THE GENERAL THEORY OF ALL SUM AND DIFFERENCE RESONANCES

IN A THREE-DIMENSIONAL MAGNETIC FIELD IN A SYNCHROTRON

(PART I)

by

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Abstract

The resonance problem has been treated in detail (Refs 18,19) for a two-dimensional magnetic field, that is without longitudinal component. The aim here is to develop a general theory valid for a three-dimensional magnetic field and simultaneously large $\delta$-variations. Ref. 13 is the only work known to the author treating this problem and this work is restricted to the linear case of transverse quadrupole fields. However, the method used in Ref. 13 encountered the difficulty of non-symplecticity of the motion equations, which are only strictly valid exactly on the resonances. In the present work, a general non-linear theory giving symplectic motion equations under all conditions is given. The first part of the report treats the linear coupling and the second part treats sum and difference resonances of any order.

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GENERAL INTRODUCTION

The usual parameters of a synchrotron, i.e. the average machine radius, the magnetic rigidity, the betatron functions and their derivatives, the phases and the tunes, are indicated in this report by the standard symbols $R$, $B_0$, $\beta_H$, $\beta_V$, $\alpha_H$, $\alpha_V$, $\mu_H$, $\mu_V$ and $Q_H$, $Q_V$. The independent variable $\theta$ used throughout is the axial distance divided by $R$ and the symbol $'$ indicates the derivative by $\theta$. The system of coordinates is given in Figure 1.

![Fig. 1. Coordinates system](image)

Looking at Fig. 1, it is obvious that the radius of curvature $\rho$ is negative in the chosen coordinate system. Thus, the magnetic rigidity $B_0$, defined as the product of the amplitude of the guiding field on the equilibrium orbit by the radius of curvature, becomes negative. In order to avoid confusion in the text, we shall always put the modules $|B_0|$ for the positive definition of this quantity, and simply $B_0$ for the negative one.
1. INTRODUCTION

For reasons of simplicity, we begin with the theory of linear coupling. This permits us to familiarize the reader with the formalism and the mathematical development, before generalizing the theory in the second part of this report for all sum and difference resonances in a three-dimensional magnetic field.

The formalism is based on a perturbation treatment of the classical Hamiltonian. In this first part, the perturbation consists of the linear forces which couple both the transverse motions of protons in a synchrotron. First we have to define these forces and then the perturbing Hamiltonian. Knowing the solution of the unperturbed motion and applying the above-mentioned perturbation treatment, the equations of the perturbed motion can be solved.

Finally it is possible to define a general coefficient of linear coupling for a machine with any form factor. This coefficient will depend on the skew gradients and the longitudinal fields existing around the machine. Thus it characterizes the effects on a proton beam of elements like skew-quadrupoles and solenoids. In the eventuality of a large, unwanted perturbation by such elements, this coefficient makes it possible to estimate which elements should be added in order to compensate the effect and what is the efficiency of this compensation.

This theory of linear coupling has been applied in the case of hypothetical solenoids in the ISR machine. These examples indicate the specific questions which have to be considered in a practical case.

2. PERTURBATION TREATMENT IN CLASSICAL DYNAMICS

In this section we summarize the treatment of perturbations in analytical mechanics\(^1\).

Let us assume that \( H_0 \) is the Hamiltonian of the unperturbed motion and \( H_t \) is the Hamiltonian of the perturbation. Without making any approximations based on the smallness of the perturbing Hamiltonian, the general problem may be stated as follows. Assuming the general solution is known of the canonical equations for unperturbed motion,

\[
\dot{q}_p = \frac{\partial H_0}{\partial p} \quad \dot{p}_p = -\frac{\partial H_0}{\partial q}
\]  

(1.2.1)
it is then necessary to set up a technique to find the motion for the perturbed Hamiltonian

\[ H = H_0(q,p,\theta) + H_1(q,p,\theta) \]  \hspace{1cm} (1.2.2)

\( \theta \) being the independent variable.

Let the solution of (1.2.1) be

\[ q_\rho = q_\rho(a_j, \theta) \hspace{1cm} p_\rho = p_\rho(a_j, \theta) \]  \hspace{1cm} (1.2.3)

where \( a_j \) stands for \( 2N \) arbitrary constants. Solving (1.2.3), we have

\[ a_j = a_j(q_\rho, p_\rho, \theta) \]  \hspace{1cm} (1.2.4)

these \( 2N \) functions being determined by the form of the Hamiltonian \( H_0(q,p,\theta) \).

Consider now the perturbed motion. The perturbation problem is reduced to the study of the way in which the \( a_j \)'s vary with \( \theta \) in agreement with the form of the perturbing Hamiltonian \( H_1(p,q,\theta) \). Since the \( a_j \)'s are the constants of the unperturbed motion, we have,

\[ a_j' = \frac{\mathrm{d}a_j}{\mathrm{d}\theta} = [a_j, H_1] = \sum_\rho \left( \frac{\partial a_j}{\partial q_\rho} \frac{\partial H_1}{\partial p_\rho} - \frac{\partial H_1}{\partial q_\rho} \frac{\partial a_j}{\partial p_\rho} \right) \]  \hspace{1cm} (1.2.5)

By virtue of (1.2.3) the right-hand side is a function of \( (a_j, \theta) \) and so we have a set of \( 2N \) equations to determine the functions \( a_j = a_j(\theta) \) and hence the perturbed motion.

