MULTI-PARTICLE PHENOMENOLOGY

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1. INTRODUCTION

The many-body problem in hadron collisions occupies a somewhat different position than in other branches of physics. Normally, either because of the weak coupling or because of the low energies involved, the many-body aspects can, to a good approximation, be isolated from what are believed to be the basic dynamics of the system. Thus, in the theory of nuclear structure, we may first study two-nucleon interactions either in the deuteron or in proton-proton scattering. The low energies of only a few MeV ensure that corrections due to multi-particle channels are unimportant. Then, deducing from these the two-nucleon potential, we may use it as input, and proceed to study many-body phenomena such as collective modes of excitation in complex nuclei. In other words, the main interest in the many-body problem is not so much the basic dynamics, but the phenomena intrinsic to the complexity of the system.

In hadron collisions, however, the situation is different. Because of both the high energies and the strong coupling involved, particles are easily created and destroyed. At 10 GeV/c, for example, identifiable two-body events account for no more than 20% of the total cross-section; the rest is made up of multi-particle final states. Since all these reactions are intimately connected by unitarity, even the simplest two-body collisions would be strongly influenced by what happens in the many-particle channels. Indeed, it is currently believed that the dominant contribution to high energy elastic scattering, known loosely as the Pomeron exchange, is nothing else than the shadow of these many-body channels. It is clear then that only when we have some knowledge of many-body reactions that the basic dynamics of hadron collisions can be fully understood. Indeed, it would be more reasonable to regard many-body collisions as an integral part of hadron dynamics. The division between two-body and many-body is merely a relic of history and is largely artificial.
Unfortunately, the intrinsic complexity of the many-body problem is not an easy obstacle to overcome. In spite of the exciting progress made in the last few years, our knowledge of multi-particle processes remains at best qualitative. The theoretical models which are available make few predictions. However, they do provide us with an increasingly clearer picture of multi-particle phenomena. In particular, the recent development in Regge type models, including the crossing symmetric version suggested by Veneziano, has yielded a means of correlating a wide variety of experimental data in terms of a few phenomenological parameters and it is the picture resulting from this that I wish in these lectures to describe.

Instead of following the historical development which would be instructive but is likely to make my lectures rather dull, I have preferred to describe the picture to you as I see it now. What I hope to achieve is a sort of an artist's impression rather than a physical theory. Although my theme is multi-particle phenomena, I wish to impress on you the fact that there is no essential difference between these and the better known two-body phenomena.

I shall therefore begin with two-body collisions, then work slowly towards reactions of higher complexity. The picture will fade in clarity as I go along, and I shall be forced to make cruder and cruder approximations. There will be many gaps also in the picture which I shall be unable to fill at present. I shall then have to rely on your indulgence and on my imagination.

2. TWO-BODY COLLISIONS

From the intense experimental and theoretical work done in the last few years, one has been able slowly to piece together a qualitative picture of the two-hadron collision amplitude. I shall summarize in this Section some of those points relevant for our subsequent discussion of the multi-particle problem.
Consider first processes where quantum numbers other than those of the vacuum are exchanged. Experiment tells us that at low energies, say $E_{\text{lab}} \sim 2 \text{ GeV}$, the reaction amplitude is dominated by resonances. When one plots the masses and spins of these resonances on a Chew-Frautschi plot, it is found that resonances with the same internal quantum numbers lie on approximate straight lines. In this low energy region, the amplitude is often well approximated by the resonances alone, say as the superposition of several Breit-Wigner states.

As the energy increases beyond 6 GeV, however, the resonance structures so prominent at low energies disappear. The cross-section takes on a smooth power behaviour of the form $E_{\text{lab}}^{-n}$ with the power $n$ strongly characteristic of the exchanged quantum numbers. The amplitude then is found to be well represented by the exchange of a few Regge poles.

Between 2 and 6 GeV laboratory energies the situation is not so straightforward. The cross-section looks like some sort of interpolation between the resonance and Regge regions just discussed. A priori, it is not clear here which representation is better - whether in terms of direct channel resonances or in terms of the crossed channel Regge exchange, or even as a superposition of both. The question has been considerably clarified by recent phenomenological studies using finite energy sum rules, which exploit the analytic properties of the amplitude. These analyses suggest that the two representations are essentially equivalent, each crudely approximating the other. Thus, in terms of the resonances, the smooth behaviour of the Regge amplitude is built up by the overlap of increasingly wider resonances, whereas in the Regge language, the loops in Argand diagrams, so characteristic of resonances, are an intrinsic property of the Regge phase. As we shall see, this so-called "principle of duality" \(^1\), though only loosely formulated, is very important for our later discussion. Its significance is particularly enhanced when coupled with crossing symmetry.
Next we consider reactions such as elastic scattering where the exchanged quantum numbers correspond to those of the vacuum. As we have already mentioned, there are strong theoretical indications that these reactions at high energy are dominated by the shadow effect of inelastic channels of which we know very little. If one insists on representing this effect in terms of Regge poles, it is found empirically that the effective Pomeranchuk trajectory has little in common with normal trajectories such as the $\rho$. Thus, for example, the experimental fact that the diffraction peak in elastic scattering either shrinks slowly or not at all, requires that the Pomeron must have a small slope ($\alpha'_p < 0.5 \text{ GeV}^{-2}$), quite distinct from all other trajectories which have a common slope of $\sim 1 \text{ GeV}^{-2}$. Such arguments strongly suggest that the Pomeron is really something outside the framework of "duality" and Reggeism. In our later discussion, therefore, we shall do our best to avoid vacuum exchanges altogether. However, when we are forced to be more explicit about this shadow effect, we shall represent it by an effective Pomeranchuk trajectory to be determined empirically.

Besides the shadow effect just discussed, elastic scattering and other reactions with vacuum exchange presumably also contain a part due to ordinary Regge exchanges. In order to study this it has been suggested that perhaps to first approximation the amplitude may be written as $A = A_{\text{Pom}} + A_{\text{Regge}}$, where $A_{\text{Regge}}$ is a "normal" dual Regge amplitude similar to that for reactions with non-vacuum exchanges. The assumption has been tested against experiment on several occasions especially recently on $\pi N$ scattering, and was found to be quite reasonable $^1$. We shall therefore also accept this tentatively for our subsequent considerations.

To summarize, we list the following properties supposedly satisfied by the two-hadron collision amplitude:

i) analyticity

ii) crossing symmetry

iii) resonances on linear trajectories

iv) Regge asymptotic behaviour
v) duality
vi) Pomeron contribution, approx. additive
vii) unitarity.

Knowing this, it is tempting for phenomenological purposes to construct a function with just these properties. The task is however far from trivial and no one has yet succeeded. In fact, the conditions are so stringent that many theoreticians would be willing to believe that if one finds a function with all the listed properties, it is most likely the exact amplitude. Failing this, then, one may try to relax some of the conditions. In view of the difficulty connected with unitarity and the shadow effect, which seems untractable at the present stage of our knowledge, it seems worth while to relax at first the conditions (vi) and (vii) and seek a solution to conditions (i)-(v). The result is then the famous Veneziano model 2).

