Membrane Scattering with M-Momentum Transfer

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Abstract

Membrane scattering in m(atrix) theory is related to dynamics in three-dimensional $SU(2)$ gauge theory, with transfer of $p^{11}$ being an instanton process. We calculate the instanton amplitude and find precise agreement with the amplitude in eleven dimensional supergravity.
1 Introduction

Matrix theory is a notable proposal for the fundamental degrees of freedom and their Hamiltonian [1]. Roughly speaking, it is proposed that of those structures known in the weakly coupled string limits, it is the Dirichlet zero-branes of the IIA string [2] and their low-energy Hamiltonian [3, 4] that actually have a much wider range of validity. More specifically, for states of large M-momentum (momentum in the eleven-direction of M-theory) these are supposed to give a complete description, even in the limit of large string coupling where eleven-dimensional Lorentz invariance reappears.

The strongest evidence that this is a step in the right direction is the connection with the supermembrane of eleven-dimensional supergravity [5, 6, 7]. Among the other tests are comparisons of various matrix theory scattering amplitudes with those of eleven-dimensional supergravity [1],[8]-[12], but only at vanishing M-momentum transfer. Processes with transfer of M-momentum are qualitatively different and seemingly much harder to calculate. In the matrix theory description, the former involve only a loop of virtual open string, while the latter require transfer of zero-branes and so depend on the still-mysterious supersymmetric bound states of zero-branes. At this point it is possible that matrix theory is some mutilation of the correct theory, which reproduces the amplitudes of vanishing M-momentum transfer but not more general amplitudes. Of course to single out one momentum mode in this way is highly nonlocal, but this is just the point: in matrix theory, locality is not at all manifest, especially in the M-direction.

In this paper we report some progress in this direction. We are able to calculate the amplitude for scattering of transverse supermembranes with exchange of one unit of M-momentum. The result is in detailed agreement with the amplitude in eleven-dimensional supergravity. The calculation is valid in a limit where the impact parameter and the string coupling are taken large together. As with other scattering amplitudes, to reach the true
M-theory limit of large string coupling at fixed impact parameter will require a supersymmetric nonrenormalization theorem.

The reason that we are able to make progress is that it appears to be simpler to boost a membrane in the M-direction than a graviton [8, 9]. The latter requires increasing the number of zero-branes in the bound state, and no simple scaling properties are yet known. The former merely increases an internal magnetic field and the Hamiltonian scales in a simple way. The scattering calculation then reduces to an instanton calculation in (2 + 1)-dimensional gauge theory.

In section 2 we obtain the matrix theory action for the two-membrane system. In section 3 we relate the matrix theory amplitude to an instanton calculation, and find detailed agreement with the eleven-dimensional result. Section 4 contains a brief discussion.

2 The Membrane Action

We consider a pair of membranes infinitely extended in the 2, 3-directions, with some separation and relative motion in the 4, . . . , 10-directions and both moving with large velocity in the 11-direction. Minkowski time is \( x^0 \) and Euclidean time \( x^1 = -ix^0 \). In the IIA string theory, the membrane degrees of freedom are described by the Dirac-Born-Infeld lagrangian [13]

\[
S = -\tau_2 \int d^3 x \text{Tr} \det^{1/2} \left( -\eta_{\mu\nu} - \partial_\mu X^i \partial_\nu X^i + 2\pi \alpha' F_{\mu\nu} \right) + O([X^i, X^j]^2) + \text{fermions}.
\] (2.1)

The precise Born-Infeld form of the commutator and fermionic terms will not be needed. Here \( \mu, \nu = 0, 2, 3 \) and \( i = 4, \ldots, 10 \). The membrane tension is [2]

\[
\tau_2 = \frac{1}{4\pi^2 \alpha'^{3/2} g}
\] (2.2)

