|[\phi_0 - \phi_* + \phi_*, \Psi - Q]\|_{L^2}^2. \end{aligned}$$

Now use $|\|f\|_{L^2}^2 - \|g\|_{L^2}^2| \leq c(\|f\|_{L^2} + \|g\|_{L^2})\|f - g\|_{L^2}$ and then apply Hölder's inequality, (3.9) and the fact that $\|\phi_*\|_{L^\infty} \leq 1$ ([JT82]).

Step two. To prove that $\tilde{d}(t)$ is continuous we notice that the continuity of $\Psi(t) - \Psi_*$ in H^1 and the inequalities of lemma A.2 imply

$$\begin{aligned} \|\tilde{P}(t) - \Psi_*\|_{L^6} &\leq \|\tilde{P}(t) - \Psi(t)\|_{L^6} + \|\Psi(t) - \Psi_*\|_{L^6} \\ &\leq c \cdot \tilde{d}(t) + c \cdot \|\Psi(t) - \Psi_*(t)\|_{H^1}. \end{aligned}$$

Now $\tilde{d}(t)$ is upper-semicontinuous since it is defined as an infimum of continuous functions. Therefore since it is non-negative it is bounded on any compact interval I , so for any such interval I there exists $c_1 > 0$ such that for $t \in I$ we have $\|\tilde{P}(t) - \Psi_*\|_{L^6} + \tilde{d}(t) \leq c_1$. Therefore by step one

$$|\tilde{d}(t_1)^2 - \tilde{d}(t_2)^2| \leq c_2 \|\Psi(t_1) - \Psi(t_2)\|_{H^1},$$

which implies the continuity of $\tilde{d}$.

Step three. The next thing we need is the Euler-Lagrange equation for the minimisation. Since $\mathcal{S}$ is a smooth submanifold this states that if $n \in T_{\tilde{P}}\mathcal{S}$ and $\psi = \Psi - \tilde{P}$ then

$$(\psi, n)_{\tilde{P}} = -(\mathcal{D}_{\tilde{P}}^* \psi, \sum_{a=0}^3 \tau^a [n_a, \psi])_{L^2}.$$

Recall that $T_{\tilde{P}}\mathcal{S}$ is described by lemma 2.9.

Claim. For $n \in T_{\tilde{P}}\mathcal{S}$

$$|(\psi, n)_{\tilde{P}}| \leq c \cdot \|\psi\|_{\tilde{P}}^2 \quad \text{for } \|n\|_{\tilde{P}} = 1.$$

This is proved by applying Cauchy-Schwarz and (A.5) to estimate the right hand side.

Step four. By lemma 2.9 we can take either $n \in \text{Ker } \mathcal{D}_{\tilde{P}}$ or $n \in T(\mathcal{G} \cdot \tilde{P})$. Taking the latter we estimate

$$\begin{aligned} W_{\tilde{P}}(\psi) &= \sup_{h \in \Gamma_{\tilde{P}}} \left(\frac{|(\psi, \mathbf{d}_{\tilde{P}} h)_{\tilde{P}}|}{\|h\|_{\tilde{P}}} \right) \\ &\leq c \|\psi\|_{\tilde{P}}^2. \end{aligned}$$

Step five. Let ν_μ be an orthonormal basis for $\text{Ker } D\mathcal{B}_{\tilde{P}} \cap \tilde{\mathcal{S}}\mathcal{L}_{\tilde{P}} \cap H_{\tilde{P}}$ then if

$$\psi = \sum_{\mu} \alpha_{\mu} \nu_{\mu} + \psi_*$$

is an orthogonal decomposition we have

$$\alpha_{\mu} = (\psi, \nu_{\mu})_{\tilde{P}} \Rightarrow |\alpha_{\mu}| \leq c \|\psi\|_{\tilde{P}}^2.$$

Since $\|\psi\|_{\tilde{P}}^2 = \|\psi_*\|_{\tilde{P}}^2 + \sum |\alpha_{\mu}|^2$ it follows that there exists δ_3 such that $\|\psi\|_{\tilde{P}} < \delta_3$ implies

$$\|\psi\|_{\tilde{P}}^2 \leq 2\|\psi_*\|_{\tilde{P}}^2.$$

Now notice that under the assumption $\|\psi\|_{\tilde{P}} < \delta_3$ lemma 3.8 implies

$$\begin{aligned} Q_{\tilde{P}}(\psi) &= Q_{\tilde{P}}(\psi_*) \geq \tilde{\gamma} \|\psi_*\|_{\tilde{P}}^2 \\ &\geq \frac{\tilde{\gamma}}{2} \|\psi\|_{\tilde{P}}^2. \end{aligned}$$