To illustrate how the model works, consider first the idealized situation where two identical scalar particles scatter via the exchange of a Regge trajectory containing again the same scalar particle. Assuming the trajectory to be strictly linear, we write for the trajectory function:

\[ \alpha(x) = \alpha_0 + \alpha' x \]  

(1)

where, as usual, \( \alpha_0 \) is the intercept and \( \alpha' \) the slope of the trajectory. Let \( p_r \) (\( r = 1,2,3,4 \)) denote the four-momenta of the external particles as in Fig. 1 where for convenience all the momenta \( p_r \) have been taken as incoming, so that the Mandelstam variables \( s, t \) and \( u \) are given as

\[ s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \]
\[ t = (p_2 + p_3)^2 = (p_1 + p_4)^2 \]
\[ u = (p_1 + p_3)^2 = (p_2 + p_4)^2 \]  

(2)
In this notation then the Veneziano amplitude for our idealized scalar-scalar scattering takes the following form:

\[ T = V(s,t) + V(t,u) + V(u,s) \]  

where

\[ V(s,t) = \beta B_4 (-\alpha(s), -\alpha(t)) \]
\[ V(t,u) = \beta B_4 (-\alpha(t), -\alpha(u)) \]
\[ V(u,s) = \beta B_4 (-\alpha(u), -\alpha(s)) \]

(4)

The function \( B_4 \) occurring in (4) is the beta function of Euler, which may be defined either in terms of the more familiar gamma function

\[ B_4(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \]  

or equivalently by the integral representation:

\[ B_4(x, y) = \int_0^1 d\mu \, \mu^{x-1} (1-\mu)^{y-1} \]  

(5)

(6)

The constant \( \beta \) in (4) is just for normalization. We shall show that this function \( T \) does have all the desired properties i) - v) listed above.

Since all the lines are identical, crossing symmetry here is equivalent to the statement that the amplitude is symmetric under any of the \( 4! = 24 \) permutations of the external lines. For convenience, we shall distinguish between i) cyclic permutations, e.g., \((1,2,3,4) \rightarrow (2,3,4,1)\), and reversions, e.g., \((1,2,3,4) \rightarrow (4,3,2,1)\); and ii) other permutations, such as \((1,2,3,4) \rightarrow (1,3,2,4)\). Orderings of the four lines which are related by type i) permutations we shall
regard as equivalent. It is then easily checked that the 24 different orderings are divided into three equivalent classes, which may be represented by the three diagrams of Fig. 2, where we adopt the convention of always counting in the counter-clockwise direction. One then notices that the three diagrams in Fig. 2 are related respectively to the three terms in (3) in such a way that the two arguments in each term are just the energy variables corresponding to the two Mandelstam channels which can occur in that diagram without changing the ordering of the external lines. Thus, for the first diagram (a), we have \(V(s,t)\) where \(s\) corresponds to the channel \(12 \to 34\), and \(t\) corresponds to the channel \(23 \to 41\). Because of the symmetry of the beta function in its arguments, e.g., \(V(s,t) = V(t,s)\) it is then clear that each diagram is invariant under a permutation of type i). Moreover, under a permutation of type ii), the diagrams transform into one another leaving the total amplitude (3) invariant. Crossing symmetry of the amplitude (3) is then fully established. The crossing properties can actually be seen quite easily in this particular case. I have, however, deliberately chosen to follow a somewhat elaborate procedure with the intention of generalizing the arguments later for multi-particle reactions.

Next, to study the analytic properties of \(T\) in (3), we first note that the gamma function \(\Gamma(z)\) is analytic in \(z\) on the whole complex \(z\) plane apart from simple poles at \(z = 0, -1, -2, ...\) and that \(\Gamma(z)\) has no zeros anywhere. Consider then \(V(s,t)\) in (4); being a ratio of \(\Gamma\)'s, \(V(s,t)\) must be analytic except possibly where the \(\Gamma\) functions in the numerator have poles, i.e., at \(\alpha(s) = 0,1,2,3, ...\) and \(\alpha(t) = 0,1,2,3, ...\) It does not have double poles, however, since when \(\alpha(s)\) and \(\alpha(t)\) are both non-negative integers, the denominator \(\Gamma(-\alpha(s) - \alpha(t))\) also has a pole to cancel it. Similar properties will hold for \(V(t,u)\) and \(V(u,s)\) and consequently also for \(T\).
What then are these poles in \( T \)? They must represent the resonances lying on our trajectory (1). Since the amplitude is completely symmetric, we need consider only one Mandelstam channel, say, e.g., the s channel. At \( \alpha(s) = \ell \), and \( t \) and \( u \) both \(<0\), there is no pole in \( V(t,u) \) but a pole in both \( V(s,t) \) and \( V(u,s) \). To find the residue of \( V(s,t) \) at \( \alpha(s) = \ell \), we can use the recurrence relation: \( \Gamma'(z+1) = z \Gamma'(z) \) repeatedly. It is then easy to see that the residue is a polynomial of degree \( \ell \) in \( t \). Remembering that \( t = -2k_s^2 (1 - \cos Q_s) \), one sees that the residue is a polynomial of degree \( \ell \) in \( \cos Q_s \). Hence the pole must represent some particle with maximum angular momentum \( \ell \). A similar argument holds for \( V(u,s) \) with \( u \) instead of \( t \). Our poles therefore do indeed represent resonances lying on a linear trajectory as required.

One notes that in terms of s channel variables, \( u = -2k_s^2 (1 + \cos Q_s) \), with a change in sign in \( \cos Q_s \) compared with \( t \). This means that the leading term in \( \cos Q_s \) in \( \text{Res} \ V(u,s) \) is \((-1)\ell\) times the leading \( \cos Q_s \) term in \( \text{Res} \ V(s,t) \). Thus for odd \( \ell \), the leading term would cancel in the full amplitude (3). This just means that our internal trajectory in the s channel has even signature.

In taking the Regge limit of the formula (3), say \( s \to \infty \) at fixed \( t \), special care must be taken, for, as we have just shown, the function \( T \) has an infinite sequence of poles in the real \( s \) axis so that the limit \( s \to \infty \) on the real axis cannot strictly exist. We shall therefore define instead as the "Regge limit" the following: \( \sigma \to \infty \) for \( s = \sigma + i \epsilon \sigma \) and some \( \epsilon > 0 \). Namely, instead of approaching \( \infty \) on the \( s \) plane strictly along the real axis, we shall approach \( \infty \) along a ray at an infinitesimal angle to the real axis. That this can indeed be taken as the Regge limit is non-trivial and needs some justification. I shall return to this later when we discuss the problem of unitarity. Once this is accepted, however, it is not difficult to show using standard textbook properties of the gamma function that the formula (3) has the limit.
\[
\lim T \cong -\beta \frac{\pi}{\sin \pi \alpha(t)} \cdot \frac{1 + e^{-i\pi \alpha(t)}}{\alpha(t)} (\alpha' s) \omega(t) 
\]

which is the standard Regge formula for the exchange of an even signature trajectory with a special form of the residue function.

Once having proved the Regge behaviour and also the existence of resonance poles in the amplitude, "duality" becomes an automatic consequence, since the same function has been shown explicitly to contain both the s channel resonance poles and t channel Regge exchange amplitude. Indeed, it may be said that the loosely formulated "principle of duality" receives in the Veneziano model its first concrete realization. All the desired properties i)-v) for the formula (3) have now then been verified.