with \( g \) the string coupling. The transverse coordinates \( X^i(x^\mu) \) and the field strength \( F_{\mu\nu} \) are 2 \( \times \) 2 matrices [3].
When the membranes are separated, with \([X^i, X^j] = 0\), then up to a
gauge transformation
\[ X^i = \begin{bmatrix} x_1^i & 0 \\ 0 & x_2^i \end{bmatrix} \] \tag{2.3}
breaks \(U(2)\) to \(U(1) \times U(1)\). The embedding coordinate for the eleventh
dimension comes from the scalar dual to a gauge field in \(2+1\) dimensions [14, 7].
Restricting to the unbroken \(U(1) \times U(1)\), treat the field strength rather
than vector potential as independent and add to the action a term
\[ S' = \frac{1}{2} \int d^3x \epsilon^{\mu\nu\rho} \lambda_r \partial_\mu F_{\nu\rho} \] \tag{2.4}
to enforce the Bianchi identity, with \(r\) indexing the two membranes. Integrating out \(F_{\nu\rho}\)
gives the Lagrange multiplier fields \(\lambda_r\) kinetic terms. From
the normalization of the kinetic term one deduces that \(\lambda = 2\pi\alpha' \tau_2 X^{11} = X^{11}/2\pi \alpha'^{1/2} g\) for each membrane. The periodicity of each \(\lambda\) is simply \(\lambda \sim \lambda + 1\). This follows because the Dirac quantization condition requires the
total \(U(1)\) flux on any membrane to be conserved mod \(2\pi\), and as a check
the periodicity \(X^{11} \sim X^{11} + 2\pi \alpha'^{1/2} g\) gives the correct relation between the
zero-brane mass \(\tau_0 = 1/\alpha'^{1/2} g\) and the Kaluza-Klein spectrum.

We will take the membranes to have equal velocity in the eleven-direction.
This corresponds to a field strength
\[ F_{23} = \frac{I f}{2\pi\alpha'} \] \tag{2.5}
with \(I\) the \(2 \times 2\) identity and \(f\) a constant. From the field equation for \(S + S'\)
one then has
\[ \dot{X}^{11} = \frac{f}{\sqrt{1 + f^2}}. \] \tag{2.6}
As a check, this gives a momentum density \(\Pi_{11} = \tau_2 f\), implying a zero-brane
charge density \(\tau_2 f/\tau_0\). This is in agreement with the coupling of the D-brane
to the Ramond-Ramond field [15], proportional to \(\tau_2 (C_{(3)} + 2\pi \alpha' F \wedge C_{(1)})\) for
the two-brane and \(\tau_0 C_{(1)}\) for each zero-brane.
The above D-brane description of the membranes is superficially different from the matrix theory description as a non-Abelian state of zero-branes [1, 6], but the equivalence should follow as in ref. [16, 9]. We thus take the above as our tentative definition of matrix theory for this system. That is, while the above description is known to be valid only for weakly coupled strings, we conjecture that for large boosts in the M-direction (\( f \gg 1 \)), it remains a valid description even for large string coupling where eleven-dimensional Lorentz invariance should reappear. We then compare scattering amplitudes with those of eleven-dimensional supergravity.

We will be studying nearly supersymmetric states, small deviations from the parallel motion (2.5). Naively one might expect the \( SU(2) \) degrees of freedom to decouple from the \( U(1) \) background, since they are neutral under the center-of-mass \( U(1) \). However, the endpoints of each open string carry equal and opposite charges. For a strong magnetic field, where the number of flux quanta per string area is large, one would expect the dynamics to be substantially affected by the field. Indeed, these complications are precisely taken into account by the DBI action [17], so we need merely do the obvious thing, to expand the DBI action (2.1) around the background (2.5). The result is

\[
S = -\tau_2 \gamma^{-1} \int d^3 x \sqrt{-\det \hat{\eta}} \text{Tr} \left\{ \frac{1}{2} \partial_{\mu} X^i \partial^{\hat{\mu}} \dot{X}^i + \pi^2 \alpha' \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right\}.
\] (2.7)

We have defined the Lorentz boost factor

\[
\gamma = \left( 1 - v_{11}^2 \right)^{-1/2} = (1 + f^2)^{1/2}
\] (2.8)

and the metric

\[
\hat{\eta}_{\mu\nu} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & \gamma^2 & 0 \\
0 & 0 & \gamma^2
\end{bmatrix}.
\] (2.9)

The hatted metric has been used to raise and lower indices, as indicated.