Step six. We now obtain a lower bound for V :

$$\begin{aligned} V(\Psi) &= V_{\min} + \int |*F_A - \nabla_A \Phi|^2 \\ &= V_{\min} + \int |\mathcal{D}_{\tilde{P}} \psi + g(\psi, \psi)|^2 + (W_{\tilde{P}}(\psi))^2 - (W_{\tilde{P}}(\psi))^2 \\ &\geq V_{\min} + \frac{1}{2} Q_{\tilde{P}}(\psi) + \text{Err}(\psi) && \text{where } |\text{Err}(\psi)| \leq c \|\psi\|_{\tilde{P}}^4 \\ &\geq V_{\min} + \frac{\tilde{\gamma}}{8} \|\psi\|_{\tilde{P}}^2 && \text{for small } \|\psi\|_{\tilde{P}}. \end{aligned}$$

Step seven. The remainder of the proof is carried out in a fashion similar to the proof of theorem 1.2. $\square$

A Spaces of functions

A.1 Quaternion notation

Unless stated explicitly all functions have range in $\mathbf{R}^4 \otimes su(2)$ in which we will identify $\mathbf{R}^4$ with the space of quaternions $\mathbf{H}$, i.e. the linear span of $1, \tau_1, \tau_2, \tau_3$ where

$$\tau_i \tau_j = -\delta_{ij} + \epsilon_{ijk} \tau_k.$$

A pair (A, Φ) consists of a quadruple of $su(2)$ -valued functions (A_1, A_2, A_3, Φ) which are combined to form the single $\mathbf{H} \times su(2)$ -valued function

$$\Phi + \sum_{j=1}^3 \tau_j A_j.$$

Thus spaces such as L^p and H^s are usually the Lebesgue and Sobolev spaces for functions having values in $\mathbf{H} \times su(2)$. Occasionally we shall use $su(2)$ -valued functions, in particular the infinitesimal gauge transformations such as $\text{Lie } \mathcal{G}$: such spaces shall be regarded as embedded in the corresponding spaces of $\mathbf{H} \times su(2)$ -valued functions via the natural inclusion of $\mathbf{R}$ in $\mathbf{H}$ as the real quaternions.

A.2 Morrey and Campanato spaces

We will use the following notation. The ball centred at x of radius r will be written $B(x, r)$, and the average of u over this ball will be written $\langle u \rangle_{x,s}$. If $x = 0$ we will write instead $B(r)$ etc. Assume that $u \in L^2(B(R))$ then for $r < R$ the Morrey norms (with $p = 2$) of u on $B(r)$ are given by

$$\|u\|_{\lambda, B(r)} \equiv \sup_{x \in B(r)} \sup_{s < R-r} s^{-\lambda} \int_{B(x,s)} |u|^2.$$

The Campanato seminorms are given by

$$[u]_{\lambda, B(r)} \equiv \sup_{x \in B(r)} \sup_{s < R-r} s^{-\lambda} \int_{B(x,s)} |u - \langle u \rangle_{x,s}|^2.$$

The relations between these are discussed in [Gia93, Chapter three]. The crucial thing for us is that if $\lambda > n$, the dimension of the space on which the functions u are defined, then $\exists c > 0$ such that

$$(A.1) \quad [u]_{C^{0,\alpha}(\overline{B(r)})} \equiv \sup_{y,z \in \overline{B(r)}} \frac{|u(y) - u(z)|}{|y - z|^\alpha} \leq c[u]_{\lambda, B(r)} \text{ with } \alpha = \frac{\lambda - n}{2}.$$

Lemma A.1 *Let $\chi(s)$ be nonnegative and nondecreasing real valued function on $[0, r_*]$. Suppose that it satisfies*

$$\chi(r) \leq A[(r/R)^\alpha + \epsilon]\chi(R) + BR^\beta \quad \text{whenever } r < R \leq r_*,$$

where A, B, α, β are nonnegative numbers and $\alpha > \beta$. Then there exists $\epsilon_0 = \epsilon_0(\alpha, \beta, A)$ such that if $\epsilon < \epsilon_0$ then

$$\chi(r) \leq c(\alpha, \beta, A)[(r/R)^\alpha \chi(R) + Br^\beta].$$

Proof. See [Gia93, p. 44]. □

A.3 Covariant Sobolev completions

For $\Psi = (A, \Phi) \in \mathcal{A}_k$ for $k = 1, 2, \dots$ we can introduce, for $l = 1, \dots, k+1$ the norms

$$(A.2) \quad \|\psi\|_{l, \Psi}^2 \equiv \sum_{|\alpha|=1}^l \|\nabla_A^\alpha \psi\|_{L^2}^2 + \|[\Phi, \psi]\|_{L^2}^2$$

on C_0^∞ . If $l = 1$ we shall write $\|\psi\|_\Psi$, omitting the integer index. These norms are equivalent (although not uniformly with respect to the base point Ψ). To see this we need the following standard results:

