Post-Newtonian Approximation for Spinning Particles

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Abstract

Using an energy-momentum tensor for spinning particles due to Dixon and Bailey-Israel, we develop the post-Newtonian approximation for N spinning particles in a self-contained manner. The equations of motion are derived directly from this energy-momentum tensor. Following the formalism of Epstein and Wagoner, we also obtain the waveform and the luminosity of the gravitational wave generated by these particles.

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I. INTRODUCTION

The possibility of detecting gravitational waves from coalescences of compact binaries in the near future by LIGO [1] and VIRGO [2] laser interferometric detectors has aroused quite a lot of research in this area. One of the main tools in analyzing these situations and in obtaining the gravitational waveforms generated is the post-Newtonian approximation [3]. As one gets closer to the coalescence, higher order terms in this approximation have to be taken into account to get results accurate enough for comparison with future observational data.

When higher order terms are considered, spin and higher multipole effects may be of significance, especially for neutron stars and Kerr black holes. For these compact objects, it is estimated that the spin-orbit and spin-spin effects are of (post)$^{3/2}$-Newtonian and (post)$^2$-Newtonian order, respectively [4]. Therefore, to get accurate results to these orders, spin effects must be included.

A number of authors have studied these spin effects in [4]-[6]. In [6], both the spin-orbit and spin-spin effects are considered in details using the multipole formalism of Blanchet, Damour, and Iyer (BDI) [7,8]. The BDI formalism is rigorous, but rather complicated in which each body is regarded as a spherically symmetric, rigidly rotating perfect fluid.

In this paper we would like to introduce another approach which is simpler and self-contained in the sense that the equations of motion are also derived along the way. In this approach we still regard the bodies as point particles, but with structures such as spin. The central quantity here is the energy-momentum tensor for these particles. In general spacetimes this energy-momentum tensor was first introduced by Dixon [9], and was later elaborated on by Bailey and Israel (BI) [10]. In [10], the Lagrangian density of a particle is expanded in a covariant Taylor expansion about another world-line, presumably the center-of-mass of some extended object. Higher order terms in this expansion correspond to spin, tidal effects and so on. From this Lagrangian one can obtain the energy-momentum tensor in the usual fashion. Therefore, in this formalism one can include structures such as spin,
quadrupole moment and so on in a systematic manner. In this paper, however, we shall concentrate only on the spin effects.

With this energy-momentum tensor for particles with spin, one can derive the equations of motion for spinning particles, that is, the Papapetrou equations, by requiring this energy-momentum tensor to be symmetric and conserving. This will be done in the next section. In Section III, we develop the post-Newtonian approximation for N spinning particles using this tensor. We follow closely the procedures in [11]. As an illustration of how our approach works, we choose the simpler case that the spin-orbit and spin-spin effects are both of post-Newtonian order. In this case, we are not dealing with compact objects, but ordinary ones. The equations of motion for positions and spins are expanded up to (post)$^2$-Newtonian order. In Section IV we use the formalism of Epstein and Wagoner (EW) [12] to obtain the waveform and the luminosity of the gravitational wave generated by the motion of these N spinning particles up to post-Newtonian order. This can be considered as an extension of the result of Wagoner and Will [13] to spinning particles. Conclusions and discussions are given in Section V.

II. ENERGY-MOMENTUM TENSOR

In [10], BI devised a general method to derive the energy-momentum tensor $T^{\mu\nu}$ for particles with structures such as spin. First the Lagrangian for a charged point particle is expanded covariantly about another world-line. For the particle to represent part of an extended object, this world-line is usually chosen to be the center-of-mass of the object. However, their formalism is rather general that one can actually choose any convenient world-line. The higher order terms in this expansion can be identified with the structures of the particle corresponding to higher gravitational and electromagnetic multipole moments. With this Lagrangian expansion, one can obtain the equations of motion and $T^{\mu\nu}$ by the usual variations.

Here we concentrate on particles with the extra structure of spin, which is related to the
gravitational dipole moment. In this case, $T^{\mu\nu}$ for a spinning particle is given by [10],

$$T^{\mu\nu} = T_{(can)}^{\mu\nu} + \nabla_\rho B^{\mu\nu\rho}, \quad (2.1)$$

where

$$T_{(can)}^{\mu\nu} = \frac{1}{\sqrt{-g}} \int d\tau p^\mu v^\nu \delta^4(x - z(\tau)), \quad (2.2)$$

$$B^{\mu\nu\rho} = -\frac{1}{2} (S^{\mu\nu\rho} + S^{\rho\mu\nu} + S^{\rho\nu\mu}), \quad (2.3)$$

$$S^{\mu\nu\rho} = \frac{1}{\sqrt{-g}} \int d\tau S^{\mu\nu\rho} v^\rho \delta^4(x - z(\tau)), \quad (2.4)$$

and $p^\mu$ is the canonical momentum, $S^{\mu\nu}$ the spin tensor. $T_{(can)}^{\mu\nu}$ is therefore the canonical energy-momentum tensor, and $B^{\mu\nu\rho}$ is the Belinfante tensor due to spin. The equations of motion can be derived in the following manner. First we require $T^{\mu\nu}$ to be symmetric in $\mu$ and $\nu$, $T^{\mu\nu} = T^{\nu\mu}$, which gives

$$\frac{1}{\sqrt{-g}} \int d\tau (p^\mu v^\nu - p^\nu v^\mu) \delta^4(x - z(\tau)) - \nabla_\rho \left[ \frac{1}{\sqrt{-g}} \int d\tau S^{\mu\nu\rho} v^\rho \delta^4(x - z(\tau)) \right] = 0. \quad (2.5)$$

