Restrictions on negative energy density in a curved spacetime

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Abstract

Recently a restriction ("quantum inequality-type relation") on the (renormalized) energy density measured by a static observer in a "globally static" (ultrastatic) spacetime has been formulated by Pfenning and Ford for the minimally coupled scalar field, in the extension of quantum inequality-type relation on flat spacetime of Ford and Roman. They found negative lower bounds for the line integrals of energy density multiplied by a sampling (weighting) function, and explicitly evaluate them for some specific spacetimes. In this paper, we study the lower bound on spacetimes whose spacelike hypersurfaces are compact and without boundary. In the short "sampling time" limit, the bound has asymptotic expansion. Although the expansion can not be represented by locally invariant quantities in general due to the nonlocal nature of the integral, we explicitly evaluate the dominant terms in the limit in terms of the invariant quantities. We also make an estimate for the bound in the long sampling time limit.

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1 Introduction

Since spacetime curvature is produced by the total stress-energy tensor of all the matter that inhabits spacetime, it is very important to investigate the conditions that stress-energy tensor $T_{\mu\nu}$ would satisfy. Indeed, "energy conditions" i.e. assumptions concerning the positivity of the locally measured energy density, play a key role in the proof of the classical theorems on the large-scale structure of spacetime [1, 2]. In quantum field theory, however, it well known that the local (i.e. pointwise) energy conditions can be violated for the expectation value of stress-energy tensor, as can be seen in the example of a free scalar field in Minkowski spacetime when its state is the vacuum plus a small admixture of two-particle state [3].

Instead of the local energy conditions, various nonlocal energy condition which may be still relevant with the theorems on large-scale structure are proposed. One of the most prominent conditions is the averaged null energy condition (ANEC) [4, 5] which requires

$$\int_\gamma < T_{\mu\nu} > k^\mu k^\nu d\tau \geq 0,$$  \hspace{1cm} (1)

where the $\gamma$ is complete null geodesic, $k^\mu$ is the tangent vector along $\gamma$, and $\tau$ is an affine parameter. $< T_{\mu\nu} >$ is the (renormalized) expectation value evaluated on a state and, for the ANEC, the inequality is required to be satisfied for any state [6]. If the $\gamma$ is timelike and $\tau$ is the observer’s proper time, the requirement of Eq.(1) is the average weak energy condition (AWEC). It has been known, however, that [7, 8] the ANEC can be violated in general due to the anomalous scaling of renormalized expectation value of energy momentum tensor under scaling transformation, and explicit examples
of violations of ANEC have been given by Visser [9, 10] [11]. Some generalizations of ANEC has been proposed; Ford and Roman evaluated the ANEC integrals (the left hand side of Eq.(1)) with a weighting ("sampling") function $f(\tau, t_0) = t_0/\pi(\tau^2 + t_0^2)$ $t_0$ being the "sampling time", and show that there exist negative lower bounds of this weighted integrals ("quantum inequality-type relation") [12, 13, 14]. Yurtsver [15, 16] considered the condition that lower bound of weighted ANEC integral in certain limit is negative but finite (generalized ANEC). He argued that this condition would hold generally in four-dimensional curved spacetime, and it may be relevant with theorems on large scale structure implying the absence of macroscopic static wormholes [16]. Flanagan and Wald [17] considered contributions from the neighborhood of the ANEC integral by introducing the "transverse smearing". They show that it is enough to consider the contributions over several Plank lengths along the transverse direction to ensure the positivity of transversely smeared ANEC, which implies traversable wormholes have "Planck scale structures" in accordance with other suggestions [16, 18]. For reviewing the status of nonlocal energy conditions of non-interacting scalar fields, see Flanagan and Wald [17].