Before going further to discuss such important questions as unitarity of the Veneziano model, it is worth pointing out for later reference that most of the properties of (3) just derived using the more familiar gamma functions can also be deduced directly from the integral representation (6) of \( B_4 \). One verifies easily that \( B_4 \) is symmetric in \( x \) and \( y \) by a change of integration variable \( u \to 1-u \). Moreover, one sees that \( B_4 \) is analytic in \( x \) and \( y \) except where the integral (6) diverges. Now the integral will diverge for \( x \leq 0 \) around \( u=0 \) (by symmetry for \( y \leq 0 \) at \( u=1 \)). To study the singularities there, we may expand the integrand in a Taylor series in \( u \), thus

\[
B_4(x,y) \sim \int_0^1 du \ u^{x-1} \left[ \sum \ell \ c_{\ell} (y) u^{\ell} \right] 
\]

where

\[
c_{\ell} (y) = \frac{1}{\ell!} \left. \frac{\partial^{\ell} \left[ (1-u)^{y-1} \right]}{\partial u^{\ell}} \right|_{u=0} 
\]
Integrating (8) term by term, one has

\[ B_4(x, y) \sim \sum_{\ell = 0}^{\infty} \frac{C_\ell(y)}{x + \ell} \]

from which it is clear that \( B_4 \) has a sequence of poles at \( x = 0, -1, -2, \ldots \). The residue at \( x = -\ell \) is just \( C_\ell(y) \), which by (9) is clearly a polynomial of degree \( \ell \) in \( y \), as required. By symmetry, then, the analytic properties in \( y \) are similar. It is particularly interesting to note how double poles are avoided in the integral representation (6). As we have seen, the poles in \( x \) occur around \( u = 0 \) while those in \( y \) around \( v = 1 - u = 0 \). Clearly, the two conditions are incompatible and cannot therefore occur simultaneously. It turns out that this simple observation is the key to later generalizations to many-body reactions.

In spite of its many beautiful features, the Veneziano model is not free from diseases. By far the most serious of these are i) the question of unitarity, and ii) the problem of ambiguities.

i) The model amplitude (3) is not unitary. This can easily be seen since the function has only poles on the real axis, whereas a unitary amplitude should have cuts also corresponding to thresholds of elastic and inelastic channels, while its poles corresponding to resonances should move off the real axis on to the unphysical sheet of its Riemann surface. For this reason, when taking the Regge limit of (3), we had to simulate this by artificially approaching \( \infty \) along a ray at an angle to the real axis. A connected question is that of the Pomeranchuk "trajectory". On the one hand we have good reason to believe that the Pomeron really represents the shadow effect of inelastic channels via unitarity. On the other, we know that the Pomeron is typically non-dual and therefore cannot be contained in such an amplitude as (3). Attempts have indeed been made to "unitarize" the Veneziano model but have as yet met with no success. For phenomenological purposes, one may simulate the effects of unitarity by artificially
introducing an imaginary part to the trajectory function $\alpha$. The poles of (15) will then move off the real axis, but their residues will now in general not be polynomials in $\cos \theta$, and our arguments given above will be only approximately valid.

ii) Consider the modification

$$V(s, t) \rightarrow V(s, t) + V'(s, t)$$

$$V'(s, t) = \beta' \frac{\Gamma(m - \alpha(s)) \Gamma(n - \alpha(t))}{\Gamma(m + n + \beta - \alpha(s) - \alpha(t))}$$

(11)

where $m, n, p$ are all non-negative integers. It can readily be seen that the new $V(s, t)$ when substituted into (3) will also possess all the properties i)-v) of the original Veneziano amplitude. It follows then that any convergent series of such terms of $V'$ when added to $V$ will give an equally good amplitude as far as properties i)-v) are concerned. These are then the so-called satellite terms. They do not in general modify the leading resonances or the leading asymptotic behaviour of the amplitude and may therefore hopefully not be too important in a first approximation for phenomenological purposes. However, the possibility of introducing satellites represents such a large degree of ambiguity, that the predictive power of the model is drastically reduced.

Because of these difficulties, theoreticians are not quite decided how seriously to take the Veneziano model. For the purpose of these lectures, however, it suffices for us to accept it as a concise formula for summarizing a lot of nice properties which we know we need in any case to describe the experimental data. Around it we hope to build a phenomenological model for a large class of hadronic processes.
Before we proceed to generalize the ideas given in this Section to more complex reactions, let us first check how they work in the two-body case in practice. For doing this, we consider the simplest realistic example, namely \( \pi \pi \) scattering. The amplitude (3) considered above was intended for the ideal case of neutral scalar-scalar scattering. Moreover, the internal trajectory contains a member with \( J = 0 \), (the residue of the first pole being a constant). For application to \( \pi \pi \) scattering, therefore, some modifications are necessary. Apart from isospin factors which are easy to work out and need not concern us here, the main modification is due to the factor that the dominant trajectory in \( \pi \pi \) scattering is the \( \rho \) trajectory the lowest member of which has \( J = 1 \). We need a formula, therefore, where the first pole in \( s \) has a residue linear in the momentum transfer \( t \) or the scattering angle \( \cos \theta \). This can be done by writing instead of (4) \(^3\)

\[
V(s, t) = \rho (\alpha(s) + \alpha(t) - 1) B_4 (1 - \alpha(s), 1 - \alpha(t))
\]

and similarly for \( V(t, u) \) and \( V(u, s) \). The function (12) is still symmetric in \( s \) and \( t \). Its poles now occur at \( \alpha(s) = 1, 2, 3, ... \), where the first pole at \( \alpha(s) = 1 \) is linear in \( t \) because of the extra factor in front. We note that such an adaptation of the idealized formula (3) to realistic system, although easy in this particular case, is in general still unsolved, and may well be a major problem in future development of the model.

The formula (12) has been applied by Lovelace \(^3\) to \( \bar{p}n \) annihilation at rest into \( \pi^+ \pi^- \pi^- \), where the \( \bar{p}n \) system has the same quantum number as a \( \pi^- \) and was treated as a pion off-mass shell. The effect of unitarity was simulated by giving the trajectory function \( \alpha \) an imaginary part fitted to the \( \rho \) width. With then essentially only one parameter, one obtains the result shown in Fig. 3, which compares quite favourably with experiment. Although there is some doubt about the real significance of this fit especially concerning the method of continuing the pion off-mass shell, this is mainly for experts. For us, it suffices to note that the general features of the Dalitz plot are reproduced, thus supporting the basic notions introduced in this Section.
3. THREE-BODY FINAL STATES

The first obvious difference between these and two-body collisions is the more complicated kinematics. Instead of two for the two-body case, one now needs five variables to describe completely the amplitude. But are there any essential differences in the basic dynamics of the system? In order to answer this question we shall turn again to experiment for inspiration. In order to avoid the complex problem of shadow scattering, we shall consider at first again only those reactions where quantum numbers other than those of the vacuum are exchanged.

At low incoming energies, the cross-section for three-body final states are not very different from two-body collisions apart from trivial threshold effects. As a function of energy, the cross-section shows prominent resonance structures. It is very likely that here also the amplitude is well represented by a superposition of Breit-Wigner shapes, the only difference being that the resonances, instead of decaying into two, now decays into three final particles.