Using the formulae,

$$\partial_\mu \sqrt{-g} = \sqrt{-g} \Gamma^\rho_{\mu\nu}, \quad (2.6)$$

and

$$v^\mu \partial_\rho \delta^4(x - z(\tau)) = -\frac{\partial}{\partial \tau} \delta^4(x - z(\tau)), \quad (2.7)$$

the derivative becomes

$$\nabla_\rho \left[ \frac{1}{\sqrt{-g}} \int d\tau S^{\mu\nu\rho} v^\rho \delta^4(x - z(\tau)) \right] = \frac{1}{\sqrt{-g}} \int d\tau \frac{DS^{\mu\nu}}{D\tau} \delta^4(x - z(\tau)), \quad (2.8)$$

where we have ignored surface terms. Here

$$\frac{DS^{\mu\nu}}{D\tau} \equiv v^\alpha \nabla_\alpha S^{\mu\nu}$$

$$= \frac{dS^{\mu\nu}}{d\tau} + v^\alpha \Gamma^\mu_{\alpha\nu} S^{\rho\mu} + v^\alpha \Gamma^\nu_{\alpha\rho} S^{\mu\rho}. \quad (2.9)$$

Therefore, Eq.(2.5) becomes
\[
\frac{1}{\sqrt{-g}} \int d\tau \left[ \frac{DS^{\mu\nu}}{D\tau} - (p^\mu v^\nu - p^\nu v^\mu) \right] \delta^4(x - z(\tau)) = 0
\]

\[
\Rightarrow \frac{DS^{\mu\nu}}{D\tau} = p^\mu v^\nu - p^\nu v^\mu.
\] (2.10)

This is the first set of the Papapetrou equations, the equations of motion for spinning particles.

Next we require that \( T^{\mu\nu} \) is conserving, \( \nabla_\mu T^{\mu\nu} = 0 \), which gives

\[
\nabla_\mu T^{\nu\mu}_{(\text{can})} + \nabla_\mu \nabla_\nu B^{\nu\mu\rho} = 0,
\] (2.11)

where we have used the symmetry property of \( T^{\mu\nu} \). Using the formulae Eqs.(2.6) and (2.7) again, we have

\[
\nabla_\mu T^{\nu\mu}_{(\text{can})} = \nabla_\mu \left[ \frac{1}{\sqrt{-g}} \int d\tau p^\nu v^\mu \delta^4(x - z(\tau)) \right] = \frac{1}{\sqrt{-g}} \int d\tau \left( \frac{Dp^\nu}{D\tau} \right) \delta^4(x - z(\tau)).
\] (2.12)

For \( \nabla_\mu \nabla_\nu B^{\nu\mu\rho} \), we use the identity

\[
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \phi^{\alpha\beta\cdots} = R^\alpha_{\xi\mu\nu} \phi^{\xi\beta\cdots} + R^\beta_{\xi\mu\nu} \phi^{\alpha\xi\cdots}
\] (2.13)

for the tensor \( \phi^{\alpha\beta\cdots} \) and the symmetry properties of the Riemann tensor \( R^\mu_{\nu\alpha\beta} \) to get

\[
\nabla_\mu \nabla_\nu B^{\nu\mu\rho} = \frac{1}{2} R^\nu_{\rho\mu\sigma} S^{\rho\mu}.\] (2.14)

Finally, Eq.(2.11) becomes

\[
\frac{1}{\sqrt{-g}} \int d\tau \left[ \left( \frac{Dp^\nu}{D\tau} \right) + \frac{1}{2} R^\nu_{\rho\mu\sigma} S^{\rho\mu} \right] \delta^4(x - z(\tau)) = 0
\]

\[
\Rightarrow \frac{Dp^\nu}{D\tau} = -\frac{1}{2} R^\nu_{\rho\mu\sigma} S^{\rho\mu}.\] (2.15)

This is the second set of the Papapetrou equations. If the spin tensor vanishes,

\[
\frac{Dp^\nu}{D\tau} = 0,
\] (2.16)

which is just the geodesic equation for a point particle. Also from Eq.(2.10), we see that the canonical momentum can be written as

5
\[ p^\mu = m v^\mu - v_\nu \frac{D S^{\mu \nu}}{D \tau}. \]  

(2.17)

The spin tensor \( S^{\mu \nu} \) is anti-symmetric in \( \mu \) and \( \nu \), giving six independent components. Three of them can be eliminated by the “spin supplementary condition”,

\[ S^{\mu \nu} v_\nu = 0. \]  

(2.18)

For \( \mu = i \),

\[ S^{i0} v_0 + S^{ij} v_j = 0 \]

\[ \Rightarrow S^{0i} = \left( \frac{v_j}{v_0} \right) S^{ij}. \]  

(2.19)

The \( \mu = 0 \) equation is then satisfied automatically. The remaining three independent components of \( S^{ij} \) can be written as a vector,

\[ S^i = \frac{1}{2} \epsilon^{ijk} S^{jk}, \]  

(2.20)

which is the spin vector of the particle.