Recently, in the extension of quantum inequality-type relation (quantum inequality) on flat spacetime of Ford and Roman [12, 13, 14], Pfenning and Ford [19] have formulated quantum inequality for a static observer on a "globally static" curved spacetime (see section 2): The integral of (renormalized) energy density of a static observer with the weighting function $f$ has been evaluated for the minimally coupled scalar field in the test field limit (i.e. without considering the back-reaction)$^\S$, and showed that it has a

$^\S$This limit will be assumed in this paper to keep the discussion manageable; However including
lower bound. The lower bound has been given in terms of a sum of mode functions and evaluate explicitly for some specific spacetimes. In the short sampling time limit where the curvature of spacetime could be neglected, the bound reduce to that of flat spacetime of Ford and Roman, to reproduce the quantum inequality in flat spacetime [13, 14].

In this paper, we will study the lower bound by applying the well established spectral theory of mathematics, to see the curvature effect in general curved case. Although the spectral theory can be used more generally, in this paper we will concentrate on the cases where the spacelike hypersurfaces of spacetime are compact and without boundary. Attentions will be focused on the short sampling time limit ($t_0 \to 0$) and the long sampling time limit ($t_0 \to \infty$). In the short sampling time limit, we will show that the bound has asymptotic expansion. Although the expansion can not be represented in general by locally invariant quantities (such as curvature) reflecting the nonlocal nature of the integral, the dominant terms in the limit turn out to be locally computable and are given explicitly in terms of the invariant quantities for 4-dimensional spacetime. In the long sampling time limit, we will estimate how fast the bound goes to 0 and reveal the nonlocal nature of the bound. In the next section, we will briefly review the derivation of quantum inequality on a globally static spacetime and introduce notations. In sections 3 and 4, the lower bound of quantum inequality will be studied in the short and long sampling time limits, respectively. The final section will be devoted for discussions.

back-reaction is necessary for application, as emphasized in Ref.[17].
2 The quantum inequality: review

In this section we will review the derivation of the quantum inequality and introduce notations which will be used in this paper. The globally static spacetime\(^1\) can be described by a metric of the form

\[
ds^2 = -dx_0^2 + g_{ij}(x)dx_idx_j, \tag{2}\]

where the \(g_{ij}\) is the metric of the spacelike hypersurface that are orthogonal to the timelike Killing vector \(\partial_{x_0}\). For the convenience of explicit evaluations, we only consider the cases where the spacelike hypersurfaces are compact and without boundary. On this spacetime, the minimally coupled scalar field \(\phi\) can be represented in terms of creation and annihilation operators

\[
\phi = \sum_\alpha (a_\alpha f_\alpha + a_\alpha^* f_\alpha^*), \tag{3}\]

where the

\[
f_\alpha = e^{-iw_\alpha x_0}U_\alpha(x) \tag{4}\]

is the positive frequency Klein-Gordon mode function. \(\alpha\) is the set of eigenvalues which characterizes the mode functions and \(f_\alpha\) is normalized to have unit Klein-Gordon norm. \(U_\alpha(x)\) thus satisfy a wave equation

\[
\nabla^i \nabla_i U_\alpha + (w_\alpha^2 - \mu^2)U_\alpha = 0, \tag{5}\]

where \(\mu\) is the constant mass and \(\nabla^i\) is the covariant derivative operator in the three-dimensional manifold \(\mathcal{M}\) of \(t=\text{constant}\) hypersurface.

\(^1\)The other name used widely in literature is the ultrastatic spacetime [20].
The energy density to a static observer is given by
\[ \rho = T_{\mu \nu} u^\mu u^\nu = T_{00} = \frac{1}{2} \left[ ( \partial_{x_0} \phi)^2 + \partial^i \phi \partial_i \phi + \mu^2 \phi^2 \right] \]  
\tag{6}
and, making use of mode expansion form of \( \phi \) in Eq.(3), it is easy to represent the energy density in terms of creation and annihilation operators. Of course, the \( T_{00} \) involves the terms which diverge upon summation and is not well defined. We may define the renormalized \( : T_{00} : \) through the normal ordering with respect to the Fock vacuum \( |0> \)
\[ : T_{00} : = T_{00} - <0|T_{00}|0> \]  
\tag{7}
where the vacuum would be defined by the globally timelike Killing vector [20, 21]. The "averaged energy density difference" is defined by the integral with weighting function \( f(x_0, t_0) \) as
\[ \hat{\rho} \equiv \frac{t_0}{\pi} \int_{-\infty}^{\infty} \frac{<\psi| : T_{00} : |\psi>}{x_0^2 + t_0^2} dx_0. \]  
\tag{8}
After some algebra along the line of Ref.[13, 12], \( \hat{\rho} \) for arbitrary \( |\psi> \) can be shown to have a lower bound as [19]
\[ \hat{\rho} \geq - \sum_{\alpha} (w^2 + \frac{1}{4} \nabla^i \nabla_i) |U_\alpha|^2 e^{-2w_\alpha t_0}. \]  
\tag{9}
This quantum inequality holds on any globally static spacetime and thus reproduce the known inequalities for static observer. For example, in 4D Minkowski spacetime it gives the quantum inequality of Ref.[12, 13, 14]. For a static observer on a circle of two-dimensional spacetime, it also reproduce the "difference inequality", Eq.(23) of Ref.[14].

