A unified BFKL and GLAP description of $F_2$ data

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Abstract

We argue that the use of the universal unintegrated gluon distribution and the $k_T$ (or high energy) factorization theorem provides the natural framework for describing observables at small $x$. We introduce a coupled pair of evolution equations for the unintegrated gluon distribution and the sea quark distribution which incorporate both the resummed leading $\ln(1/x)$ BFKL contributions and the resummed leading $\ln(Q^2)$ GLAP contributions. We solve these unified equations in the perturbative QCD domain using simple parametric forms of the nonperturbative part of the integrated distributions. With only two (physically motivated) input parameters we find that this $k_T$ factorization approach gives an excellent description of the measurements of $F_2(x, Q^2)$ at HERA. In this way the unified evolution equations allow us to determine the gluon and sea quark distributions and, moreover, to see the $x$ domain where the resummed $\ln(1/x)$ effects become significant. We use $k_T$ factorization to predict the longitudinal structure function $F_L(x, Q^2)$ and the charm component of $F_2(x, Q^2)$. 
1. Introduction

The experiments at HERA have opened up the small Bjorken $x$ regime. One of the most striking features of the data is the strong rise of the structure function $F_2$ as $x$ decreases from $10^{-2}$ to below $10^{-4}$ [1]. At first sight it appeared that the rise was due to the (BFKL) resummation of leading $\ln(1/x)$ contributions [2, 3]. However, with an appropriate choice of input distributions and of the starting scale for the $Q^2$ evolution, the observed growth can also be reproduced within the conventional GLAP framework which just sums up the leading (and next-to-leading) $\ln Q^2$ contributions. Indeed GLAP global fits exist which give a good description of the small $x$ measurements of $F_2$ in the $x$ range accessible at HERA [4, 5], see also [6, 7]. Is it possible to conclude that there will be no significant BFKL-type contributions to $F_2$ in the HERA small $x$ regime or does a physically reasonable alternative description exist with sizeable $\ln(1/x)$ resummation contributions? Here we address this question. Clearly the specification of the non-perturbative input to the QCD evolution will be crucial.

Recall that the basic dynamical quantity at low $x$ is the gluon distribution $f(x,k_T^2)$ unintegrated over its transverse momentum $k_T$. It is related to the conventional gluon density $g(x,Q^2)$ by

$$xg(x,Q^2) = \int_Q^{Q^2} \frac{dk_T^2}{k_T^2} f(x,k_T^2).$$

(1)

In the leading $\ln(1/x)$ approximation $f(x,k_T^2)$ satisfies the BFKL equation and exhibits an $x^{-\omega_0}$ growth and a diffusion in $\ln k_T^2$ as $x \to 0$, where the BFKL or so-called hard Pomeron intercept $\omega_0 = (3\alpha_S/\pi)4 \ln 2$ for fixed $\alpha_S$. The observable quantities are computed in terms of $f$ via the $k_T$ (or high energy) factorization prescription [8, 9]. For example, the structure functions $F_i$ are given by

$$F_i = F_i^{\gamma g} \otimes f$$

(2)

where $\otimes$ denotes a convolution in transverse, as well as longitudinal, momentum. Here $F_i^{\gamma g}$ are the off-shell gluon structure functions, which at lowest order are determined by the quark box (and crossed-box) contributions to photon-gluon fusion, see Fig. 1.

The BFKL gluon $f(x,k_T^2)$ and $k_T$ (or high energy) factorization theorem were used [10] to predict the small $x$ behaviour of $F_2$ prior to the measurements at HERA. The method used a starting distribution $f(x_0,k_T^2)$ at, say, $x_0 = 0.01$ deduced from an integrated gluon $g(x_0,Q^2)$ which had been determined in a global parton analysis of fixed-target deep inelastic and related scattering data. With this input the BFKL equation was solved to determine $f(x,k_T^2)$ in the small $x$ domain ($x < x_0$). A major uncertainty in this procedure to predict $F_2$ is the treatment of the infrared region, $k_T^2 < k_0^2$. A recent study [11] using this approach finds that the BFKL predictions for $F_2$ increase too steeply with decreasing $x$ in comparison with the HERA measurements, and concludes that there is no evidence for the effects of the resummation of $\ln(1/x)$ terms. Before we accept such a conclusion we should note the limitations of this form of the test of the BFKL $k_T$-factorization approach.
It is assumed [11, 12] that in the infrared region \( f(x, k_T^2) \) has the form

\[
f(x, k_T^2 < k_0^2) = \frac{k_T^2}{k_T^2 + k_a^2} \left( \frac{k_0^2}{k_T^2 + k_a^2} \right) f(x, k_0^2)
\]

(3)

with \( k_0^2 = 1 \, \text{GeV}^2 \) say, and where \( k_a^2 \) is an adjustable parameter. It turns out that there is a sizeable contribution from the infrared non-perturbative region which, since it is tied to \( f(x, k_0^2) \), is forced to have the BFKL growth with decreasing \( x \). The formalism does not include the GLAP leading \( \ln Q^2 \) resummations which go beyond the leading \( \ln(1/x) \) approximation. Finally it should be noted that the BFKL equation for \( f(x, k_T^2) \) only resums the leading order \( \ln(1/x) \) terms. Sub-leading \( \ln(1/x) \) effects are expected to reduce the growth of \( f \) with decreasing \( x \).

Clearly this simplified procedure provides only a crude test of the underlying dynamics in the small \( x \) (HERA) domain. The main deficiencies are the treatment of the infrared region and the lack of a unified approach which incorporates both the BFKL \( \ln(1/x) \) and the GLAP \( \ln(Q^2) \) resummations. An important development is the demonstration by [8, 9, 10] that at the leading twist level the BFKL \( k_T \) factorization approach can be reduced to the conventional collinear GLAP factorization in which the anomalous dimensions and coefficient functions are extended to include the full resummation of leading \( \ln(1/x) \) terms. This motivated an informative study by Thorne [13]. He has shown (in a scheme independent way) that the inclusion of the \( \ln(1/x) \) terms within the collinear factorization approach gives a satisfactory, and even an improved, description of the \( F_2 \) data. He assumes non-perturbative components to the input distributions for the observables at scale \( Q_0^2 \) which are ‘flat’ at small \( x \), and which are the **only** contributions at the scale \( (A_{LL} \lesssim 1 \text{GeV}^2) \) that denotes the boundary between the perturbative and non-perturbative regions. In this way he demands that the rise of the structure functions with decreasing \( x \) must entirely come from perturbative effects. In summary Thorne finds, within the collinear factorization framework, that the ‘BFKL’ \( \ln(1/x) \) terms are not excluded, but rather are favoured, by the \( F_2 \) data.

The agreement with the data obtained by Thorne [13] is contrary to the conclusion of ref. [11] which was based on the BFKL gluon and \( k_T \) factorization. This suggests that the application of the \( k_T \) factorization approach was too simplistic. Here we re-examine and improve the determination of \( f(x, k_T^2) \) and, via \( k_T \) factorization, obtain a more realistic description of \( F_2 \).

The first improvement is that we study a ‘unified’ equation for \( f(x, k_T^2) \) which incorporates BFKL and GLAP evolution on an equal footing [14]. To be precise we solve a coupled pair of integral equations for the gluon and sea quark distributions, as well as allowing for the effects of the valence quarks. In this way we eliminate the problems of matching at \( x = x_0 \). A second improvement is a more physical treatment of the non-perturbative (or infrared) contributions to the BFKL equation and the \( k_T \) factorization integrals. We shall see that the former can be specified entirely by the integrated gluon distribution at the scale \( Q^2 = k_0^2 \) which marks the boundary of the perturbative and non-perturbative regions, whereas the integrals also need a non-perturbative component of the sea. In fact we find that an excellent description of the HERA measurements of \( F_2 \) is possible in terms of just two physically motivated parameters which fully determine these infrared contributions.
There is considerable merit in going back from the collinear to the $k_T$ factorization approach for the description of small $x$ deep inelastic scattering. Indeed, in the reduction to collinear form we loose some of the physical structure which is contained in the gluon ladder and $k_T$ factorization. We discuss these limitations of the collinear approach below.

First we note that the high energy (or low $x$) behaviour of the structure functions is driven by the BFKL gluon ladder coupled (through $k_T$ factorization) to the photon via the quark box. The corresponding Feynman diagram has a calculable perturbative contribution for all $Q^2$ (except for $Q^2 \rightarrow 0$ for massless quarks). The ‘hard’ or ‘QCD’ pomeron contribution generated by this ladder is thus present for all $Q^2$. In particular there is a known perturbative contribution to the structure functions at $Q^2 = k_T^2$ coming from configurations in which the gluon transverse momenta within the ladder lie in the perturbative domain $k_T^2 > k_0^2$. This contrasts with the collinear factorization approach in which it is contrived to describe the observables in terms of a purely non-perturbative contribution at some scale, say $Q^2 = k_0^2$. The perturbative component, which must be present at $Q^2 = k_0^2$, will be evident in the $k_T$ factorization approach that we introduce below.