### III. POST-NEWTONIAN APPROXIMATION

In this section we develop the post-Newtonian expansion for \( N \) spinning particles using the energy-momentum tensor \( T^{\mu \nu} \) introduced in the last section. We follow closely the procedure in [11]. In the post-Newtonian approximation, it is assumed that the velocities of the particles are small,

\[ v \sim \epsilon^{1/2} \ll 1. \]  

(3.1)

Moreover, the gravitational potential \( \phi \) is comparable to the kinetic energy,

\[ \phi \sim v^2 \sim \epsilon. \]  

(3.2)

Here we take the spin \( S^i \) to be of order \( \epsilon^{1/2} \), the same as velocity. Under this assumption, the spin-orbit and spin-spin effects are both of order \( \epsilon \). This is valid for ordinary situations but
not compact objects for which these effects will come in only at higher orders. In this paper we wish to treat the former case, which is simpler, as an illustration of how our approach works.

To begin we expand the spacetime metric,

\[ g_{00} = -1 + g_{00}^{(2)} + g_{00}^{(4)} + \cdots, \quad (3.3) \]
\[ g_{0i} = g_{0i}^{(3)} + \cdots, \quad (3.4) \]
\[ g_{ij} = \delta_{ij} + g_{ij}^{(2)} + \cdots, \quad (3.5) \]

where

\[ g_{00}^{(2)} = -2\phi, \quad (3.6) \]
\[ g_{00}^{(4)} = -2(\phi^2 + \psi), \quad (3.7) \]
\[ g_{0i}^{(3)} = \zeta_i, \quad (3.8) \]
\[ g_{ij}^{(2)} = -2\phi\delta_{ij}. \quad (3.9) \]

Here the superscript denotes the order in the post-Newtonian expansion, for example, \( g_{00}^{(i)} \) is the (post) \( i/2 \)-Newtonian order of \( g_{00} \). Through the Einstein equations the potentials \( \phi, \zeta_i, \) and \( \psi \) can be expressed in terms of the various components of \( T^{\mu\nu} \) in the appropriate order,

\[ \phi(\vec{x}, t) = -\int d^3x' \frac{(T^{00}(\vec{x}', t))^{(0)}}{|\vec{x} - \vec{x}'|}, \quad (3.10) \]
\[ \zeta_i(\vec{x}, t) = -4\int d^3x' \frac{(T^{0i}(\vec{x}', t))^{(1)}}{|\vec{x} - \vec{x}'|}, \quad (3.11) \]
\[ \psi(\vec{x}, t) = \frac{\partial^2 \chi}{\partial t^2} - \int d^3x' \left[ (T^{00}(\vec{x}', t))^{(2)} + (T^{ii}(\vec{x}', t))^{(2)} \right], \quad (3.12) \]

where \( \nabla^2 \chi = \phi \). Note that we have used the harmonic gauge \[ g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0. \quad (3.13) \]
in deriving the above expressions.

As for \( T^{\mu\nu} \), we use the one given in the last section for N spinning particles,

\[ T^{\mu\nu} = T^{\mu\nu}_{(can)} + \nabla_\rho B^{\mu\nu\rho}, \quad (3.14) \]
where

\[ T^{\mu \nu}_{(can)} = \frac{1}{\sqrt{-g}} \sum_A p_A^\mu v_A^\nu \delta^3(\vec{x} - \vec{x}_A), \]  
\[ B^{\mu \nu \rho} = -\frac{1}{2} (S^{\mu \nu \rho} + S^{\nu \mu \rho} + S^{\mu \rho \nu}), \]  
\[ S^{\mu \nu \rho} = \frac{1}{\sqrt{-g}} \sum_A S_A^{\mu \nu \rho} \delta^3(\vec{x} - \vec{x}_A), \]  
\[ p_A^\mu = m_A v_A^\mu - v_A^\rho \frac{DS_A^{\mu \nu}}{D_{\tau A}}, \]

and \( A = 1, ..., N \). Assuming that the spin part is of the order of \( \epsilon^{1/2} \), we can expand \( T^{00}, T^{0i}, T^{i0} \) and \( T^{ij} \),