Introduce the new coordinate \( y^m \),

\[
y^0 = x^0, \quad y^{2,3} = \gamma x^{2,3}
\] (2.10)
in terms of which the metric is just $\eta_{mn}$. Define $X^i = 2\pi\alpha'\phi^i$, and convert from $SU(2)$ matrix notation to vector notation, $\phi^i = \frac{1}{2}\sigma^a\phi^{ai}$. The action becomes

$$S = -\frac{\alpha'^{1/2}}{2g\gamma} \int d^3x \left\{ \frac{1}{2} \partial_m \phi^{ai} \partial^m \phi^{ai} + \frac{1}{4} F^a_{mn} F^{amn} \right\}, \quad (2.11)$$

an ordinary $SU(2)$ Yang-Mills-Higgs action with coupling $e^2 = 2g\gamma/\alpha'^{1/2}$.

Thus far we have used the natural string parameters $\alpha'$ and $g$. The relation to M theory parameters is

$$R_{11} = \alpha'^{1/2}g, \quad M_{11}^{-3} = \alpha'^{3/2}g. \quad (2.12)$$

Here $R_{11}$ is the radius of the M-direction, and $M_{11}$ is the eleven-dimensional Planck scale up to a numerical factor.

### 3 Membrane Scattering

For scattering with impact parameter $X^i \sim b$, the effective dimensionless coupling is

$$\frac{e^2}{\phi} \sim \frac{\gamma g \alpha'^{1/2}}{b} \sim \frac{\gamma R_{11}}{b}. \quad (3.1)$$

This is large in the M theory limit of large $R_{11}$ with fixed $b$, so the effective $SU(2)$ Yang-Mills theory is strongly coupled. Recently, beginning with refs. [18], many exact results have been derived for three-dimensional gauge theories. We began this work with the hope that these exact results might be used directly to test matrix theory predictions for membrane scattering. However, the exact results are primarily for $v^2$ terms in $d = 3$, $N = 4$ supersymmetry, whereas the two-membrane system has $N = 8$ supersymmetry where the leading terms are $v^4$. As with other tests of matrix theory scattering [1],[8]-[12], we must wait for a more complete understanding of the constraints that supersymmetry places on these $v^4$ terms.

One might hope to study matrix theory questions with less supersymmetry. The simplest way to reduce the supersymmetry (and specifically to
remove the adjoint hypermultiplet) is to consider membranes trapped at the
fixed point of a K3 orbifold. However, this changes the M theory picture
substantially. Blowing up the fixed point slightly, a trapped D two-brane is
evidently a D four-brane wrapped on the collapsed two-sphere. In M theory
this becomes a five-brane, extended in the M direction. So the membrane
is no longer a localized probe in this direction. This defeats our present
purpose; it may be interesting to return to this point later.

What we shall do is to consider instead the regime \( b \gg R_{11} \gamma \) where
reliable calculations can be made. As with other matrix theory scattering
calculations, we will have to assume that nonrenormalization theorems allow
the result to be extended to the M theory limit. However, we will already be
able to see local eleven-dimensional physics in the region of validity of the
calculation.

3.1 Instanton Calculation

The relation between magnetic flux on the membrane and M-momentum
means that in a scattering with one unit of M-momentum transfer, the change
in the flux is

\[
\int dx^2 dx^3 \left[ F_{23}(x^0 = \infty) - F_{23}(x^0 = -\infty) \right] = 2\pi \sigma^3 .
\]  

(3.2)

That is, the integral over the sphere at infinity of the flux in the unbroken
\( U(1) \subset SU(2) \) is non-zero. This is therefore an \( SU(2) \) instanton process,
since the instanton of three-dimensional gauge theory is the same as the
magnetic monopole in three spatial dimensions [19]. Instanton effects were
considered in \( d = 3 \ N = 1 \) and \( N = 2 \) supersymmetry in ref. [20]. Just as
this is being written, an explicit calculation for \( N = 4 \) supersymmetry has
appeared [21], which has substantial overlap with the calculation below.