A more subtle limitation of the Renormalisation Group (RG) and collinear factorization approach at small $x$ concerns the treatment of the running of $\alpha_S$. We solve the BFKL equation with running, rather than fixed, coupling where $\alpha_S$ depends on the local scale $k_T^2$ along the ladder. This way of implementing the running of $\alpha_S$ is supported by the calculation of next-to-leading $\ln(1/x)$ effects [15, 16]. The solution of the BFKL equation with running $\alpha_S$ can be reduced to the conventional RG form using saddle point techniques. However, the saddle point approximation is not applicable for arbitrarily small values of $x$ [17, 3, 15, 16, 18, 19].

A third advantage of the $k_T$ factorization approach is that it allows us to appropriately constrain the transverse momenta of the emitted gluons along the BFKL ladder. We are therefore able to quantify the effect of imposing this constraint. (Recall that in the usual application of the BFKL equation the gluon transverse momenta are taken to be unlimited.) We will see that the kinematic constraint largely subsumes the angular ordering constraint which is the basis of the ‘CCFM’ equation [20, 21]. The CCFM equation also incorporates both BFKL and GLAP evolution.

Another difference is that the BFKL contribution is a sum over all twists, whereas when it is reduced to collinear form only the leading twist is conventionally retained. Finally $k_T$ factorization is much simpler to implement at small $x$ than collinear factorization. We deal with dynamical quantities (namely the BFKL kernel and the structure function of the off-shell gluon) which can be calculated perturbatively. We calculate them to first order in $\alpha_S$. Essentially the $\alpha_S \ln(1/x)$ terms are effectively resummed by simply integrating over the entire $k_T^2$ phase space allowed for the gluon ladder and the $k_T$ factorization integrals.

In summary, the natural framework with which to describe observables at small $x$ is the \textit{unintegrated} gluon density $f(x,k_T^2)$ together with the $k_T$ factorization theorem. (Here we use it to calculate the observable deep inelastic scattering structure functions $F_2$ and $F_L$.) That is, at small $x$ the distribution $f(x,k_T^2)$ is the basic, universal quantity which can be taken from
process to process. If we were to reduce this framework to a collinear factorized form then the “simple”, but rich, physics structure of the gluon ladder is not fully taken into account and even may be distorted. It could be argued that most of the effects occur at subleading order in \( \ln(1/x) \) and since only the leading order is completely known, little is lost. However, the effects have a direct physical origin. They are clearly present and are expected to be the dominant corrections in a more complete analysis.

Since we shall unify the BFKL and GLAP formalisms, the resulting equation for \( f(x, k_T^2) \) is valid both for small \( x \) and large \( x \). Moreover the region where the ‘BFKL’ \( \ln(1/x) \) effects become significant will be decided by the underlying dynamics (QCD). We use the formalism to fit to deep inelastic data and so we are able to quantify the importance of BFKL effects.

The outline of the paper is as follows. In Section 2 we make modifications to the BFKL equation for the unintegrated gluon distribution \( f(x, k^2) \) which allow for the GLAP leading \( \ln Q^2 \) contributions and which enable all the non-perturbative effects to be encapsulated in an input distribution for the integrated gluon, \( xg(x, k_0^2) \). In Section 3 we introduce the equation for the quark singlet (momentum) distribution \( \Sigma(x, Q^2) \), again paying particular attention to isolate the contribution to the non-perturbative region and to ensure that the perturbative terms are allowed to contribute for all \( Q^2 \). In Section 4 we numerically solve the coupled integral equations for \( f \) and \( \Sigma \). The \( k_T \) factorization theorem is used to calculate \( F_2(x, Q^2) \) as a function of the two parameters that specify the input gluon (which we take to be ‘flat’ at small \( x \)). Optimum fits to the available \( F_2 \) data at small \( x \) are presented, and predictions are made for the charm component \( F_2^c \) and for the longitudinal structure function \( F_L \). Finally, in Section 5 we present our conclusions.

2. Unified BFKL and GLAP equation for the gluon

We start from the BFKL equation for the unintegrated gluon distribution

\[
\begin{align*}
    f(x, k^2) &= f^{(0)}(x, k^2) \\
    &+ \alpha_S(k^2) k^2 \int_x^1 \frac{dz}{z} \int \frac{dk'^2}{k'^2} \left\{ f\left(\frac{x}{z}, k'^2\right) - f\left(\frac{x}{z}, k^2\right) \right\} + f\left(\frac{x}{z}, k^2\right) [4k'^4 + k^4]^{\frac{1}{2}} \right\}
\end{align*}
\]

where \( \alpha_S = 3\alpha_S/\pi \) and \( k \equiv k_T, k' \equiv k'_T \) denote the transverse momenta of the gluons, see Fig. 1. The equation corresponds to the leading \( \ln(1/x) \) approximation.

2.1 From the BFKL to the unified equation

In order to make the BFKL equation for the gluon more realistic and to extend its validity to cover the full range of \( x \) we make the following modifications. First, to incorporate leading order GLAP evolution, we add on to the right-hand side of (4) the term [14]

\[
\alpha_S(k^2) \int_x^1 \frac{dz}{z} \left( \frac{z}{6} P_{gg}(z) - 1 \right) \frac{x}{z} g\left(\frac{x}{z}, k^2\right)
\]
\[ \equiv \alpha_s(k^2) \int_x^1 \frac{dz}{z} \left( \frac{z}{6} P_{gg}(z) - 1 \right) \left\{ \frac{x}{z} g \left( \frac{x}{z}, k_0^2 \right) + \int_{k_0^2}^{k^2} \frac{dk'^2}{k'^2} f \left( \frac{x}{z}, k'^2 \right) \right\}, \]

where we have used (1). The \(-1\) allows for the contribution which is already included in the BFKL equation. The inclusion of the additional term (5) gives contributions to the gluon anomalous dimension \(\gamma_{gg}\) which are subleading in \(\alpha_s \ln(1/x)\) but leading in \(\alpha_s\). These standard leading order GLAP contributions have an impact on the gluon intercept. They soften the small \(x\) rise of the gluon distribution and also change its overall normalisation.

The second modification to (4) is the introduction of the kinematic constraint [22, 23]

\[ k'^2 < \frac{k^2}{z} \]

on the real gluon emission term, that is, on the integral over \(f(x/z, k'^2)\). The origin of the constraint is the requirement that the virtuality of the exchanged gluon is dominated by its transverse momentum, \(|k'^2| \approx k_T^2\). For clarity we have restored the subscript \(T\) in this equation. The constraint is another physically motivated, subleading correction in \(\alpha_s \ln(1/x)\).

Thirdly, we notice that the integration region over \(k'^2\) in (4) extends down to \(k'^2 = 0\) where we expect that non-perturbative effects will affect the behaviour of \(f(x, k'^2)\). We are only going to solve equation (4) in the perturbative region, defined by \(k^2 > k_0^2\), so we only have to worry about the infrared contribution due to the real emission term from the interval \(0 < k'^2 < k_0^2\).

We may rewrite this infrared contribution in the form

\[ k^2 \int_{0}^{k_0^2} \frac{dk'^2}{k'^2 - k^2} f \left( \frac{x}{z}, k'^2 \right) \simeq \int_{0}^{k_0^2} \frac{dk'^2}{k'^2} f \left( \frac{x}{z}, k'^2 \right) \equiv \frac{x}{z} g \left( \frac{x}{z}, k_0^2 \right). \]

The parameter \(k_0^2(\equiv Q_0^2)\) denotes the border between the perturbative and non-perturbative regions. Its magnitude will be taken to be around 1GeV\(^2\).