In this paper, we will consider only the four-dimensional spacetime. For the convenience of explicit evaluation, we will concentrate ourselves on the cases where the spacelike hypersurface \( M \) of the spacetime is compact and without boundary.
If we denote the Laplacian \((-\nabla^i \nabla_i)\) on \(\mathcal{M}\) as \(\triangle\), the \(w_\alpha^2\) is an eigenvalue of \(\triangle + \mu^2\) and \(U_\alpha\) is the corresponding eigenfunction. Since the lower bound, the right hand side (r.h.s.) of Eq.(9), can be given in terms of mode functions on \(\mathcal{M}\), instead of \(\{w_\alpha^2, U_\alpha\}\) we will use the spectral resolution \(\{\lambda_i, \phi_i\}\) of

\[
(\triangle + \mu^2)\phi_i = \lambda_i \phi_i, \quad \int_{\mathcal{M}} \phi_i^* \phi_j d\text{vol}(g) = \delta_{ij},
\]

(10)

for explicit evaluation.

The zero mode has no contribution to the bound since it can not survive the differentiation, and the difference in the resolutions is only in the normalization. Taking these points into considerations, the inequality can be written as

\[
\hat{\rho} \geq -\frac{1}{2} \sum_{i=1}^{\infty} \left( \sqrt{\lambda_i} - \frac{1}{4\sqrt{\lambda_i}} \triangle \right) |\phi_i|^2 e^{-2\sqrt{\lambda_i} t_0}.
\]

(11)

In the summation, the smallest value of \(i\) is given as 1 to denote that zero mode case is excluded.

By the exponential factor, it is clear that the lower bound of \(\hat{\rho}\) converges for any positive \(t_0\) and reduces to 0 in the long sampling time limit.

3 The short sampling time limit

In this section we will examine the lower bound in the short sampling time limit. For this purpose, it is useful to study the pseudo(\(\Psi DO\))-differential operator \(P := \sqrt{\triangle + \mu^2}\) and its heat kernel. The heat kernel for \(P\) in the coincident limit (or, on diagonal) is written
as

\[ h(x, P)(t) := \sum_i e^{-\sqrt{x}t} |\phi_i(x)|^2. \] (12)

In the following subsection, we will evaluate the asymptotic expansion form of \( h(x, P)(t) \) as \( t \) goes to 0, relying on the well-known facts in spectral theory of mathematics. The results will be then applied in the subsection 3.2, to find some explicit form of the asymptotic expansion of the bound.

### 3.1 Heat kernel method

For the \( \Psi \)DO-differential operator of order 1, it has been known that \( h(x, P)(t) \) has the asymptotic expansion as \( t \) goes to 0 [22]

\[ h(x, P)(t) \sim \sum_{n=0}^{\infty} a_n(x, P)t^{n-m} + \sum_{k=1}^{\infty} b_k(x, P)t^k \ln t. \] (13)

Here the \( m \) is the dimension of the manifold \( M \) on which the operator \( P \) acts. For the 4-dimensional spacetime, \( m \), the dimension of spacelike hypersurface, is 3.