The equations (1.2.5) can be put in a different form. Using (1.2.3) \( H_1 \) is expressible as function of \( (a_j, \theta) \):

\[ H_1(q,p,\theta) = U(a_j, \theta) \]  \hspace{1cm} (1.2.6)

(1.2.5) can then be rewritten;

\[ \frac{\mathrm{d}a_j}{\mathrm{d}\theta} = \sum_\lambda \left[ a_j, a_{\lambda\lambda} \right] \frac{\partial U}{\partial a_{\lambda\lambda}} \]  \hspace{1cm} (1.2.7)

where

\[ \left[ a_j, a_{\lambda\lambda} \right] = \sum_\rho \left( \frac{\partial a_j}{\partial q_\rho} \frac{\partial a_{\lambda\lambda}}{\partial p_\rho} - \frac{\partial a_{\lambda\lambda}}{\partial q_\rho} \frac{\partial a_j}{\partial p_\rho} \right) . \]
So far everything is rigorous and (1.2.7) gives the exact solution of the motion. But if the derivatives of \( H \) or of \( U \) are small, then the right-hand side of (1.2.7) is small and there may be the possibility of approximating the perturbed motion by keeping only the dominant terms of the function \( U \).

3. DESCRIPTION OF TRANSVERSE MOTIONS
OF THE PROTONS

3.1 Unperturbed motion and the Hamiltonian

The equations of the unperturbed transverse motion are well known and can be written as follows:

\[
\begin{align*}
x'' + K_1(\theta)x &= 0 \\
z'' + K_2(\theta)z &= 0
\end{align*}
\]  

(1.3.1)

where the derivatives are taken w.r.t. \( \theta \). The functions \( K_1 \) and \( K_2 \) are the forces exerted on the particles by the magnetic field gradients.

Knowing the equations (1.3.1), it is easy to define the Hamiltonian of the unperturbed motion:

\[
H_0 = \frac{i}{2} \left[ K_1 x^2 + K_2 z^2 + p_x^2 + p_z^2 \right]
\]  

(1.3.2)

This is easily checked by applying the canonical equations (1.2.1) and verifying that the equations are equivalent to (1.3.1).

The general solutions of equations (1.3.1), by virtue of the Floquet's theorem\(^2\) are,

\[
\begin{align*}
x &= a_1 u(\theta) e^{iQ_1 \theta} + \overline{a_1 \overline{u}(\theta)} e^{-iQ_1 \theta} \\
z &= a_2 v(\theta) e^{iQ_2 \theta} + \overline{a_2 \overline{v}(\theta)} e^{-iQ_2 \theta}
\end{align*}
\]  

(1.3.3)

where \( u \) and \( v \) are the Floquet's functions\(^2,3\)

\[
\begin{align*}
u(\theta) &= \sqrt{\frac{\partial I(\theta)}{2R}} \exp \left[ i \int_0^\theta \left( \frac{R}{\partial I(\theta')} - Q_1 \right) d\theta' \right] \\
v(\theta) &= \sqrt{\frac{\partial V(\theta)}{2R}} \exp \left[ i \int_0^\theta \left( \frac{R}{\partial V(\theta')} - Q_2 \right) d\theta' \right]
\end{align*}
\]  

(1.3.4)

\( \overline{u} \) and \( \overline{v} \) represent the complex conjugates of the functions \( u \) and \( v \).
a₁ and a₂ and their complex conjugates are the constants of the motion. They represent the complex amplitudes of the transverse oscillations of the protons.

3.2 Motions perturbed by linear coupling and the Hamiltonian

A three-dimensional magnetic field can couple both the transverse motions of the protons in the following way:

\[
x'' + K₁(θ)x = \frac{R²}{|B₀|} \frac{∂B_z}{∂z} z - \frac{R}{|B₀|} B_s z' \\
z'' + K₂(θ)z = - \frac{R²}{|B₀|} \frac{∂B_x}{∂x} x + \frac{R}{|B₀|} B_s x'
\]  

(1.3.5)

where \( B_x, B_z, B_s \) are the three field components and \( |B₀| \) is the magnetic rigidity (always positive).

In order to simplify the notation, we can introduce the following functions characterizing the magnetic field:

\[
K(θ) = \frac{1}{|B₀|} \left( \frac{∂B_x}{∂x} + \frac{∂B_z}{∂z} \right) \\
M(θ) = \frac{R}{|B₀|} B_s
\]  

(1.3.6)

Remembering that,

\[
\text{div} \ B = 0 \\
\frac{∂B_x}{∂x} + \frac{∂B_z}{∂z} + \frac{∂B_s}{∂s} = 0
\]  

(1.3.7)

then, we can write the following equations

\[
\frac{R²}{|B₀|} \frac{∂B_x}{∂x} = K - \frac{1}{2} M' \\
- \frac{R²}{|B₀|} \frac{∂B_z}{∂z} = K + \frac{1}{2} M'
\]  

(1.3.8)

where the derivative of \( M \) is taken with \( θ \).

Putting the relations (1.3.6) and (1.3.8) in the motion equations (1.3.5), gives

\[
x'' + K₁x = - (K - \frac{1}{2} M') z - Mz' \\
z'' + K₂z = - (K - \frac{1}{2} M') x + Mx'
\]  

(1.3.9)
Two specific examples of magnetic elements which can give coupling are,

1) skew-quadrupole lenses for which we have:

\[
M = 0
\]

\[
K = \frac{R^2}{|B_p|} \frac{\partial B_x}{\partial x} = -\frac{R^2}{|B_p|} \frac{\partial B_z}{\partial z}
\]

(1.3.10)

2) solenoids for which we have:

\[
M = \frac{R}{|B_p|} B_s
\]

and \(K\) describing the end effects in agreement with (1.3.6).

Let us put for instance\(^4\)

\[
\frac{\partial B_x}{\partial x} = -(1-a_s) B'_s \quad \frac{\partial B_z}{\partial z} = -a_s B'_s
\]

(1.3.11)

then, 
\[a_s = \frac{1}{2}\] if the solenoid has no end plates and 
\[a_s = 1\] if the solenoid has end plates with a horizontal slot.