Finally we must of course add to the right-hand side of (4) the term which allows the quarks to contribute to the evolution of the gluon, that is

\[ \frac{\alpha_s(k^2)}{2\pi} \int_x^1 \frac{dz}{z} P_{gg}(z) \Sigma \left( \frac{x}{z}, k^2 \right) \]

where \(\Sigma\) is the singlet quark momentum distribution. To be explicit

\[ \Sigma(x, k^2) = \sum_{q=u,d,s} x(q + \bar{q}) + x(c + \bar{c}) \]

\[ \equiv V(x, k^2) + S_{uds}(x, k^2) + S_c(x, k^2) \]

where \(V, S_{uds}\) and \(S_c\) denote the valence, the light sea quark and the charm quark contributions respectively. We discuss the evolution equation for \(\Sigma(x, k^2)\) in the next section.
Gathering together all the above modifications, equation (4) for the gluon becomes

\[
f(x, k^2) = \tilde{f}(0)(x, k^2) + \alpha_s(k^2) k^2 \int_x^1 \frac{dz}{z} \int_{k_0^2}^{k^2} \frac{dk'^2}{k'^2} \left\{ \frac{f\left(\frac{x}{z}, k'^2\right) \Theta\left(\frac{k^2}{z} - k'^2\right)}{|k'^2 - k^2|} - f\left(\frac{x}{z}, k^2\right) \right\} + \frac{f\left(\frac{x}{z}, k^2\right)}{[4k'^4 + k^2]^\frac{1}{2}}
\]

\[
+ \alpha_s(k^2) \int_x^1 \frac{dz}{z} \left( \frac{z}{6} P_{gg}(z) - 1 \right) \int_{k_0^2}^{k^2} \frac{dk'^2}{k'^2} f\left(\frac{x}{z}, k'^2\right) + \frac{\alpha_s(k^2)}{2\pi} \int_x^1 dz P_{gg}(z) \frac{x}{z} g\left(\frac{x}{z}, k^2\right)
\]

where now the driving term has the form

\[
\tilde{f}(0)(x, k^2) = f(0)(x, k^2) + \frac{\alpha_s(k^2)}{2\pi} \int_x^1 dz P_{gg}(z) x g\left(\frac{x}{z}, k^2\right).
\]

In (10) we include the constraint \(k'^2 > k_0^2\) on the virtual, as well as the real, contributions in order to avoid spurious singularities at \(k^2 = k_0^2\). In the perturbative region, \(k^2 > k_0^2\), we may safely neglect the genuinely non-perturbative contribution \(f(0)(x, k^2)\) which is expected to decrease strongly with increasing \(k^2\). It is important to note that we have avoided the necessity to parametrize \(f(x, k^2)\) in the non-perturbative region. Equation (10) only involves \(f(x, k^2)\) in the perturbative domain, \(k^2 > k_0^2\). The input (11) is provided by the conventional gluon at the scale \(k_0^2\). That is the input to our ‘unified BFKL + GLAP’ equation is determined by the same distribution as in conventional GLAP evolution. The modifications to (4) allow us to overcome the serious limitations discussed in the introduction. Surprisingly, we find that we can achieve an excellent description of all the deep inelastic data using the most economical parametrization of the input gluon

\[xg(x, k_0^2) = N(1 - x)^β.\]

In particular the observed growth in \(F_2(x, Q^2)\) with decreasing \(x\) is generated entirely by perturbative (\(\ln(1/x)\) and \(\ln Q^2\)) dynamics.

It is easy to see how eq. (10) reduces to the conventional GLAP evolution equation for the gluon in the leading \(\ln Q^2\) (or rather \(\ln k^2\)) approximation. The leading \(\ln k^2\) terms arise from the strongly ordered configuration, \(k_0^2 \ll k^2 \ll k^2\), for the real emission contributions and to the neglect of the virtual contributions. Then (10) becomes

\[
f(x, k^2) = \frac{\alpha_s(k^2)}{2\pi} \int_x^1 dz P_{gg}(z) \left[ \frac{x}{z} g\left(\frac{x}{z}, k_0^2\right) + \int_{k_0^2}^{k^2} \frac{dk'^2}{k'^2} f\left(\frac{x}{z}, k'^2\right) \right]
\]

\[
+ \frac{\alpha_s(k^2)}{2\pi} \int_x^1 dz P_{gg}(z) \Sigma\left(\frac{x}{z}, k^2\right),
\]

(12)
where we have taken into account (11) and the remarks concerning the omission of \( f^{(0)} \). Upon using (1) we see that (12) becomes

\[
\frac{\partial (xg(x,k^2))}{\partial \ln k^2} = \frac{\alpha_S(k^2)}{2\pi} \int_x^1 dz \left[ P_{gg}(z) \frac{x}{z} g \left( \frac{x}{z}, k^2 \right) + P_{gq}(z) \Sigma \left( \frac{x}{z}, k^2 \right) \right],
\]

which is simply the GLAP evolution equation for the gluon.

### 2.2 Anomalous dimension of the gluon.

We will solve (10) for the unintegrated gluon. However first we anticipate the general behaviour of the anomalous dimension of the gluon which will come from this equation. To do this we rewrite the equation in terms of the moment function

\[
f(\omega,k^2) = \int_0^1 dx x^{-1} f(x,k^2).
\]

We have

\[
\mathcal{F}(\omega,k^2) = \mathcal{F}^{(0)}(\omega,k^2) + \frac{\alpha_S(k^2)}{\omega} k^2 \int_{k_0^2}^{\infty} \frac{dk^2}{k'^2} \left\{ \begin{array}{c}
\mathcal{F}(\omega,k'^2) \left[ \Theta(k'^2 - k^2) + \frac{k'^2/k^2}{\omega} \Theta(k'^2 - k^2) \right] - \mathcal{F}(\omega,k^2) \\
\end{array} \right\} + \frac{\mathcal{F}(\omega,k^2)}{(4k'^4 + k^4)^{1/2}}
\]

\[
+ \frac{\alpha_S(k^2)}{\omega} P(\omega) \int_{k_0^2}^{k^2} \frac{dk'^2}{k'^2} \mathcal{F}(\omega,k'^2)
\]

where we have neglected, for simplicity, the contribution coming from the quarks and where \( P(\omega) \) is the moment function of \((zP_{gg}(z)/6 - 1)\). The term in square brackets is due to the kinematic constraint. Without this constraint we would have 1 instead of \((k'^2/k^2)^{1/2}\), and the two \( \Theta \) functions would simply sum to unity.

For large \( k^2 \) the moment function behaves as

\[
\mathcal{F}(\omega,k^2) \sim \left( \frac{k^2}{k_0^2} \right)^{\gamma_{gg}(\omega)}
\]

where, for illustration, we take fixed \( \alpha_S \). The quantity \( \gamma_{gg} \) is the anomalous dimension of the gluon. If we insert (16) into (15) then we find, after some algebra, the following implicit equation for \( \gamma_{gg} \)

\[
1 - \frac{\alpha_S}{\omega} K(\gamma_{gg},\omega) - \frac{\alpha_S}{\gamma_{gg}} P(\omega) = 0
\]

where \( K \), the double moment of the kernel in (15), is given by

\[
K(\gamma,\omega) = \int_0^{\infty} d\rho \left\{ \frac{[\rho^\gamma(1-\rho) + \rho^{\gamma-\omega}\Theta(\rho - 1)] - 1}{|\rho - 1|} + \frac{1}{[4\rho^2 + 1]^{1/2}} \right\}.
\]
If $\omega = 0$ then the expression in square brackets reduce to $\rho^\gamma$ and we have the familiar BKFL result

$$K(\gamma, \omega = 0) = [2\Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma)]$$

(19)

where $\Psi$ is the logarithmic derivative of the Euler gamma function.

It is clear that $\gamma_{gg}$, which satisfies (17), is of the form

$$\gamma_{gg}(\omega, \bar{\alpha}_S) = \gamma^{\text{BFKL}} \left( \frac{\alpha_S}{\omega} \right) + \bar{\alpha}_S P(\omega) + \text{higher order terms},$$

(20)

where $\gamma^{\text{BFKL}}$ satisfies the usual equation

$$1 - \frac{\bar{\alpha}_S}{\omega} K\left( \gamma^{\text{BFKL}}, \omega = 0 \right) = 0.$$ 

(21)

The higher order terms include contributions which are subleading in $\bar{\alpha}_S/\omega$ as well as in $\bar{\alpha}_S$. The anomalous dimension $\gamma_{gg}$ has a branch point singularity in the $\omega$ plane, whose position $\omega = \omega_0(\bar{\alpha}_S)$ controls the small $x$ behaviour of the gluon distribution. The inverse of (14) gives

$$f(x, k^2) \sim x^{-\omega_0(\bar{\alpha}_S)}.$$ 

(22)

The value of $\omega_0$ is obtained from the requirement that

$$\frac{\partial}{\partial \gamma} \left\{ 1 - \frac{\bar{\alpha}_S}{\omega_0} K(\gamma, \omega_0) - \frac{\bar{\alpha}_S}{\gamma} P(\omega_0) \right\} = 0,$$ 

(23)

together with the equation

$$1 - \frac{\bar{\alpha}_S}{\omega_0} K(\gamma, \omega_0) - \frac{\bar{\alpha}_S}{\gamma} P(\omega_0) = 0,$$ 

(24)

see (17). Recall that in the leading $\ln(1/x)$ (or leading $1/\omega$) approximation (23) reduces to

$$\frac{\partial}{\partial \gamma} K(\gamma, 0) = 0,$$ 

(25)

which is satisfied when $\gamma = \frac{1}{2}$. Thus from (21) we obtain the well-known BFKL result that

$$\omega_0 = \bar{\alpha}_S K(\gamma = \frac{1}{2}, \omega = 0) = \bar{\alpha}_S 4 \ln 2.$$ 