To determine the coefficients \( a_n(x, P), b_k(x, P) \), it is useful to define \( \zeta(x, P, s) \)

\[ \zeta(x, P, s) := \sum_i \lambda_i^{-s/2} |\phi_i(x)|^2 \] (14)

which is related to \( h(x, P)(t) \) through the Mellin transformation

\[ \Gamma(s)\zeta(x, P, s) = \int_0^\infty t^{s-1}h(x, P)(t)dt. \] (15)

As shown in appendix A, the coefficients can be read from the pole structures of \( \Gamma(s)\zeta(x, P, s) \) as

\[ a_n(x, P) = \text{Res}_{s=m-n} \Gamma(s)\zeta(x, P, s) \quad (= \text{residue of } \Gamma(s)\zeta(x, P, s) \text{ at } s = m - n)(16) \]
\[ b_k(x, P) = \text{Res}_{s=-k} (s + k) \Gamma(s) \zeta(x, P, s). \] (17)

In fact, it has been known [22, 23, 24] that the asymptotic expansion of \( h(x, P)(t) \) determines the pole structure of \( \Gamma(s) \zeta(x, P, s) \) (conversely the pole structure of \( \Gamma(s) \zeta(x, P, s) \) determines the asymptotic expansion of \( h(x, P)(t) \)) to give the following equality:

\[
\Gamma(s) \zeta(x, P, s) = \sum_{n=0}^{N} \frac{a_n(x, P)}{s + (n - m)} - \sum_{k=1}^{N} \frac{b_k(x, P)}{(s + k)^2} + \text{hol}, \quad (18)
\]

where \( N \) is large enough integer number and \( \text{hol} \) represents some holomorphic function of \( s \) which depends on \( N \) [25].

A remark in order is that, as explained in appendix B (see, also Ref.[23]), the \( b_k(P) \) is actually 0 for even \( m \) and there is no log term in the expansion of \( h(x, P)(t) \). This implies [through the similar reasons given in the next subsection] that, there is no log term in the inequality of odd dimensional spacetime if it is expanded in terms of short sampling time, as can be seen in an explicit example [19]. In the following we will consider only the odd \( m \) cases.

It is useful to introduce well understood \( \zeta \)-function for the differential operator \( D := \Delta + \mu^2 \) which is of order 2

\[ \zeta(x, D, s) := \sum_i \lambda_i^{-s} |\phi_i(x)|^2. \] (19)

The pole structure of \( \Gamma(s) \zeta(x, D, s) \) is known as

\[ \Gamma(s) \zeta(x, D, s) = \sum_{n=0}^{N} \frac{a_n(x, D)}{s + \frac{n-m}{2}} + \text{hol}. \] (20)

\( a_n(x, D) \) has been computed in terms of locally invariant quantities [These quantities defined independent of the coordinate system are written as covariant derivatives of the
curvature tensor. An example would be the scalar curvature or the norm of the Ricci curvature. For compact manifold without boundary $a_n(x, D)$ is zero for odd $n$.

A simple relation of two $\zeta$-functions is available from the definitions;

$$\zeta(x, P, s) = \zeta(x, D, s/2). \quad (21)$$

which yields

$$\Gamma(s)\zeta(x, P, s) = \frac{\Gamma(s)}{\Gamma(s/2)} \zeta(x, D, s/2) = 2 \frac{\Gamma(s)}{\Gamma(s/2)} \left( \sum_{n=0}^{\infty} \frac{a_n(x, D)}{s + (n - m)} + hol \right). \quad (22)$$

Thus the coefficient $a_n(x, P)$ can be written as

$$a_n(x, P) = \text{Res}_{s=m-n} \left[ 2 \frac{\Gamma(s)}{\Gamma(s/2)} \left( \sum_{l=0}^{\infty} \frac{a_l(x, D)}{s + l - m} + hol \right) \right]. \quad (23)$$

For $n \leq m$, $a_n(x, P)$ can be easily written in terms of $a_n(x, D)$

$$a_n(x, P) = \left[ 2 \frac{\Gamma(s)}{\Gamma(s/2)} \right]_{s=m-n} a_n(x, D) \quad \text{for } m > n, \quad (24)$$

$$a_m(x, P) = a_m(x, D). \quad (25)$$

For $n > m$, due to poles in gamma function the $hol$ term can make contribution to the residue; For even $n$, $a_n(x, P)$ given as

$$a_n(x, P) = \text{Res}_{s=m-n} \left[ 2 \frac{\Gamma(s)}{\Gamma(s/2)} hol \right] \quad (26)$$

which is recently proved to be not locally computable (Theorem 1.7 of Ref.[24]; For the application of the general theorem of Gilkey and Grubb to our case, see appendix C.).

