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GREEN'S FUNCTION APPROACH TO INFRARED FACTORIZATION AND FINITE EIKONAL CORRECTIONS

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G. C. Gellas, A. I. Karanikas

University of Athens, Department of Physics
Nuclear and Particle Physics Section
Panepistimiopolis
GR-15771 Athens, Greece

and

C. N. Ktorides

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

and

University of Athens, Department of Physics
Nuclear and Particle Physics Section
Panepistimiopolis
GR-15771 Athens, Greece

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Abstract

The infrared sector of a generic gauge theory with spin- $\frac{1}{2}$ matter fields and, for simplicity, only one mass scale, is factored out via a procedure which relies on a path integral (worldline) casting of the field system. The basic idea is to employ a velocity expansion which imposes the spin- $\frac{1}{2}$ particle's mass as a cutoff for the factorized sector. Anomalous dimensions characterizing the infrared regime are derived in connection with two- and three-point Green's functions. Finally, an off mass shell expansion of the propagator is achieved which contains genuine corrections to the eikonal approximation.

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1. Introduction

Factorization constitutes a powerful methodological tool in present day applications of field theory, QCD in particular, to elementary particle processes. Utilizing the basic instruction of renormalization theory [1] that the description of physical systems with a, seemingly, unmanageable number of degrees of freedom should be organized on a scale by scale basis, the factorization strategy amounts to justifying a separation between short and long distance physics, entering a given process, via the introduction of an intermediary mass scale μ . The most advertised situation where factorization holds the key to one's ability to conduct quantitative analyses in QCD is the treatment of hard processes, such as inclusive deep inelastic scattering, where one separates the so-called collinear from short distance physics. In effecting such factorization programmes one relies heavily on operator formalisms, with Wilson's operator product expansion (OPE) [2] providing the most relevant tool.

On the other hand, the cleanest and most unambiguous factorization formula of them all has been known for a long time and certainly much before Wilson's work. We are referring, of course, to the factorization (and exponentiation) of infrared (IR) divergencies in QED discovered 35 years ago by Yennie, Frautchi and Suura [3].

It so happens that the incoherent emission of soft photons in the abelian gauge theory of QED allows one to reach the factorization result for IR divergencies via straight forward field theoretical methods. A simple way to depict this abelian result is through an eikonal setting wherein one views the charged particle, e.g. electron, as moving fast along a straight trajectory (amenable to sudden derailments which involve some momentum transfer Q) and interacting on a no-impulse basis with a soft photon cloud.

In the non-abelian case, where gauge fields themselves carry charge, uncorrelated emission of soft gluons¹ is no longer justified. Nevertheless, it has become possible to pursue the aforementioned eikonal picture in the operator formalism by taxing all the effects of the interaction of the eikonal matter field's trail with the soft gluon cloud on Wilson lines. Specifically, the Wilson line operator enters the gauge invariant computation of hadronic matrix elements, which evaluate soft contributions to cross-sections and/or amplitudes, as a phase attached to a renormalized quark field ψ :

$$\psi(x) \longrightarrow \Psi(x) = \psi(x) \mathcal{P} \exp \left[i g \int_0^\infty ds u \cdot \mathcal{A}(x + su) \right], \quad (1)$$

where u_μ is a 'velocity' vector along the straight line trajectory, $\mathcal{A}_\mu = A_\mu^a t^a$ (t^a group generators in the natural representation) and $\mathcal{P}$ stands for path ordering with respect to group indices. The expansion of Ψ -operator products provides the basis for the formal Feynman rules associated with eikonal lines and vertices (see, e.g., [4]).

The above operator approach emulates the corresponding treatment of collinear singularities via the OPE strategy and utilizes well known results pertaining to renormalization properties of Wilson loop (and, to a lesser degree, line) operators [5-9]. The aim of this paper, by juxtaposition, is to place eikonal-related computations in the traditional vein of quantum field theory, namely one that is based directly on the theory's Green's functions. The importance of soft contributions to a host of situations where perturbative, but also non-perturbative, QCD is employed, coupled with the result [10] that factorization (but

¹For simplicity we shall refer to non-abelian gauge field quanta as "gluons" from now on.

not necessarily exponentiation) of IR singular terms applies to non-abelian systems as well, makes the development of alternative methods of confronting the IR sector of gauge theories both relevant and worth pursuing.

Given our stated objective the following question immediately arises: How can eikonal lines be associated with the Green's functions of a field theoretical system? Addressing this very issue constitutes, in fact, the quintessence of our present effort. We shall rely centrally on the so-called worldline, or particle-based, description of field systems [11-14]. According to this approach functional integrals which furnish the field theoretical Green's functions are recast into path integral form. Within such a framework it becomes natural to think in terms of trajectories of particle-like objects in spacetime, a particular case of which are eikonal lines. As it will turn out, an immediate bonus stemming from our effort is the emergence of finite correction terms resulting from small deformations of eikonal contours.

Deferring to the next section the presentation of the worldline approach to field systems let us close the introductory discussion by displaying the organization of our paper. As already mentioned, in the next section we shall be giving a (short) account of the worldline approach to field theory with exclusive emphasis on gauge theories with spin- $\frac{1}{2}$ matter fields and then subsequently proceed to furnish relevant expressions for the full fermionic propagator and the vertex function. In section 3 we shall introduce a velocity functional expansion in the path integral whose 0th order contribution amounts to the eikonal result. A technical aside of our analysis in this section is our dimensional regularization handling of IR divergencies. In section 4 we derive anomalous dimensions pertaining to the IR regime for the full fermionic propagator and the vertex function. In section 5 we obtain finite, off-mass-shell, corrections to the propagator coming from small deformations of the eikonal contour. Finally, in section 6 we formulate our concluding remarks and furnish commentary on perspectives and outlook of our work.

2. Worldline Casting of Field Systems

The recent emergence of worldline methodology in field theory is, really, a feedback from string theory. It was Polyakov[15] who, after using the worldline path integral description of a relativistic point particle as a paradigm for quantizing the string, turned things around by employing first quantization methods in their own right to study propagation properties of point-like excitations in Euclidean spacetimes [16]. To be sure, the particle, as opposed to the field, mode of description of relativistic quantum systems is not a new idea [17]. The spinning relativistic particle, in particular, has attracted the attention of many investigators, see, e.g., Refs [18-20].

Direct connections between quantum field theories and (Euclidean) path integrals have been pursued by Strassler [11], who was subsequently followed by Schmidt and Schubert [12], on the influence of the string-based work of Bern and Kosower [21] and, independently, by two of the present authors on the inspiration of the abovementioned work by Polyakov. The basic methodology involved is straight forward and can be illustrated as follows. Let $S = \int d^d x \mathcal{L}$ be some field theoretical action appropriately formulated in d Euclidean dimensions. One now constructs a partition function or, equivalently, a free energy for the system according to

$$-\log Z = \log \left\{ \int \mathcal{D}\phi e^{-S[\phi]} \right\}, \quad (2)$$

where $\mathcal{D}\phi$ is an appropriately defined functional measure for the collection of fields ϕ .