(26)

The relevant domain for solving the pair of equations (17) and (21) is $0 < \gamma < 1$ and $\omega > 0$. In this region the Mellin transform of the non-singular part of the gluon splitting function satisfies

$$P(\omega) < 0,$$ 

(27)

and moreover

$$K(\gamma, \omega) < K(\gamma, 0).$$ 

(28)

\[1\text{If we were to replace } \gamma^{\text{BFKL}} \text{ simply by the term which is leading order in } \alpha_S, \text{ that is } \bar{\alpha}_S/\omega, \text{ then the sum of the first two terms of (20) gives the conventional GLAP anomalous dimension.}\]
Thus both the additional non-singular part of the GLAP splitting function \((P_{gg} - 6/z)\) and the kinematic constraint (which takes \(K(\gamma, 0)\) to \(K(\gamma, \omega)\)) tend to reduce the magnitude of \(\omega_0\) from the BFKL value shown in (26). These corrections are of course subleading in \(\ln(1/x)\). Our numerical analysis with running \(\alpha_S\) reflects this softening of the \(x^{-\omega_0}\) singular behaviour.

2.3 The CCFM equation

A more general treatment of the gluon ladder, which follows from the BKFL formalism is provided by the Catani-Ciafaloni-Fiorani-Marchesini (CCFM) equation based on angular ordering of the gluon emission along the chain [20, 21, 24]. The equation embodies both the BKFL equation at small \(x\) and the conventional GLAP evolution at large \(x\). The unintegrated gluon distribution \(f\) now acquires a dependence on an additional scale (which we may take to be \(Q^2\)) that specifies the maximal angle of gluon emission. The CCFM equation has the form

\[ f(x, k^2, Q^2) = f^{(0)}(x, k^2, Q^2) + \alpha_S \int_x^1 \frac{dz}{z} \int \frac{d^2q}{\pi q^2} \Theta(Q - qz) \Delta_R(z, k^2, q^2) \frac{k^2}{(q + k)^2} f\left(\frac{x}{z}, (q + k)^2, q^2\right) \]  

where the theta function \(\Theta(Q - qz)\) reflects the angular ordering constraint on the emitted gluon. The so-called ‘non-Sudakov’ form-factor \(\Delta_R\) is given by

\[ \Delta_R(z, k^2, q^2) = \exp\left[-\alpha_S \int_z^1 \frac{dz'}{z'} \int \frac{dq^2}{q'^2} \Theta(q' - z'q) \Theta(k^2 - q'^2)\right]. \]

Eq. (29) contains only the singular \(1/z\) term of the \(g \to gg\) splitting function (which is screened by the virtual corrections contained in \(\Delta_R\)). Its generalisation to include the remaining parts of this vertex (as well as the quark contributions) is possible. Eq. (29) has been solved numerically in the small \(x\) domain and the solution for \(f(x, k^2, Q^2)\) was presented in [21]. The CCFM equation, which is a generalisation of the BFKL equation, generates a steep \(x^{-\omega_0}\) type of behaviour\(^2\) but \(\omega_0\) now acquires significant subleading \(\ln(1/x)\) corrections which come from the angular ordering constraint [26]. The constraint also introduces subleading terms in the anomalous dimension

\[ \gamma_{gg} = \gamma_{gg}^{\text{BFKL}} \left(\frac{\alpha_S}{\omega}\right) + \alpha_S \left(\frac{\alpha_S}{\omega}\right) + \ldots \]

so the angular ordering constraint which gives rise to the CCFM equation and the kinematic constraint (7) lead to similar effects – both give subleading \(\ln(1/x)\) corrections to the “QCD Pomeron intercept” \(\omega_0\) and to the gluon anomalous dimension \(\gamma_{gg}\). We found that the kinematic constraint overrides the angular ordering constraint except possibly in the large \(x\) domain when \(Q^2 < k^2\) [22], see also [23]. Thus in our formulation we neglect the angular ordering constraint altogether.

3. The equation for the quark distribution

\(^2\)The effect on \(F_2\) is considered in [25].
At small $x$ the gluon drives the sea quark (momentum) distribution $S$ via the $g \to q\bar{q}$ transition, see Fig. 1. We evaluate the effect using the $k_T$ factorization theorem. To be precise we use the $k_T$ factorization prescription to calculate observables (such as $F_2$) directly from the unintegrated gluon distribution $f(x,k_T^2)$. For $F_2$ we interpret the result in terms of the sea quark distributions, implicitly assuming the DIS scheme. The total sea is the sum of the individual quark contributions

$$S(x,Q^2) = \sum_q S_q(x,Q^2).$$

At small $x$ the factorization theorem gives

$$S_q(x,Q^2) = \int_x^1 \frac{dz}{z} \int \frac{dk^2}{k^2} S_{box}^q (z,k^2,Q^2) f\left(\frac{x}{z},k^2\right)$$

(31)

where $S_{box}$ describes the quark box (and crossed box) contribution shown in Fig. 1. $S_{box}$ implicitly includes an integration over the transverse momentum, $\kappa$, of the exchanged quark. Indeed, evaluating the box contributions we find

$$S_q(x,Q^2) = \frac{Q^2}{4\pi^2} \int \frac{dk^2}{k^2} \int_0^1 d\beta \int d^2\kappa' \alpha_S \left\{ [\beta^2 + (1-\beta)^2] \left( \frac{\kappa}{D_{1q}} - \frac{\kappa-k}{D_{2q}} \right)^2 \right. $$

$$+ \left. [m_q^2 + 4Q^2\beta(1-\beta)^2] \left( \frac{1}{D_{1q}} - \frac{1}{D_{2q}} \right)^2 \right\} f\left(\frac{x}{z},k^2\right) \Theta \left(1-\frac{x}{z}\right)$$

(32)

where $\kappa' = \kappa - (1-\beta)k$ and

$$D_{1q} = \kappa^2 + \beta(1-\beta)Q^2 + m_q^2$$

$$D_{2q} = (\kappa-k)^2 + \beta(1-\beta)Q^2 + m_q^2$$

$$z = \left[ 1 + \frac{\kappa^2 + m_q^2}{\beta(1-\beta)Q^2} + \frac{k^2}{Q^2} \right]^{-1}. $$

(33)

The argument of $\alpha_S$ is taken as $(k^2+\kappa'^2)+m_q^2$. We set the quark masses to be $m_u = m_d = m_s = 0$ and $m_c = 1.4$ GeV.

3.1 The light quark component of the sea

We first discuss the calculation of the contribution of the “massless” $u,d,s$ quarks to the total sea distribution $S$. It is necessary to consider three different regions of the $k$ and $\kappa'$ integrations of (32).

(a) The contribution from the non-perturbative region $k^2,\kappa'^2 < k_0^2$ is evaluated phenomenologically assuming that it is dominated by “soft” Pomeron exchange [27]. The contribution is parametrized by the form

$$S^{(a)} = S_u^{IP} + S_d^{IP} + S_s^{IP} $$

(34)

10
where
\[ S_u/F = S_d/F = 2S_s/F = C_{IP} x^{-0.08} (1 - x)^8. \] (35)

The coefficient \( C_{IP} \) is independent of \( Q^2 \) (in the large \( Q^2 \) region) since the contribution arises from the region in which the struck quarks have limited transverse momentum, \( \kappa^2 < k_0^2 \).

(b) In the region \( k^2 < k_0^2 < \kappa^2 \) we apply the strong-ordered approximation at the quark-gluon vertex and take [28]
\[ S_{\text{box}} \rightarrow S_{\text{box}}^{(b)} (z, k^2 = 0, Q^2). \] (36)

Then the contribution to (31) from this domain becomes
\[
S^{(b)} = \int_x^1 \frac{dz}{z} S_{\text{box}}^{(b)} (z, k^2 = 0, Q^2) \int_0^{k_0^2} \frac{dk^2}{k^2} f \left( \frac{x}{z}, k^2 \right)
\]

\[ = \int_x^1 \frac{dz}{z} S_{\text{box}}^{(b)} (z, k^2 = 0, Q^2) \frac{x}{z} g \left( \frac{x}{z}, k_0^2 \right) \] (37)

where the summation over \( u, d, s \) is implicitly assumed. The potential collinear singularities in the on-shell structure function \( S_{\text{box}} \) are regulated by the cut-off \( k_0^2 \). Recall that \( \kappa^2 \simeq \kappa' > k_0^2 \).

(c) In the remaining region, \( k^2 > k_0^2 \), eq. (32) is left unchanged. To be precise we use the perturbative expression for \( S_q(x, Q^2) \).