Lemma A.2 *Assume $\Psi = (A, \Phi) \in \mathcal{A}$ and ψ, u, v are smooth, compactly supported $\mathbf{H} \otimes su(2)$ -valued functions. Then the following statements are true:*

(i) *Kato's inequality states that if $|\nabla_A \psi| \in L^p$ then*

$$(A.3) \quad \|\nabla|\psi|\|_{L^p} \leq \|\nabla_A \psi\|_{L^p}.$$

Together with the three dimensional Sobolev inequality this implies

$$(A.4) \quad \|\psi\|_{L^6} \leq c\|\nabla_A \psi\|_{L^2}.$$

(ii) *Let i, j be indices for the four quaternion components of u, v , then*

$$(A.5) \quad \| [u^i, v^j] \|_{L^2} \leq c(\Psi) \|u\|_\Psi \|v\|_\Psi$$

where c depends only on the Lebesgue measure of the set $\{|\Phi| < \frac{1}{2}\}$ and is thus finite for $\Psi \in \mathcal{A}$.

(iii) *For smooth compactly supported ψ and $A \in H_{loc}^1$ there exists a number $c = c(\|F_A\|_{L^2})$ such that*

$$(A.6) \quad \|\nabla_A \nabla_A \psi\|_{L^2} + \|\nabla_A \psi\|_{H^1} \leq c(\|\nabla_A \psi\|_{L^2} + \|\Delta_A \psi\|_{L^2}).$$

which combined with the Sobolev inequality and Kato's inequality gives

$$(A.7) \quad \|\psi\|_{L^\infty} \leq c(\|\psi\|_{L^6} + \|\nabla|\psi|\|_{L^6}) \leq c(\|\nabla_A \psi\|_{L^2} + \|\Delta_A \psi\|_{L^2}).$$

Proof. See [JT82, Sections VI.6 and V.8] and [Tau82, Lemma 6.6]. $\square$

Lemma A.3 For $\Psi = (A, \Phi) \in \mathcal{A}$ all the norms $\|\cdot\|_\Psi$ are equivalent on H_* . For $n = 2, 3, \dots$ if $\Psi \in \mathcal{A}_{n-1}$ the norms $\|\cdot\|_{k,*}$ and $\|\cdot\|_{k,\Psi}$ are equivalent on $H_{k,*}$ for $k = 1, \dots, n$. Inequalities (A.3-A.5) are valid on H_* while (A.6-A.7) hold on $H_{2,*}$.

Proof. First of all notice that

$$\nabla_A \psi = \nabla_* \psi + [A - a_*, \psi] \text{ and } [\Phi, \psi] = [\phi_*, \psi] + [\Phi - \phi_*, \psi].$$

From (A.5) and the definitions of the norms it follows that if $\Psi \in \mathcal{A}$ then

$$\|\Psi - \Psi_*\|_\Psi \leq c(\Psi) \|\Psi - \Psi_*\|_*(1 + \|\Psi - \Psi_*\|_*),$$

which implies in turn that for smooth compactly supported ψ there exists $c = c(\Psi)$ such that

$$c^{-1} \|\psi\|_* \leq \|\psi\|_\Psi \leq c \|\psi\|_*.$$

For $n = 2, 3, \dots$ the argument is similar; consider the case $n = 2$ to start with. If we expand out $\nabla_A^2 \psi$ there are terms of the following four types:

$$\nabla_*^2 \psi \quad [A - a_*, \nabla_* \psi] \quad [\nabla_*(A - a_*), \psi] \quad [A - a_*, [A - a_*, \psi]]$$

the first is estimated directly in terms of $\|\psi\|_{2,*}$. The second is estimated, using (A.5), by $\|\Psi - \Psi_*\|_* \|\psi\|_{2,*}$. The third may be estimated by $\|\Psi - \Psi_*\|_* \|\psi\|_\infty$, which is in turn estimated by $\|\Psi - \Psi_*\|_* \|\psi\|_{2,*}$ by virtue of (A.6-A.7). The fourth term is estimated directly by $\|A - a_*\|_*^2 \|\psi\|_*$, on account of (A.4), and Hölder's inequality.