If it so happens that the Lagrangian density is quadratic in the fields, which is usually the case for spin- $\frac{1}{2}$ field terms, then the formal result of the functional integration is a determinant to some negative (bosons) or positive (fermions) power. Explicitly,

$$\log Z = \pm a \text{Tr} \log[D], \quad (3)$$

where $[D]$ is the relevant local, differential operator residing in the action and a counts the independent field modes that have been integrated.

Schwinger's proper time representation is subsequently employed to write

$$\log Z = \pm a \text{Tr} \int_0^\infty \frac{dT}{T} e^{-T[D]}. \quad (4)$$

To the extent that $[D]$ is hermitean, the trace operation turns the exponential into a path integral with coinciding initial and final points, i.e. to be performed over closed paths in Euclidean spacetime. Analogous considerations applied to Green's functions lead, as we shall have ample opportunity to witness in the sequel, to path integrals over open contours.

With these general remarks in mind let us witness this procedure in action for the class of field theories we are truly interested in, namely gauge theories with spin- $\frac{1}{2}$ matter fields. The simplest, non-trivial, path integral expression that can be worked out, in which all field integrations are executed, corresponds to the situation where the gauge fields are external. In this case the free energy (effective action in Minkowski version) will register as a functional of the classical gauge potential ². Following Ref. [14] we carry out the Gaussian-Grassmann integrations over the spinorial fields having first performed a careful regularization (one that avoids doubling problems) of the field system. This is achieved via an averaging procedure in the continuum which introduces a resolution distance ϵ . The resulting expression is

$$\begin{aligned} \Gamma[A] = & \text{tr} \int_c \frac{dT}{T} e^{-Tm_0} \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x(\tau) e^{-\frac{T}{2\epsilon}} \mathcal{P} \exp \left[\frac{1}{\epsilon} \int_0^T d\tau \gamma_\mu \dot{x}_\mu(\tau) \right] \\ & \times \exp \left[-\frac{1}{2\epsilon} \int_0^T d\tau \dot{x}(\tau)^2 \right] \exp \left\{ ig \int_0^T d\tau \dot{x}_\mu(\tau) \mathcal{A}_\mu(x(\tau)) \right\}, \end{aligned} \quad (5)$$

where $\mathcal{N}$ is a normalization factor and where $\lim c \rightarrow 0$ is implied throughout ³.

The first thing one notices in the above result is the appearance of a Wilson loop. Even though not *a priori* invited its automatic entrance into worldline expressions presents an interest of its own. The most important feature of (5), however, is that the original functional integral for the field system has been reduced to particle modes which circulate on a one dimensional closed contour. One recognizes a 'kinetic' term $\dot{x}(\tau)^2$ in one of the exponentials⁴ for the one dimensional particle. The latter is indigenous to the worldline casting of field systems and has its analogue in the so-called z -fields which enter (by hand) Wilson loop studies in the operator formalism [5].

²Our work in Ref. [14], see also [22], has exclusively focused on such effective actions.

³ c plays the role of an ultraviolet cutoff and must be larger, or at most equal to the resolution scale ϵ .

⁴It should be pointed out that whereas in our case this kinetic term arises through our regularization procedure, as applied to the Dirac sector of the gauge theory in its customary first order form, in Strassler's approach it results via a treatment of the Dirac operator in its second order form.

Were it not for the term $\gamma \cdot \dot{x}$ entering the preceeding exponential one would be thinking in terms of a system wherein a one dimensional quantum mechanical particle, living on a loop, interacts with the external potential $A_\mu(x)$. The presence of the γ -matrices, which account for the spin- $\frac{1}{2}$ nature of the particle, poses the interesting question concerning the symbiosis of geometry and dynamics on the, closed in this case, line. To face this issue Polyakov introduced the so-called spin factor [16], see Ref. [14] for its field theoretical derivation, while an alternative approach is to use supersymmetric coordinates for the one-dimensional particle [18-20],[12]. For open lines, on the other hand, the matter remains effectively open, at least as far as interacting particles are concerned.

Fortunately, there is a way to resolve the interference between spin-geometry and dynamics for open lines and this is by extending the path integral formula to phase space. In doing so, we shall continue to refer to the overall approach as 'worldline' even though the characterization 'particle-based' might be more appropriate.

The phase space transcription of the path integral for $\Gamma[A]$ is accomplished by using the following identities

$$\mathcal{N} \exp\left[-\frac{1}{2\epsilon} \int_0^T d\tau \dot{x}(\tau)^2\right] = \int \mathcal{D}p(\tau) \exp\left[i \int_0^T d\tau p(\tau) \cdot \dot{x}(\tau)\right] \exp\left[-\frac{\epsilon}{2} \int_0^T d\tau p(\tau)^2\right], \quad (6)$$

$$\begin{aligned} & \mathcal{P} \exp\left[\frac{1}{\epsilon} \int_0^T d\tau \gamma_\mu \dot{x}_\mu(\tau)\right] \exp\left[i \int_0^T d\tau p_\mu(\tau) \dot{x}_\mu(\tau)\right] \\ &= \mathcal{P} \exp\left[-\frac{i}{\epsilon} \int_0^T d\tau \gamma_\mu \frac{\delta}{\delta p_\mu(\tau)}\right] \exp\left[i \int_0^T d\tau p_\mu(\tau) \dot{x}_\mu(\tau)\right], \end{aligned} \quad (7)$$

and, finally,

$$\begin{aligned} & \mathcal{P} \exp\left[\frac{i}{\epsilon} \int_0^T d\tau \gamma_\mu \frac{\delta}{\delta p_\mu(\tau)}\right] \exp\left[-\frac{\epsilon}{2} \int_0^T d\tau p(\tau)^2\right] \\ &= \mathcal{P} \exp\left[-i \int_0^T d\tau \gamma_\mu p_\mu(\tau)\right] \exp\left[-\frac{\epsilon}{2} \int_0^T d\tau p_\mu(\tau)^2\right] e^{T/2\epsilon}. \end{aligned} \quad (8)$$

Upon inserting the above identities into (5) we obtain

$$\begin{aligned} \Gamma[A] &= -\text{tr} \int_c^\infty \frac{dT}{T} e^{-Tm} \int_{x(T)=x(0)} \mathcal{D}x(\tau) \int \mathcal{D}p(\tau) \exp\left(i \int_0^T d\tau \dot{x} \cdot p\right) \\ &\quad \times \mathcal{P} \exp\left(-i \int_0^T d\tau \gamma \cdot p\right) \exp\left(ig \int_0^T d\tau \dot{x} \cdot A\right), \end{aligned} \quad (9)$$

where the trace is over discrete indices. One notices the disappearance of the resolution scale ϵ from the above expression as it has been traded with the spreading that naturally occurs in phase space. For this reason we replace c by 0^+ from hereon.

The Green's functions of the theory are retrieved through the generating functional of the collection of sources $\{\eta, \bar{\eta}, J\}$

$$Z[\eta, \bar{\eta}, J] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu \exp[iS_d + iS_\xi + i \int d^d x (\bar{\eta} \psi + \bar{\psi} \eta + A \cdot J)], \quad (10)$$

where S_d and S_ξ are Dirac and gauge field action terms, respectively (the latter supplemented by a covariant gauge fixing term with ξ as its coefficient).