3.2 The charm component

The calculation of the charm component of the sea follows perturbative QCD in all regions. To evaluate \( S_{q=c} \) we divide the integration over \( k^2 \) into the regions \( k^2 < k_0^2 \) and \( k^2 > k_0^2 \). For \( k^2 < k_0^2 \), which we denote region (b), we use the on-shell approximation to evaluate \( S_{\text{box}} \). That is we calculate \( S_{\text{box}}(z, k^2 = 0, Q^2; m_c^2) \), which is finite due to \( m_c \neq 0 \). Then (31) gives
\[
S_{q=c}^{(b)} (x, Q^2) = \int_x^a \frac{dz}{z} S_{\text{box}}(z, k^2 = 0, Q^2; m_c^2) \int_0^{k_0^2} \frac{dk^2}{k^2} f \left( \frac{x}{z}, k^2 \right)
\] (38)

where \( a = (1 + 4m_c^2/Q^2)^{-1} \), see (33). For \( k^2 > k_0^2 \), which we call region (c), we use the full perturbative formula. Thus adding the two contributions we have
\[
S_{q=c} (x, Q^2) = \int_x^a \frac{dz}{z} S_{\text{box}}(z, k^2 = 0, Q^2; m_c^2) \frac{x}{z} g \left( \frac{x}{z}, k_0^2 \right)
\]
\[ + \int_x^a \frac{dz}{z} \int_{k_0^2}^{k_0^2} \frac{dk^2}{k^2} S_{\text{box}}(z, k^2, Q^2; m_c^2) f \left( \frac{x}{z}, k^2 \right), \] (39)

where we have used (1) which enables \( S_{q=c} \) is to be specified in terms of the conventional gluon input distribution.
3.3 The equation for the quark singlet distribution

Besides $S$, the singlet momentum distribution $\Sigma$ also contains a valence quark contribution $V$, which is taken from a known set of partons. Thus in summary the singlet distribution is

$$\Sigma = \left( S^{(a)} + S^{(b)} + S^{(c)} \right)_{uds} + (S^{(b)} + S^{(c)})_{q=c} + V,$$  \hspace{1cm} (40)

where $S^{(a)}$ is phenomenologically parametrized in terms of “soft” Pomeron exchange and the $S^{(b)}$ terms are determined perturbatively except for the (non-perturbative) input gluon distribution at scale $k_0^2$. The $S^{(c)}$ terms are defined entirely in terms of the unintegrated gluon distribution $f$ in the perturbative region $k^2 > k_0^2$. Finally, $V = x (u_{val} + d_{val})$ is the valence quark contribution.

In order to see the connection with the GLAP evolution of the (light) quark sea we first note that

$$S_q(x, Q^2) = S_q(x, k_0^2) + \int_{k_0^2}^{Q^2} \frac{dQ^2}{Q^2} S_q(x, Q^2),$$  \hspace{1cm} (41)

where here $S$ denotes the sum over just the $u$, $d$ and $s$ quarks. We next recall that the leading twist part of the $k_T$ factorization formula (31), written in the form

$$Q^2 \frac{\partial S_q(x, Q^2)}{\partial Q^2} = \int_x^1 \frac{dz}{z} \int \frac{dk^2}{k^2} Q^2 \frac{\partial S_{box}^q(z, k^2, Q^2)}{\partial Q^2} f \left( \frac{x}{z}, k^2 \right),$$  \hspace{1cm} (42)

can be reduced to the collinear form \[18\]

$$Q^2 \frac{\partial S_q(x, Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 dz \int P_{qg}(z, \alpha_s(Q^2)) \frac{x}{z} g \left( \frac{x}{z}, Q^2 \right)$$  \hspace{1cm} (43)

which incorporates leading $\ln 1/x$ resummation effects in both the splitting function $P_{qg}$ and in the integrated gluon distribution $g$. Thus (41) may be written in the form

$$S_q(x, Q^2) = S_q(x, k_0^2) + \int_{k_0^2}^{Q^2} \frac{dQ^2}{Q^2} \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 dz \left[ P_{qg}(z, \alpha_s(Q^2)) \frac{x}{z} g \left( \frac{x}{z}, Q^2 \right) + P_{qq}(z, \alpha_s(Q^2)) S_q \left( \frac{x}{z}, Q^2 \right) \right],$$  \hspace{1cm} (44)

where for consistency we have included the $S \rightarrow S$ contribution to the evolution. This additional term is needed to ensure the correct GLAP structure. Of course, at small $x$ we expect $S$ to be dominantly driven by the gluon. Equation (44) is simply the integral form of the GLAP evolution equation for the (light) sea quark (momentum) distribution, $S$.

Guided by the GLAP structure, it is clear that we should also add the $S \rightarrow S$ contribution to the complete equation (40) based on $k_T$ factorization. Then (40) becomes

$$\Sigma(x, k^2) = S^{(a)}(x) + \sum_q \int_x^a \frac{dz}{z} S^q_{box}(z, k^2 = 0, k^2; m_q^2) \frac{x}{z} g \left( \frac{x}{z}, k^2 \right) + V(x, k^2)$$

\[+ \sum_q \int_{k_0^2}^{\infty} \frac{dk^2}{k^2} \int_x^1 \frac{dz}{z} S^q_{box}(z, k^2, k^2; m_q^2) f \left( \frac{x}{z}, k^2 \right) \]

\[+ \int_{k_0^2}^{k^2} \frac{dk^2}{k^2} \frac{\alpha_s(k^2)}{2\pi} \int_x^1 dz P_{qq}(z) S_{uds} \left( \frac{x}{z}, k^2 \right) \]  \hspace{1cm} (45)

12
where $S(a)$ is given by (34) and where the $uds$ subscript indicates that the additional $S \rightarrow S$ term is only included for the light quarks. This equation for the singlet quark distribution $\Sigma$, together with eq. (10) for the gluon, form the pair of coupled equations which we solve. In this way we can specify the structure function $F_2$ in terms of the parameters of the input distributions, and hence determine the values of the parameters by fitting to the data for $F_2$, see sections 4-6.

3.4 $k^T$ versus collinear factorization and $P_{qg}$

As we have already mentioned, the leading-twist part of the $k_T$ factorization formula can be rewritten in a collinear factorization form. Once the unintegrated gluon distribution is taken as a solution of the BFKL equation and the $k_T$ factorization integral is performed over the entire available phase-space (i.e. not only over the region corresponding to the strongly ordered transverse momenta) then the leading small $x$ effects are automatically resummed in the splitting functions and in the coefficient functions. The $k_T$ factorization theorem can in fact be used as the tool for calculating these quantities [8, 9].

We illustrate this point by using the example of the calculation of the splitting function $P_{qg}$. For simplicity we assume that the coupling $\alpha_S$ is fixed and that the quarks are massless. We begin from the $k_T$ factorization formula (42) written in moment space

$$Q^2 \frac{\partial S(\omega, Q^2)}{\partial Q^2} = \int \frac{dk^2}{k^2} Q^2 \frac{\partial S_{box}(\omega, Q^2/k^2)}{\partial Q^2} \tilde{f}(\omega, k^2).$$

(46)

where we have noted, for massless quarks, that $S_{box}$ is a function of the ratio $Q^2/k^2$. Thus we may factorise the convolution over $k^2$ by taking moments. We find

$$Q^2 \frac{\partial \bar{S}(\omega, Q^2)}{\partial Q^2} = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\gamma \gamma \bar{S}_{box}(\omega, \gamma) \tilde{f}(\omega, \gamma) \left( \frac{Q^2}{k^2} \right)^{\gamma}$$

(47)

where $\bar{S}_{box}(\omega, \gamma)$ and $\tilde{f}(\omega, \gamma)$ are the Mellin transform of the moment functions $S_{q}^{box}(\omega, k^2, Q^2)$ and $\tilde{f}(\omega, k^2)$ i.e.

$$\bar{S}_{box}(\omega, k^2, Q^2) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\gamma \gamma \bar{S}_{box}(\omega, \gamma) \left( \frac{Q^2}{k^2} \right)^{\gamma}$$

(48)

and

$$\tilde{f}(\omega, k^2) = \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\gamma \tilde{f}(\omega, \gamma) \left( \frac{k^2}{k_0^2} \right)^{\gamma}.$$  

(49)

Retaining the leading pole $\gamma_{gg}$ contribution of $\tilde{f}(\omega, \gamma)$ in the $\gamma$ plane, the integrals (47) and (49) can be evaluated to give

$$Q^2 \frac{\partial S(\omega, Q^2)}{\partial Q^2} = \gamma_{gg} \bar{S}_{box}(\omega, \gamma_{gg}) C(\omega, \alpha_S) \left( \frac{Q^2}{k_0^2} \right)^{\gamma_{gg}}$$