The general case ($n \geq 3$) is very similar, but the notation is complicated. Define the symbol $I(f)$ by $I(f)\psi = [f, \psi]$. Then on expanding out $\nabla_A^n \psi$ we get a sum of terms which can be written schematically as

$$\prod_{i=1}^M I(\nabla_*^{j_i}(A - a_*)) \nabla_*^{n-k} \psi \quad \text{with} \quad \sum_{i=1}^M (1 + j_i) = k$$

where M, j_i, k are positive integers less than or equal to k . We treat these terms in various ways depending upon k :

- (i) if $k = 0$ the single term $\nabla_*^n \psi$ is bounded directly by $\|\psi\|_{n,*}$.
- (ii) if $k = 1$ the term $[A - a_*, \nabla_*^{n-1} \psi]$ is estimated by $\|A - a_*\|_* \|\psi\|_{n,*}$ by (A.5).
- (iii) if $k = 2, \dots, n-1$ then each $j_i \leq n-2$. We consider two cases. (a) If each $j_i \leq n-3$ (A.6-A.7) imply that all the commutator terms can be bounded in L^∞ , and so we estimate as follows:

$$\prod_{i=1}^M \|\nabla_*^{j_i}(A - a_*)\|_{L^\infty} \|\nabla_*^{n-k} \psi\|_{L^2} \leq \|A - a_*\|_{n-1,*}^M \|\psi\|_{n,*}.$$

- (b) If there is an i with $j_i = n-2$ then there is only one type of term of the form $[\nabla_*^{n-2}(A - a_*), \nabla_* \psi]$, which is bounded by $\|A - a_*\|_{n-1,*} \|\psi\|_{2,*}$.
- (iv) the final case $k = n$ is almost the same as the previous one except that we cannot bound ψ in L^2 . We consider three cases. (a) If all $j_i \leq n-3$ we estimate as follows:

$$\prod_{i=1}^{M-1} \|\nabla_*^{j_i}(A - a_*)\|_{L^\infty} \|\nabla_*^{j_M}(A - a_*)\|_* \|\psi\|_*,$$

and since $j_M \leq n-3$ this will do since $\|\nabla_*^{j_M}(A-a_*)\|_* \leq \|A-a_*\|_{n-2,*}$. (b) If there is an i with $j_i = n-2$ we have to bound a single term of the form $[A-a_*, [\nabla_*^{n-2}(A-a_*), \psi]]$ which is estimated by $\|A-a_*\|_{L^\infty} \|A-a_*\|_{n-1,*} \|\psi\|_*$. (c) If there is an i with $j_i = n-1$ we have a single term of the form $[\nabla_*^{n-1}(A-a_*), \psi]$ which is bounded by $\|\psi\|_{L^\infty} \|A-a_*\|_{n-1,*} \leq c \|\psi\|_{2,*} \|A-a_*\|_{n-1,*}$. $\square$

Remark It is clear from the proof that the equivalence is not uniform with respect to Ψ .

A.4 Dual norms

The norm dual to $\|\cdot\|_\Psi$ is defined by

$$(A.8) \quad \|\psi\|_{(-\Psi)} = \sup_{f \in C_0^\infty} \frac{|(f, \psi)_{L^2}|}{\|f\|_\Psi}$$

and for $\Psi \in \mathcal{A}$ we let $X(\Psi)$ be the proper subset of L^2 defined by

$$X(\Psi) \equiv \{\psi \in L^2 : \|\psi\|_{X(\Psi)} < \infty\} \text{ where } \|\psi\|_{X(\Psi)} \equiv \|\psi\|_{(-\Psi)} + \|\psi\|_{L^2}$$

For the case $\Psi = \Psi_*$ we shall write $X = X(\Psi_*)$. It follows from lemma A.3 that the norms $\|\cdot\|_{X(\Psi)}$ are all equivalent and $X(\Psi) = X$ for $\Psi \in \mathcal{A}$.

B Operators

The present section is mostly a summary of results from [Tau82, Tau83, Stu94] with some technical changes. If $\Psi = (A, \Phi)$ then in quaternionic notation the operator $\mathcal{D}_\Psi$ is given by

$$\mathcal{D}_\Psi = -\tau^i (\nabla_A)_i + [\Phi, \cdot]$$

with summation convention. The formal L^2 adjoint will be written $\mathcal{D}_\Psi^*$. A direct calculation shows that Ψ satisfies (1.2) then

$$(B.1) \quad \|\mathcal{D}_\Psi^* \psi\|_{L^2} = \|\psi\|_\Psi.$$

The next lemma summarises the analytic properties of $\mathcal{D}_\Psi$ and $\mathcal{D}_\Psi^*$.

Lemma B.1 *Assume that $\Psi \in \mathcal{S}$, then the following statements hold.*

(i) *The differential operators $\mathcal{D}_\Psi$ and $\mathcal{D}_\Psi^*$ are Fredholm from H_* into L^2 , and*

$$\text{Ker} \mathcal{D}_\Psi^* \cap H_* = \text{Ker} \mathcal{D}_\Psi^* \cap L^2 = \{0\}$$

while $\text{Ker} \mathcal{D}_\Psi \cap H_$ is a $4N$ -dimensional vector space (where N is the winding number).*