As the energy increases, cross-sections again become fairly smooth as functions of the energy, with an energy dependence similar to that of two-body collisions. However, because of the larger degree of freedom, the more complex features of the amplitude are revealed. As an example, consider the reaction

$$k^- p \rightarrow \pi^- \pi^+ \Lambda$$

One notes that in the effective mass plot of say $\pi^+ \Lambda$, the low mass region is dominated by hyperon resonances; similarly, in the effective mass plot of $\pi^+ \pi^-$, the low mass region is dominated by the $\rho$. If one selects only those events say in the $\gamma_1^*(1385)$ region and study these for varying incoming energy, one finds that the quasi two-body amplitude Reggeizes behaving in every way like a proper two-body amplitude, as shown in Fig. 4.
Next, one may ask what happens when one looks at those events where the $\pi^+ \Lambda$ effective mass becomes large also. Here the situation is less clear, since the cross-section becomes rather small. None the less, one can claim quite definitely that events in this region show double-peripheral features, with $\pi^-$ going forward, $\Lambda$ backward, and $\pi^+$ remaining somewhere in between. Indeed, it will not be a bad idea to describe events in this region by a double-Regge diagram as shown in Fig. 5.

In short, the experimental indications are that the main features of two-body collisions which we have discussed in the last Section are here repeated. The only difference is that here the resonance Regge picture seems to apply not only for the incoming energy but also for two-body effective masses of the final state. Whenever the masses or energies are low, we have resonance dominance; when these are high, we have Reggeization. The fact that the rule seems to apply also for effective masses in the final state is not at all surprising because of crossing symmetry. This states that the same analytic function has to describe all the reactions with the same five external lines, whichever particles are taken as the incoming particles. If we were able to study experimentally $\pi^+ \Lambda$ scattering, no one will be surprised to see the low energy region dominated by the hyperon resonance $Y_1^*(1385)$. In theoretical language, this means at $m_{\pi^+ \Lambda} = 1385$ there is a pole in the amplitude. The same function, however, is supposed by crossing to describe also the reaction (13). The pole at $m_{\pi^+ \Lambda} = 1385$ will then naturally give rise to a peak corresponding to $Y_1^*$ in the effective mass plot.

In view of these considerations, it is tempting to take over completely the listed properties i)-v) for two-body collision and assume that the same properties hold as well for processes with five external lines. Better still, one could try to construct a generalization of the Veneziano model which will embody all these properties.
To do this, let us consider first again the idealized case when all five external particles are scalars, interacting via the exchange of the same Regge trajectory containing the scalar particle. In analogy to the four-line case we first write the amplitude as the sum of 12 terms where the 12 terms correspond to the 12 non-equivalent orderings of the five external lines, thus

\[ T = \sum_P V(1, 2, 3, 4, 5) \]  \hspace{1cm} (14)

We require that each term in (14) is invariant under a cyclic permutation or reversion of its indices. Any permutations other than these will carry a term in (14) into another term in the sum so that complete crossing symmetry of (14) will be guaranteed. Provided these conditions are satisfied, we need consider then only one such term, say, for example \( V(1, 2, 3, 4, 5) \).

What other properties do we need for \( V(1, 2, 3, 4, 5) \)? It should be analytic apart from poles corresponding to resonances which should lie on a linear trajectory. Such poles should occur in every Mandelstam channel which can be formed without changing the ordering of the external lines. Moreover, in the proper limit the function should have double Regge behaviour.

Such a function can indeed be constructed \(^4\) by a generalization of the integral representation given in (6). First we introduce the following generalized Mandelstam variables

\[
\begin{align*}
S_{42} &= (p_1 + p_2)^\pm = (p_3 + p_4 + p_5)^\pm, \\
S_{23} &= (p_2 + p_3)^\pm = (p_4 + p_5 + p_1)^\pm, \\
S_{34} &= (p_3 + p_4)^\pm = (p_5 + p_1 + p_2)^\pm, \\
S_{45} &= (p_4 + p_5)^\pm = (p_1 + p_2 + p_3)^\pm, \\
S_{51} &= (p_5 + p_1)^\pm = (p_2 + p_3 + p_4)^\pm
\end{align*}
\]  \hspace{1cm} (15)
It is readily checked that these are independent, being unrelated by energy momentum conservation. They can therefore be used for a complete description of our amplitude.

In addition we introduce auxiliary variables \( u_{12}, u_{23}, u_{34}, u_{45}, \) and \( u_{51} \) which will be the generalizations of the integration variables \( u \) and \( v = 1 - u \) in the formula (6). We shall call the variable \( u_{ij} \) the conjugate to the variable \( s_{ij} \). These variables \( u_{ij} \) are not all independent - we shall restrict them by the following algebraic conditions:

\[
\begin{align*}
\mu_{12} &= 1 - \mu_{51} \mu_{23} \\
\mu_{23} &= 1 - \mu_{12} \mu_{34} \\
\mu_{34} &= 1 - \mu_{23} \mu_{45} \\
\mu_{45} &= 1 - \mu_{34} \mu_{51} \\
\mu_{51} &= 1 - \mu_{45} \mu_{12} .
\end{align*}
\]

(16)

the significance of which will soon be apparent. Although (16) represents altogether five equations, not all of them are independent. In fact, it can be readily checked by direct substitution that all five equations can be solved in terms of two independent variables, e.g., \( u_{12} \) and \( u_{45} \):

\[
\begin{align*}
\mu_{23} &= \frac{1 - \mu_{12}}{1 - \mu_{12} \mu_{45}} , \\
\mu_{34} &= \frac{1 - \mu_{45}}{1 - \mu_{12} \mu_{45}} \\
\mu_{51} &= 1 - \mu_{12} \mu_{45} .
\end{align*}
\]

(17)

(Notice that the two independent variables must not share a common index.)
Define now

\[ V(1, 2, 3, 4, 5) = \beta B_5 (-\alpha_{12}, -\alpha_{23}, -\alpha_{34}, -\alpha_{45}, -\alpha_{51}) \]  

(18)

where

\[ \alpha_{ij} = \alpha_0 + \alpha' S_{ij} \]  

(19)

and

\[ B_5 (x_{12}, x_{23}, x_{34}, x_{45}, x_{51}) \]

\[ = \int_0^1 du_{12} \int_0^1 du_{45} \frac{1}{u_{51}} \prod_{i<j} u_{ij}^{x_{ij} - 1} \]

(20)

In (20) the product is to be carried over all five \( u_{ij} \), where \( u_{23}, u_{34} \) and \( u_{51} \) are given by (17) in terms of \( u_{12} \) and \( u_{45} \). The function \( B_5 \) is called the generalized beta function of the fifth order, and bears a close resemblance to the ordinary beta-function given in (6). We shall show that \( V(1, 2, 3, 4, 5) \) so defined does satisfy all the requirements of the Veneziano model.

First we note that the integral (20) is invariant under a cyclic change of the indices provided we make also a similar change in the integration variables, thus

\[ u_{12} \rightarrow u_{12} = u_{35} ; \quad u_{45} \rightarrow u_{45} = u_{51} \text{ etc.} \]

Clearly, since (16) is cyclic symmetric, solving \( v_{23}, v_{34} \) and \( v_{51} \) in terms of \( v_{12} \) and \( v_{45} \) would give the same solution as (17) with \( u \) replaced by \( v \). Thus the product in (20) remains invariant.