The Euclidean \( N = 8 \) Yang-Mills action, now including the commutator
and fermionic terms, can be conveniently written as the dimensional reduc-
tion of the $d = 10$ theory,

$$S = \frac{1}{e^2} \int d^3y \left\{ \frac{1}{4} F_{MN} F_{MN} + \frac{i}{2} \bar{\psi} \gamma_M D_M \psi \right\} . \quad (3.3)$$

Here $M, N$ run over $m = 1, 2, 3$ and $i = 4, \ldots, 10$, with $A_M \equiv (A_m, \phi_i)$. The $SU(2)$ vector index $a$ is suppressed. The gamma matrices can be taken as

$$\gamma_m = \sigma_m \otimes 1 \otimes \tau^1, \quad \gamma_i = 1 \otimes \Gamma_i \otimes \tau^2 . \quad (3.4)$$

Here $\sigma_m$ and $\tau^{1,2}$ are both used to denote the Pauli matrices, but in the first and third factors respectively, while $\Gamma_4, \ldots, \Gamma_{10}$ are the (pseudo-Majorana) $8 \times 8 SO(7)$ gamma matrices. The Weyl condition is $\psi = \tau^3 \psi$, while the Majorana property means that $\bar{\psi} = \psi^T \gamma_2$.

The supersymmetry variation of $\psi$ is

$$\delta \psi^a = \frac{i}{2} F^a_{MN} \gamma_M \gamma_N \epsilon$$

$$= -(B^a_m + D_m \phi^a_i \Gamma_i) \sigma_m \epsilon \quad (3.5)$$

with $\tau^3 \epsilon = \epsilon$. Here $B^a_m = \frac{1}{2} \epsilon_{mnp} F_{np}$. To be specific let us consider first membranes separated in the 4-direction with no transverse velocity, so that $\phi_4$ breaks the gauge $SU(2)$ to $U(1)$ and the transverse $SO(7)$ to $SO(6)$. The instanton lies in the same $SO(7)$ direction, $\phi^a_4$. It follows from the supersymmetry variation (3.6) that for an instanton satisfying the BPS condition [22]

$$B^a_m = D_m \phi^a_4 \quad (3.7)$$

the eight supersymmetries with $\Gamma_4 \epsilon = -\epsilon$ are unbroken. The other eight supersymmetries give fermionic zero modes,

$$\psi^a = B^a_m \sigma_m \epsilon \quad (3.8)$$

for $\Gamma_4 \epsilon = +\epsilon$. The instanton action is $S_{cl} = s_0/e^2$ where

$$s_0 = \frac{1}{2} \int d^3y \left\{ B^2 + (D \phi)^2 \right\} = \int d^3y D_m (B^a_m \phi^a_4) = 4\pi \phi_0 , \quad (3.9)$$

\[1\] Earlier $\sigma^a$ was also used for the gauge $SU(2)$. 

8
with $\phi_0$ the asymptotic value of $(\phi_0^2 \phi_0^2)^{1/2}$.

Expand around the background, $A_M = A_M^{\text{cl}} + a_M$, and add to the Lagrangian the background gauge-fixing term $(D_M a_M)^2/2e^2$. The action becomes

$$S = \frac{s_0}{e^2} + \frac{1}{2e^2} \int d^3y \left\{ a_M \Delta_{MN} a_N + i \bar{\psi} \gamma_M D_M \psi - bD^2c \right\} + O(a^3) \quad (3.10)$$

with $b$ and $c$ the Fadeev-Popov ghosts and

$$\Delta_{MN} a_N = -D^2 a_M + 2F_{MN} a_N \quad . \quad (3.11)$$

The square of the Dirac operator is

$$(i \gamma_M D_M)^2 = -D^2 + i(1 + \Gamma_4) B_m \sigma_m \quad . \quad (3.12)$$

That is, this sixteen-component operator reduces to eight copies of the one-component operator $-D^2$ and four copies of the two-component operator

$$\Delta = -D^2 + 2iB_m \sigma_m \quad . \quad (3.13)$$

Also, if one assembles the bosonic fluctuations $a_{1,\ldots,4}$ into a matrix $M(a_M) = a_4 - ia_m \sigma_m$, one finds that

$$M(\Delta_{MN} a_N) = \Delta M(a_M) \quad , \quad (3.14)$$

so that the bosonic fields give two copies of $\Delta$, from the the columns of $M$.