Furthermore if $n \in \text{ker} \mathcal{D}_\Psi$ there exists f such that

$$(B.2) \quad n = \mathcal{D}_\Psi f \text{ with } f \in L^2 \bigcap H_*.$$

(ii) *Let ψ be orthogonal to all $n \in \text{ker} \mathcal{D}_\Psi$ with respect to either the L^2 inner product or the H_Ψ inner product. Then there exists a constant γ such that*

$$\|\mathcal{D}_\Psi \psi\|_{L^2} \geq \gamma \|\psi\|_\Psi$$

(iii) *Let $f \in L^2$ then there exists $\psi \in H_\Psi$ such that $\mathcal{D}_\Psi \psi = f$.*

(iiib) *Let $f \in L^2$ be L^2 orthogonal to $\text{Ker} \mathcal{D}_\Psi$ then there exists $\psi \in H_\Psi$ such that $\mathcal{D}_\Psi^* \psi = f$.*

(iv) *$\mathcal{D}_\Psi$ defines a bounded linear surjection $H_* \cap L^2 \rightarrow X$ where X was defined in section A.4.*

Proof. *Proof of (i)* The Fredholm property was proved in [Tau83] for $\Psi \in \mathcal{S} \cap C^\infty$. To extend this to $\Psi \in \mathcal{S}$ we use the fact ([JT82, Theorem V.1.1]) that for any $\Psi \in \mathcal{S}$ there exists a Hölder continuous gauge transformation $g : \mathbf{R}^3 \rightarrow \text{SU}(2)$ such that $\Psi' = (A', \Phi') = (g^{-1}dg + g^{-1}Ag, g^{-1}\Phi g)$ is smooth. (In addition g lies in H_{loc}^2 because it satisfies

$$dg = gA' - Ag$$

so that since A' is smooth and $A \in H_{loc}^1 \cap L^6$ it follows that $dg \in L_{loc}^6$. From this we deduce in turn that $d(gA' - Ag) \in L_{loc}^2$ and hence that $g \in H_{loc}^2$.) Now for $\psi \in H_\Psi$ the gauge transformation acts according to $\psi \rightarrow \psi' = g^{-1}\psi g$; this is an isometry in L^2 and maps H_Ψ isometrically to $H_{\Psi'}$. In addition $\mathcal{D}_{\Psi'}\psi' = g^{-1}\mathcal{D}_\Psi\psi g$, so that all inequalities involving L^2 norms of $\mathcal{D}_\Psi$, (or its adjoint), and $\|\cdot\|_\Psi$ on the space H_Ψ will remain valid for $\Psi \in \mathcal{S}$.

Proof of (ii) The H_Ψ case is proved in [Tau83] and the L^2 case in [Stu94]. The same remarks as in (i) apply to the problem of extending this to $\Psi \in \mathcal{S}$.

Proof of (iii) Since $\mathcal{D}_\Psi$ is a densely defined operator $L^2 \rightarrow L^2$ with domain $H_\Psi \cap L^2$ we know that

$$(B.3) \quad L^2 = \overline{\text{Ran } \mathcal{D}_\Psi} \oplus \text{Ker } \mathcal{D}_\Psi^*$$

so since $\text{Ker } \mathcal{D}_\Psi^* = \{0\}$ we see that the closure of the range is all of L^2 . The result then follows from lemma B.1. The corresponding result for $\mathcal{D}_\Psi^*$ is proved similarly except that $\text{Ker } \mathcal{D}_\Psi$ is non-empty necessitating the Fredholm conditions.

Proof of (iv) Step one By (B.3) since $\text{Ker } \mathcal{D}_\Psi^* \cap L^2 = \{0\}$ we see that $\mathcal{D}_\Psi(H_\Psi \cap L^2)$ is dense in L^2 . But if $\psi \in H_\Psi \cap L^2$ then $\mathcal{D}_\Psi\psi \in X$ since $\mathcal{D}_\Psi\psi \in L^2$ and

$$\|\mathcal{D}_\Psi\psi\|_{(-\Psi)} \leq \|\psi\|_{L^2}.$$

In fact more can be said if we assume that ψ is L^2 -orthogonal to $\text{Ker } \mathcal{D}_\Psi$ since in that case we can find $f \in H_\Psi$ such that $\psi = \mathcal{D}_\Psi^* f$ and inserting this into the definition of dual norm (A.8) and using (B.1) gives

$$(B.4) \quad \|\mathcal{D}_\Psi\psi\|_{(-\Psi)} = \|\psi\|_{L^2}.$$

Step two Now consider $\psi \in L^2 \cap H_\Psi$ which is L^2 orthogonal to $\text{Ker } \mathcal{D}_\Psi$. Then there exists $c > 0$ such that

$$\|\mathcal{D}_\Psi\psi\|_X \geq c(\|\psi\|_\Psi + \|\psi\|_{L^2}).$$

This follows from (B.4) combined with lemma B.1. It follows that $\mathcal{D}(H_\Psi \cap L^2)$ is closed in X .