Changing now the integration variables to \( v_{12} = u_{23} \) and \( v_{45} = u_{51} \), we note first that the limits of integration remain unchanged. Moreover, the Jacobian of the transformation gives
\[
\frac{\omega(u_{12}, u_{45})}{\omega(v_{12}, v_{45})} = \frac{u_{51}}{u_{12}} = \frac{u_{51}}{v_{51}}
\]

which leaves then the whole integral in (20) invariant. A similar result can be shown for reversions in ordering of the five external lines. This then establishes the crossing symmetry of the full amplitude in (14).

The symmetry of the function just discussed simplifies considerably our investigation of its analytic properties. We need from now on only consider one Mandelstam channel, say $12 \to 345$ corresponding to the variable $\alpha_{12}$; the other channels will automatically have the same properties. As in $B_4$, then, one notes first that the integral is analytic in $x_{12}$ except perhaps where the integral diverges. This can occur only for $x_{12} \leq 0$ around $u_{12} = 0$. To study the analytic properties of $B_5$ in $x_{12}$ for $x_{12} \leq 0$, we again expand the integrand in a Taylor series about $u_{12} = 0$ obtaining

\[
B_5 \propto \int_0^1 \frac{du_{12}}{u_{12}^{x_{12}-1}} \sum \mathcal{C}_\ell u_{12}^\ell
\]

where

\[
\mathcal{C}_\ell = \frac{1}{\ell!} \frac{\partial^\ell}{\partial u_{12}^\ell} \int_0^1 du_{45} \frac{u_{45}^{x_{45}-1} u_{23}^{x_{23}-1} u_{51}^{x_{51}-1} u_{34}^{x_{34}-1}}{u_{12} = 0}
\]

the variables $u_{23}$, $u_{51}$ and $u_{34}$ being given in terms of $u_{12}$ and $u_{45}$ by (17). Integrating (22) term by term, one sees as before for $B_4$, that one gets a sequence of poles at $x_{12} = 0, -1, -2, -3, \ldots$, with residue at $x_{12} = -\ell$ given by $\mathcal{C}_\ell$. We wish to show that these
poles at $\alpha_{12} = -x_{12} = \ell$ correspond to particles of maximum spin $\ell$, or in other words that their residues $C_\ell$ are polynomials in the momentum transfers, namely $s_{23}$ and $s_{51}$, of total degree $\ell$. That this is so can be seen by direct differentiation as in (23). Thus for example,

$$\frac{\partial}{\partial u_{12}} u_{23}^{x_{23} - 1} = (x_{23} - 1) \frac{\partial u_{23}}{\partial u_{12}} u_{23}^{x_{23} - 2}$$

(24)

On putting $u_{12} = 0$, one notes by (17) that $u_{23} = 1$, so that one has

$$\frac{\partial}{\partial u_{12}} u_{23}^{x_{23} - 1} \bigg|_{u_{12} = 0} = (x_{23} - 1) \frac{\partial u_{23}}{\partial u_{12}} \bigg|_{u_{12} = 0}$$

(25)

which is linear in $x_{23}$. Repeated application of $\partial/\partial u_{12}$ will bring each time an extra power in $x_{23}$ leading thus to a polynomial. Since by (17) $u_{12} = 0$ also implies $u_{51} = 1$, a similar situation holds also for $x_{51}$. Also, since in $C_\ell$, the operator $\partial/\partial u_{12}$ occurs only $\ell$ times, the maximum total degree of the polynomial in $x_{23}$ and $x_{51}$ is indeed $\ell$, thus proving our previous statement.

There are two further points worth noting:

i) the preceding argument does not hold for $x_{45}$ and $x_{34}$ since the corresponding conjugate variables $u_{45}$ and $u_{34} \neq 1$ when $u_{12} = 0$. Thus $C_\ell$ need not be polynomial in $x_{45}$ and $x_{34}$. However, these variables in the channel $12$ are effective masses and not momentum transfers, and does not therefore affect our conclusion about the spin of the intermediate state;

ii) there are no double poles between channels which share a common index, e.g., $\alpha_{12}$ and $\alpha_{23}$. The poles of $\alpha_{12}$ can occur in (20) only around $u_{12} = 0$ and those of $\alpha_{23}$ only around $u_{23} = 0$. 

As we have seen, however, \( u_{12} = 0 \Rightarrow u_{23} = 1 \), excluding thus this possibility. This is physically important; otherwise one would have an enormous enhancement in the overlap region between two resonance bands on the Dalitz plot. However, a double pole can occur in channels without a common index, e.g., \( \alpha_{12} \) and \( \alpha_{45} \). This would correspond to the Feynman diagram shown in Fig. 6. This may be interpreted as the formation, in 1+2 collision, of a resonance which then decays by cascade.

Finally, as in the four-point case, it can be shown that \( B_5 \) does have the proper Regge limit provided that one approaches \( \infty \) along a ray at an angle to the real axis. The arguments are however, too long to be presented here. From this then it follows automatically that "duality" also holds, and the validity of the amplitude (14) is fully established.

There is yet another very attractive feature of this generalized Veneziano model which is far from trivial. This concerns its consistency with our discussion of two-body reactions in the previous Section. There, we have proposed that scalar-scalar scattering be represented essentially just by the ordinary beta function \( B_4 \). Now, in our generalized model, the amplitude contains poles corresponding to the scalar particle, e.g., in the 12 channel, represented by the diagram in Fig. 7. The residue at the pole in Fig. 7 is then just a four-line graph for scalar-scalar scattering which for consistency should be just the \( B_4 \) function. That this is indeed the case can be seen from (23) for \( \xi = 0 \). On putting \( u_{12} = 0 \), one has by (17) \( u_{23} = 1 \), \( u_{51} = 1 \), \( u_{34} = 1 - u_{45} \), giving thus

\[
C_0 = \int_0^1 d\mu \mu^{x_{45} - 1} (1 - \mu)^{x_{34} - 1}
\]

which is exactly \( B_4(x_{34}, x_{45}) \). We shall call this consistency condition the "bootstrap consistency" of the Veneziano model.
Pursuing this consistence idea further, one is forced to consider the residue $C_{\ell}$ at $\alpha_{12} = \ell$ as the amplitude for the four-line reaction where one line has spin $\ell$. The model, if accepted, thus predicts unambiguously the amplitude for the production of a resonance of any spin lying on our trajectory. Indeed one need not consider quasi-particles at all unless one chooses to, since their production mechanism as well as their decays are already contained in the five-point function.

Theoretically, therefore, the beauty of the generalized Veneziano model, is undeniable. It is a different question, however, whether the model has anything to do with physics. In order to answer this, there are many obstacles yet to overcome:

i) the problems of unitarity and ambiguity of satellites of course remain. On the phenomenological level, we shall assume as in the four-point function that for the first approximation, satellite terms are unimportant, and we shall simulate the effects of unitarity by giving our trajectory function $\alpha$ an imaginary part;

ii) our model does not apply in cases where vacuum exchanges occur. Without modifications, therefore, it is applicable only to a small class of rather special reactions.

iii) the model has been written down only for the idealized case where all external particles are scalars and all internal trajectories have negative intercepts (i.e., lowest member has spin 0). To apply to realistic cases, modifications are necessary. The problem of adaptation of the formula to real reactions has been solved only in a few special cases where the spins of baryons are neglected. This fact again restricts the range of application at present.
One example in which such questions have been worked out is the reaction (13), which we shall describe in some detail. We start first by drawing diagrams with the 12 non-equivalent orderings of the external lines, corresponding to the terms in the sum (14). For convenience, we shall again take all the lines as incoming. We remember that the generalized beta function for any given ordering has poles between every neighbouring pair of external lines corresponding to resonances in that channel. If a certain diagram is such that some neighbouring pair has "forbidden" quantum numbers such as $Q = 2$ and $B = 0$, where no resonance is known to exist, then the diagram is disallowed. An example is the diagram shown in Fig. 8 which contains "resonances" between the $K^-\pi^-$ lines with $S = -1$ and $Q = -2$, which are clearly forbidden. Excluding all such cases, then, it turns out that only six terms remain in the sum (14), corresponding to the diagrams shown in Fig. 9.