The function \(K\) then has the following form:

\[
K = (a_s - \frac{1}{2}) M'
\]

(1.3.12)

Knowing the motion equations (1.3.9) it is not difficult to find the associated Hamiltonian\(^4\):

\[
H = \frac{1}{2} \left[ K_1 x'^2 + K_2 z'^2 + 2Kxz + (p_x - iMz)^2 + (p_z + iMp_x)^2 \right]
\]

(1.3.13)

As before, we can check the form of \(H\) using the canonical equations (1.2.1) and comparing with the relations (1.3.9).

Subtracting \(H_0\) given by (1.3.2) from \(H\), we will obtain the perturbing Hamiltonian \(H_1\) associated with linear coupling:

\[
H_1 = Kxz - \frac{1}{4} Mp_x + \frac{1}{2} Mp_z \quad + \frac{1}{4} M^2 z^2 + \frac{1}{4} M^2 x^2
\]

(1.3.14)

4. GENERAL EQUATION OF THE PERTURBED MOTION OF PROTONS

Using the description given in section 3 of the perturbation problem, we are now able to apply the treatment summarized in section 2.

Firstly the solution of the unperturbed motion in a form equivalent to (1.2.3) is required. The expressions for \(x\) and \(z\) are already given in (1.3.3). In order to get the expressions in \(p_x\) and \(p_z\), we use the first set of canonical equations of (1.2.1):
\[ \frac{dx}{d\theta} = \frac{\partial H_2}{\partial p_X} = p_X \]  \hspace{1cm} (1.4.1)
\[ \frac{dz}{d\theta} = \frac{\partial H_2}{\partial p_2} = p_2 \]

Taking the equations (1.3.3) and differentiating, we obtain the solution equivalent to (1.2.3):

\[ x = a_1 \ u \ e^{iQ_3 \theta} + \bar{a}_1 \ \bar{u} \ e^{-iQ_3 \theta} \]
\[ z = a_2 \ v \ e^{iQ_3 \theta} + \bar{a}_2 \ \bar{v} \ e^{-iQ_3 \theta} \]  \hspace{1cm} (1.4.2)

\[ p_X = a_1 \ (u' + iQ_3 u) \ e^{iQ_3 \theta} + \bar{a}_1 \ (\bar{u}' - iQ_3 \bar{u}) \ e^{-iQ_3 \theta} \]
\[ p_2 = a_2 \ (v' + iQ_3 v) \ e^{iQ_3 \theta} + \bar{a}_2 \ (\bar{v}' - iQ_3 \bar{v}) \ e^{-iQ_3 \theta} \]

Solving (1.4.2), we have the equivalent of (1.2.4):

\[ a_1 = \frac{1}{W(u)} \left[ (\bar{u}' - iQ_3 \bar{u}) \ x - \bar{u} p_X \right] \ e^{-iQ_3 \theta} \]
\[ \bar{a}_1 = -\frac{1}{W(u)} \left[ (u' + iQ_3 u) \ x - u p_X \right] \ e^{iQ_3 \theta} \]  \hspace{1cm} (1.4.3)
\[ a_2 = \frac{1}{W(v)} \left[ (\bar{v}' - iQ_3 \bar{v}) \ z - \bar{v} p_2 \right] \ e^{-iQ_3 \theta} \]
\[ \bar{a}_2 = -\frac{1}{W(v)} \left[ (v' + iQ_3 v) \ z - v p_2 \right] \ e^{iQ_3 \theta} \]

where \( W(u) \) and \( W(v) \) are the Wronskians associated with Floquet's functions\(^2\):

\[ W(u) = u \ (\bar{u}' - iQ_3 \bar{u}) - (u' + iQ_3 u) \ \bar{u} = -i \]  \hspace{1cm} (1.4.4)
\[ W(v) = v \ (\bar{v}' - iQ_3 \bar{v}) - (v' + iQ_3 v) \ \bar{v} = -i \]

Using the relations (1.2.7), we are now able to write the general equation of the perturbed motion:

\[ \frac{da_1}{d\theta} = \frac{1}{W(u)} \ \frac{\partial U}{\partial a_1} \]  \hspace{1cm} (1.4.5)
\[ \frac{\bar{a}_1}{d\theta} = -\frac{1}{W(u)} \ \frac{\partial U}{\partial \bar{a}_1} \]

and the equivalent equations for the vertical motion with \( a_2, \bar{a}_2 \).
By virtue of (1.4.4), these equations of motion become:

\[
\frac{da_1}{d\theta} = i \frac{\partial U}{\partial a_1},
\]
\[
\frac{da_1}{d\theta} = -i \frac{\partial U}{\partial a_1},
\]
\[
\frac{da_2}{d\theta} = i \frac{\partial U}{\partial a_2},
\]
\[
\frac{da_2}{d\theta} = -i \frac{\partial U}{\partial a_2}.
\] (1.4.6)

These are exact motion equations associated with the analytical forms (1.4.2) and with the perturbation \( U \) [or \( H_1 \), from (1.2.6)].

5. SOLUTION OF THE PERTURBATION EQUATIONS FOR LINEAR COUPLING

5.1 Discussion of the assumptions

The problem is now reduced to the solution of the equations (1.4.6) for the particular case of linear coupling, which is described by the Hamiltonian \( H_1 \) given in (1.3.14).

We can first express \( H_1 \) as function of \( a_j \), by putting (1.4.2) in (1.3.14):

\[
U(a_1, a_2, a_3, a_4, \theta) = H_1(x, z, p_x, p_z, \theta)
\] (1.5.1)

The explicit form of \( U \) is given in Appendix 1. As the synchrotron is a periodic machine, it is possible to develop in Fourier series the function \( U \):

\[
U = \sum_{j,k,l,m} \sum_{\infty}^{2} h^{(2)}_{jklm} a_1^{j-k} a_2^{l-k} a_3^{m-k} \exp\left[i[(j-k)Q_1 + (l-m)Q_2 + m]\theta\right]
\] (1.5.2)

the coefficients \( h^{(2)} \) being explicitly given in Appendix 1.