\[ (T^{00})^{(0)} = \sum_A m_A \delta^3(\vec{x} - \vec{x}_A), \]
\[ (T^{00})^{(2)} = \sum_A m_A (\phi_A + \frac{1}{2} v_A^2) \delta^3(\vec{x} - \vec{x}_A) - \sum_A \epsilon_{ijk} v_A^j S_A^k \partial_j \delta^3(\vec{x} - \vec{x}_A), \]
\[ (T^{0i})^{(1)} = \sum_A m_A v_A^i \delta^3(\vec{x} - \vec{x}_A) + \frac{1}{2} \sum_A \epsilon_{ijk} S_A^k \partial_j \delta^3(\vec{x} - \vec{x}_A) \]
\[ = (T^{00})^{(1)} \]
\[ (T^{0i})^{(3)} = \sum_A m_A (\phi_A + \frac{1}{2} v_A^2) v_A^i \delta^3(\vec{x} - \vec{x}_A) + \sum_A \epsilon_{ijk} v_A^j \left( \frac{dS_A^k}{dt} \right)^{(2)} \delta^3(\vec{x} - \vec{x}_A) \]
\[ + \sum_A \epsilon_{ijk} \left( \frac{dv_A^j}{dt} \right)^{(2)} S_A^k \delta^3(\vec{x} - \vec{x}_A) + \sum_A \epsilon_{ijk} S_A^k \phi_A \partial_j \delta^3(\vec{x} - \vec{x}_A) \]
\[ - \frac{1}{2} \sum_A \epsilon_{ikl} v_A^i v_A^j S_A^l \partial_j \delta^3(\vec{x} - \vec{x}_A) - \frac{1}{2} \sum_A \epsilon_{ikl} v_A^i v_A^j S_A^l \partial_j \delta^3(\vec{x} - \vec{x}_A) \]
\[ = (T^{00})^{(3)} + \sum_A \epsilon_{ijk} v_A^j \left( \frac{dS_A^k}{dt} \right)^{(2)} \delta^3(\vec{x} - \vec{x}_A), \]
\[ (T^{ij})^{(2)} = \sum_A m_A v_A^i v_A^j \delta^3(\vec{x} - \vec{x}_A) - \frac{1}{2} \sum_A \epsilon_{ijk} \left( \frac{dS_A^k}{dt} \right)^{(2)} \delta^3(\vec{x} - \vec{x}_A) \]
\[ + \sum_A \epsilon_{ikl} v_A^i S_A^l \partial_k \delta^3(\vec{x} - \vec{x}_A), \]
\[ (T^{ij})^{(4)} = \sum_A m_A (\phi_A + \frac{1}{2} v_A^2) v_A^i v_A^j \delta^3(\vec{x} - \vec{x}_A) - \frac{1}{2} \sum_A \epsilon_{ijk} \left( \frac{dS_A^k}{dt} \right)^{(4)} \delta^3(\vec{x} - \vec{x}_A) \]
\[ + \sum_A \epsilon_{ikl} \left( \frac{dv_A^j}{dt} \right)^{(2)} S_A^l \delta^3(\vec{x} - \vec{x}_A) + \sum_A \epsilon_{ikl} v_A^j \left( \frac{\partial \phi_A}{\partial t} \right) \delta^3(\vec{x} - \vec{x}_A) \]
requiring where we have used the convention for symmetrization of indices, On the other hand, for $T$ respectively, can be evaluated to be,

\[
T = \sum_A \epsilon^{ijkl} v^k_A S^l_A \phi_A \delta^3(\vec{x} - \vec{x}_A) - \sum_A \epsilon^{ijk} v^k_A S^m_A \rho_A \delta^3(\vec{x} - \vec{x}_A)
\]

\[
+ \sum \epsilon^{ijkl} S^l_A v^k_A (\partial^I \phi_A) \delta^3(\vec{x} - \vec{x}_A) - 2 \sum A \epsilon^{jkl} S^l_A (\partial^I \phi_A) \delta^3(\vec{x} - \vec{x}_A)
\]

\[
- \frac{1}{2} \sum A \epsilon^{jkl} S^l_A (\partial^I \zeta_A) \delta^3(\vec{x} - \vec{x}_A) + \frac{1}{2} \sum A \epsilon^{jkl} S^l_A (\partial^A \zeta_A) \delta^3(\vec{x} - \vec{x}_A),
\]

(3.24)

where we have used the convention for symmetrization of indices,

\[
A^{iBj} \equiv \frac{1}{2} (A^iB^j + A^jB^i)
\]

(3.25)

From this expansion of $T^{\mu\nu}$, the potentials $\phi$, $\zeta$, and $\psi$ in Eqs.(3.10), (3.11) and (3.12), respectively, can be evaluated to be,

\[
\phi = - \sum A \frac{m_A}{|\vec{x} - \vec{x}_A|},
\]

(3.26)

\[
\zeta = -4 \sum A \frac{m_A \vec{v}_A}{|\vec{x} - \vec{x}_A|} + \sum A \frac{(\vec{x} - \vec{x}_A) \times \vec{S}_A}{|\vec{x} - \vec{x}_A|^3},
\]

(3.27)

\[
\psi = \sum A \sum_{B \neq A} \frac{m_A m_B}{|\vec{x} - \vec{x}_A| |\vec{x}_A - \vec{x}_B|} - \frac{1}{2} \sum A \sum_{B \neq A} \frac{m_A (\vec{x} - \vec{x}_A)}{|\vec{x} - \vec{x}_A|} \cdot \frac{m_B (\vec{x}_A - \vec{x}_B)}{|\vec{x}_A - \vec{x}_B|^3} + \frac{1}{2} \sum A \frac{m_A |\vec{v}_A \cdot (\vec{x} - \vec{x}_A)|^2}{|\vec{x} - \vec{x}_A|^3} - 2 \sum A \frac{m_A v^2_A}{|\vec{x} - \vec{x}_A|} - 2 \sum A \frac{(\vec{x} - \vec{x}_A) \cdot (\vec{v}_A \times \vec{S}_A)}{|\vec{x} - \vec{x}_A|^3}.
\]

(3.28)

The equations of motion for $\vec{x}_A$ and $\vec{S}_A$ can be derived, as discussed in Section II, by requiring $T^{\mu\nu}$ to be symmetric and conserving. For $T^{\mu\nu}$ to be symmetric, we have the conditions,

\[
T^{0i} = T^{i0},
\]

(3.29)

\[
T^{ij} = T^{ji}.
\]

(3.30)

On the other hand, for $T^{\mu\nu}$ to be conserving, we have

\[
\nabla_\mu T^{\mu\nu} = 0
\]

\[
\Rightarrow \frac{\partial}{\partial t} T^{00} + \partial_j T^{j0} + 2\Gamma^{00}_{i0} T^{00} + 3\Gamma^{00}_{i0} T^{00} + \Gamma^{00}_{i0} T^{00} + \Gamma^{j0} T^{ij} + \Gamma^{ji} T^{ij} = 0,
\]