The components $a_{5,\ldots,10}$ give six copies of $-D^2$. Thus all determinants involve only two differential operators [23], namely $-D^2$ (with no zero modes) and $\Delta$ (with two zero modes [24]). Defining the measure so that the gaussian path integral of

$$\frac{1}{2e^2} \int d^3 y \left\{ \phi_0^2 a_M a_M + \phi_0 \bar{\psi} \psi + \phi_0^2 bc \right\} \quad (3.15)$$

is unity, the nonzero mode determinants are

$$(\det -\phi_0^{-2} D^2)^{-3+2+1}(\det' \phi_0^{-2} \Delta)^{-1+1+0} = 1 \quad . \quad (3.16)$$
the exponents coming respectively from the bosons, fermions, and ghosts.\footnote{Of course the final amplitude cannot depend on the definition of the measure, but to see this in detail depends on a subtle non-cancellation of the nonzero modes. Happily, the authors of ref. [21] have spared us from explaining this.}

According to the above discussion, there will be four bosonic zero modes. Three are from translations,

\[ a_4^{(m)} = D_m \phi_{\text{cl}}, \quad a_n^{(m)} = F_{mn} \phi_{\text{cl}}. \tag{3.17} \]

The fourth is a rotation in the unbroken $U(1)$ with gauge parameter $\lambda^a = \phi_4^a / \phi_0$,

\[ a_4^{(0)} = 0, \quad a_n^{(0)} = \phi_0^{-1} D_n \phi_{\text{cl}}. \tag{3.18} \]

This is not a gauge rotation, being nontrivial at infinity, and so gives rise to a normalizable zero mode. The finite transformation is trivial at infinity for $\alpha = 2\pi$. Using the BPS condition (3.7), one finds that

\[ \int d^3 y a_M^{(m)} a_M^{(n)} = s_0 \delta_{nm} \]
\[ \int d^3 y a_M^{(0)} a_M^{(0)} = \phi_0^{-2} s_0. \tag{3.19} \]

The gaussian normalization thus determines the bosonic zero mode measure to be

\[ \frac{\phi_0^3 s_0^2}{4\pi^2 e^4} \int d^3 y \frac{8\pi \phi_0^5}{e^4} d^3 y, \tag{3.20} \]

where the integral over $\alpha$ just gives its range $2\pi$.

Similarly the eight fermionic zero modes (3.8) are normalized

\[ \frac{e^8}{\phi_0^5 s_0^8} \prod_{I=1}^8 d\theta_I. \tag{3.21} \]

The simplest nonzero amplitude, with eight massless fermions, is then

\[ \left\langle \prod_{s=1}^8 \bar{\zeta}_s \psi_s(y_s) \right\rangle = \frac{e^4 e^{-4\pi \phi_0 / e^2}}{32\pi^3 \phi_0^3} \int d^3 y \det K_{st}(y) \tag{3.22} \]
where $K_{sI}(y) = \bar{\zeta}_s \sigma_m \epsilon_I \tilde{B}_m (y_s - y)$. The $\epsilon_I$ run over the eight spinors with $\Gamma_4 = +1$. We use a tilde to denote contraction with the normalize Higgs field, e. g. $\tilde{\psi} = \phi^a \psi^a / \phi_0$.

To express the result as an effective operator, note that the fermionic propagator is $e^2 y_m \sigma_m / 4\pi y^3$ while $\tilde{B}_m = y_m / y^3$. The effective operator is thus

$$
\frac{2^{11} \pi^5}{e^{12}} \int d^3 y \phi^{-3} e^{-4\pi \phi / e^2 + i\lambda} \prod_{\beta=1}^8 \psi_\beta .
$$

The product runs over the eight spinor components with $\Gamma_4 = +1$. We have included the dependence on the dual scalar $\lambda$ [19, 20].

To compare with the gravitational result, it is more convenient to consider the operator with no fermions but four powers of the membrane velocity. This is related to the above by supersymmetry, but we will obtain it directly from the instanton calculation. The instanton solution depends on the asymptotic values $\tilde{\phi}_i$ of the moduli. Let $\varphi_{cl}(y, y', \tilde{\phi})$ refer to the classical value of a generic field $\varphi(y)$ in the instanton solution that is centered at $y'$ with asymptotic moduli $\tilde{\phi}$. We expand around the quasisolution

$$
\varphi_{cl}(y, y', \tilde{\phi}(y^1))
$$

where $\tilde{\phi}_i(y^1) = b_i + u_i y^1$. Thus $b_i$ is the impact parameter and $u_i$ the Euclidean velocity, which we take to be small. Away from the instanton this reduces to linear motion of the membranes. We can choose a frame in which $b_i$ is in the 4-direction and $u_i$ in the 5-direction.