For odd $n$, $a_n(x, P)$ is still locally computable, i.e. can be represented in terms of locally invariant quantities.
Through similar analysis, one can also find that

\[ b_k(x, P) = -2a_{k+m}(x, D) \text{Res}_{s=-k} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} \]  

(27)

which is nonvanishing only for odd \( k \).

Making use of the well-known coefficient \( a_n(x, D) \), we can find some of the coefficients. For \( m = 3 \) which corresponds to four-dimensional spacetime,

\[
\begin{align*}
  a_0(x, P) &= \frac{2\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} \bigg|_{s=3} \quad a_0(x, D) = \frac{1}{\pi^2} \\
  a_2(x, P) &= \frac{1}{4\pi^2} \left(\frac{\tau}{6} - \mu^2\right) \\
  b_1(x, P) &= \frac{1}{2880\pi^2} \left(180\mu^4 - 60\tau\mu^2 + 12\nabla_i\nabla^i\tau + 5\tau^2 - 2|\rho|^2 + 2|R|^2\right),
\end{align*}
\]

(28, 29, 30)

where the Ricci tensor, the scalar curvature, and the norms of the Ricci and full curvature tensors are defined as

\[
\rho_{ij} := R_{kij}^k, \quad \tau := \rho_i^i \quad \text{and} \quad |\rho|^2 := \rho_{ij}\rho^{ij}, \quad |R|^2 := R_{ijkl}\tilde{R}^{ijkl}.
\]

(31, 32)

As already explained \( a_1 = a_3 = 0 \) and \( a_4(x, P) \) is not locally computable.

In summary,

\[
h(x, P)(t) = \frac{1}{\pi^2 t^3} + \frac{a_2(x, P)}{t} + b_1(x, P) t \ln t + \text{terms vanishing in the } t \to 0 \text{ limit.}
\]

(33)

Generally, the terms vanishing in the limit \( t \to 0 \) could be locally noncomputable.
3.2 The asymptotic expansion of the lower bound

For the evaluation of the lower bound, we need to evaluate the summations

$$H_{\pm} = \sum_{i=1}^{\infty} (\lambda_i)^{\pm \frac{1}{2}} e^{-t\sqrt{\lambda_i}} |\phi_i|^2. \quad (34)$$

As in the previous subsection, these summation may be studied by investigating the pole structures of $\Gamma(s)S_{\pm}$ where

$$S_{\pm} := \sum_{i=1}^{\infty} (\lambda_i)^{-s\pm \frac{1}{2}} |\phi_i|^2. \quad (35)$$

Instead, we find the asymptotic expansion form of $H_+$

$$H_+ = -\frac{3}{\pi^2 t^4} - \frac{a_2(x, P)}{t^2} + b_1(x, P) \ln t + \text{finite or vanishing terms in the } t \to 0 \text{ limit} \quad (36)$$

here by differentiating the r.h.s. of Eq.(33) with respect to $t$. [Such operation may be justified by that the series converge in the $C^\infty$ topology for any finite $t$ as implied by the Lemma 1.6.5 of Ref.[25]]