(50)

and

$$\tilde{f}(\omega, k^2) = \gamma_{gg} C(\omega, \alpha_S) \left( \frac{k_0^2}{k^2} \right)^{\gamma_{gg}}.$$  

(51)
where $\gamma_{gg} C$ is the residue of the pole. The function $\gamma_{gg}(\bar{\alpha}_S, \omega)$ is the (leading twist) gluon anomalous dimension. From (51) we see that the integrated gluon is given by

$$\bar{g}(\omega, Q^2) = C(\omega, \alpha_S) \left( \frac{Q^2}{k_0^2} \right)^{\gamma_{gg}}$$

(52)

where $\bar{g}(\omega, Q^2)$ is the moment function of the (leading twist part) of the gluon distribution. Thus by comparing (50) with the conventional GLAP form

$$Q^2 \partial \bar{S}_q(\omega, Q^2) = \frac{\alpha_S}{2\pi} P_{qq}(\omega, \alpha_S) \bar{g}(\omega, Q^2)$$

(53)

we can identify the moment of the $P_{qq}$ splitting function to be

$$\frac{\alpha_S}{2\pi} P_{qq}(\omega, \alpha_S) = \gamma_{gg}^2(\bar{\alpha}_S, \omega) \bar{S}_q(\omega, \gamma_{gg}(\bar{\alpha}_S, \omega))$$

(54)

in the so-called $Q^2_0$ regularization and DIS scheme [9] which we implicitly adopt.

In the leading $\ln(1/x)$ approximation we have

$$\frac{\alpha_S}{2\pi} \bar{P}_{qq}(\omega, \alpha_S) = (\gamma_{BFKL}^2) \bar{S}_q(\omega = 0, \gamma_{BFKL})$$

(55)

The anomalous dimension $\gamma_{BFKL}$ has the following expansion [29]

$$\gamma_{BFKL}(\bar{\alpha}_S, \omega) = \sum_{n=1}^{\infty} c_n \left( \frac{\bar{\alpha}_S}{\omega} \right)^n$$

(56)

which in turn gives for the splitting function $P_{gg}$

$$z P_{gg}(z, \alpha_S) = \sum_{n=1}^{\infty} c_n \left[ \frac{\bar{\alpha}_S \ln(1/z)^{n-1}}{(n-1)!} \right]$$

(57)

whereas representation (55) generates the following expansion of the splitting function $P_{qq}(z, \alpha_S)$ at small $z$

$$z P_{qq}(z, \alpha_S) = z P_{qq}^{(0)}(z) + \bar{\alpha}_S \sum_{n=1}^{\infty} b_n \left( \frac{\bar{\alpha}_S \ln(1/z)}{(n-1)!} \right)^{n-1}$$

(58)

The first term on the right hand side vanishes at $z = 0$. It should be noted that the splitting function $P_{qq}$ is formally non-leading at small $z$ when compared with the splitting function $P_{gg}$. For moderately small values of $z$ however, when the first few terms in the expansions (58) and (57) dominate, the BFKL effects can be much more important in $P_{qq}$ than in $P_{gg}$.

This comes from the fact that all coefficients $b_n$ in (58) are different from zero, while in (57) we have $c_2 = c_3 = 0$ [29]. The small $x$ resummation effects within the conventional QCD evolution formalism have recently been discussed in refs. [30, 31, 32, 33, 34]. These studies already emphasize this point, namely that at the moderately small values of $x$ which
are relevant for the HERA measurements, the \( \ln(1/x) \) resummation effects in the splitting function \( P_{gg} \) have a much stronger impact on \( F_2 \) than do those in the splitting function \( P_{qg} \). In particular we should also recall that the BFKL effects in the splitting function \( P_{qg} \) can significantly affect the extraction of the gluon distribution from the experimental data on the slope of the structure function \( F_2 \): 

\[
Q^2 \frac{\partial F_2(x, Q^2)}{\partial Q^2} \simeq \sum_q e^2 \alpha_s(Q^2) \int_x^1 dz P_{qg}(z, \alpha_s(Q^2)) \frac{x}{z} g \left( \frac{x}{z}, Q^2 \right). 
\] (59)

Here we also include the subleading \( \ln(1/x) \) terms which would come from the subleading terms in \( \gamma_{gg} \) etc. Keeping the exact \( k_T \) factorisation (and not just its large \( Q^2 \) limit) we also include the non-leading twist contributions to \( F_2 \). They would formally be generated by the contributions given by the (non-leading) twist anomalous dimensions.

4. Numerical analysis and the description of \( F_2 \)

We now have a closed system of two coupled integral equations for two unknowns. Namely equation (10) of Section 2 for the unintegrated gluon distribution \( f(x, k^2) \) and equation (45) of Section 3 for the integrated quark singlet (momentum) distribution \( \Sigma(x, k^2) \). The effect of the gluon in the perturbative region, \( k^2 > k_0^2 \), is of special interest. It is the ‘dynamo’ which drives small \( x \) physics.

The advantage of this formulation of the unified BFKL/GLAP equation is that the input is well-controlled. We emphasized in Section 2 that the equation for \( f(x, k^2) \) required only the specification of an input form for the integrated gluon,

\[
xg(x, k_0^2) = N(1 - x)^\beta, 
\] (60)

say. Moreover, the equation for the singlet \( \Sigma(x, k^2) \) requires as input only the non-perturbative sea contribution whose form we assume is given by the “soft” Pomeron

\[
S^{(a)} = C_P x^{-0.08} (1 - x)^8 
\] (61)

and the contributions \( S^{(b)} \) of (37) and (38) which depend on \( xg(x, k_0^2) \) of (60). The choice of the exponent \(-0.08\) is motivated by the Regge pomeron intercept found in the analysis of total cross section data [27]. We choose the exponent of \((1 - x)\) to be \(8\), typical of the behaviour of the sea distribution. In our small \( x \) analysis any similar choice would be equally good and would not change the quality of the description.

The valence quark contribution \( V(x, k^2) \) in (45), which is determined mainly by fixed target deep inelastic data, is taken from the leading order GRV set of partons [6]. We are therefore able to self-consistently determine \( f(x, k^2) \) and \( \Sigma(x, k^2) \) as functions of a small number of physically motivated parameters. In fact, we have only the two parameters, namely \( N \) and \( \beta \) determining the input gluon distribution (60). The momentum sum rule fixes the value of \( C_P \), which determines the input sea, (61). The presence of BFKL-like terms means that the
momentum sum rule is not exactly conserved. However the violation is small. For example, after evolution to $Q^2 = 50\text{GeV}^2$ we find that the sum of the momentum fractions carried by the gluon and the light quarks is only increased from 1 to 1.007. We neglect this small violation of momentum conservation.

4.1 The optimum description of the $F_2$ data at small $x$

We determine the values of the input parameters by fitting to the HERA measurements of the proton structure function $F_2$ using

$$F_2 = \sum_q e_q^2 (S_q + V_q),$$

which holds in the DIS scheme. We thus have to calculate $S_q(x, Q^2)$ in terms of the input gluon parameters $N$ and $\beta$. To do this we solve the pair of equations (10) and (45) for $f(x, k^2)$ and $\Sigma(x, k^2)$ using an extension of the method proposed in [35]. This method incorporates the interpolation in two variables $x$ and $Q^2$ with orthogonal polynomials. Thus the coupled integral equations can be transformed into the set of linear algebraic equations and readily solved. In this way we can express $F_2(x, Q^2)$ in terms of $N$ and $\beta$. We then determine the optimum values of these parameters by fitting to the HERA [1] and fixed-target [36] data for $F_2(x, Q^2)$ that are available in the small $x$ domain, $x < 0.1$. We take a running coupling which satisfies $\alpha_S(M_Z^2) = 0.12$.

We actually show the results of two fits. The first is the ‘realistic’ fit with the kinematic constraint imposed (which requires the virtuality of the exchanged gluons along the ladder to satisfy $|k'^2| \simeq k_T^2$). Then for comparison we repeat the analysis without imposing the kinematic constraint, that is we omit $\Theta$ function in (10). The quality of the fits are shown in Figs. 2 and 3, and the parameters given in Table 1. To be precise Figs. 2 and 3 respectively show the description of the H1 and ZEUS data [1] together with those fixed-target data that occur at the same values of $Q^2$.

The fit with the kinematic constraint included (continuous curves) is significantly better than that in which it is omitted (shown by the dashed curves). Without the constraint the predicted rise of $F_2$ is a little too steep at the smallest values of $x$ and $Q^2$. Over the remainder of the $x, Q^2$ domain the fit (fit 2) gives a good description of $F_2$. It is far better, for example, than that shown in ref. [11].