Functional differentiation with respect to the sources produces corresponding Green's functions. In this paper we shall be concerned with the full one-fermion propagator $G(x, y) = \delta^2 Z / \delta \bar{\eta}(x) \delta \eta(y)|_{\bar{\eta}=0, \eta=0, J=0}$ and the vertex function $\mathcal{G}_\mu(x, y, z) = \delta^3 \log Z / \delta \bar{\eta}(x) \delta \eta(y) \delta J_\mu(z)|_{\bar{\eta}=0, \eta=0, J=0}$. Applying a similar procedure on regularized Euclidean versions of these quantities, as was done with the effective action $\Gamma[A]$, we obtain path integral expressions for the propagator

$$\begin{aligned} G^{ij}(x, y) &= \int_{0^+}^\infty dT \int_{x(T)=y}^{x(0)=x} \mathcal{D}x(\tau) \mathcal{D}p(\tau) \exp\left[i \int_0^T d\tau p(\tau) \cdot \dot{x}(\tau)\right] \\ &\quad \times \mathcal{P} \exp\left\{-\int_0^T d\tau [i\gamma \cdot p(\tau) + m]\right\} \left\langle \mathcal{P} \exp\left\{ig \int_0^T d\tau \dot{x}(\tau) \cdot A(x(\tau))\right\} \right\rangle_A^{ij} \end{aligned} \quad (11)$$

and for the vertex function

$$\begin{aligned} \mathcal{G}_\mu^{ij}(x, y, z) &= ig \int_{0^+}^\infty dT e^{-Tm} \int_0^T ds \int_{x(T)=y}^{x(0)=x} \mathcal{D}x(\tau) \mathcal{D}p(\tau) \dot{x}_\mu(s) \delta[x(s) - z] \exp\left[i \int_0^T d\tau p(\tau) \cdot \dot{x}(\tau)\right] \\ &\quad \times \mathcal{P} \exp\left\{-\int_0^T d\tau [i\gamma \cdot p(\tau) + m]\right\} \left\langle \exp\left\{ig \int_0^T d\tau \dot{x}(\tau) \cdot A(x(\tau))\right\} \right\rangle_A^{ij}, \end{aligned} \quad (12)$$

where ⁵

$$\langle \mathcal{O}(A) \rangle_A = \int \mathcal{D}A_\mu(x) \mathcal{O}(A) \det[G^{-1}(x, y|A)] \exp\{-S_\xi[A]\}, \quad (13)$$

with $G^{-1}(x, y|A)$ standing for the Euclidean Dirac operator in the presence of the potential A_μ . Our considerations, in what follows, will be based on the above worldline expressions for the propagator and the vertex.

3. Velocity Expansions in the Path Integrals

Consider expression (11) for the propagator whose full handling demands the execution of the functional integration with respect to the gauge fields. Given our interest in the IR regime of the theory we adopt from the outset the quenched approximation according to which the (active) gauge field quanta do not have enough energy to produce fermion pairs. Under this circumstance let us first observe that in the abelian case we get

$$\left\langle \exp\left\{ig \int_0^T d\tau \dot{x}(\tau) \cdot A(x(\tau))\right\} \right\rangle_A = \exp\left\{-\frac{1}{2}g^2 \int_0^T d\tau \int_0^T d\tau' \dot{x}_\mu(\tau) \dot{x}_\nu(\tau') < A_\mu(x(\tau)) A_\nu(x(\tau')) >_A\right\}. \quad (14)$$

Being able to relate the expectation value of the Wilson line operator to the exponential of the gauge field correlator forms the basis on which the eventual exponentiation of the IR divergent factors in QED can be understood. The derivation of this result within the worldline context has been given in [23].

For the non-abelian case the best we can do is to perform a perturbative, *albeit* not necessarily Feynman diagrammatic [11,12], expansion. In this context, the dimensionally

⁵Ghost integrations are assumed to reside within the measure $\mathcal{D}A_\mu(x)$.

regularized gauge field correlation function reads (in the Feynman gauge)

$$\begin{aligned} \langle \mathcal{A}_\mu(x(\tau)) \mathcal{A}_\nu(x(\tau')) \rangle_A^{ij} &= \delta_{ij} c_F \mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{\delta_{\mu\nu}}{k^2} e^{-ik[x(\tau)-x(\tau')]} \\ &= \delta_{\mu\nu} \delta_{ij} c_F \frac{\mu^{4-D}}{4\pi^{D/2}} \Gamma\left(\frac{D}{2} - 1\right) |x(\tau) - x(\tau')|^{2-D}, \end{aligned} \quad (15)$$

with $D = 4 - 2\epsilon$ ($\epsilon \rightarrow 0$) and where c_F is the eigenvalue of the Casimir operator in the fundamental representation of the gauge group.

The important feature of the expression that resulted from the gauge field integration is that it fully aligns itself with the path integration inherited from the matter field sector and which awaits to be performed. In particular, the following functional has made its entrance, along with some constant coefficients, through the integration over the gauge fields

$$\begin{aligned} F[\dot{x}] &= \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) |x(\tau_2) - x(\tau_1)|^{2-D} = \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) \left| \int_0^{\tau_2} d\tau \dot{x}(\tau) - \int_0^{\tau_1} d\tau \dot{x}(\tau) \right|^{2-D} \\ &= \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) \left| \int_0^T d\tau \dot{x}(\tau) [\theta(\tau_2 - \tau) - \theta(\tau_1 - \tau)] \right|^{2-D}. \end{aligned} \quad (16)$$

Now, the above expression has come from the perturbative expansion of the expectation value of a Wilson line operator. Recall, on the other hand, that in the operator formalism the eikonal description calls for Wilson lines taken along straight paths. This leads us to consider a functional expansion of $F[\dot{x}]$ around a constant 'velocity' vector u_μ , pointing along a straight path, as follows:

$$\begin{aligned} F[\dot{x}] &= F[u] + \int_0^T d\sigma [\dot{x}_\mu(\sigma) - u_\mu] \frac{\delta F[\dot{x}]}{\delta \dot{x}_\mu(\sigma)} \Big|_{\dot{x}=u} \\ &+ \frac{1}{2} \int_0^T d\sigma_1 \int_0^T d\sigma_2 [\dot{x}_\mu(\sigma_1) - u_\mu] [\dot{x}_\nu(\sigma_2) - u_\nu] \frac{\delta^2 F[\dot{x}]}{\delta \dot{x}_\mu(\sigma_1) \delta \dot{x}_\nu(\sigma_2)} \Big|_{\dot{x}=u} + \dots \end{aligned} \quad (17)$$

To 0th order in this expansion, where the actual propagation takes place strictly along straight lines, u is the spin- $\frac{1}{2}$ physical particle's velocity, which means $u^2 = 1$. Higher order terms, on the other hand, bring in deformations of the straight line paths which imply an off the mass shell course. We shall, therefore, keep u^2 in our expressions adjusting to on or off mass shellness according to the prevailing situation.