(3.31)
\[ \nabla_{\mu} T^{\mu i} = 0 \]
\[ \Rightarrow \frac{\partial}{\partial t} T^{0i} + \partial_j T^{ij} + \Gamma^0_{00} T^{0i} + \Gamma^i_{0j} T^{0j} + \Gamma^k_{k0} T^{0i} \]
\[ + \Gamma^k_{kj} T^{ij} + \Gamma^i_{00} T^{00} + 2 \Gamma^i_{0j} T^{0j} + \Gamma^i_{jk} T^{jk} = 0. \] (3.32)

The lowest order of Eq.(3.29) is satisfied automatically because \((T^{0i})^{(1)} = (T^{i0})^{(1)}\) as given in Eq.(3.21). From Eq.(3.30), we have
\[ (T^{ij})^{(2)} = (T^{ji})^{(2)} \]
\[ \Rightarrow \left( \frac{d\vec{S}_A}{dt} \right)^{(2)} = 0, \] (3.33)
which is the equation of motion for spins to the lowest order. From Eq.(3.31), we have
\[ \left( \frac{\partial}{\partial t} T^{00} \right)^{(0)} + \partial_i (T^{0i})^{(1)} = 0, \] (3.34)
which is also satisfied automatically, and so there is no new constraint. From Eq.(3.32),
\[ \left( \frac{\partial}{\partial t} T^{0i} \right)^{(1)} + \partial_j (T^{ij})^{(2)} + (\Gamma^i_{00})^{(2)} (T^{00})^{(0)} = 0 \]
\[ \Rightarrow \left( \frac{d\vec{v}_A}{dt} \right)^{(2)} = -\nabla^A \phi_A = - \sum_{B \neq A} \frac{m_B \vec{r}_{AB}}{r^3_{AB}}, \] (3.35)
where we have denoted \(\vec{x}_A - \vec{x}_B\) by \(\vec{r}_{AB}\). This equation is just the Newtonian equation of motion for point particles. The various components of the Christoffel symbol \((\Gamma^\mu_{\alpha\beta})^{(i)}\) can be found in [11].

Now we go to the next order. From Eq.(3.29), we have
\[ (T^{0i})^{(3)} = (T^{i0})^{(3)}, \] (3.36)
which is satisfied automatically as seen from Eqs.(3.22) and (3.33). Then from Eq.(3.30), we have
\[ (T^{ij})^{(4)} = (T^{ji})^{(4)} \]
\[ \frac{dS_A}{dt}^{(4)} = \left(2 \left( \frac{\partial \phi_A}{\partial t} \right) + \vec{v}_A \cdot \vec{\nabla}_A \phi_A \right) \vec{S}_A - (\vec{S}_A \cdot \vec{v}_A) \vec{\nabla}_A \phi_A + 2(\vec{S}_A \cdot \vec{\nabla}_A \phi_A) \vec{v}_A - \frac{1}{2} (\vec{S}_A \cdot \vec{\nabla}_A) \vec{A} + \frac{1}{2} \vec{S}_A (\vec{A} \cdot \vec{\nabla}_A) \]

\[ = \sum_{B \neq A} \frac{m_B \vec{r}_{AB} \cdot (\vec{v}_A - 2 \vec{v}_B)}{r_{AB}^3} \vec{S}_A - \sum_{B \neq A} \frac{m_B [(\vec{v}_A - 2 \vec{v}_B) \cdot \vec{S}_A] \vec{r}_{AB}}{r_{AB}^3} + 2 \sum_{B \neq A} \frac{\vec{S}_A \times \vec{S}_B}{r_{AB}^3} \vec{r}_{AB} + 3 \sum_{B \neq A} \frac{(\vec{r}_{AB} \cdot \vec{S}_A)}{r_{AB}^5} (\vec{r}_{AB} \times \vec{S}_B), \tag{3.37} \]

where we have used the results in Eqs. (3.26) to (3.28) for the potentials.

Next from Eq. (3.31),

\[ \left( \frac{\partial}{\partial t} T^{00} \right)^{(2)} + \partial_i (T^{0i})^{(3)} + 2 (\Gamma_0^0)^{(3)} (T^{00})^{(0)} + 3(\Gamma_0^0)^{(2)} (T^{00})^{(1)} + (\Gamma^0_i)^{(3)} (T^{0i})^{(0)} + (\Gamma^0_i)^{(2)} (T^{0i})^{(1)} = 0, \tag{3.38} \]

which is satisfied automatically because of the Newtonian equation of motion as in Eq. (3.35).

Then from Eq. (3.32),

\[ \left( \frac{\partial}{\partial t} T^{0i} \right)^{(4)} + \partial_j (T^{ij})^{(4)} + (\Gamma_0^0)^{(3)} (T^{0i})^{(1)} + (\Gamma_0^0)^{(2)} (T^{ij})^{(2)} + (\Gamma_i^j)^{(3)} (T^{0j})^{(0)} + (\Gamma_i^j)^{(2)} (T^{0j})^{(1)} = 0, \tag{3.39} \]

which gives

\[ \frac{d\vec{v}_A}{dt}^{(4)} = 3\vec{v}_A \left( \frac{\partial \phi_A}{\partial t} \right) - v_A^2 (\vec{\nabla}_A \phi_A) + 4\vec{v}_A (\vec{v}_A \cdot \vec{\nabla}_A \phi_A) - 4\phi_A (\vec{\nabla}_A \phi_A) - \vec{\nabla}_A \phi_A - \frac{\partial \vec{\zeta}_A}{\partial t} + \vec{v}_A \times (\vec{\nabla}_A \times \vec{\zeta}_A)
\]