The graphs (e) and (f) in Fig. 9 for high incoming energy in $K^-p$ scattering gives Regge graphs with double baryon exchange, as shown in Fig. 10. Since at large incoming energy, one or the other of these baryon trajectories must Reggeize, and since the intercepts of baryon trajectories are low, these graphs will give small contributions. It can easily be checked that at 3 GeV/c they are already quite negligible. In any case this approximation is not at all essential.

For the remaining four graphs, we shall in each case take into account only the leading trajectory in each channel, as shown in Fig. 9, where by $Y_1^*$ we mean $Y_1^*(1385)$. There is a slight ambiguity in some channels, such as the $Y^*$ in $K^-p$ channel; we are not sure whether the $Y_1^*$ or $Y_0^*$ is more important. The calculation discussed below, used the $Y_1^*$, but it has been checked that $Y_0^*$ would do just as well.

One notices that in all four graphs in Fig. 9 the leading resonance on the trajectory in each channel is a $p$ wave resonance. One needs thus a modification of the formula (18), similar to that needed for adopting the $B_4$ function for $\pi\pi$
scattering as discussed in the last Section. It turns out that provided one neglects the baryon spin, such an adaptation is possible. Indeed if one writes instead of (18),

\[ V(1,2,3,4,5) = \beta \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma B_5 \]

(27)

with the arguments of \( B_5 \) shifted to give poles in the right position, one has a formula with the desired properties. Thus, for example, the first pole in the 12-channel has a residue

\[ \beta \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma \]

which is clearly spin 1. Moreover, the formula (27) retains the desired cyclic symmetry, since

\[ \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_5^\sigma = - \epsilon_{\mu\nu\rho\sigma} p_2^\mu p_3^\nu p_5^\rho (p_1 + p_3 + p_4 + p_5) \]

(28)

using momentum conservation, which by the antisymmetry of \( \epsilon_{\mu\nu\rho\sigma} \) is easily seen to be identical to the original factor in (27).

Since, in distinction to the idealized case considered previously, our external lines here are not identical, the amplitude \( T \) need not be symmetric under all permutations of the external lines. In general, then \( T \) may be a linear combination of the four graphs we are considering, namely (a), (b), (c), (d) of Fig. 9, where the relative weights between them may be regarded as phenomenological parameters. It turns out, however, that in this particular case, these parameters are readily determined by arguments connected with exchange degeneracy.

One remembers first in the four-point case that the residues at \( \alpha(s) = \ell \) for \( V(s,t) \) and \( V(u,s) \) differ only by a sign \((-1)^L\) in the leading term in \( \cos \theta_s \). If one takes \( V(s,t) \) alone, say, then one has particles for every \( L \) on the leading trajectory, which has thus mixed signature, and is exchange degenerate.
On the other hand, if one takes the sum $V(s,t) + V(u,s)$ then all the particles with odd $I$ on the leading trajectory are removed, which has then definite (even) signature. One notes in addition that $V(s,t) = V(1,2,3,4)$ and $V(u,s) = V(1,2,4,3)$ differ only by interchanging 3 and 4.

A similar situation holds also for the five-point amplitude. It can be seen that the graphs (a) and (c) of Fig. 9 differ only by interchanging $p$ and $\vec{\Lambda}$, whereas (b) and (d) differ only by interchanging $K^-$ and $\pi^+$. If one assumes that $K^*$ is exchange degenerate, which seems true empirically and is implied by other theoretical considerations, one would require by the preceding arguments that we keep only either graph (a) or graph (c) but not both. A similar condition holds for (b) and (d). Next one observes in the actual data for this reaction that $Y_1^{*-}(1385)$ production is prominent, but there is no evidence for the production of its first recurrence. This may be taken as an indication that in this process the $Y_1^{*-}$ trajectory has definite signature. As argued above, this can happen in our model only when one combines with equal weights two graphs which differ only by interchanging $\pi^- \vec{\Lambda}$. It is then easily seen that this leaves us only one possible choice, namely

$$\tau = (a) - (b)$$

(29)

where (a) and (b) refer to the diagrams in Fig. 9, the minus sign between being due to the antisymmetry of the $\varepsilon_{\mu \nu \gamma \sigma}$ factor in (27).

Taking then the intercepts and imaginary parts of our trajectories by fitting the masses and widths of existing resonances as taken say from Rosenfeld Tables, one is then left with only one parameter, namely the over-all normalization with which to fit the data for all incoming energies. For further detailed technical points, I refer the reader to the original paper of Petersson and Törnqvist 5).
Examples of the result of this calculation are compared with the data in Figs. 11-15. The agreement is remarkable considering the small amount of freedom in the model. Doubts may indeed be raised, especially concerning i) the baryon trajectories which have been crudely treated neglecting their spin, and ii) the arguments concerning exchange degeneracy which are at best approximate. However, the calculation has opened up exciting possibilities which may revolutionize multi-particle phenomenology.

None the less, one example does not prove a theory, however good the fit. A lot of work need yet be done to check the consistency of the approach, by application to a wide class of processes. Such work is indeed in progress at CERN, and presumably elsewhere. Although no new calculation has yet been completed, the preliminary results are extremely encouraging.

There are yet two more important points to note in connection with applications of the five-point Veneziano model:

i) One important feature of the model is that it is supposed to be crossing symmetric. If so, the same function should be able to describe all reactions which are related by crossing. Thus, for example, one is supposed to fit all the following reactions with the same formula and same parameters:

\[ \kappa^+ \bar{p} \rightarrow \kappa^0 \pi^+ \bar{p} \]
\[ \kappa^- \bar{p} \rightarrow \kappa^0 \pi^- \bar{p} \]
\[ \pi^- \bar{p} \rightarrow \kappa^0 \kappa^- \bar{p} \]
\[ \pi^+ \bar{p} \rightarrow \kappa^0 \kappa^+ \bar{p} \]
\[ \bar{p} \bar{p} \rightarrow \kappa^0 \kappa^- \pi^+ \]

(30)
Attempts along this direction are already being made and show a lot of promise.

ii) The proper Veneziano model applies strictly only to reactions where Pomeron effects are negligible. It is however not excluded that one may be able to construct a hybrid model combining some Veneziano ideas with a phenomenological Pomeron for studying reactions such as \( \pi^+ p \rightarrow \pi^+ \pi^0 p \).

Where such attempts will lead to, only time will tell.

4. MANY-BODY FINAL STATES

For the examples given then, the dual resonance Regge picture we have considered is apparently quite good. It is logical therefore to try extending it to reactions with more particles. To crystallize our ideas, it would be best to construct a Veneziano model for an arbitrary number of external lines, which will then summarize for us the properties which we think the multi-particle amplitudes possess.