For evaluation of the $H_-$, we will integrate $h(x, P)$;

$$H_- = \int_{t}^{\infty} h(x, P)(y) \, dy \quad (37)$$

$$= \int_{t}^{\delta} h(x, P)(y) \, dy + \int_{\delta}^{\infty} h(x, P)(y) \, dy, \quad (38)$$

for $0 < t < \delta < 1$. From the fact that the heat kernel decays exponentially fast as $t$ goes to $\infty$, one can find that the second integral of the r.h.s. of Eq.(38) is finite. Making use of the asymptotic expansion of $h(x, P)(t)$ in Eq.(33), from the first integral we can find the terms which diverges in the $t \to 0$ limit

$$H_- = \frac{1}{2\pi^2 t^2} - a_2(x, P) \ln t + \text{finite or vanishing terms in the } t \to 0 \text{ limit}. \quad (39)$$
Thus the quantum inequality of Eq.(11) can be written in the short sampling limit as

\[ \hat{\rho} \geq -\frac{3}{32\pi^2}\left[\frac{1}{t_0^4} + \frac{1}{3t_0^2}\left(\frac{\tau}{6} - \mu^2\right)\right] + \ln\left(E_0t_0\right)\left(180\mu^4 - 60\tau\mu^2 - 18\nabla_i\nabla^i\tau + 5\tau^2 - 2|\rho|^2 + 2|R|^2\right) \]

+ finite or vanishing terms in the \( t_0 \to 0 \) limit. 

(40)

Here, a constant \( E_0 \) of mass dimension has been introduced to make the argument of logarithmic function dimensionless. As is clear from it’s derivation, the finite or vanishing terms of Eq.(40) in the limit can not be represented by the locally invariant quantities in general.

In passing, we note that, the short sampling time expansion of the bound for odd dimensional spacetime can be represented by the locally invariant quantities.

For the case that \( \mathcal{M} \) is \( S^3 \) of radius \( a \), the locally invariant quantities are given as

\[ \tau = \frac{6}{a^2}, \quad |\rho|^2 = |R|^2 = \frac{12}{a^4}. \]

(41)

The quantum inequality is thus given as

\[ \hat{\rho} \geq -\frac{3}{32\pi^2}\left[\frac{1}{t_0^4} + \frac{1}{3t_0^2}\left(\frac{1}{a^2} - \mu^2\right)\right] + \frac{\ln(E_0t_0)}{540}\left(180\mu^4 - 60\tau\mu^2 - 18\nabla_i\nabla^i\tau + 5\tau^2 - 2|\rho|^2 + 2|R|^2\right) \]

+ finite or vanishing terms in the \( t_0 \to 0 \) limit.

(42)

in agreement with the results of Ref.[19].

\(^{\dagger\dagger}\)In circulating the manuscript, I’ve been informed that the same formula has been obtained by Pfennning and Ford through different method [26].
4 The long sampling time limit

The lower bound reduces to zero in the long sampling time limit. Even though the $T_{00}$ is normal ordered with respect to the vacuum defined by the globally timelike killing vector, it implies at least that the generalized AWEC in the sense of Ref.[16] is satisfied for the timelike geodesic of static observer.

In this section, we will study how fast the lower bound will go to zero in the limit. This will, as a byproduct, expose the non-local nature of the lower bound, as already has been encountered through the fact that the finite or vanishing term in the short sampling time limit can not be represented by the locally invariant quantities in general.

For positive $v$ and $y$ ($\geq \lambda_1 > 0$), the upper bound of the function $y^je^{-u\sqrt{y}}$ can be given as $C(j, v, \lambda_1)$. The following relation is thus true for positive $\alpha$ ($< 1$)

$$\sum_{i=1}^{\infty} \lambda_i^j e^{-u\sqrt{\lambda_i}} = \sum_{i=1}^{\infty} \lambda_i^j e^{-u\alpha\sqrt{\lambda_i}} e^{-u(1-\alpha)\sqrt{\lambda_i}}$$

$$\leq C(j, u\alpha, \lambda_1) \sum_{i=1}^{\infty} e^{-u\sqrt{\lambda_i}} = C(j, u\alpha, \lambda_1) e^{-u\sqrt{\lambda_1}(1-\alpha)} \sum_{i=1}^{\infty} e^{-u(1-\alpha)(\sqrt{\lambda_i}-\sqrt{\lambda_1})}.$$ (43)

Furthermore since the summation $\sum_{i=1}^{\infty} e^{-u(1-\alpha)(\sqrt{\lambda_i}-\sqrt{\lambda_1})}$ is finite for positive $\alpha$ ($< 1$), one can find a lemma

$$\sum_{i=1}^{\infty} \lambda_i^j e^{-u\sqrt{\lambda_i}} \leq C'(j, d_0, \lambda_1)e^{-u\sqrt{\lambda_1}}$$ (44)

in the $u \to \infty$ limit with fixed $d_0 (= u\alpha)$. Here, note that the $C'(j, d_0, \lambda_1)$ do not depend on $u$.