The crucial point of the velocity expansion is that it serves to factorize the IR sector of the theory with point of reference the physical mass for the matter field particles⁶. If we were to keep only the 0th order term, then the aforementioned physical mass scale literally plays the role of a "UV" cutoff for the (IR) sector of the theory we have isolated. Indeed, matter particles appear to be too heavy to suffer any kind of fluctuating motion around the straight line course, despite their interactions with the, obviously, soft gluons which reside in this sector. Note, at the same time, that the massless one dimensional particle modes continue to exist on the eikonal contour. They too, however, cannot cause any off-the-straight-line

⁶To simplify things we shall agree that we are dealing with a gauge system that contains only one such mass scale.

course maneuvering. Bringing into consideration higher order terms in the velocity expansion allows us to venture a bit beyond this mass level. What we shall discover in this context, for the propagator at least, will be unveiled in section 5.

From (16) it follows immediately that

$$F[u] = u^{4-D} |\tau_2 - \tau_1|^{2-D}. \quad (18)$$

Combining the first term of the velocity expansion with the 0th (unit) term of the Wilson exponential we form the quantity

$$\begin{aligned} G_0^{(2)}(x, y|u) &= \int_{0^+}^\infty dT e^{-Tm} \int \mathcal{D}p(\tau) \mathcal{P} \exp \left\{ -i \int_0^T d\tau p(\tau) \cdot \gamma \right\} \int_{x(0)=x}^{x(T)=y} \mathcal{D}x(\tau) \exp \left\{ i \int_0^T d\tau p(\tau) \cdot \dot{x}(\tau) \right\} \\ &\times (1 - g^2 C u^{4-D} \int_0^T d\tau_1 \int_{\tau_1}^T d\tau_2 |\tau_2 - \tau_1|^2), \end{aligned} \quad (19)$$

where C stands for the various constant coefficients entering (15).

Now,

$$\int_0^T d\tau_1 \int_{\tau_1}^T d\tau_2 |\tau_2 - \tau_1|^{2-D} = -\frac{1}{4-D} \frac{1}{D-3} T^{4-D}, \quad (20)$$

a result which exhibits ultraviolet (UV) divergencies. The latter, even though not genuine with respect to the full microscopic theory, appear to be so from the perspective of the low energy sector we have factorized. This occurrence is also true in connection with the factorization of collinear divergencies in QCD where the anomalous dimensions of the various operators entering the OPE induce a renormalization group running of distribution functions [24]. We too are interested in the derivation, within the framework of our approach, of corresponding anomalous dimensions for the 'IR factorization' presently under study. Leaving this task for the next section, let us now turn our attention to the IR problems that are lurking in our expressions⁷.

Looking at (19) it becomes immediately obvious that what controls the IR behavior of the propagator function $G_0^{(2)}(x, y|u)$, or any Green function in our low energy sector for that matter, is the upper limit in the T -integral. Clearly, the longer the length of the path over which we count the presence of active degrees of freedom (soft gluons and massless one-dimensional particle modes) the longer is the range of admitted wavelengths. We proceed to regularize these infrared divergencies via the following replacement

$$\int_{0^+}^\infty dT \longrightarrow \lambda^{D-4} \int_{0^+}^\infty dT T^{D-4}, \quad (21)$$

where λ is an arbitrarily small mass, certainly satisfying $\lambda \ll \mu \leq m$.

Turning our attention to the vertex function we readily determine that its structure is, to order g^3 , of the form

$$\mathcal{G}_{\mu,ij}^a(x, y, w) = \int d^4 z D_{\mu\nu}^{ab}(w - z) G_{\nu,ij}^b(x, y, z) + \Delta_{\mu,ij}^a(x, y, w), \quad (22)$$

where $D_{\mu\nu}^{ab}$ is the gluon propagator. The second term on the right hand side, involving the 3-gluon coupling, vanishes in the Feynman gauge, which we are adopting throughout, so this

⁷Clearly, even the IR sector of the full microscopic theory we have isolated has its own "IR limit"!

leaves $G_{\mu,ij}^a(x, y, z)$ as our quantity of real interest. It represents a fermionic worldline being "struck" at the point z and reads as follows

$$G_{\mu,ij}^a(x, y, z) = ig t_{ij}^a \lambda^{D-4} \int_0^\infty dT T^{D-4} e^{-Tm} \int_0^T ds \int_{x(0)=x}^{x(T)=y} \mathcal{D}x(\tau) \int \mathcal{D}p(\tau) \dot{x}_\mu(s) \delta[x(s) - z] \\ \times \exp[-i \int_0^T d\tau (p(\tau) \cdot \gamma - p(\tau) \cdot \dot{x}(\tau))] \left\{ 1 - g^2 C_F \frac{c_F}{2} \left[\int_0^s d\tau_1 \int_0^s d\tau_2 \right. \right. \\ \left. \left. + \int_s^T d\tau_1 \int_s^T d\tau_2 + 2 \int_0^s d\tau_1 \int_s^T d\tau_2 \right] \dot{x}_\rho(\tau_1) \dot{x}_\sigma(\tau_2) D_{\rho\sigma}(x(\tau_2) - x(\tau_1)) \right\}, \quad (23)$$

where $D_{\mu\nu}$ is given by (15) for, $i = j$. Note should be taken of the fact that we have amputated the external gluon propagator appearing in (22).

We now dispense with the presence of $\dot{x}_\mu(s)$ inside the path integral as follows. Write

$$\mathcal{P} e^{-i \int_0^T d\tau p \cdot \gamma} \dot{x}_\mu(s) e^{i \int_0^T d\tau p \cdot \dot{x}} = \mathcal{P} e^{-i \int_0^T d\tau p \cdot \gamma} \left(-i \frac{\delta}{\delta p_\mu(s)} \right) e^{i \int_0^T d\tau p \cdot \dot{x}} \\ = \mathcal{P} e^{-i \int_0^T d\tau p \cdot \gamma} \gamma_\mu \mathcal{P} e^{-i \int_s^T d\tau p \cdot \gamma} e^{i \int_0^T d\tau p \cdot \dot{x}}, \quad (24)$$

where the last step is a result of an integration by parts, given that the above manipulations take place inside integrals. We note, in passing, that the handling of $\dot{x}_\mu(s)$ -velocity vectors along paths for open line configurations is what causes complications in the original worldline formalism. The phase space formulation of the path integral, on the other hand, simply trades the $\dot{x}$'s with γ 's with simultaneous splittings of the exponential $\exp[-i \int_0^T d\tau p \cdot \gamma]$.

Returning to (23), let us designate the three, order g^3 , terms entering the last curly brackets as $V_1^{(2)}$, $V_2^{(2)}$ and $V_{12}^{(2)}$, respectively. The first two are, really, independent one fermion propagators, of the form we have already discussed. With respect to the parametric description, they form two separate branches, one from 0 to s and the other from s to T , which introduce, in the velocity expansion, two directions with corresponding velocity vectors $u_{1\mu}$ and $u_{2\mu}$. The third term is associated with the total, broken at point z , line configuration. It amounts to an interference, or crossed, term which brings along the velocity functional

$$\Phi[\dot{x}] = \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) \left| \int_{\tau_1}^s d\tau \dot{x}(\tau) + \int_s^{\tau_2} d\tau \dot{x}(\tau) \right|^{2-D}, \quad (25)$$

to be appropriately expanded around the velocity pair (u_1, u_2) .