Now we can make two assumptions and consequently the solution will no longer be exact and will only be valid for small coupling, that is for a perturbation:

1) We can first neglect the \( M^2 \) term in the Hamiltonian (1.3.14) compared with the \( K \) and \( M \) terms.

2) We can keep only the low frequency part of the function (1.5.2), which gives the slow but important variations of the variables \( a_j \).
For this second assumption, it is necessary to define which specific resonance will be looked at. In our particular case of linear coupling, the Hamiltonian function $U$ (Appendix I) indicates that four resonances can be excited:

\[
\begin{align*}
Q_H + Q_V &= p \\
2Q_H &= p \\
2Q_V &= p \\
Q_H - Q_V &= p
\end{align*}
\]

\text{sun resonances}

(1.5.3)

\text{difference resonance}

$p$ being an integer.

The mono-dimensional resonances vanish if the $M^2$ terms of the Hamiltonian are neglected. Thus, the first assumption cannot be accepted for the study of these mono-dimensional resonances. On the other hand, all the terms of the Hamiltonian contribute to the two-dimensional resonances and hence we can apply both assumptions mentioned above.

5.2 Solution of the equations for the resonances $2Q_H = p$ and $2Q_V = p$

Starting from the motion equations (1.4.6) and from the Hamiltonian (1.5.2) and keeping the low-frequency part in agreement with the second assumption of the section 5.1, we can write the explicit equation (Appendix I) for $2Q_H = p$:

\[
\frac{da_1}{d\theta} = i(a_1\lambda + \tilde{a}_1 2\kappa e^{-i\Delta \theta})
\]

(1.5.4)

where

\[
\kappa = \frac{h(z)}{2000-p}
\]

\[
\lambda = \frac{h(z)}{11000}
\]

(see Appendix I)

\[
\Delta = 2Q_H - p = \text{distance from the resonance}
\]

Putting $a_1 = r_1 e^{i\phi_1}$ in the relation (1.5.4) and separating the imaginary part from the real part, we will get:

\[
\frac{dr_1}{d\theta} = 2r_1 |\kappa| \sin (2\phi_1 + \phi_\kappa + \Delta \theta) - |\lambda| r_1 \sin (\phi_\lambda)
\]

\[
\frac{d\phi_1}{d\theta} = 2|\kappa| \cos (2\phi_1 + \phi_\kappa + \Delta \theta) + |\lambda| \cos (\phi_\lambda)
\]

(1.5.5)

with \[
\kappa = |\kappa| e^{-i\phi_\kappa} \quad \lambda = |\lambda| e^{-i\phi_\lambda}
\]

Basically, the second equation gives $\phi_1(\theta)$ and the first one will then give $r_1(\theta)$ by integration. Assuming now that $\Delta$ is small and that the contribution of $\lambda$ is small, because it is a zero-harmonic, we can find an approximate solution for the second equation (1.5.5) and then put it in the first one:
\[ \phi_1 = \text{arctg} \left[ \exp \left( \frac{4i|\kappa|}{\Delta} \sin \theta \Delta + C_1 \right) \right] - \frac{\phi_0}{\zeta} - \frac{\pi}{4} \]  
\[ r_1 = r_{10} \exp \left[ 2i |\kappa| \int_0^\Theta \sin (2\phi_1 + \phi_0 + \theta \Delta) \, d\theta \right] \]  
\[ \text{with } \kappa = \frac{1}{32\pi R} \int_0^{2\pi} M^2 \delta_H(\theta) e^{-i\theta} \int_0^\Theta \left( \frac{\delta_H}{\delta_H} - Q_H \right) \, d\theta \]  

Putting (1.5.6) in the first equation (1.3.3) will give the explicit solution of the horizontal motion. The form of (1.5.6) shows that \( r_1 \) remains finite.

The same type of solution can be applied to the vertical motion for the resonance \( 2Q_H = p \).

5.3 Solutions of the equations for the resonance \( Q_H + Q_V = p \)

Using again the equations (1.4.6) and (1.5.2), taking account of both the assumptions given in the section 5.1, the explicit equations are the following (Appendix 1):

\[ \frac{da_1}{d\theta} = i \kappa a_1 e^{-i\Delta \theta} \]  
\[ \frac{d^2a_1}{d\theta^2} = i \kappa a_1 e^{-i\Delta \theta} \]  
\[ \text{with } \kappa = \text{h}_{1010-p} \]  
(see Appendix 1)

\[ \Delta = Q_H + Q_V - p = \text{distance from the resonance} \]

Defining the new variable \( a_2 = a_1 e^{i\Delta \theta} \) and putting it in (1.5.7), we have:

\[ \frac{da_2}{d\theta} = i \kappa a_2 \]  
\[ \frac{d^2a_2}{d\theta^2} = -i \kappa a_2 - i\Delta \frac{d^2a_2}{d\theta^2} \]  
\[ \text{instead of } (1.5.8), \text{it is possible to write a second order equation for } a_2^{\text{v}} \text{ by differentiating,} \]

\[ \frac{d^2a_2}{d\theta^2} + i \Delta \frac{da_2}{d\theta} - \kappa \kappa a_2^{\text{v}} = 0 \]  
(1.5.9)
The general solution of (1.5.9) is then:

\[ \bar{a}_2 = A_+ e^{i\omega_+ \theta} + A_- e^{i\omega_- \theta} \]  

(1.5.10)

where \( \omega_\pm = -\frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 - |\kappa|^2} \)

Knowing the explicit form of \( \bar{a}_2 \), the expressions for \( a_1 \) and \( a_2 \) are,

\[ a_1 = \frac{1}{\kappa} \left( A_+ \frac{1}{\omega_+} e^{i\omega_+ \theta} + A_- \frac{1}{\omega_-} e^{i\omega_- \theta} \right) \]  