This is indeed possible for the idealized case of scalar interactions for any number \( N \) of external lines. In addition to properties i)-v) the \( N \) point Veneziano model satisfies the "bootstrap consistency condition" mentioned in the last Section. However, this generalized model is a bit too complicated to be described in this course of lectures. I shall only illustrate its intriguing properties by considering the "simple" case for six external lines and refer for further details to my recent review article on the subject 6).

Quite in analogy to the four and five-line case, the six-point function for scalar interaction is written as a sum of terms corresponding to the various non-equivalent orderings of the six external lines. Each term is proportional to a generalized beta function of sixth order, which we shall denote by \( B_6 \). Consider
in particular the term corresponding to the diagram of Fig. 16. The situation here is somewhat more complicated than in the four and five-line case, for, in addition to six two-body channels, such as $12 \rightarrow 3456$ corresponding to the Mandelstam variable

$$S_{12} = (p_1 + p_2)^2 = (p_3 + p_4 + p_5 + p_6)^2$$

(31)

there are also three three-body channels, such as $123 \rightarrow 456$ corresponding to the Mandelstam variable

$$S_{123} = (p_1 + p_2 + p_3)^2 = (p_4 + p_5 + p_6)^2$$

(32)

$B_6 (123456)$ is thus defined as a function of nine variables. Again, as for $B_5$, it is most convenient to write $B_6$ in an integral representation similar to (20) for $B_5$. This $B_6$ has then poles corresponding to resonances on linear trajectories for every one of the nine Mandelstam channels and has in addition the correct multi-Regge behaviour. It thus incorporates all the diagrams contained in Fig. 17. Namely, this single function is supposed to describe simultaneously (a) formation of a four-body resonance, (b) double resonance production via Regge exchange, (c) single resonance production via double-Regge exchange, (d) production of a three-body resonance, and (e) triple-Regge mechanism. The resonance in each case may be of any integral spin, their decay correlations being also automatically given.

It is easy to see then why theoreticians are so excited about this model, and why some are tempted to hope that this may be the beginning of a new theory of strong interactions. Most of the excitement for the $N$ point function for arbitrary $N$ is however only on the theoretical level at present. On the phenomenological level, it appears likely that we shall have to remain at $N = 5$ for some time. The reasons are as follows:
i) for multiplicities greater than three, it becomes increasingly
difficult to find reactions where the vacuum exchange does not
occur;

ii) no general method has been found to adapt the N scalar ampli-
tude to realistic cases. In fact the difficulties are so acute
that they may be deeply rooted, perhaps connected with our
ultimate problem - unitarity!

iii) even if all theoretical difficulties are overcome, we still need
a dramatic improvement in our calculational technique. The
present Monte Carlo calculations if applied to N > 5 will
overload even the CERN CDC.

It seems then that future hopes of application lie in
finding some efficient means of approximating the generalized beta
functions. This is unfortunately not easy without destroying a lot
of their many desirable properties.

There do exist however some earlier models, which may
in a rough sense be regarded as approximations to the picture so
far discussed. Though in general rather crude, they may be useful
for gaining an over-all impression of high multiplicity reactions.
For my illustrative purpose, I shall use here the so-called CMA
model 7) which is probably the crudest of the crude, but has the
virtue of being simple and extensively applied. This retains the
Regge behaviour whenever any energy or effective mass is large.
However, the low effective mass region, instead of being dominated
by resonances as we believe it to be, is smoothed over by an
effective constant. Clearly, in doing this, one loses a very at-
tractive feature of the picture advocated in the last two Sections.
None the less, when averaged over many degrees of freedom, as in
plotting single-particle distributions, the constant or phase space
approximation may not be too bad. This assumption is similar to
the statistical model of Fermi for production processes at low
energies, only extended also to low effective masses.
The actual parametrization of the amplitude in this model reads as follows

$$|A| \sim \prod_{i=1}^{n-1} \left( \frac{q_i s_i + c a}{s_i + a} \right) \left( \frac{s_i + a}{a} \right)^{x_i} \left( \frac{s_i + b_i}{b_i} \right)^{t_i}$$

(33)

where the variables $s_i$ and $t_i$ are indicated in Fig. 18. In the region of phase space where all $s_i$ are $\gg a$ and $b_i$, the amplitude (33) becomes

$$\prod_i q_i \left( \frac{s_i}{a} \right)^{x_i} \exp \left[ B_i + \log s_i \right] t_i$$

(34)

which is of the multi-Regge form. For $s_i \to 0$, however, the $i$th term in the product (33) becomes a constant $c$. The high and low mass regions were then interpolated in a simple-minded manner. Further simplifications such as the neglect of the particle charge reduces the $n$ particle amplitude finally to only three or four parameters which have since been used to correlate a wide range of experimental data. With the same values of the parameters, one seems able to describe quite consistently not only cross-sections and single-particle distributions for varying incoming energies and final multiplicities, but also qualitative differences between reactions initiated by different particles. I shall quote a few examples to illustrate the picture so obtained.

To be specific, consider the reactions

$$\pi^+ \pi \rightarrow \pi + (n-1) \pi$$

(35)

If one ignores the charge of the pion, one has to compute the diagrams shown in Fig. 19, where a single line denotes a meson, and a double line a baryon. Each term in Fig. 19 is to be parametrized by a formula like (33), and the sum is taken incoherently,
neglecting interference. Clearly, one cannot avoid exchanging here the quantum numbers of the vacuum. When this occurs, we shall approximate it by an effective Pomeron trajectory, as explained in Section 2. Apart then from quantities such as intercepts of trajectories which may be taken say from two-body collisions, the amplitude depends only on the following parameters: i) $a$, the energy scale which demarcates the high and low mass regions, ii) $c$, the constant which governs the strength of low mass interaction, iii) $b$, which governs the dependence on $t$ of the internal vertices, and iv) $e_N/e_M$, the relative strength between baryon and meson exchange. With these parameters, one is supposed to describe all single-particle distributions for varying multiplicity $n$ and incoming energy $s$, as well as the variations of the cross-sections for fixed $n$ as functions of the incoming energy.

Consider then what happens when one keeps the incoming energy $s$ fixed at some high value, and vary the final multiplicity. When $n$ is low, the incoming energy $s$ is shared among only a few $s_i$. The diagrams of (33) will then have lots of chance to Reggeize, giving prominent multiperipheral features. Since the baryon trajectories have much lower intercepts than mesons, when Reggeized they give negligible contributions. The first diagram in Fig. 19 will thus dominate, pushing the proton far backwards. As the multiplicity increases, however, the same incoming energy has to be shared among more and more $s_i$, and cluster formation sets it. Peripheral features are rapidly lost, until at extreme multiplicities, there will be so little energy available that the whole system forms a single cluster, with almost phase space distributions of final particles. That this picture is approximately valid can be seen in the longitudinal momentum distributions of Fig. 20 and Fig. 21, in which the calculational results are compared with experimental data.

Next consider what happens when one keeps the multiplicity $n$ fixed and increases the incoming energy. Clearly, the reverse of the situation described above will take place. When the
energy is low, cluster formation is dominant; at higher energies the amplitude will Reggeize. One expects then increasingly more prominent peripheral features for increasing energy, which is well borne out by experiment as can be seen in Fig. 22.