Introducing the notation $u_\mu(\rho) \equiv u_{1\mu}\theta(s - \rho) + u_{2\mu}\theta(\rho - s)$ we write

$$\Phi[\dot{x}] = \Phi(u_1, u_2) + \int_0^T d\rho [\dot{x}_\mu(\rho) - u_\mu(\rho)] \frac{\delta \Phi[\dot{x}]}{\delta \dot{x}_\mu(\rho)} \Big|_{\dot{x}(\rho)=u(\rho)} + \dots, \quad (26)$$

with the 0th order term given by

$$\Phi(u_1, u_2) = u_1 \cdot u_2 [(s - \tau_1)u_1 + (\tau_2 - s)u_2]^{2-D}. \quad (27)$$

In the following section we shall extract anomalous dimensions, pertaining to the infrared factorization, from the leading terms in the velocity expansion.

4. Anomalous Dimensions of the Infrared Sector

Consider the quantity $G_0^{(2)}(x, y|u)$, cf. eq (19), which comprises the order g^2 piece of the propagator determined solely by the constant term of the velocity expansion. Upon inserting the expression given by (20), the remaining task is to perform the path integral that contains only free particle terms. Let us verify that the familiar free Dirac propagator is obtained as a result. To begin, we find

$$\int_{x(0)=x}^{x(T)=y} \mathcal{D}(x) e^{i \int_0^T d\tau p(\tau) \cdot \dot{x}(\tau)} = \int \mathcal{D}e(\tau) e^{\int_0^T d\tau e(\tau) \cdot p(\tau)} \int_{x(0)=x}^{x(T)=y} \mathcal{D}(\tau) \delta[e(\tau) - \dot{x}(\tau)] \\ = \int \frac{d^4 q}{(2\pi)^4} e^{-iq(x-y)} \int [d\alpha e(\tau)] e^{i \int_0^T d\tau e(\tau) \cdot [p(\tau) - q]}, \quad (28)$$

where $[d\alpha e(\tau)]$ serves to denote a measure of the form $\lim_{N \rightarrow \infty} \prod_{i=1}^N d\epsilon_i$ resulting from the division of $[0, T]$ into N segments ($\alpha \equiv T/N$). The corresponding integration yields a product of delta-functions, we can denote it by $\prod_{\tau \in [0, T]} (2\pi)^4 \delta[p(\tau) - q]$, which, upon insertion into the remaining path integral over $\mathcal{D}p(\tau)$, yields a constant momentum q over the path.

Putting everything together we finally determine, once reading off (28) the Fourier transform of the (Euclidean) propagator,

$$\tilde{G}_E^{(0)}(q) = \int_0^T e^{-T(m+i\gamma \cdot q)} = \frac{1}{m + i\gamma \cdot q}, \quad (29)$$

where m is the *physical* mass of the spin- $\frac{1}{2}$ particle(s).

The transcription to Minkowski space reads (metric: $+- --$)

$$\tilde{G}_{Mink}^{(0)}(q) = -i \tilde{G}_E^{(0)}(iq) = \frac{i}{\gamma_\mu q^\mu - m}, \quad (30)$$

to be amended by the appropriate $i\epsilon$ prescription, according to desired boundary conditions.

Our final result, therefore, becomes, in Minkowski space,

$$\tilde{G}_{0,Mink}^{(2)}(q|u) = \left(1 + \frac{g^2}{4\pi^{D/2}} c_F \frac{\Gamma(\frac{D}{2} - 1)}{(4 - D)(D - 3)} \left(\frac{\mu}{\lambda} \right)^{4-D} u^{4-D} \right) \frac{i}{\gamma_\mu q^\mu - m + i\epsilon}. \quad (31)$$

We record the emergence of a wave function renormalization constant which can be taxed on the massless one dimensional particle modes residing on the eikonal line. We shall have further comments on this matter shortly.

Higher order terms in the velocity expansion do not introduce divergencies, an occurrence that will be explicitly confirmed in section 5.

From (31) we immediately read an anomalous dimension $\gamma(\alpha_s)$ associated with the IR domain of the theory in 4 dimensions. The on mass shell situation presently in hand leads to the following expression for $\gamma(\alpha_s)$

$$\gamma(\alpha_s) = -\frac{\alpha_s}{2\pi} c_F + O(\alpha_s^2). \quad (32)$$

We now move on to the cusped line configuration which presents more interest. First, let us dispense with the kinematical factor that accompanies the unit term in the expansion of the

Wilson exponential, cf eq (23). The functional integration over $\mathcal{D}(x)$ can be manipulated as follows

$$\int_{x(0)=x}^{x(T)=y} \mathcal{D}(x) \delta[x(s) - z] e^{\int_0^T d\tau p \cdot \dot{x}} = \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} e^{-iq_1 \cdot (x-z) - iq_2 \cdot (z-y)} \times \int [\alpha d\epsilon(\tau)] e^{i \int_0^T d\tau \epsilon(\tau) \cdot [p(\tau) - q_1]} e^{i \int_0^T d\tau \epsilon(\tau) \cdot [p(\tau) - q_2]} \quad (33)$$

Off the q integrations we read the Fourier transform $\tilde{V}_\mu^{(0)}(q_1, q_2)$, while the functional integration produces the string of delta functions $\prod_{\tau \in [0, s]} (2\pi)^4 \delta[p(\tau) - q_1] \prod_{\tau \in [s, T]} (2\pi)^4 \delta[p(\tau) - q_2]$. The functional integration with respect to $\mathcal{D}p(\tau)$ can now be carried out to give

$$\tilde{V}_\mu^{(0)}(q_1, q_2) = ig \int_0^\infty dT T \int_0^1 ds e^{-Ts(iq_1 \cdot \gamma + m)} \gamma_\mu e^{-T(1-s)(iq_2 \cdot \gamma + m)} = ig \frac{1}{iq_1 \cdot \gamma + m} \gamma_\mu \frac{1}{iq_2 \cdot \gamma + m}, \quad (34)$$

where we agree to drop the ij -indices from now on as superfluous at the order in which we are calculating. The Minkowski result follows automatically.

It is obvious now that the terms we have designated as $V_1^{(2)}$ and $V_2^{(2)}$ in (23) will contribute, per branch, the same factor as the one entering (31). The relevance of these contributions has been discussed by Weinberg [25] in the context of the following question. How consistent is it to correct "external" lines entering IR type situations such as the present? Eikonal lines seem to have the characteristics of external lines (particle on mass shell). The matter particles moving along these lines are, however, in constant interaction with a soft gluon/photon cloud. From this viewpoint their propagation is amenable to renormalization. As already mentioned, in the worldline scheme there is an agent which absorbs these renormalization effects, namely the massless one-dimensional particle modes. This occurrence serves to underline the conceptual advantage of the worldline approach to eikonal based descriptions: It correctly regards the matter particle propagation lines as worldlines and *not* as lines entering a Feynman diagram. This, in fact, constitutes the basis of our original expectation that a Green's function approach to eikonal physics is persuasive within the worldline formalism.