(1.5.11)

\[ a_2 = \bar{A}_+ e^{i\omega_+ \theta} + \bar{A}_- e^{i\omega_- \theta} \]

where, \( A_+ , A_- \) are complex constants of the motion

\( \omega_\pm \) are the frequencies given in (1.5.10)

and \( \kappa \) is given by

\[ \kappa = \frac{1}{4\pi R} \int_0^{2\pi} \sqrt{\frac{\alpha_+ \alpha_-}{\beta_+ \beta_-}} \left[ K + MR \left( \frac{\alpha_+}{\beta_+} - \frac{\alpha_-}{\beta_-} \right) - \frac{i}{2} \frac{MR}{2} \left( \frac{1}{\beta_+} - \frac{1}{\beta_-} \right) \right] \]

(1.5.12)

\[ \times \left( \int_0^\theta \left( \frac{R}{\beta_+} - Q_+ \right) d\theta' + i \int_0^\theta \left( \frac{R}{\beta_-} - Q_- \right) d\theta' \right) e^{ip\theta} d\theta \]

Putting (1.5.11) in the equations (1.3.3) will give the explicit solution of the perturbed motion. The initial conditions for \( x, x', z, z' \) will define the constants \( A_+ \) and \( A_- \).

Depending on the amplitude of \( |\kappa| \), the motion can be stable or unstable. It is stable if \( \omega_\pm \) (1.5.10) is real. In other terms this means:

If \( |\kappa| \leq \frac{|\Delta|}{2} \) the motion is stable

(1.5.13)

If \( |\kappa| > \frac{|\Delta|}{2} \) the motion is unstable

On the resonance \( (\Delta = 0) \), the motion is never stable for any small perturbation \( \kappa \).

5.4 Solutions of the equations for the resonance \( Q_+ - Q_- \neq 0 \)

Using (1.4.6), (1.5.2) and the assumptions given in 5.1, the explicit equations are (Appendix 1):
\[
\frac{da_1}{d\theta} = i \kappa a_2 e^{-i\Delta} \\
\frac{da_2}{d\theta} = i \kappa a_1 e^{i\Delta}
\]

with \( \kappa = \frac{h_{001}-p}{4\pi} \) (see Appendix 1)

\( \Delta = Q_H - Q_Y - p \) = distance from the resonance

Defining the new variable \( a_2^- = a_2 e^{-i\Delta} \), putting it in (1.5.14) and then differentiating, the equation for \( a_2^- \) becomes a second order equation:

\[
\frac{d^2 a_2^-}{d\theta^2} + i \Delta \frac{da_2^-}{d\theta} + \kappa \kappa a_2^- = 0
\]

The general solution of (1.5.15) is then:

\[
a_2^- = A_+ e^{i\omega_+ \theta} + A_- e^{i\omega_- \theta}
\]

where \( \omega_\pm = -\frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + |\kappa|^2} \)

The expressions for \( a_1 \) and \( a_2 \) are then

\[
a_1 = \kappa \left( A_+ e^{i\omega_+ \theta} + A_- e^{i\omega_- \theta} \right) \\
a_2 = \left( A_+ e^{i\omega_+ \theta} + A_- e^{i\omega_- \theta} \right) e^{i\Delta}
\]

where \( A_+, A_- \) are complex constants of the motion

\( \omega_\pm \) are the frequencies given in (1.5.16)

and \( \kappa \) is given by

\[
\kappa = \frac{1}{4\pi} \int_0^{2\pi} \sqrt{\frac{R_H - Q_H}{R_Y - Q_Y}} \left[ K + MR \left( \frac{R}{R_H} - \frac{Q}{Q_Y} \right) - \frac{i}{2} MR \left( \frac{1}{E_H} + \frac{1}{E_Y} \right) \right] \\
+ i \int_0^{\varphi} \left( \frac{R}{R_H} - Q_H \right) ds', -i \int_0^{\varphi} \left( \frac{R}{R_Y} - Q_Y \right) ds', e^{i\varphi} ds
\]

(1.5.18)

Both (1.5.17) and (1.3.3) will give the explicit solution of the perturbed motion, the constants \( A_+ \) and \( A_- \) being defined by the initial conditions for \( x, x', z, z' \).

For this case, the motion is always stable, because the frequencies (1.5.16) are real for any \( \Delta \) and \( |\kappa| \).
6. **LINEAR COUPLING COEFFICIENTS**

6.1 **Definition of linear coupling coefficients**

The most important resonances excited by linear coupling are the two-dimensional ones, which depend linearly on $K$ and $M$. The characteristic parameter $\kappa$ of the perturbed motion is given for these resonances in (1.5.12) and (1.5.18) respectively. It is possible to summarise both expressions as follows:

For $\Delta = Q_H + nQ_V - p$, $n = \pm 1$

$$\kappa = \frac{1}{4\pi R} \int_0^{2\pi} \sqrt{\beta_H \beta_V} \left[ K + MR \left( \frac{a_H}{\beta_H} - \frac{a_V}{\beta_V} \right) - \frac{i}{2} MR \left( \frac{1}{\beta_H} - \frac{1}{\beta_V} \right) \right]$$

$$\times \int_0^{2\pi} \left( \frac{R}{\beta_H} - Q_H \right) d\theta + i \int_0^{2\pi} \left( \frac{R}{\beta_V} - Q_V \right) d\theta + \int_0^{2\pi} e^{i p \theta} d\theta$$

(1.6.1)

$\kappa$ is the sum of two terms which are basically orthogonal in the complex plane, if we do not consider the phases associated with the exponentials. The first term represents the skew quadrupole field effects and the second one represents the longitudinal field effects (see section 3.2). So we can split $\kappa$ in two parts and define two (or three) linear coupling coefficients:

$$C_{tot} = C_q + C_b = 2\kappa$$

$$C_q = \frac{1}{2\pi R} \int_0^{2\pi} \sqrt{\beta_H \beta_V} \left[ K + MR \left( \frac{a_H}{\beta_H} - \frac{a_V}{\beta_V} \right) \right]$$

$$\exp \left\{ i \left[ \left( \mu_H - Q_H \theta \right) + n(\mu_V - Q_V \theta) + p \theta \right] \right\} d\theta$$

(1.6.2)

$$C_b = -\frac{1}{4\pi} i \int_0^{2\pi} \sqrt{\beta_H \beta_V} \left( \frac{1}{\beta_H} - \frac{1}{\beta_V} \right)$$

$$\exp \left\{ i \left[ \left( \mu_H - Q_H \theta \right) + n(\mu_V - Q_V \theta) + p \theta \right] \right\} d\theta$$

The coefficient $C_q$ for skew fields contains not only the pure skew term $K$ in agreement with (1.3.10) or (1.3.12), but also a skew term due to the longitudinal changes of the $\beta$ functions.