\[ + \frac{1}{m_A} \vec{S}_A \times (\vec{\nabla}_A \left( \frac{\partial \phi_A}{\partial t} \right)) + \frac{1}{m_A} \vec{S}_A \times \vec{\nabla}(\vec{v}_A \cdot \vec{\nabla}_A \phi_A)
\]

\[ - \frac{2}{m_A} \vec{v}_A \left[ (\vec{v}_A \times \vec{S}_A) \cdot \vec{\nabla}_A \phi_A \right] + \frac{1}{2m_A} \vec{\nabla} \left[ \vec{S}_A \cdot (\vec{\nabla}_A \times \vec{\zeta}_A) \right]
\]

\[ = \sum_{B \neq A} \frac{m_B \vec{r}_{AB}}{r_{AB}^3} \left[ 4 \sum_{C \neq A} \frac{m_C}{r_{AC}^3} + \sum_{C \neq A, B} \frac{m_C}{r_{BC}^3} \right] \left( 1 - \frac{\vec{r}_{AB} \cdot \vec{r}_{BC}}{r_{BC}^2} \right) \]
in Eqs. (3.19) to (3.24) can be simplified to ordinary particles. Hence, Eqs. (3.33), (3.35), (3.37), and (3.40) form a complete set of equations of motion for \( \vec{x}_A(t) \) and \( \vec{S}_A(t) \) for spinning particles to the (post)\(^2\)-Newtonian order. This can be considered as an extension to the well-known Einstein-Infeld-Hoffmann equations [14] for ordinary particles.

Using these equations of motion, the expressions for the energy-momentum tensor \( T^{\mu\nu} \) in Eqs. (3.19) to (3.24) can be simplified to

\[
(T^{00})^{(0)} = \sum_A m_A \delta^3(\vec{x} - \vec{x}_A),
\]

\[
(T^{00})^{(2)} = \sum_A m_A (\phi_A + \frac{1}{2} v_A^2) \delta^3(\vec{x} - \vec{x}_A) - \sum_A \epsilon^{ijk} v_A^i S_A^j \partial_k \delta^3(\vec{x} - \vec{x}_A),
\]

\[
(T^{0i})^{(1)} = (T^{0i})^{(1)}
\]

\[
= \sum_A m_A v_A^i \delta^3(\vec{x} - \vec{x}_A) + \frac{1}{2} \sum_A \epsilon^{ijk} S_A^j A \partial_k \delta^3(\vec{x} - \vec{x}_A)
\]

\[
(T^{0i})^{(3)} = (T^{0i})^{(3)}
\]

\[
= \sum_A m_A (\phi_A + \frac{1}{2} v_A^2) v_A^i \delta^3(\vec{x} - \vec{x}_A)
\]

\[
+ \sum_A \epsilon^{ijk} S_A^j (\partial_k^A \phi_A) \delta^3(\vec{x} - \vec{x}_A) - \sum_A \epsilon^{ijk} S_A^j \partial_k \phi_A \delta^3(\vec{x} - \vec{x}_A)
\]

\[
- \frac{1}{2} \sum_A \epsilon^{ikl} v_A^i v_A^k S_A^l A \partial_j \delta^3(\vec{x} - \vec{x}_A) - \frac{1}{2} \sum_A \epsilon^{ikl} v_A^i v_A^k S_A^l A \partial_j \delta^3(\vec{x} - \vec{x}_A),
\]

\[
(T^{ij})^{(2)} = \sum_A m_A v_A^i v_A^j \delta^3(\vec{x} - \vec{x}_A) + \sum_A \epsilon^{ikl} v_A^i S_A^j \partial_k \delta^3(\vec{x} - \vec{x}_A),
\]
\[ (T^{ij})^{(4)} = \sum_A m_A(\phi_A + \frac{1}{2} v_A^2) v_A^i v_A^j \delta^3(\vec{x} - \vec{x}_A) \]
\[ + 2 \sum_A e^{ikl} v_A^k S_A^l (\phi_A \partial_k \delta^3(\vec{x} - \vec{x}_A)) - 4 \sum_A e^{ikl} v_A^k S_A^l (\partial_k \phi_A) \delta^3(\vec{x} - \vec{x}_A) \]
\[ + \sum_A e^{ikl} v_A^k S_A^l (\partial_A^j \phi_A) \delta^3(\vec{x} - \vec{x}_A) - \sum_A e^{ikm} v_A^k S_A^m v_A^l \partial_l \delta^3(\vec{x} - \vec{x}_A) \]
\[ - \frac{1}{2} \sum_A e^{ikl} S_A^l (\partial_A^j \phi_A) \delta^3(\vec{x} - \vec{x}_A) + \frac{1}{2} \sum_A e^{ikl} S_A^l (\partial_A^j \phi_A) \delta^3(\vec{x} - \vec{x}_A), \quad (3.46) \]

and the expressions for the potentials \( \phi_A \) and \( \zeta_A^i \) are those given in Eqs.(3.26) and (3.27).

These components of the energy-momentum tensor \( T^{\mu\nu} \) to various post-Newtonian order will be used in the next section to find the gravitational wave generated by a system of \( N \) spinning particles in the post-Newtonian approximation.