Since $|\phi_i|^2$ or it’s covariant derivative is finite, indeed this lemma dictates that the lower bound reduce to 0 as fast as $-|f(\lambda)|e^{-2\nu\sqrt{\lambda_1}}$ in the long sampling time limit, where the $\lambda_1$ is the smallest one among positive eigenvalues of $D$. Note that $f$ does not depend
on \( t_0 \). In general, the spectrum of an operator contains informations on global property of manifold and can not be represented only by the locally invariant quantities. This behavior of the bound in the long sampling time limit thus clearly shows the nonlocal nature of the bound.

As a simple test, for the massless theory on \( \mathcal{M} = S^3 \) (of radius \( a \)), our estimate yields that the lower bound is proportional to \( -e^{-2t_0\sqrt{3}/a} \) in the long sampling time limit, which is in agreement with the explicit calculation of Ref.[19]. For the massive theory, the estimate simply gives that the bound is proportional to \( -e^{-2\mu t_0} \) in the limit for any \( \mathcal{M} \).

5 Discussion

The quantum inequality or weighted AWEC exposes the distribution of allowed negative energy density which could be neglected in simple average along time. The weighting function thus necessarily has peak(s) around which main contribution to the integral is made. As in the application of quantum inequality for the wormhole geometry via extrapolation [18], in the semiclassical approximation of gravity it seems unavoidable to compare the weighted integral with the local invariant quantities of spacetime for applications. Our results in previous sections show a possible nature of the weighted integral: it could be nonlocal. [In fact, in the test field limit without considering back-reaction, only the energy tensors have quantum corrections which contain information on global structure of spacetime.] We think, this fact would play a role in finding useful
weighting (sampling) functions. In the sense that the dominant terms of the lower bound of quantum inequality in the short sampling time limit are locally invariant, the sampling function of Ford and Roman would be still useful in general globally static spacetime, although applications of the inequality are not intended in this paper.

The case that spacelike hypersurface is noncompact, could be physically interesting. Although we have not treat the case in this paper, the heat kernel method could be useful for the evaluation of the bound if the hypersurface is complete.

For the case that the hypersurface is compact and with boundary, the spacetime is not globally hyperbolic. However, the spectral theory itself has been well developed and can be used in analyzing the bound given in Eq.(11), as in the boundaryless case we have considered. In both of the boundary conditions Neumann and Dirichlet, the $a_0(x, P)$ is same to that of Eq.(28) for four-dimensional spacetime, and the leading divergence of the bound in the short sampling time limit thus does not depend on the boundary condition. The sub-leading divergences, however, crucially depend on the boundary conditions; For example, in general $a_1(x, P)$ is not zero and depends on boundary condition [25], so that the term proportional to $1/t^3$ could appear in the short sampling time expansion of the bound.

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Appendix A

In this appendix, we wish to expose explicitly the relations (Eqs.16-18) between the asymptotic expansion of $h(x,P)(t)$ and pole structure of $\Gamma(s)\zeta(x,P,s)$ [22, 23, 24, 25]. Mellin transformation of Eq.(15) gives the following equality:

$$\Gamma(s)\zeta(x,P,s) = \int_0^1 t^{s-1}h(x,P)(t)dt + \int_1^\infty t^{s-1}h(x,P)(t)dt. \quad (A1)$$