Our true interest, of course, lies with the interference term to which we now turn our attention. We start with the following remark. According to (23) the two eikonal branches, on each side of the point of deflection, are interfering along their full extent. Nothing prevents us, on the other hand, from restricting the region of interference around a small segment on each side of the derailment point z^b . This is actually consistent with our intention to consider the cusped matter-particle contour as a part of a vertex configuration. By contrast, the smaller segment around the cusp where the interference does takes place corresponds to a form factor. Let us, then, introduce a small length scale σ such that $\frac{\lambda^2}{m^2} \sim \frac{\sigma}{T} \equiv \delta \ll 1$ so that interference occurs between the segments $[s - \sigma, s]$ and $[s, s + \sigma]$ to the left and right of z , respectively.

With the above in mind we form, by referring to (27), the quantity

$$I_D = \int_{s-\sigma}^s d\tau_1 \int_s^{s+\sigma} d\tau_2 \frac{u_1 \cdot u_2}{|(s - \tau_1)u_1 + (\tau_2 - s)u_2|^2 - D}$$

⁸One should not lose sight of the fact that we are dealing with worldline paths and *not* Feynman diagrams.

$$= \delta^{4-D} \frac{2\theta}{4-D} T^{4-D} |u|^{4-D} \int_0^1 dx (x^2 + 2x\theta + 1)^{\frac{2-D}{2}}, \quad (35)$$

where $\theta \equiv \frac{u_1 \cdot u_2}{|u_1| |u_2|}$. The four velocity u has been introduced on the basis of $|u_1| = |u_2| \equiv u$. This is, of course an identity for the on mass shell situation presently in hand. Under more general circumstances it would still be necessary if one demanded a uniform off mass shell parametrization of the form factor. Other possibilities, however, *would* be permissible, such as one branch being on mass shell and the other not.

Proceeding further we find

$$\int_0^1 dx (x^2 + 2x\theta + 1)^{\frac{2-D}{2}} = \frac{1}{D-3} \left[\theta F\left(1, \frac{D}{2} - 1; \frac{D-1}{2}; 1 - \theta^2\right) - 2^{\frac{2-D}{2}} (1 + \theta)^{2-\frac{D}{2}} \times F\left(1, \frac{D}{2} - 1; \frac{D-1}{2}; \frac{1 - \theta^2}{2(\theta + 1)}\right) \right] \equiv J_D(\theta). \quad (36)$$

The last equality in (35) along with the above result are given in an Appendix.

Following the familiar, by now, routine where we perform the path integral over $\mathcal{D}x(\tau)$, obtain a string of delta functions, then carry out the $\mathcal{D}p(\tau)$ integration and finally read off the Fourier transform, we obtain

$$\tilde{V}_{12,\mu}^{(2)}(q_1, q_2 | u_1, u_2) = -ig^3 c_F \frac{\Gamma(\frac{D}{2} - 1)}{4\pi^{D/2}} \left(\frac{\mu}{\lambda}\right)^{4-D} \delta^{4-D} |u|^{4-D} \frac{2\theta}{4-D} J_D(\theta) \times \int_{0^+}^\infty dT e^{-Tm} T \int_0^1 ds e^{isT q_1 \cdot \gamma} \gamma_\mu e^{-i(1-s)T q_2 \cdot \gamma}. \quad (37)$$

To isolate the singularity structure residing in the broken fermionic line we simply focus on the term $D=4$. We find

$$2\theta J_4(\theta) = \frac{\theta}{\sqrt{1 - \theta^2}} \arctg \frac{\sqrt{1 - \theta^2}}{\theta}. \quad (38)$$

The above expression registers as the coefficient of the singular factor $\frac{1}{4-D}$. There are finite terms lurking in the Laurent expansion of (37) in powers of $(4 - D)$, an occurrence which, we feel, illustrates the enrichment of the worldline over the operator formalism. Preliminary indications point to the fact that, for QED at least, the first finite term furnishes the Sudakov form factor, a result consistent with the observation [12], in connection with closed loops, that the worldline casting of field systems has the capacity of simultaneously accommodating all possible ways of gauge field exchange between fermionic lines. For the purposes of the present paper we shall fix our attention on the IR related anomalous dimensions. Designating the anomalous dimension just determined by Γ_{cusp} and transcribing to Minkowski space, where $\theta \geq 1$, we write

$$\Gamma_{cusp} = c_F \frac{\alpha_s}{\pi} \frac{1}{2\beta} \ln \frac{1 + \beta}{1 - \beta}, \quad (39)$$

where $\beta = (1 - \frac{1}{\theta^2})^{\frac{1}{2}} = (1 - \frac{q_1^2 q_2^2}{(q_1 \cdot q_2)^2})^{\frac{1}{2}}$ is the relative velocity.

On the mass shell we can relate Γ_{cusp} with the momentum transfer Q as follows

$$-Q^2 = (q_1 - q_2)^2 = 2m^2 - 2m_2\theta \quad (40)$$

from which we deduce $\theta = 1 + \frac{Q^2}{2m^2}$. But, since $\theta = 1/\sqrt{1-\beta^2}$, we obtain

$$\frac{1}{2\beta} \ln \frac{1+\beta}{1-\beta} = 1 + B \left(\frac{Q^2}{m^2} \right), \quad (41)$$

where

$$B \left(\frac{Q^2}{m^2} \right) = \frac{1 + \frac{2m^2}{Q^2}}{\sqrt{1 + 4m^2/Q^2}} \ln \left[\frac{\sqrt{1 + 4m^2/Q^2} + 1}{\sqrt{1 + 4m^2/Q^2} - 1} \right] - 1 \quad (42)$$

is the well known bremsstrahlung function.

In the operator formalism the above result is obtained [26] within a set up according to which the Wilson lines extend to infinity (and by doing so they actually comprise a loop). In the worldline formalism besides the conceptual advantage of associating the line contour with particle propagation and the accompanying picture of a sudden impulse acting on it, there are dividends of practical importance having to do with our ability to *deform* the eikonal lines. This, obviously, comes about once higher order terms in the velocity expansion are taken into account. We shall demonstrate this occurrence in the next section in connection with the propagator.

Let us bring the present section to a conclusion with a final comment. Had we not amputated the gluonic propagator entering (22) we would be in position to derive an anomalous dimension assigning itself to the vertex operator and which is given, on mass shell, by $-\frac{2}{3}c_A$, where c_A is the adjoint representation Casimir eigenvalue. This we have demonstrated in a recent paper [27] where we applied the worldline approach to straight paths strictly described by 'physical' four velocity vectors, $u^2 = 1$, while simultaneously adjusting our description to heavy quark theory.

5. Off Mass Shell Expansion of the Propagator

In this section we intend to deal with corrections brought about by the higher order terms in the velocity expansion. In preparation for this job we find it useful to briefly assess our work to this point.