These definitions (1.6.2) of coupling coefficients are consistent with the previous definitions given in the sinusoidal approximation. If we put $\beta_H = \beta_V = \text{constant}$, $a_H = a_V = 0$ and $\mu = Q\theta$ in (1.6.2), we find, using the definitions of $K$ (1.3.10) and $M$ (1.3.6),
- 13 -

\[ |C_q| = \frac{R^2}{Q} \frac{1}{|B_p|} \frac{\partial B_x}{\partial x} \]  \hspace{1cm} (1.6.3)

\[ |C_b| = \frac{R}{|B_c|} B_s \]

In our definition (1.6.2), we are taking account of the amplitude modulation around the machine and of the phases. The phases \( \nu_H \) and \( \nu_V \) are very important. Their presence means that \( C_q \) and \( C_b \) are not always orthogonal. It is therefore possible to compensate the \( C_b \) term with the \( C_q \) term provided the phases are well chosen.

6.2 Calculation of the linear coupling coefficients

To avoid beam blow-up, we would like to have \( |\kappa| = 0 \). First we evaluate the coupling coefficients with the existing fields and then, if necessary, with compensating fields in order to suppress the perturbation.

\[ |C_{\text{tot}}| = 2|\kappa| = \sqrt{C_q^2 + C_b^2 + C_q C_b + C_q C_b} \]  \hspace{1cm} (1.6.4)

To find the amplitudes of the coefficients \( C_q, C_b \) and \( C_{\text{tot}} \) for a given practical problem, it is necessary to approximate the integrals (1.6.2) with summations, splitting the magnetic elements in pieces. Assuming we have \( N_e \) elements cut in \( N_c \) parts, we can calculate any of these amplitudes as indicated below.

The functions \( K, M \) being defined by (1.3.6), the amplitudes and phases of the coupling coefficients for one piece of element can be deduced from (1.6.2):

\[ |C_q|_{jk} = \left| \frac{1}{2\pi R} (\nu H_B V)_{jk} \left[ K_{jk} + K_{jk} \left( \frac{\nu H}{B_V} - \frac{\nu V}{B_H} \right) \right] \right| \]

\[ \Psi_{q}_{jk} = \nu_{qjk} \pm \frac{\pi}{2} \]

\[ |C_b|_{jk} = \frac{1}{4\pi R} M_{jk} \left( \frac{\nu_H}{B_V} - \frac{\nu V}{B_H} \right) \]

\[ \Psi_{b}_{jk} = \Psi_{qjk} \pm \frac{\pi}{2} \]

\[ |C_{\text{tot}}|_{jk} = \sqrt{C_q^2_{jk} + C_b^2_{jk}} \]

\[ \Psi_{\text{tot}}_{jk} = \Psi_{qjk} \pm \arctg \left( \frac{|C_b|_{jk}}{|C_q|_{jk}} \right) \]

with \( n = \pm 1 \) depending on the resonance \( Q_H + nQ_V = p \)

\( j = 1, \ldots, N_e \) no. of elements

\( k = 1, \ldots, N_c \) no. of cuts

\( l = \) length of one piece of elements

The signs appearing in \( \Psi_b \) and \( \Psi_{\text{tot}} \) depend on the signs of \( C_q \) and \( C_b \), and the phase shift \( \pi \) is present in \( \Psi_q \) only for negative \( C_q \).
Knowing the amplitudes and phases, we can sum the contributions of each piece in each element:

\[
C = \sum_{j=1}^{N_e} \sum_{k=1}^{N_c} |C|_{jk} e^{i\psi_{jk}}
\]

\[
|C|^2 = \sum_{j=1}^{N_e} \sum_{k=1}^{N_c} |C|_{jk}^2 + \sum_{j=1}^{N_e} \sum_{k=1}^{N_c} \sum_{n=1}^{N_c} 2|C|_{jk}|C|_{jn} \cos(\psi_{jk} - \psi_{jn})
\]

\[+ \sum_{j=1}^{N_e-1} \sum_{n=j+1}^{N_e} \sum_{k=1}^{N_c} \sum_{n=1}^{N_c} 2|C|_{jk}|C|_{mn} \cos(\psi_{jk} - \psi_{mn})\] (1.6.6)

The relation (1.6.6) is valid for any one of the coefficients \(C_q, C_b, C_{tot}\). Using (1.6.5) and (1.6.6) \(|C_q|, |C_b| \) or \(|C_{tot}|\) can be found.

A computer program has been written to calculate these coupling coefficients and to determine the necessary field levels in elements such as skew quadrupoles or solenoids to minimise \(|C_{tot}|\).

7. APPLICATION TO THE CERN STORAGE RINGS

Some skew field always exists in the ISR due to random tilts of the magnet units and for this reason skew quadrupoles have been included in the lattice\(^7\). A project to install a solenoid at one intersection to analyse secondary particles is now under way. This will be the first axial field element in the ISR machine. To check experimentally the effects of a solenoid on the beams, a test solenoid was installed.