To saturate the fermionic path integral we need four powers of the action

$$
\delta S = \frac{i}{e^2} \int d^3 y \bar{\psi} \gamma_M \delta D_M \psi
$$

where the velocity enters into the action through the change in the Dirac operator, $\delta D_M$. The relevant overlap integrals are then

$$
\frac{i u_i}{e^2} \int d^3 y y^1 \bar{\psi} \gamma_M \delta D_M \frac{\delta \tilde{\phi}_i}{\delta \phi} \psi
$$
where again the fermionic zero modes are $\psi^a_I = B^a_m \sigma_m \epsilon^I$. Since $\gamma_M D_M \psi_J = 0$ we have also

$$\frac{\delta}{\delta \phi_i} (\gamma_M D_M \psi_J) = 0 . \quad (3.27)$$

We can therefore move the modular derivative over to $\psi_J$ and then integrate by parts to turn the overlap integral into

$$\frac{i u_i}{e^2} \int d^3 y \, \bar{\psi} I \gamma_1 \frac{\delta}{\delta \phi_i} \psi_J . \quad (3.28)$$

Now let us note that for spinors $\epsilon$ and $\epsilon'$ having $\Gamma_4 = +1$, the overlap $\bar{\epsilon} \gamma_1 \epsilon'$ vanishes. The nonzero contribution to the overlap (3.28) must come from the rotation of $\epsilon_J$. The rate of angular rotation in the 45-plane is $u_\perp / \phi$, so the overlap is this, times $\frac{1}{2}$ from the spinor rotation matrix, times $s_0$ from the spatial integral as before. Each spinor has a nonzero overlap with the rotation of exactly one other, giving the overlap to the fourth power. Rejoining the earlier calculation after eq. (3.21), we have the final amplitude

$$\frac{\pi e^4}{2} \int d^3 y (\frac{u_\perp^2}{\phi^4})^2 e^{-4 \pi \phi / e^2 + i \lambda} . \quad (3.29)$$

To conclude this section we use the results of section 2 to express the amplitude in terms of M theory quantities,

$$\frac{1}{16 R_{11}^3 M_{11}^3} \int d^3 x \, \frac{(\dot{X}_\perp^2)^2}{X^3} e^{-(X/\gamma - iX^{11}) / R_{11}} \quad (3.30)$$

where $X$ is the transverse separation in ten dimensions.

### 3.2 Supergravity Calculation

The matrix theory amplitude is to be compared with the scattering in low energy supergravity. One can consider the action for one membrane moving in the field of another. Because $R_{11}$ remains finite, we must include also the fields of the images. The ‘source’ membrane and its images are at

$$x^{11} = v_{11} x^0 + 2\pi n R_{11} . \quad (3.31)$$
We find it convenient to boost to the rest-frame in the M-direction,

\[ x'^{11} = \gamma (x^{11} - v_{11} x^0) = 2\pi n \gamma R_{11} \]
\[ x'^{0} = \gamma (x^{0} - v_{11} x^{11}) = \gamma^{-1} x^{0} - 2\pi n \gamma R_{11} v_{11}. \]  

(3.32)

In this frame only the transverse velocity of the ‘test’ membrane remains. The action for the test membrane is

\[ -\tau_2 \int d^3 x' \det^{1/2}(h_{\mu\nu}) + i\mu_2 \int H \]  

with the induced metric

\[ h_{\mu\nu} = g_{\mu\nu} + \partial_\mu x^i \partial_\nu x^j g_{ij}. \]  

(3.34)

The metric of the source membrane is [25]

\[ g_{\mu\nu} = f^{-2/3}(r) \eta_{\mu\nu}, \quad g_{ij} = f^{1/3}(r) \delta_{ij}, \]
\[ f(r) = 1 + \frac{r_0^6}{r^6}, \]  

(3.35)

with \( r \) the transverse separation.