Step three Finally we must show that $\mathcal{D}_\Psi : H_\Psi \cap L^2 \rightarrow X$ is onto. To see this we use the fact that if $f \in L^2$ we can find $\psi \in H_\Psi$ such that $\mathcal{D}_\Psi\psi = f$; but if in addition $f \in X$ then (B.4) implies that $\psi \in L^2$. $\square$

Lemma B.2 *If $g \in \mathcal{G}_k$ and $\Delta_* g \in H_{k-1,*}$ then $g \in \mathcal{G}_{k+1}$.*

Proof. This follows from lemma A.3 and (A.6). $\square$

Lemma B.3 *For $\Psi \in \mathcal{A}$ the operator $M_\Psi = -\Delta_A - [\Phi, [\Phi, \cdot]]$ defines a bounded linear surjection $H_{2,*} \rightarrow X$. More generally for $k = 1, 2, \dots$ if $\Psi \in \mathcal{A}_{k+1}$ it defines a bounded linear surjection $H_{k+2,*} \rightarrow X \cap H_{k,*}$. (The space X was defined in section A.4.)*

Proof. Recall that by lemma A.3 we can work with the norm $\|\cdot\|_{k,\Psi}$ instead of $\|\cdot\|_{k,*}$, which is more convenient. The equation

$$-\Delta_A \psi - [\Phi, [\Phi, \psi]] = f \quad \text{for } f \in X$$

has a solution $\psi \in H_\Psi$ by Riesz representation since $X(\Psi) = X$. To complete the proof for $k = 0$ we first bound $[\Phi, [\Phi, \psi]]$ in L^2 as follows: write

$$[\Phi, [\Phi, \psi]] = [\Phi - \phi_*, [\Phi - \phi_*, \psi]] + [\Phi - \phi_*, [\phi_*, \psi]] + [\phi_*, [\Phi, \psi]]$$

then by Hölder's inequality and (A.2) we obtain

$$||[\Phi, [\Phi, \psi]]||_{L^2} \leq c. (||\Phi - \phi_*||_{L^6}^2 ||\psi||_{L^6} + ||\Phi - \phi_*||_{L^3} ||\psi||_{L^6} + ||\psi||_{\Psi})$$

Now we get a bound for $||\Delta_A \psi||_{L^2}$ and hence also for $||\psi||_{2, \Psi}$ by applying (A.6). There will be an estimate of the form

$$||\psi||_{2, \Psi} \leq c ||f||_{X(\Psi)}$$

Now for $k = 1$ differentiate the above equation, to give an equation which we write schematically (supressing all indices) as

$$\begin{aligned} M_{\Psi} \nabla_A \psi &= -[\nabla_A F, \psi] - 2[F, \nabla_A \psi] + [\nabla_A \Phi, [\Phi, \psi]] + [\Phi, [\nabla_A \Phi, \psi]] + \nabla_A f \\ &= j_1 \end{aligned}$$

Now we claim that

$$||j_1||_{X(\Psi)} \leq c. (||f||_{L^2} + ||f||_{\Psi} + (||F_A||_{L^2} + ||F_A||_{\Psi} + ||\Phi||_{L^\infty} ||\nabla_A \Phi||_{L^2}) ||\psi||_{2, \Psi})$$

To see this consider, for example, the first term in j_1 and estimate using (A.5) as follows

$$\begin{aligned} ||[\nabla_A F, \psi]||_{L^2} &\leq ||\psi||_{L^\infty} ||\nabla_A F||_{L^2} \\ ||[\nabla_A F, \psi]||_{(-\Psi)} &\leq c ||\nabla_A F||_{L^2} ||\psi||_{\Psi} \end{aligned}$$

the other terms are treated similarly. On further differentiation a similar structure appears. $\square$

C Proofs of some lemmas

Proof of lemma 2.2 Let $\psi = (\beta, \eta) \in \mathcal{A}_k$ then we decompose this as

$$(C.1) \quad (\beta, \eta) = (\nabla_A h, [\Phi, h]) + (b, e)$$

where h is the solution to the equation

$$d_A^* d_A h + [\Phi, [\Phi, h]] = d_A^* \beta + [\Phi, \eta].$$

This equation has a unique solution in $H(\Psi)$ by lemma B.3 with $h \in \text{Lie}\mathcal{G}_{k+1}$. $\square$