### IV. GRAVITATIONAL WAVES GENERATION

In this section we follow the works of EW [12] and Wagoner and Will [13] to consider the gravitational wave generation of \( N \) spinning particles in the post-Newtonian approximation. As in [12], we derive the waveform and the luminosity of the gravitational waves up to post-Newtonian order.

The waveform of the gravitational wave in the radiation zone is given by [12]

\[ h_{TT}^{ij} = \frac{2}{R} \frac{\partial^2}{\partial t^2} \left[ I^{ij}(t - R) + n_k I^{jkl}(t - R) + n_k n_l I^{ijkl}(t - R) \right]_{TT}, \quad (4.1) \]

where \( TT \) denotes the transverse-traceless part, \( n_k \equiv R_k/R \), and

\[ I^{ij} = \int \tau^{00} x^i x^j d^3x, \quad (4.2) \]
\[ I^{jkl} = \int (2\tau^0 x^j x^k - \tau^{0k} x^i x^j) d^3x, \quad (4.3) \]
\[ I^{ijkl} = \int \tau^{ijkl} x^i x^j d^3x, \quad (4.4) \]

where \( \tau^{\mu\nu} \) is the total energy-momentum tensor,

\[ \tau^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu}, \quad (4.5) \]

which includes the matter part \( T^{\mu\nu} \) and the gravitational part \( t^{\mu\nu} \).
Here, for spinning particles, we use the energy-momentum tensor $T^{\mu\nu}$ discussed in the last section. While $t^{\mu\nu}$, up to post-Newtonian order, is given in [13]. Therefore,

$$\tau^{00} = \sum_A m_A(1 + \phi_A + \frac{1}{2}v_A^2)\delta^3(\vec{x} - \vec{x}_A) - \sum_A \epsilon^{ijk}v_A^jS_A^k\partial_i\delta^3(\vec{x} - \vec{x}_A)$$
$$- \frac{1}{8\pi}(4\phi\partial^2\phi + 3\partial_i\phi\partial_i\phi), \quad (4.6)$$

$$\tau^{0i} = \sum_A m_A v_A^i \delta^3(\vec{x} - \vec{x}_A) - \frac{1}{2} \sum_A \epsilon^{ijk} S_A^i \partial_k \delta^3(\vec{x} - \vec{x}_A), \quad (4.7)$$

$$\tau^{ij} = \sum_A m_A v_A^i v_A^j \delta^3(\vec{x} - \vec{x}_A) + \sum_A \epsilon^{(ikl} S_A^j \partial_k \delta^3(\vec{x} - \vec{x}_A)$$
$$+ \frac{1}{8\pi} \left[-2\partial_i\phi\partial_j\phi + 4\phi\partial_i\phi\partial_j\phi + \delta_{ij}(4\phi\partial^2\phi + 3\partial_k\phi\partial_k\phi)\right], \quad (4.8)$$

up to post-Newtonian order, where the last terms in Eq.(4.6) and Eq.(4.8) come from the contributions of $t^{00}$ and $t^{ij}$, respectively. Then the integrals in Eqs.(4.2) to (4.4) can be evaluated,

$$I^{ij} = \sum_A m_A x_A^i x_A^j \left(1 + \frac{1}{2}v_A^2 - \frac{1}{2} \frac{m_B}{r_{AB}} \right) + 2 \sum_A \epsilon^{(ikl} S_A^j x_A^i, \quad (4.9)$$

$$I^{ijk} = \sum_A m_A (2v_A^i x_A^j x_A^k - v_A^i x_A^j x_A^k) + 2 \sum_A \epsilon^{(ikl} x_A^j S_A^i, \quad (4.10)$$

$$I^{ijkl} = \sum_A m_A v_A^i v_A^j x_A^k x_A^l - \frac{1}{12} \sum_A \sum_{B \neq A} \frac{m_A m_B r_A^i r_A^j}{r_{AB}} \left[\delta^{kl} + \frac{1}{r_{AB}} (-r_A^k r_A^l + 6 x_A^k x_A^l) \right]$$
$$- \sum_A \left(\epsilon^{(ikm} x_A^j x_A^l + \epsilon^{(ilm} x_A^j x_A^k) S_A^m, \quad (4.11)\right)$$

In obtaining these expressions, we have discarded terms with vanishing transverse-traceless parts, of which some are actually divergent, because those terms will not contribute to the generation of gravitational waves [13]. For example,

$$\langle\delta_{ij}\rangle_{TT} = 0, \quad (4.12)$$
$$\langle n^i f^j \rangle_{TT} = 0, \quad (4.13)$$

for some function $f^i$. Also we have used the formula,

$$\int \phi \partial_i \phi \partial_j x_A^k x_A^l \delta^3 x$$
$$= \sum_A \sum_{B \neq A} \frac{\pi m_A m_B r_A^i r_A^j}{3r_{AB}} \left[\delta^{kl} + \frac{1}{r_{AB}} \left(2r_A^k r_A^l - 6 v_A^k x_A^l + 6 x_A^k x_A^l\right) \right], \quad (4.14)$$
which can be proved [13] by ignoring terms which have no transverse-traceless parts in a similar way.