Since there exist positive constants $c$ and $k$ satisfying

$$h(x,P)(t) \leq ke^{-ct} \text{ for } t \geq 1,$$

the integral $\int_1^\infty t^{s-1}h(x,P)(t)dt$ gives only holomorphic function of $s$. Therefore, the poles of $\Gamma(s)\zeta(x,P,s)$ should come from the first integral on the r.h.s. of Eq.(A1). Making use of the asymptotic expansion in Eq.(13), the first integral can be calculated as

$$\Gamma(s)\zeta(x,P,s) = \sum_{n=0} a_n \int_0^1 t^{s+n-m-1}dt + \sum_{k=1} b_k \int_0^1 t^{s+k-1}\ln t \ dt + \text{hol.} \quad (A2)$$

Reflecting the fact that $\Gamma(s)\zeta(x,P,s)$ has meromorphic extension to whole complex plane, except some values of $s$, the integral of Eq.(A2) can be carried out to give the relation of Eq.(18).
Appendix B

In this appendix, when $m$ is even, we wish to explicitly reprove that there is no log term in the asymptotic expansion of $h(x, P)(t)$ [23]. We first study $\zeta(x, D, s)$ at $s = -1/2, -1, -3/2, \cdots$. By the Eq.(20), $\zeta(x, D, s)$ can be written as

$$\zeta(x, D, s) = \Gamma(s)^{-1} \sum_{n=0}^{\infty} \frac{a_n(x, D)}{s+n-m} + \Gamma(s)^{-1} h_{ol}.$$  \hspace{1cm} (B1)

Since $\Gamma(s)^{-1}$ has simple zero when $s$ is negative integer, Eq.(B1) shows that $\zeta(x, D, s)$ is regular at $s = -1, -2, \cdots$. For $s = -1/2, -3/2, \cdots$, $\Gamma(s)^{-1}$ is regular and $m - 2s$ is odd. If there exist simple pole of $\zeta(x, D, s)$ for the negative half-integer $s$, then it should be proportional to $a_{m-2s}(x, D)$ which is zero because of the odd $m - 2s$. Now we have the fact that $\zeta(x, D, s)$ is regular at $s = -1/2, -1, -3/2, \cdots$.

If we assume that $b_k(x, P)$ is not zero, the Eqs.(17,21) then lead to a wrong conclusion that $\zeta(x, D, s/2)$ has simple poles for negative integer $s$, to show $b_k(x, P)=0$. $h(x, P)(t)$ thus does not have the log term in the asymptotic expansion for even $m$.

Appendix C

In order to see how the general theorem of Gilkey and Grubb mentioned in the text works in our case, let’s consider the scaling transformation

$$g_{ij} \rightarrow c^{-2} g_{ij}, \quad \mu^2 \rightarrow c^2 \mu^2$$
with positive $c$. The $P(g(c^{-2}), c^2\mu^2)$ denotes the scale-transformed operator from the original $P (= P(g, \mu^2))$. The spectral resolution for $P(g(c^{-2}), c^2\mu^2)$ is then $\{c\sqrt{\lambda_i}, c^{m/2}\phi_i\}$, with the equality $P(g(c^{-2}), c^2\mu^2) = cP(g, \mu^2)$. From these facts, one can find the relation

$$h(x, P(g(c^{-2}), c^2\mu^2))(t) = c^{-m}h(x, P(g, \mu^2))(t)$$

(C1)

which yields the equality

$$a_{m+k}(x, P(g(c^{-2}), c^2\mu^2)) =$$

$$c^{m+k}[a_{m+k}(x, P(g, \mu^2)) + b_k(x, P(g, \mu^2)) \ln c] \text{ for } k \geq 1.$$  

(C2)

For the odd $m$ and odd $k$ where $b_k$ is not zero, the equality of (C1) contradicts the Eq.(1.8) of Ref.[24] which comes from the assumption that $a_{m+k}$ is locally computable (See also Ref.[25], and note the slight difference in notations.). This contradiction thus shows that $a_{m+k}(x, P)$ is not locally computable for odd $m$ and $k$, as in the original proof of the theorem.
References


[11] In Ref.[9, 10], Visser generalized the ANEC by considering the integrals on generic null curves \( \gamma \). The integrations were then done past the event horizon, and this energy condition was called truncated ANEC. For various energy conditions, see M. Visser, *Lorenzian Wormholes - From Einstein to Hawking* (American Institute of Physics, New York, 1995).