Proof of lemma 3.6 We decompose exactly as in (C.1). This leads to the equation

$$\delta_P M_P d_P h = -\delta_P M_P \psi.$$

to be solved in the weak sense i.e. we require that

$$(d_P h, d_P g)_P = -(\psi, d_P g)_P$$

for all $g \in \Gamma_P$. This is solved by the Riesz representation theorem on the space Γ_P of measurable functions $h : \mathbf{R}^3 \rightarrow su(2)$ such that $d_P h \in H_P$, on which we use the norm

$$||h||_{(P)} = ||D_P^* d_P \psi||_{L^2} = ||d_P \psi||_P.$$

$\square$

Proof of lemma 2.9 We wish to prove that the zeros of the Bogomol'nyi equations (1.2) define a submanifold in $\mathcal{A}_k$. This will be assured if we can check that at each $\Psi \in \mathcal{S}$ the derivative of the equations defines a surjection whose kernel admits a closed complement ([Lan85, Section II.2]). The latter condition follows immediately in Hilbert space, so we need only check the surjectivity. We need to be careful about function spaces; recall that X consists of functions taking values in $su(2) \otimes \mathbf{H}$. Let Y denote the sub-space of X consisting of those functions which

are pure imaginary in $\mathbf{H}$. Then one can check using (A.5) that the three components of (1.2) lie in Y and the function

$$\begin{aligned} \mathcal{B} : \quad \mathcal{A} &\rightarrow \quad Y \\ (A, \Phi) &\mapsto \quad *F_A - d_A \Phi \end{aligned}$$

is smooth. Notice that $D\mathcal{B}(A, \Phi)(\psi)$, the derivative of $\mathcal{B}$ at (A, Φ) in the direction $\psi = \eta + \tau^i \beta_i$, is just the three pure imaginary components of

$$(C.2) \quad \mathcal{D}_{(A, \Phi)} \psi = 0$$

But since this operator is surjective from $T_\Psi \mathcal{A}$ to X by lemma B.1 the surjectivity of $D\mathcal{B}(A, \Phi) : T_\Psi \mathcal{A} \rightarrow Y$ follows. Now gauge invariance implies that $T(\mathcal{G} \cdot (A, \Phi)) \subset \text{Ker} D\mathcal{B}(A, \Phi)$ and the fourth component of (C.2) is just the slice condition (2.2). Therefore,

$$\text{Ker} \mathcal{B}(A, \Phi) = T(\mathcal{G} \cdot (A, \Phi)) \oplus \text{Ker} \mathcal{D}_{(A, \Phi)}$$

which are closed subspaces whose orthogonal complements provide complementary closed subspaces. The formulae for tangent and normal spaces follow. $\square$

Lemma C.1 (Slice) *Let $B_s(Q, r)$ be the ball*

$$\{P \in \mathcal{A}_s : \|P - Q\|_{s, Q} + \|P - Q\|_{L^2} < r\}.$$

For $k = 1, 2 \dots$ fix $Q_0 \in \mathcal{A}_{k+1}$ then there exists $\delta_1, r_1 > 0$ such that if $Q = Q_0 + v$ with

$$r < r_1 \text{ and } \|v\|_{k+1, Q_0} < \delta_1$$

there is a diffeomorphism

$$\Xi_Q : B_{k+1}(Q, r) \rightarrow U_2 \times U_3$$

where U_2 is a neighbourhood of 0 in $\mathcal{SL}_Q \cap H_{k+1, Q}$ and U_3 is an open neighbourhood of id in $\mathcal{G}_{k+2, Q}$. This diffeomorphism depends smoothly on Q in the following sense; write the two components of Ξ_Q as $\Xi_Q = (\lambda_Q, \mu_Q)$, with $Q = Q_0 + v$ with $v \in B_{k+1}(0, \delta_1)$ then the functions

$$\begin{aligned} s_1 : B_{k+1}(0, \delta_1) \times B_{k+1}(0, r) &\rightarrow H_{k+1, *} & s_1(v, P - (Q_0 + v)) &= \lambda_{Q_0+v}(P) \\ s_2 : B_{k+1}(0, \delta_1) \times B_{k+1}(0, r) &\rightarrow \mathcal{G}_{k+2} & s_2(v, P - (Q_0 + v)) &= \mu_{Q_0+v}(P) \end{aligned}$$

are smooth.

Proof of lemma C.1 The proof is an application of the implicit function theorem as follows. The idea is that given $P \in B_{k+1}(Q_0 + v, r)$ we try to find a gauge transformation g such that

$$g \cdot P - (Q_0 + v) \in \mathcal{SL}(Q_0 + v)$$

If we write $P = (A, \Phi)$, $Q_0 = (a, \phi)$ and $v = (\beta, \eta)$ then this leads to an equation

$$d_{a+\beta}^*(g^{-1}dg + g^{-1}Ag - a - \beta) + [\phi + \eta, g^{-1}\Phi g - \phi - \eta] = 0$$

for g . It is convenient to write this as an equation for $\xi \in \text{Lie } \mathcal{G}_{k+2}$ via the exponential mapping Exp :

$$g = \text{Exp} \xi \quad \text{Exp} : \text{Lie } \mathcal{G}_{k+2} \rightarrow \mathcal{G}_{k+2}.$$