Putting these together the waveform of the gravitational wave, up to post-Newtonian order, is

\[
    h_{ij}^{TT} = \frac{2}{R} \frac{\partial^2}{\partial t^2} \left( \sum_A m_A x_A^i x_A^j \left( 1 - \hat{n} \cdot \vec{v}_A + \frac{1}{2} v_A^2 - \frac{1}{2} \sum_B m_B \frac{1}{r_{AB}} \right) + 2 \sum_A m_A v_A^i x_A^j (\hat{n} \cdot \vec{x}_A)^2 \right.
\]

\[
    - \frac{1}{12} \sum_A \sum_{B \neq A} \frac{m_A m_B r_{AB}^i}{r_{AB}} \left[ 1 + \frac{1}{r_{AB}^2} \left( - (\hat{n} \cdot \vec{r}_{AB})^2 + 6 (\hat{n} \cdot \vec{x}_A)^2 \right) \right]
\]

\[
    + 2 \sum_A x_A^i (\vec{v}_A \times \vec{S}_A)^i - 2 \sum_A x_A^i (\hat{n} \times \vec{S}_A)^i - 2 \sum_A (\hat{n} \cdot \vec{x}_A) v_A^i (\hat{n} \times \vec{S}_A)^i \right)_{TT}.
\]

(4.15)

Hence, together with the equations of motion in Eqs.(3.33), (3.35), (3.37) and (3.40), one can obtain the gravitational waves generated by N spinning particles.

The total luminosity of the gravitational wave is also given in [12] as,

\[
    L = \frac{1}{5} N_{ij} N_{ij} + \frac{1}{105} (11 N_{ijk} N_{ijk} - 6 N_{ijj} N_{ikk}
\]

\[
    - 6 N_{ij} N_{ik} + 22 N_{ij} N_{ikj} - 24 N_{ij} N_{ikij} + \cdots),
\]

(4.16)

where the bracket denotes an average over several characteristic wavelengths, and

\[
    N_{ijk \cdots k_m} \equiv \frac{d^3}{dt^3} \left( I_{ijk \cdots k_m} - \frac{1}{3} \delta^{ij} I_{ilk \cdots k_m} \right).
\]

(4.17)

Up to post-Newtonian order, \( L \) can be expanded to

\[
    L = \frac{1}{5} N_{ij}^{(0)} N_{ij}^{(0)} + \frac{1}{21} (42 N_{ij}^{(0)} N_{ij}^{(2)} + 11 N_{ijk}^{(1)} N_{ijk}^{(1)} - 6 N_{ijj}^{(1)} N_{ikk}^{(1)}
\]

\[
    - 6 N_{ijk}^{(1)} N_{ikj}^{(1)} + 22 N_{ij}^{(0)} N_{ij}^{(0)} + 24 N_{ij}^{(0)} N_{ikij}^{(2)}).
\]

(4.18)

Using the results for \( I^{ij} \), \( I^{ijk} \), and \( I^{ijkl} \) in Eqs.(4.9), (4.10), and (4.11), respectively, one can obtain the various orders of \( N_{ijk \cdots k_m} \) as follows.

\[
    N_{ij}^{(0)} = \frac{d^3}{dt^3} \left( \sum_A m_A \left( x_A^i x_A^j - \frac{1}{3} \delta^{ij} x_A^2 \right) \right),
\]

(4.19)
This completes our consideration of the gravitational wave generated by the motion of N spinning particles to the post-Newtonian order.

V. CONCLUSIONS AND DISCUSSIONS

In this work we consider a method to develop the post-Newtonian expansion for both the equations of motion and the generation of gravitational wave of N spinning particles. In this approach the energy-momentum tensor $T_{\mu\nu}$ for spinning particles, due to Dixon and BI, is introduced first. Using this $T_{\mu\nu}$, one can derive the Papapetrou equations by requiring it to be symmetric and conserving. By the formalism of EW, we can then obtain the waveform and the luminosity of the gravitational wave generated by the motion of these spinning particles.

This approach is straight forward and much simpler than the BDI multipole formalism since we still regard the particles as point-like. It is self-contained in the sense that once we write down $T_{\mu\nu}$, the equations of motion can also be derived from it. Another merit is that the BI formalism is very general so that structures other than spin, such as tidal effects,
can also be taken into account systematically by introducing terms corresponding to these structures in $T^\mu\nu$.

However, there are several points in our calculation that we must be careful with. First, the EW formalism is less rigorous than the BDI formalism in which integrals like those in Eqs.(4.2) to (4.4) have divergent parts. Fortunately, these divergent parts have no transverse-traceless parts so they will not contribute to the generation of gravitational waves. We therefore expect the two formalisms to give the same results.

Another point to note is that the harmonic coordinate condition used in Section III, as well as in [11], is not the same as the gauge condition used in the EW formalism. It is nevertheless shown in [15] that, at least to the post-Newtonian order that we are considering here, the extra terms coming from these different gauge choices all have vanishing transverse-traceless parts. Therefore, there is no inconsistency in our investigation in this paper concerning gauge conditions. On the other hand, if we want to extend our consideration to higher post-Newtonian orders, we must be more careful with gauge choices.

In our calculation we have assumed that spin $\vec{S}$ is of post-Newtonian order, same as velocity $\vec{v}$. Then the spin-orbit and spin-spin effects come in both at the post-Newtonian order. Since these effects for compact objects come in at even higher orders, we are really dealing with ordinary objects. We have chosen this simpler case just as an illustration of how our formalism works for spinning particles while staying at the post-Newtonian level. Of course, the compact case is more interesting because of the possibility of detection of their coalescences by LIGO and VIRGO in the near future. We plan to consider that situation in a separate work.

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