Having set for ourselves the task of factorizing the IR sector of a gauge field theory, with spin- $\frac{1}{2}$ matter fields (and, for simplicity, with only one physical mass scale m) we employed the worldline formalism in order to project out that part of the theory where the matter particles appear as extremely heavy and move along straight line contours. The alive degrees of freedom left are, (a) gluons/photons that look up to m as a cutoff and (b) one dimensional massless particles living on these contours. Also, by introducing, through our regularization procedure, a small mass scale λ below which the presence of gluons is not acknowledged (presumably they are instrumental in forming the vacuum structure of the theory)⁹ we were led to sharply defined on mass shell expressions such as the one appearing in (31).

The question now is the following. What kind of corrections do we expect from the higher order terms in the velocity expansion? A reasonable guess would be: Terms in a series of

⁹In QED, if one were to by-pass the introduction of λ , then one would discover the well known near mass shell result for the propagator $\frac{1}{(\gamma_\mu q^\mu - m)^2 + \lambda^2}$, where α is a gauge dependent quantity. This behavior displays the beginning of a cut that starts at the physical mass threshold. By contrast, our result (31), which is equally good for QED, displays an explicit isolation of the mass shell due to the protection by λ .

powers of $1/m$ which progressively compensate for the extreme view $m \rightarrow \infty$ associated with the 0th order term. This turns out to be only a part of the story. Besides the possibility of organizing the corrections in powers of $1/m$ we shall discover that a series is obtained which organizes itself in powers of off mass shellness. With these comments in mind let us proceed with the technical aspects of the derivation. Admittedly, the computations are both lengthy and involved so, we shall present only a bare minimum of formalism.

Going back to (17) and looking at the 1st order term in the velocity expansion, we determine

$$\frac{\delta F[\hat{x}]}{\delta \hat{x}_\mu(\sigma)} \Big|_{\hat{x}=\bar{x}} = 2u_\mu |u|^{2-D} |\tau_1 - \tau_2|^{2-D} \delta(\tau_2 - \sigma) - (D-2)u_\mu |u|^{2-D} |\tau_2 - \tau_1|^{1-D} [\theta(\tau_2 - \sigma) - \theta(\tau_1 - \sigma)] \equiv f_\mu. \quad (43)$$

Referring to the full one fermion propagator we now find ourselves having to deal with the following expression

$$G_1(x, y|u) = -g^2 \frac{1}{4\pi^{D/2}} c_F \left(\frac{\mu}{\lambda} \right)^{4-D} \Gamma\left(\frac{D}{2} - 1\right) \int_0^\infty dT e^{-Tm} T^{D-4} \int_0^T d\tau_1 \int_{\tau_1}^T d\tau_2 \int_0^T d\sigma f_\mu(u) \\ \times \int \mathcal{D}p(\tau) \mathcal{P} e^{-i \int_0^\sigma d\tau p \cdot \gamma} (\gamma_\mu - u_\mu) \mathcal{P} e^{-i \int_\sigma^T d\tau p \cdot \gamma} \int_{x(T)=y}^{x(0)=x} \mathcal{D}x(\tau) e^{i \int_0^T d\tau p \cdot \dot{x}}, \quad (44)$$

where, by the previously displayed method, we have traded $\dot{x}_\mu(\sigma)$ with γ_μ . Performing the functional integral over $\mathcal{D}x(\tau)$, dispensing with the resulting string of delta functions via the $\mathcal{D}p(\tau)$ integration and reading off the Fourier transform we obtain, after substituting f_μ from (43),

$$\tilde{G}_1(q|u) = -g^2 \frac{1}{4\pi^{D/2}} c_F \left(\frac{\mu}{\lambda} \right)^{4-D} \Gamma\left(\frac{D}{2} - 1\right) \int_0^\infty dT T^{D-4} e^{-T(m+iq \cdot \gamma)} \\ \times (\gamma \cdot u + u^2) u^{2-D} (4-D) \int_0^T d\tau_1 \tau_1^T d\tau_2 |\tau_2 - \tau_1|^{2-D} \\ = g^2 \frac{1}{4\pi^{D/2}} c_F \left(\frac{\mu}{\lambda} \right)^{4-D} \Gamma\left(\frac{D}{2} - 1\right) \frac{1}{D-3} u^{2-D} (\gamma \cdot u + u^2) \frac{1}{i\gamma \cdot q + m}. \quad (45)$$

Note the absence of singular terms in the above expression. The latter are exclusively associated with straight line paths.

Now, as we have already remarked, cf. comments following (17), in the strict eikonal limit the "heavy" particle is on its mass shell and its only relevant attribute is the four-velocity u_μ which satisfies the relation $q_\mu = mu_\mu$. But as long as we are not far removed from the eikonal set up we can still set $u_\mu = q_\mu/m$, only now u_μ is not bound by the condition $u^2 = 1$ and q^2 is not necessarily on mass shell. Noting that in the Euclidean version the corresponding identification is $u_\mu = -iq_\mu/m$, we obtain, upon insertion into (45), the four-dimensional result

$$\tilde{G}_1(q) = \frac{\alpha_s}{\pi} \left(-\frac{m^2}{q^2} \right) \left(-i\frac{\gamma \cdot q}{m} + \frac{q^2}{m^2} \right) \frac{1}{i\gamma \cdot q + m}. \quad (46)$$

The Minkowski version of the above reads

$$\tilde{G}_{1,Mink}(q) = \frac{\alpha_s}{\pi} \left(\frac{i}{\gamma_\mu q^\mu - m + i\epsilon} \right) \frac{m^2}{q^2} \left(\frac{\gamma_\mu q^\mu}{m} - \frac{q^2}{m^2} \right). \quad (47)$$

The next order contribution involves lengthy, *albeit* not terribly tricky computations. Sparing the reader of any in between detail we quote the final result, in four dimensions, which includes all contributions up to second order in the velocity expansion,

$$\begin{aligned} \tilde{G}_{2,Mink}^{(2)}(q) = & \frac{i}{\gamma_\mu q^\mu - m + i\epsilon} \left[1 + \frac{\alpha_s}{\pi} c_F \frac{m^2}{q^2} \frac{1}{D-4} \right] + \frac{i}{\gamma_\mu q^\mu - m + i\epsilon} \\ & \times \frac{\alpha_s}{\pi} c_F \left\{ \frac{m^2}{q^2} \left(\frac{\gamma_\mu q^\mu}{m} - \frac{q^2}{m^2} \right) - \frac{1}{2} \left[\frac{m^2}{q^2} \left(\frac{\gamma_\mu q^\mu}{m} - \frac{q^2}{m^2} \right) \right]^2 + \dots \right\}. \quad (48) \end{aligned}$$

Notice that the expression in the curly brackets vanishes if the four momentum q_μ is on mass shell, in which case the propagator reduces to (31). The finite off mass shell corrections are a genuine new feature stemming from our approach. Practical utilization of the above result has to await similar considerations to higher order Green's functions. We hope that we shall soon be in position to report on progress in this direction with respect to the vertex function.