It is interesting to apply the present theory of linear coupling to both of these solenoids. As current ISR working lines are close to the diagonal, only the difference resonance \(Q_H - Q_V = 0\) is of importance.

Table 1 summarises the amplitudes of the coupling coefficients associated with the detector solenoid in intersection 1. The characteristics of this solenoid are the following:

Length = 1.8 m

\[\int B_s \, ds = 2.7 \, \text{Tm}\]

Both ends with slots so that \(a_s = 1\) (1.3.12)

The \(|C|\) amplitudes have been calculated for four currently used working lines\(^8\),\(^9\), for a steel low-\(B_V\) section\(^10\) and for a superconducting low-\(B_V\) section\(^11\). Both low-\(B_V\) sections are planned for the same intersection as the solenoid.
### TABLE 1

Linear coupling coefficients for the detector solenoid in the ISR at 26 GeV/c

| Working conditions | $\beta_H$ (m) | $\beta_V$ (m) | $\alpha_H$ | $\alpha_V$ | $C_q$ (10^{-3}) | $|C_b|$ (10^{-3}) | $|C_{tot}|$ (10^{-3}) |
|------------------|--------------|-------------|-----------|--------|-----------------|-----------------|-------------------|
| FP               | 22.6         | 14.6        | .232      | -.131  | 2.26            | 4.96            | 3.95              |
| TW               | 24.0         | 18.4        | .474      | -.217  | 3.83            | 4.89            | 5.49              |
| SC               | 22.7         | 14.5        | .236      | -.130  | 2.29            | 4.96            | 3.95              |
| ELSA             | 20.9         | 12.2        | .189      | -.106  | 2.24            | 5.02            | 3.54              |
| steel low $\beta_V$ | 47.5       | 2.6         | -.214     | .100   | 12.31           | 10.52           | 1.98              |
| supercon. low $\beta_V$ | 4.7       | .3          | .134      | .219   | 7.34            | 4.91            | 8.22              |

Two interesting points arise from these results:

1) The steel low $\beta_V$ section gives the largest $|C_q|$ and $|C_b|$ but also the lowest $|C_{tot}|$. This means that the coefficients $C_b$ and $C_q$ are roughly colinear and opposed.

2) The $|C_b|$ amplitude for the superconducting low $\beta_V$ section is roughly half that for the steel section, although the $\beta_H/\beta_V$ ratios look very similar. This is due to the fact that the $\beta_V$ is increasing far more rapidly inside the solenoid in the superconducting section, so that after integration the average ratio $\beta_H/\beta_V$ is smaller.

Table 2 summarises the amplitudes of the coupling coefficients associated with the test solenoid. The characteristics of this solenoid are the following:

$$\text{Length} = 0.5 \text{ m}$$

$$\int B_s \, dx = 0.34 \text{ Tm}$$

The $|C|$ amplitudes have been calculated for the SC working line, but for different end conditions (solenoid without end plates, with two slots and with one slot only).
TABLE 2

Linear coupling coefficients for the test solenoid in the ISR at 26 GeV/c

Working line: 8C \( a_\| = 57.1 \) \( a_\perp = 12.4 \) \( \alpha_\| = 0.253 \) \( \alpha_\perp = 0.061 \)

| End plates | \( |C_q| \) \( (10^{-3}) \) | \( |C_b| \) \( (10^{-3}) \) | \( |C_{\text{tot}}| \) \( (10^{-3}) \) |
|------------|----------------|----------------|----------------|
| \( a_{s1} = a_{s2} = 0.5 \) | 0.06 | 0.71 | 0.71 |
| \( a_{s1} = a_{s2} = 1 \) | 0.45 | 0.71 | 0.29 |
| \( a_{s1} = 1 \) \( a_{s2} = 0.5 \) | 22.40 | 0.71 | 22.40 |

Two points are of interest in this table:

1) With two slots, the test solenoid also excites co-linear and opposed coefficients, \( C_b \) and \( C_q \).

2) With one slot only, the \( C_q \) dominates (\( \sim 30 \) times larger than \( C_b \)). For comparison, the normal machine with the 8C line at 26 GeV/c has a residual \( C_q \) of \( \sim 1 \cdot 10^{-7} \).

Both these tables illustrate the main points:

1) A solenoid clearly contributes to the \( C_b \) coefficient but it also contributes to the \( C_q \) term in two different ways: firstly because the \( \gamma \) functions change inside the solenoid (\( \alpha \) contribution) and secondly because of the skew fields in the end plates (\( X \times M' \) contribution). Finally, \( C_q \) can be as important as \( C_b \) for such an element.

2) It can happen that the slots of the end plates partially compensate the effects of the centre of the solenoid.

3) It is possible to compensate the effects of a longitudinal field with skew fields by virtue of the phase terms. In other words, a set of skew quadrupoles can compensate both the random tilts of the main magnets and the solenoids' contribution.

8. CONCLUSIONS

An analytical formalism able to treat the problem of the linear coupling of the transverse motions in a synchrotron has been developed. This formalism enabled us to give the explicit solutions of the motion equations in case of linear coupling for the different resonances excited. Linear coupling coefficients for a machine with a high form factor were defined. These coefficients characterise the perturbed motions and make it possible to analyse the effects of skew and longitudinal fields. This is of use when introducing new elements and when studying compensation schemes.
Finally it is interesting to see the relation of this theory with the work already existing. Basically, three methods have been used. The first one refers to transfer matrices and does not give the differential equations and their explicit solutions. The second one uses the Hamiltonian formalism, but was only applied to a transverse field. The third one starts by substituting the differential equations of the unperturbed motion in the differential equations and then to deduce the equations for the constants. This method was successful in the case of pure skew quadrupole fields and in the sinusoidal approximation, but it was not successful in the case of longitudinal field and large $\beta$ variations. For example ref. 13 obtains the equations (1.8.1) which are similar to those derived in this report (1.5.14).

\[ \frac{da_1}{d\vartheta} = Q_z a_2 e^{-i\Delta \theta} \]  
(1.8.1)

\[ \frac{da_2}{d\vartheta} = Q_x a_1 e^{i\Delta \theta} \]

with $Q_z \neq -Q_x$.