The kinematic constraint, which corresponds to subleading $\ln(1/x)$ corrections, lowers the ‘hard’ pomeron intercept and improves the description of the data, particularly at the smaller values of $x$. In fact the resulting description of $F_2(x, Q^2)$ with just two free parameters ($N$ and $\beta$) is excellent, and is comparable, even a little better than, to that achieved in the global parton analyses, see, for example, the $\chi^2$ listed in Table 1. Moreover, the overall behaviour of the gluon is much more realistic than that of the fit without the kinematic constraint. It gives an acceptable description of the WA70 prompt photon data [37], which directly sample the gluon at $x \simeq 0.4$. These data were not used to constrain the gluon. For fit 1 the prediction is
some 30% above WA70 data, which is within the QCD scale uncertainties, whereas the gluon of
fit 2 gives a prediction which is a factor of about 2.5 above the data. It is not surprising that the
gluons are so different in the two fits since they are both contrived to give satisfactory description
of the measurements of $F_2$ at small $x$, despite the fact that the kinematic constraint significantly
reduces the gluon intercept $\omega_0$. It is encouraging that it is the description with the kinematic
constraint that gives the acceptable large $x$ behaviour of the gluon. For completeness we use
our determination of the unintegrated gluon to compute the conventional gluon distribution
$xg(x,Q^2)$ and compare the result with the gluons of recent sets of partons obtained in GLAP-
based global analyses of deep inelastic and related data. To be specific the continuous curves
in Fig. 4 compare the gluon calculated from $f(x,k_T^2)$ of the fit 1 (via eq. (1)) with the gluon
distributions of the MRS(R2) [4] and GRV [6] set of partons (shown by the dashed and dotted
curves respectively). We see that the behaviour of the integrated gluon is very similar to that
of MRS(R2). This may be expected since the MRS analysis used to the same HERA data
as those fitted in the present work, whereas these data were not available at the time of the
GRV analysis. However, we emphasize the different underlying structure of the present analysis
and the pure GLAP-based descriptions. We shall see below that in the unified BFKL/GLAP
approach the rise of $F_2$ is generated essentially by $\ln(1/x)$ effects in the off-shell gluon structure,$\ F_2^{\gamma g}$ of Fig. 1.

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<td>MRS(R2)</td>
<td></td>
<td></td>
<td>1.12</td>
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Table 1: The parameters $N$ and $\beta$ determined in the optimum fit to the available data [1, 36]
for $F_2$ with $x < 0.05$ and $Q^2 > 1.5 GeV^2$, without and with the inclusion of the kinematic
constraint along the gluon ladder. The value of $C_{IP}$ of (35) is also shown, although this is fixed
in terms of $N$ and $\beta$ by the momentum sum rule. For comparison we also show the $\chi^2$
for the same set of HERA and fixed-target data obtained in a recent next-to-leading order GLAP
global parton analysis [4]. For both fit 1 and 2 the gluon carries 45% of the proton’s momentum
at the input scale $k_0^2 = 1 GeV^2$.

4.2 The effect of the $\ln(1/x)$ resummation on the gluon

Fig. 5 shows the behaviour of the unintegrated gluon distribution $f(x,k^2)$ as a function of
$k_0^2$ for $x = 10^{-3}$ and $10^{-4}$. Three different determinations are shown, each of which start from
the input

$$xg(x,k_0^2) = 1.57(1-x)^{2.5}$$
of fit 1 of Table 1. The continuous and dashed curves correspond, respectively, to the behaviour with and without the kinematic constraint. The dotted curve is obtained from a GLAP determination in which the BFKL kernel in (10) is replaced by the leading order $P_{gg}$ function. That is (10) is replaced by

$$f(x, k^2) = \frac{\alpha_S(k^2)}{2\pi} \left[ \int_x^1 dz P_{gg}(z) \frac{x}{z^2} g \left( \frac{x}{z}, k_0^2 \right) + \int_x^1 dz P_{gg}(z) \int_{k_0^2}^{k^2} \frac{dk'}{k'^2} f \left( \frac{x}{z}, k'^2 \right) + \int_x^1 dz P_{gg}(z) \Sigma \left( \frac{x}{z}, k^2 \right) \right],$$

(63)

where $P_{gg}$ has the usual form

$$P_{gg}(z) = 6 \left[ \frac{1-z}{z} + z(1-z) + \frac{z}{(1-z)^3} + \frac{11}{12} \delta(1-z) \right] - \frac{N_f}{3} \delta(1-z).$$

(64)

The comparison of the dashed and dotted curves shows that the differences between the BFKL and GLAP approaches are not very big, even for the values of $x$ as low as $10^{-3}$. The differences become more obvious when one considers smaller values of $x$, around $10^{-4}$. Even then the discrepancies are only visible at lower values of $k^2$. This effect can be explained in terms of power series expansion in $\alpha_S/\omega$ of the gluon anomalous dimension

$$\gamma_{BFKL} = \frac{\alpha_S}{\omega} + c_4 \left( \frac{\alpha_S}{\omega} \right)^4 + \ldots$$

see (56), where $c_2 = c_3 = c_5 = 0$. The first term of the expansion, which is common to BFKL and GLAP, is clearly dominant for the smaller values of $\alpha_S(k^2)$. Thus we confirm the well known result that, in the region of moderately small values of $x$ relevant for the HERA measurements, $\ln(1/x)$ resummation has little effect on the gluon distribution. If the gluon input were adjusted to correspond to the optimum fit with the kinematic constraint imposed, then the continuous curve would be comparable to the other two. However, a common input is used to show the impact of the kinematic constraint. We see that the resulting gluon is smaller and less steep. This implies that subleading $\ln(1/x)$ corrections are significant.

### 4.3 Effect of $\ln(1/x)$ resummation on the structure function $F_2$

To investigate the various effects of the $\ln(1/x)$ terms we compute $F_2(x, Q^2)$ using four different procedures but with a common input\(^3\)

$$xg(x, k_0^2) = 1.57(1 - x)^{2.5},$$

corresponding to fit 1. The four different determinations are shown in Fig. 6 and correspond to

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\(^3\)In this calculation we have only included the light quarks $u, d, s$ (which we treat as massless), since we want to avoid any dependence on the choice of scale for the heavy quarks. Such a dependence would spoil the clarity of the explanation of some effects which we want to stress. Everywhere else we include also the charm quark.

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(i) The full unified BFKL + GLAP calculation with the kinematic constraint included, eqs. (10) and (45), shown as a continuous curve.

(ii) Analogous to (i) but without the kinematic constraint (dashed curve).

(iii) Replace (10) by the pure GLAP equation in the gluon sector, eq. (63), but keep the full $k_T$ factorization for the quarks (dotted curve).

(iv) Pure GLAP evolution for both the gluons and the quarks (dot-dashed curve). That is instead of (45) we use

$$
\Sigma(x, k^2) = S^{(a)}(x) + S^{(b)}(x, k^2) + V(x, k^2) + \int_x^1 dz P_{qq}(z) \int_{k_0^2}^{k^2} \frac{dk_2^2}{k_2^2} f \left( \frac{x}{z}, k_2^2 \right) \xi(k_2^2, k^2) \tag{65}
$$

where $\xi(k_2^2, k^2)$ is the evolution length and is defined by,

$$
\xi(k_2^2, k^2) = \int_{k^2}^{k_2^2} \frac{dq^2}{q^2} \alpha_S(q^2). \tag{66}
$$

One can again see from Fig. 6 that the differences between BFKL (with no kinematical constraint) and GLAP evolution in the gluon sector are not very big. The calculations start to differ only at $x \approx 10^{-4}$ (dashed and dotted lines). On the other hand when we compare the pure GLAP evolution (with the $P_{qq}$ splitting function) with the equations where the entire phase space has been taken into account then the differences are much bigger. This implies that the leading order terms in $\alpha_S \ln 1/x$ present in the gluon off-shell structure function ($F_2^{\gamma g}$) are much more important than the terms in the gluon anomalous dimension resulting from the BFKL equation. The effect of the kinematic constraint is again evident. It leads to the change from the dashed to the continuous curves. Fig. 6 also enables us to see the $x$ values at which the effect of the $\ln(1/x)$ summation effects become important.

4.4 Predictions for $F_2^c$ and $F_L$

Once we have determined the parton distributions we can predict the values of other hard scattering observables. At small $x$ we see, via the $k_T$ factorization theorem, that the observables are ‘driven’ by the unintegrated gluon distribution $f(x, k^2)$. Here we calculate $F_2^c$ and $F_L$.