6. Concluding Remarks

In this paper we have employed the worldline formalism in order to factor out the IR sector of a generic gauge theory with massive, spin- $\frac{1}{2}$ matter fields (only a single mass scale assumed). The overall strategy was based on two specifications¹⁰: (a) Quenched setting, (b) path integral calculations to be based around straight line contours. The first instruction assures us that the photons/gluons are not overly energetic (they cannot produce fermion pairs), while the second sets up the spin- $\frac{1}{2}$ particle's mass as a cutoff.

The projected onto sector of the full microscopic theory has its own rules: UV divergencies, anomalous dimensions, IR limit, etc. Working at the level of the propagator and the vertex function to order α_s for non-abelian theories (in QED it is possible to work at all orders [23]), we were able to retrieve quantities, such as anomalous dimensions, with relative ease and through straight forward manipulations. Where computations *do* get harder is in connection with the off mass shell expansion, (our present efforts did not go further than the propagator) which involves going *beyond* the strict eikonal approximation. The compensating reward is that they produced concrete final expressions. Even the possibility for such expansions had not been envisaged before -on the basis, one might guess, that the very nature of the eikonal line is conceived to be purely classical. In the worldline scope, by contrast, it simply enters IR descriptions as the dominant contour in the path integral whose contribution *can* be improved upon through the inclusion of well defined correction terms (deformations-velocity expansion).

Despite an initial intimidation from the starting path integral expressions, once familiarity sets in one soon discovers that practical computations often become simpler and much more economical in the worldline, versus other, formulation. The work in Refs. [21], [11] and [12] especially attests to that. In this connection, we should stress that going to higher order *perturbative* corrections is, at least, no more difficult than the corresponding task in the operator formalism. For example, the two-loop correction for Γ_{cusp} in [26] can be carried

¹⁰Our feeling is that (b) should automatically imply (a) but, as long as we have no formal proof, we are obliged to keep the two instructions separate.

over to our case (after all, a Wilson loop is nothing but a Euclidean spacetime closed contour traced by infinitely heavy fermions) without any modification. On the other hand, the ability of the worldline perturbative expansion to accomodate many Feynman diagrams at once, which is the gist of the work in the references mentioned three sentences earlier, encourages the expectation that one should be able to cut corners by using the worldline approach.

In summary, simplicity along with conceptual clarity and, most importantly, capacity for new type of computations, such as the off mass shell expansion introduced in this paper, are virtues which can be attributed to the worldline casting of field systems in general. With respect to IR descriptions, in particular, we intend to pursue the specific line of investigation in this work to off mass shell finite vertex corrections and subsequently test its applicability to physical situations where IR contributions become important (e.g., phase space edges where gluon emission is suppressed or where it becomes overwhelming).

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Appendix

In this appendix we trace the steps which furnish quantity I_D entering (35) by the second equality in the same equation and subsequently establish (36). To begin, we bring, via some trivial manipulations on the variables ($\tau_1 \rightarrow \frac{|u|}{\sigma}(\tau_1 - s + \sigma)$, $\tau_2 \rightarrow \frac{|u|}{\sigma}(\tau_2 - s)$), I_D in the form

$$I_D = \delta^{4-D} T^{4-D} \theta \int_0^{|u|} d\tau_1 \int_0^{|u|} d\tau_2 (\tau_1^2 + \tau_2^2 + 2\tau_1 \tau_2 \theta)^{\frac{2-D}{2}}, \quad (\text{A.1})$$

where the various symbols stand as they have been defined in the text.

Using the identity

$$\left(\tau_1 \frac{\partial}{\partial \tau_1} + \tau_2 \frac{\partial}{\partial \tau_2} \right) (\tau_1^2 + \tau_2^2 + 2\tau_1 \tau_2 \theta)^t = 2t(\tau_1^2 + \tau_2^2 + 2\tau_1 \tau_2 \theta)^t, \quad (\text{A.2})$$

we can reduce the double integral into a sum of two single ones. The latter combine to give

$$I_D = \frac{2\theta}{4-D} \delta^{4-D} T^{4-D} |u| \int_0^{|u|} d\tau (\tau^2 + |u|^2 + 2\tau|u|\theta)^{\frac{2-D}{2}}, \quad (\text{A.3})$$

which, after the rescaling $\tau = x|u|$, leads to the last leg of (35).

Our next task is to establish (36). Noting that $\frac{2-D}{2} = -1 + \epsilon$ we write

$$J_D(\theta) \equiv \int_0^1 dx (x^2 + 2x\theta + 1)^{-1+\epsilon} = \frac{1}{\theta(1-\epsilon)} \int_0^\infty d\xi \xi^{-\epsilon} \int_0^1 dx e^{-\xi(x^2 + 2x\theta + 1)}. \quad (\text{A.4})$$

Changing variables according to $x = y - \theta$ we obtain

$$J_D(\theta) = \frac{1}{\Gamma(1-\epsilon)} \int_0^\infty d\xi \xi^{-\epsilon} e^{-\xi(1-\theta^2)} \int_\theta^{1+\theta} dy e^{-\xi y^2}. \quad (\text{A.5})$$

Our next move is to make the change $y \rightarrow -\xi y^2$ in the last integral, whereupon we determine

$$\begin{aligned} \frac{1}{2\xi^{\frac{1}{2}}} \int_{\xi\theta^2}^{\xi(1+\theta)^2} dy y^{-\frac{1}{2}} e^{-\xi y^2} &= \frac{1}{2\xi^{\frac{1}{2}}} \left[\int_{\xi\theta^2}^\infty dy y^{-\frac{1}{2}} e^{-\xi y^2} - \int_{\xi(1+\theta)^2}^\infty dy y^{-\frac{1}{2}} e^{-\xi y^2} \right] \\ &= \frac{1}{2\xi^{\frac{1}{2}}} \left[\Gamma\left(\frac{1}{2}, \xi\theta^2\right) - \Gamma\left(\frac{1}{2}, \xi(1+\theta)^2\right) \right], \end{aligned}$$

where $\Gamma(a, b)$ denotes an incomplete Gamma function. But

$$\int_0^\infty d\xi \xi^{-(1+\frac{1}{2})} \Gamma\left(\frac{1}{2}, \xi\theta^2\right) = 2\theta \frac{\Gamma(1-\epsilon)}{1-2\epsilon} F\left(1, 1-\epsilon; \frac{3}{2}-\epsilon; 1-\theta^2\right) \quad (\text{A.6})$$

and

$$\int_0^\infty d\xi \xi^{-(1+\frac{1}{2})} \Gamma\left(\frac{1}{2}, \xi(1+\theta)^2\right) e^{-\xi(1-\theta^2)} = \frac{2(1+\theta)}{1-2\epsilon} \frac{\Gamma(1-\epsilon)}{[2(1+\theta)]^{1-\epsilon}} F\left(1, 1-\epsilon; \frac{3}{2}-\epsilon; \frac{1-\theta^2}{2(1+\theta)}\right). \quad (\text{A.7})$$

Substituting in the full expression for J_D one recovers eq (36) given in the text.

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