The problem which arises is that the equations (1.8.1) admit the possibility of having coupling much stronger one way than the other, which is non-symplectic and forbidden. This was realised by Kolomensky who calculated $Q_z + Q_x^{13}$ and showed that this is small if $\Delta$ is small (in the vicinity of the resonance). This is restrictive and not a typical ISR operating condition. In the present report in (1.5.14) we have the identity $i\kappa = (-i\kappa)$, which means the equations are symplectic under all conditions. Finally the comparison of $\kappa$ with $Q_z$ and $Q_x$ of ref. 13 suggests that some terms are missing in $Q_z$ and $Q_x$.

* * *

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APPENDIX I

The Hamiltonian of the perturbation due to linear coupling

The relation (1.3.14) gives the form of the Hamiltonian of the perturbation due to linear coupling. In order to have the expression of $H_i$ as function of $(a_j, \delta)$, we have to put the expressions (1.4.2) in (1.3.14). Doing so we get:

$$U(a_1, \bar{a}_1, a_2, \bar{a}_2, \theta) = \sum_{j, k, \lambda, \mu} h_{jk\lambda\mu}^{(2)} a_1^j \bar{a}_1^k \mu^a \bar{a}_2^\lambda \mu^b \exp \left\{ i \left( j-k \right) Q_{\lambda \mu} + (\lambda-\mu) Q_{\lambda \mu} \right\} \delta$$  \hspace{1cm} (Al.1)

with the following definitions:

$$h_{2000}^{(2)} = \frac{M^2}{8} u^2$$
$$h_{0020}^{(2)} = \frac{M^2}{8} v^2$$
$$h_{0002}^{(2)} = \frac{M^2}{8} \bar{u}^2$$
$$h_{0011} = \frac{M^2}{4} \bar{v}$$
$$h_{1100} = \frac{M^2}{4} \bar{u}$$

$$h_{1010} = K u v + \frac{M}{2} \left[ u(v' + iQ_y v) - v(u' + iQ_x u) \right]$$
$$h_{0101} = K \bar{u} \bar{v} + \frac{M}{2} \left[ \bar{u}(\bar{v}' - iQ_y \bar{v}) - \bar{v}(\bar{u}' - iQ_x \bar{u}) \right]$$
$$h_{1001} = K u \bar{v} + \frac{M}{2} \left[ u(\bar{v}' - iQ_y \bar{v}) - \bar{v}(u' + iQ_x u) \right]$$
$$h_{0110} = K v \bar{u} + \frac{M}{2} \left[ v(\bar{v}' + iQ_y \bar{v}) - \bar{v}(v' - iQ_x v) \right]$$

$u$ and $v$ being the Floquet's functions (1.3.4).

As the synchrotron is a periodic machine, it is possible to develop in Fourier series the function $U$ or, which is equivalent, the coefficients $h_{jk\lambda\mu}^{(2)}$:

$$h_{jk\lambda\mu}^{(2)}(\theta) = \sum_{q=-\infty}^{+\infty} h_{jk\lambda\mu}^{(2)} e^{iq\theta}$$  \hspace{1cm} (Al.3)

with

$$h_{jk\lambda\mu}^{(2)} = \frac{1}{2\pi} \int_0^{2\pi} h_{jk\lambda\mu}^{(2)}(\theta) e^{-iq\theta} \, d\theta$$
Putting (A1.3) in (A1.1) will give the expression (1.5.2).

As we have seen in section 3, we have to keep the low-frequency part of the Hamiltonian function $U$ corresponding to the resonance we want to look at:

$$n_1 Q_{\theta} + n_2 Q_{\lambda} = p$$  \hspace{1cm} (A1.4)

$n_1, n_2, p$ being integers and, in the present case, $|n_1| + |n_2| = 2$.

This low-frequency part of $U$ can be written as follows:

$$U(a_{j,k}, \delta) = h_{11000}^{(1)} a_{a_1} + h_{00110}^{(1)} a_{a_2}$$

$$+ h_{jklm-p}^{(1)} a_{a_1} a_{a_2} e^{i(n_1 Q_{\theta} + n_2 Q_{\lambda} - p)\delta}$$

$$+ h_{jklm-p}^{(1)} a_{a_1} a_{a_2} e^{-i(n_1 Q_{\theta} + n_2 Q_{\lambda} - p)\delta}$$  \hspace{1cm} (A1.5)

where $j, k, l, m$ can take the following values:

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$j$</th>
<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
| 0     | 2     | 0   | 0   | 2   | 0   |  \hspace{1cm} (A1.6)

The form of $U$ as given in (A1.5) and (A1.6) is just adequate for being used in the motion equations (1.4.6). In general:

$$\frac{da_{a_1}}{d\delta} = i h_{11000}^{(1)} a_{a_1} + i k h_{jklm-p}^{(1)} a_{a_1} a_{a_2} e^{i\delta}$$

$$+ i j h_{jklm-p}^{(1)} a_{a_1} a_{a_2} e^{-i\delta}$$

$$+ i j k l m h_{jklm-p}^{(1)} a_{a_1} a_{a_2} e^{-i\delta}$$  \hspace{1cm} (A1.7)

For simplifying, we will define the parameter $\kappa$:

$$\kappa = h_{jklm-p}^{(1)} \hspace{1cm} \bar{\kappa} = h_{jklm-p}^{(1)}$$

with

$$\kappa = \frac{1}{2\pi} \int_{0}^{2\pi} h_{jklm-p}^{(1)}(\theta) e^{ip\theta} d\theta$$  \hspace{1cm} (A1.8)

These last two relations are used in sections 5.2, 5.3 and 5.4.