Treat this as an equation $F(\xi, v, P - Q) = 0$ to be solved for ξ as a function of $(v, P - Q)$ in a neighbourhood of $(0, 0, 0)$ by the implicit function theorem. Notice that

$$F : \text{Lie } \mathcal{G}_{k+2} \times H_{k+1, (a, \phi)} \times H_{k+1, (a, \phi)} \rightarrow H_{k, (a, \phi)} \cap X$$

is a smooth function. To check the local solvability for g we are interested in the first partial derivative $D_1 F(0, 0, 0)$ which is the operator

$$-\Delta_a - [\phi, [\phi, \cdot]]$$

which is an isomorphism $H_{k+2, (a, \phi)}$ onto $H_{k, (a, \phi)} \cap X$ by lemma B.3. This ensures the applicability of the inverse function theorem. Since $Q_0 \in \mathcal{A}_{k+1}$ the norms $\|\cdot\|_{s, (a, \phi)}$ are equivalent to $\|\cdot\|_{s, *}$ for $1 \leq s \leq k+2$. The construction of the local diffeomorphism is then exactly as in [FU84, Chapter three] and will not be given. $\square$

D Acknowledgements

I would like to thank S. Demoulini for helpful conversations which led to theorem 1.3, and clarified other issues. I would like to thank the IHES for support and hospitality.

References

- [AH88] M. Atiyah and N. Hitchin. *Geometry and Dynamics of Magnetic Monopoles*. Princeton University Press, Princeton N.J., 1988.
- [BO86] J. Burzlaff and N. O'Murchadha. Global existence of time-dependent Yang-Mills-Higgs monopoles. *Communications in Mathematical Physics*, 105:85–98, 1986.
- [Bog76] E. Bogomolny. Stability of classical solutions. *Soviet Journal of Nuclear Physics*, 24:861–870, 1976.
- [Die76] J. Dieudonne. *Treatise on Analysis II*. Academic Press, New York, 1976.
- [Don84] S. Donaldson. Nahm's equations and the classification of monopoles. *Communications in Mathematical Physics*, 96:387–407, 1984.
- [DS96] S. Demoulini and D. Stuart. Gradient flow of the superconducting Ginzburg-Landau functional on the plane. To appear in *Communications in Analysis and Geometry*, 1996.
- [EM82] D.M. Eardley and V. Moncrief. Global existence of Yang-Mills-Higgs fields in 4-dimensional minkowski space. *Communications in Mathematical Physics*, 83:171–212, 1982.
- [FU84] D. Freed and K. Uhlenbeck. *Instantons and Four Manifolds*. Springer, New York, 1984.
- [Gia93] M. Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems*. Birkhauser, Basel, 1993.
- [Gro84] D. Groisser. Integrality of the monopole number. *Communications in Mathematical Physics*, 93:367–378, 1984.
- [GV81] J. Ginibre and G. Velo. The cauchy problem for coupled yang-mills and scalar fields in temporal gauge. *Communications in Mathematical Physics*, 82:1–28, 1981.
- [JT82] A. Jaffe and C. Taubes. *Vortices and Monopoles*. Birkhauser, Boston, Mass, 1982.
- [Kee95] M. Keel. Princeton PhD thesis, 1995.
- [KM95] S. Klainerman and M. Machedon. Finite energy solutions of the Yang-Mills equations in $\mathbf{R}^{3+1}$. *Annals of Mathematics*, 142,1:39–119, 1995.
- [Lan85] S. Lang. *Differential Manifolds*. Springer, New York, 1985.
- [Maj84] A. J. Majda. *Compressible Fluid Flow and Systems of Conservation Laws*. Springer, New York, 1984.
- [Man82] N. Manton. A remark on scattering of BPS monopoles. *Physics Letters*, 110B:54–56, 1982.
- [Str89] W. Strauss. *Nonlinear wave equations*. AMS, Providence, R.I., 1989.
- [Stu94] D. Stuart. The Geodesic Approximation for the Yang-Mills-Higgs equations. *Commun. Math. Phys.*, 166:149–191, 1994.

- [SU82] R. Schoen and K. Uhlenbeck. A regularity theory for harmonic maps. *J. Diff. Geo.*, 17:307–335, 1982.
- [SU84] R. Schoen and K. Uhlenbeck. Regularity of minimising harmonic maps into the sphere. *Inv. Math.*, 78:89–100, 1984.
- [Tau82] C. Taubes. Existence of a non-minimal solution to $SU(2)$ Yang-Mills equations on $\mathbf{R}^3$, parts one and two. *Communications in Mathematical Physics*, 86:257–320, 1982.
- [Tau83] C. Taubes. Stability in Yang-Mills theories. *Communications in Mathematical Physics*, 91:235–263, 1983.