Expanding in to fourth order in the velocities \( v^i = \partial_0 x^i \),

\[ \det^{1/2}(h_{\mu\nu}) = f(r)^{-1} - \frac{1}{2} v^2 - \frac{1}{8} f(r)(v^2)^2. \]  

(3.36)

The velocity-independent term cancels the antisymmetric tensor interaction. The \( v^2 \) term is position-independent, so the leading interaction is the \( v^4 \) term

\[ \frac{\tau_2 r_0^6}{8r^6} (v^2)^2. \]  

(3.37)

In order to determine the value of \( \tau_2 r_0^6 \) we compare with the gravitational contribution to the static force between D two-branes in the IIA string [2], \( 3\alpha'/2X^5 \). To compare to the velocity-independent term \( \tau_2 r_0^6 / r^6 \) in eleven dimensions, we note that at long distance the interaction in the IIA string comes from the smeared sum over the periodic images,

\[ \frac{3\alpha'}{2X^5} = \int \frac{dx^{11}}{2\pi R_{11}} \frac{\tau_2 r_0^6}{(X^2 + R_{11}^2)^3} \]  

(3.38)
or \( \tau_2 r_0^6 = 8\alpha' R_{11} = 8M_{11}^{-3} \).

The \( v^4 \) interaction is then

\[
\frac{1}{M_{11}^3} \int d^3 x' (\partial'_0 X^i \partial'_0 X^i)^2 \sum_{n=-\infty}^{\infty} \left( X^2 + [2\pi n R_{11} - X^{11}]^2 \gamma^2 \right)^{1/3}.
\]

To pick out the term with one unit of M-momentum, multiply by \( e^{-iX^{11}/R_{11}} \) and average from 0 to \( 2\pi R_{11} \). The sum on \( n \) can be used to extend the integral from \(-\infty\) to \( \infty \), with the leading result at large \( r \) being

\[
\frac{1}{16\gamma^3 R_{11}^3 M_{11}^3} \int d^3 x' \left( \frac{\partial'_0 X^i \partial'_0 X^i}{X^3} \right)^2 e^{-\left( X/\gamma - iX^{11} \right)/R_{11}}.
\]

Boosting back to the lab frame introduces a factor of \( \gamma^3 \), giving precise agreement with the matrix theory result (3.30).\(^3\) In particular, the exponential suppression of the instanton result has a simple spacetime origin. At long distance the fields of the periodic images overlap, and so the \( x^{11} \) dependence falls exponentially.

### 4 Discussion

We have tested the eleven-dimensional Lorentz invariance of matrix theory beyond the known results on the classical Lorentz invariance of the supermembrane action and on scattering at zero M-momentum transfer.

It is important to determine the extent to which the result is restricted by supersymmetry. It is clear that the instanton calculation is unchanged if additional massive degrees of freedom are added to the theory, so these cannot be excluded. It may be that supersymmetry determines fully the form of the amplitude, given its \( X^{11} \)-dependence and the perturbative \( v^4 \) term. Supersymmetry alone however cannot determine the absolute normalization,

\(^3\)The instanton amplitude involves only the transverse velocity, whereas the supergravity calculation involves the total velocity. However, terms with the radial velocity can be converted by parts into second-derivative terms, to which the instanton calculation is insensitive.
since it linearizes to this order. However, supersymmetry plus a nonsingularity condition (which would appear at $X \to 0$, deep in the M theory regime) might do so. The $N = 4$ case [21] is a clear illustration of this.

It may be useful to study further the physics in the regime of validity of the present calculation, $X \gg R_{11} \gamma$. Although the full Lorentz invariance is not visible there, any complete theory must include this regime, and at least some matrix-theory processes are weakly coupled. Thus one should be able to consider less supersymmetric processes, e. g. [11].

It remains to be seen whether this result for membranes gives any insight into the corresponding graviton scattering amplitudes. It is interesting to note that whereas the zero-brane system is strongly interacting, the fluctuations of a single membrane are weakly interacting in the infrared. It may be that (compact) membrane-like states are in some sense attractive in the large-$N$ limit of the zero-brane system, so that the zero-brane bound state becomes in some sense ‘membrane-dominated.’

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References