The charm component $F_2^c$ of $F_2$ is given by

$$
F_2^c(x, Q^2) = e_c^2 S_{q=c}(x, Q^2)
$$

where the charm sea $S_{q=c}$ is calculated from (39) in terms of the gluon. It is the second term on the right-hand side of (39) which drives the small $x$ behaviour. The predictions are compared with the H1 measurements [38] of $F_2^c$ in Fig. 7. The percentage of charm in the deep inelastic structure function is shown in Fig. 8. At small $x$ we see that $F_2^c$ is an appreciable fraction of
Recall that in the massless charm limit the fraction would be 0.4, provided that we are below the bottom quark threshold.

The predictions of the longitudinal structure function $F_L$ are shown in Fig. 9. For $F_L$ the $k_T$ factorization formula can be written in the form \[28, 39\]

$$F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{\pi} \left[ \frac{4}{3} \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 F_2(y, Q^2) + \sum_q e_q^2 \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 \left( 1 - \frac{x}{y} \right) yg(y, k_0^2) \right]$$

$$+ \sum_q e_q^2 Q^4 \int_{k_0^2}^{Q^2} \frac{dk^2}{k^4} \int_0^1 d\beta \beta^2 (1 - \beta)^2 \int d^2 \kappa' \alpha_s \left[ \frac{1}{D_{1q}} - \frac{1}{D_{2q}} \right]^2 f \left( \frac{x}{z}, k^2 \right), \quad (67)$$

where the quark box variables $D_{1q}$ and $\kappa'$ are defined in (33). The behaviour of $F_L$ is driven by the gluon through the last term. The argument $\alpha_s$ is taken to be $k^2 + \kappa'^2 + m_q^2$ as before.

### 5. Conclusions

The natural framework for describing ‘hard’ scattering observables at small $x$ Bjorken $x$ is provided by the gluon distribution $f(x, k_T^2)$ un unintegrated over its transverse momentum, together with the $k_T$ factorization theorem. At small $x$ it is only to be expected that the $k_T$ dependence should be treated explicitly.

In the leading $\ln(1/x)$ limit, $f(x, k_T^2)$ satisfies the BFKL equation. To make a smooth transition to the larger $x$ domain we have studied a modified equation which treats the leading $\ln(1/x)$ and the leading $\ln(Q^2)$ terms on an equal footing. Moreover, we arrange the equation so that we need only to consider $f(x, k_T^2)$ in the perturbative domain, $k_T^2 > k_0^2$. The integrated gluon distribution $xg(x, k_0^2)$ is the only non-perturbative input that is required. At small $x$, the singlet quark distribution $\Sigma(x, Q^2)$ is controlled by $f(x, k_T^2)$ through the $g \to q\bar{q}$ splitting. We therefore extend the formalism to a pair of coupled integral equations for $f$ and $\Sigma$ which embrace both the BFKL leading $\ln(1/x)$ and GLAP leading $\ln(Q^2)$ contributions in a consistent way. A notable feature of the formalism is that we can retain the full perturbative contribution of the quark box which contributes to the sea distribution for all $Q^2$. In this way we can isolate the non-perturbative contribution to a (scaling) sea contribution whose general form is known from ‘soft’ physics.

An alternative way to unify the BFKL and GLAP formalisms is based on collinear factorization. It has been shown that we can reduce the (leading twist part of the) BFKL behaviour to collinear form in which the splitting and coefficient functions contain explicit calculable series of $\alpha_s^n (\alpha_s \ln(1/x))^n$ terms. This approach has attracted much interest. However, in the introduction we stressed the advantage of working with the unintegrated gluon distribution and using the $k_T$ factorization theorem, and we mentioned some of the limitations of the reduction of the BFKL equation to collinear form. Here we simply state some of the points to consider. In the ‘ unintegrated’ formalism it is straightforward to identify the perturbative contributions which contribute at all $Q^2$ and so to avoid subsuming them in the input distributions. Second,
there is a natural way to introduce the running of $\alpha_S$ in the BFKL formalism, that has increasing theoretical support, which for sufficiently small $x$ goes beyond the Renormalisation Group behaviour (and so is difficult to implement in the collinear factorization approach). Thirdly, the kinematic constraint along the BFKL ladder is easy to implement in the ‘unintegrated’ formalism. Another point is that the BFKL formalism contains all twists, whereas only the leading twist is retained in the collinear approach. Last, but not least, the $k_T$ factorization approach, which we may symbolically write as $F_2 = F_2^{\gamma g} \otimes f$, is easier to implement. The BFKL kernel and the off-shell gluon structure function $F_2^{\gamma g}$ are calculable perturbatively. We simply use leading order in $\alpha_S$ expressions. The $\ln(1/x)$ summations are implicit in the integration over the entire $k_T^2$ phase space of the gluon ladder and in the $k_T$ factorization integrals.

We solved numerically the coupled integral equations for $f(x, k_T^2)$ and $\Sigma(x, k_T^2)$, and we then used the $k_T$ factorization theorem to calculate $F_2$ in terms of a two-parameter input form for the gluon, $xg(x, k_0^2) = N(1 - x)^{\beta}$. The parameters $N$ and $\beta$ are determined by a fit to the available small $x$ data for $F_2$. An excellent description is obtained. The data at the smallest values of $x$ give support for the presence of the kinematic constraint, as does the extrapolation of the gluon to describe the WA70 prompt photon data at $x \approx 0.4$. Notice that the rise of $F_2$ with decreasing $x$ is purely of perturbative origin in our description, and that we find a significant ‘BFKL’ component.

The fact that we achieve an excellent two-parameter fit of the small $x$ data for $F_2$ is not, in itself, remarkable. Other equally good phenomenological fits have been obtained. What is encouraging is that we have a theoretically well-grounded and consistent formalism which, with the minimum of non-perturbative input, is able to give a good perturbative description of the observed structure of $F_2$. Moreover the BFKL/GLAP components of $F_2$ are decided by dynamics. In this way we have made a determination of the universal gluon distribution, $f(x, k_T^2)$, which can be used, via $k_T$ factorization, to predict the behaviour of other small $x$ observables. We showed the predictions for $F_2^c$ and $F_L$.

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4The codes for calculating $F_2$ are available upon request from a.m.stasto@durham.ac.uk
References


[38] H1 collaboration: C. Adloff et al., *Z. Phys.* **C72** (1996) 593.

Figure captions

Fig. 1 The diagrammatic representation of the $k_T$ factorization formula $F_i = F_i^\gamma g \otimes f$. At lowest order in $\alpha_s$, the photon-gluon fusion processes (or to be precise the structure functions $F_i^\gamma g$ of the virtual gluon) are given by the quark box shown (together with the crossed box).

Fig. 2 The two-parameter fit to the $F_2$ data at small $x$ using eq. (10) for $f(x, k^2)$ with (continuous curves) and without (dashed curves) the kinematic constraint. The optimum values of the parameters $N$ and $\beta$, which describe the input form of the gluon, are given in Table 1. The figure shows the H1 data [1] together with the E665 and NMC measurements [36] which occur at the same values of $Q^2$.

Fig. 3 As for Fig. 1, but for the ZEUS measurements [1] of $F_2$, together with the E665, NMC and BCDMS data [36] which occur at the same values of $Q^2$.

Fig. 4 The continuous curves show the behaviour of the conventional gluon distribution $xg(x, Q^2)$ corresponding to fit 1, and calculated using eq. (1). For comparison we also show the gluon distributions of the MRS (R2) [4] (dashed curve) and GRV [6] (dotted curve) sets of partons.

Fig. 5 The unintegrated gluon distribution $f(x, k^2)$ as a function of $k^2$ for $x = 10^{-4}$ and $10^{-3}$ obtained by solving the simultaneous equations for $f(x, k^2)$ and $\Sigma(x, k^2)$. The continuous and dashed curves are obtained by using the unified BFKL/GLAP equation (10) for $f(x, k^2)$ with and without the kinematic constraint respectively. The dotted curve corresponds to using GLAP evolution for $f$, eq. (63). In each case the input $xg(x, k_0^2) = 1.57(1-x)^{2.5}$ is used, where $k_0^2 = 1 GeV^2$.

Fig. 6 The light quark contribution to $F_2(x, Q^2)$ for various $Q^2$ values obtained from solving different sets of coupled equations for the gluon $f$ and the quark singlet $\Sigma$ with, in each case, the input $xg(x, k_0^2) = 1.57(1-x)^{2.5}$ where $k_0^2 = 1 GeV^2$. The continuous and dashed curves come from solving (10, 45) with and without the kinematic constraint. The dotted curve is obtained using GLAP in the gluon sector, that is (63,45), whereas the dot-dashed curve corresponds to pure GLAP evolution, (63,65).

Fig. 7 The predictions for $F_2^c$, compared with H1 charm data, obtained from the optimum fit (fit 1).

Fig. 8 Ratio $F_2^c/F_2$ for different values of $Q^2$ obtained from fit 1.

Fig. 9 The prediction for the structure function $F_L$ as a function of $x$ for different values of $Q^2$ using the parameters of fit 1.