As an example of how particle distributions may depend on quantum numbers, let us consider the reaction

\[ K^- p \rightarrow \Lambda + (n - 1) \pi \]  

(36)

The main difference of this reaction from the previous one is the fact that here only strange mesons can be exchanged. Since strange meson trajectories have much lower intercepts than either the Pomeron or non-strange meson trajectories such as the \( \rho \), one expects that the baryon exchange graphs in Fig. 19 will be relatively more important in (36) than in (35), yielding thus more baryons going forward in the c.m. system. That this is indeed the case experimentally, can be seen by comparing Fig. 23 and Fig. 24, where the theoretical curves shown have been calculated with exactly the same parameters except for the difference in intercepts of the meson trajectories.

The same type of analysis has been extensively applied to various reactions over a wide range of energies and multiplicities. A qualitative agreement between the model calculations with experiment is consistently maintained as far as single particle distributions are concerned. Where the model drastically fails is in correlation effects between final particles, such as the Goldhaber effect and resonance productions. This is to be expected since the model neglects both resonance dominance and the phase of the amplitude, in which such correlation effects are known to depend. Indeed, more elaborate calculations including some resonance production are able to reproduce correlations qualitatively similar to those observed experimentally.
5. CONCLUDING REMARKS

Judging from the few example I have seen, I believe that the Veneziano model is going to be quite reliable for the cases where it can be applied, namely five-line processes with no vacuum exchange. Also, it seems likely that for reactions such as $\pi^+ p \to \pi^+ \pi^0 p$, hybrid models may be constructed by combining a phenomenological effective Pomeron with some dual resonance-Regge ideas. The extension of such models to actual analysis of higher multiplicities will be difficult, though not entirely excluded. None the less, the results of such investigations as with the QHA model may be considered as a fair indication that our basic picture is probably correct even for these reactions. Remembering how little we knew about multi-particle processes, just two or three years ago, the outlook is thus most gratifying.

However, it seems to me that such a picture is only valuable for intermediate energies, say up to 15 GeV or so. Above that, these reactions for which our model strictly applies will die out completely, leaving only those which are dominated by vacuum exchanges. Besides the multiplicity will be so high on the average, that it makes little sense any more to study each reaction in detail. For example, we know that the total cross-section seems to remain constant at high energy. If our model is indeed correct, then calculating the cross-section for each channel one by one, and adding them all up, one should obtain a constant result. But even if we were able to do this, we would still have missed the main point, for surely the constancy of the total cross-section has a deeper physical meaning than that. Even at the phenomenological level, therefore, our emphasis will be different for these energies; we shall be looking at other features and other regularities. We shall then need models of a completely different type, such as those being considered by Yang and Feynman. However, since I know nothing about these models, I better stop.
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5) B. Petersson and N. Törnqvist, CERN preprint TH.1040 (1969), to be published in Nuclear Phys. B.


Figure 3. The Dalitz plot obtained experimentally for the annihilation at rest \( \bar{p}n \rightarrow \pi^+ \pi^- \pi^- \) is compared with the calculation by Lovelace 3).

Figure 11. The total cross-section for the reaction \( K^-p \rightarrow \Lambda \pi^+ \pi^- \) as a function of the incoming momentum. The partial cross-sections for \( \Upsilon^* \) productions are also shown 5).

Figure 12. The effective mass plots for the reaction \( K^-p \rightarrow \Lambda \pi^+ \pi^- \) at 3.5 GeV/c 5).

Figure 13. The effective mass plots for the reaction \( K^-p \rightarrow \Lambda \pi^+ \pi^- \) at 5.5 GeV/c 5).

Figure 14. The distributions in t and \( cc* \) for the production of \( \phi \) and \( \Upsilon^{*+}(1385) \) in the reaction \( K^-p \rightarrow \Lambda \pi^+ \pi^- \) at 5.5 GeV/c 5).

Figure 15. The Van Hove hexagonal plot for the reaction \( K^-p \rightarrow \Lambda \pi^+ \pi^- \) at 10 GeV/c 5).

Figure 20. Percentage distributions of protons in the c.m. longitudinal momentum \( p_L^* \) from the reactions \( \pi^+ p \rightarrow p + (n-1)\pi \) at 8 GeV/c for \( n \) ranging from three to nine:

\[ \text{a) } p \pi^+ \pi^0, \text{ b) } p_2 \pi^+ \pi^- \text{, c) } p_2 \pi^+ \pi^- \pi^0, \]
\[ \text{d) } p_3 \pi^+ 2 \pi^-, \text{ e) } p_3 \pi^+ 2 \pi^- \pi^0, \text{ f) } p_4 \pi^+ 3 \pi^-, \]
\[ \text{g) } p_4 \pi^+ 3 \pi^- \pi^0, \text{ (Ref. 7).} \]

Figure 21. Percentage distributions of pions in the c.m. longitudinal momentum \( p_L^* \) from the reactions \( \pi^+ p \rightarrow p + (n-1)\pi \) for \( n \) ranging from three to eight:

\[ \text{a) } p \pi^+ \pi^0, \text{ b) } p_2 \pi^+ \pi^-, \text{ c) } p_2 \pi^+ \pi^- \pi^0, \]
\[ \text{d) } p_3 \pi^+ 2 \pi^-, \text{ e) } p_3 \pi^+ 2 \pi^- \pi^0, \text{ f) } p_4 \pi^+ 3 \pi^-, \text{ (Ref. 7).} \]
Figure 22  Percentage distribution of protons and pions in the cosine of the c.m. production angle $\theta^*$ from the reactions
$\pi^\pm p \rightarrow p3\pi^\pm 2\pi^\mp$ at 5.5, 8 and 16 GeV/c:
a) protons, $\pi^- p$, 16 GeV/c,  b) protons, $\pi^+ p$, 8 GeV/c,
c) protons, $\pi^- p$, 5.5 GeV/c,  d) pions, $\pi^- p$, 16 GeV/c,
e) pions, $\pi^+ p$, 8 GeV/c,  f) pions, $\pi^- p$, 5.5 GeV/c.
(Ref. 7).

Figure 23  Percentage distributions of protons in the cosine of the c.m. production angle $\theta^*$ from the reactions $\pi^+ p \rightarrow p + (n-1)\pi$
at 8 GeV/c for $n$ ranging from three to nine:
a) $p \pi^+ \pi^0$,   b) $p2\pi^+ \pi^-$,  c) $p2\pi^+ \pi^- \pi^0$,
d) $p3\pi^+ 2\pi^-$,  e) $p3\pi^+ 2\pi^- \pi^0$,  f) $p4\pi^+ 3\pi^-$,
g) $p4\pi^+ 3\pi^- \pi^0$.  (Ref. 7).

Figure 24  Percentage distributions of $\Lambda$'s in the cosine of the c.m. production angle $\theta^*$ from the reactions $K^- p \rightarrow \Lambda + (n-1)\pi$
at 6 GeV/c for $n$ ranging from three to six:
a) $\Lambda \pi^+ \pi^-$,  b) $\Lambda \pi^+ \pi^- \pi^0$,  c) $2\pi^+ 2\pi^-$,
d) $\Lambda 2\pi^+ 2\pi^- \pi^0$.  (Ref. 7).
\[ \bar{p}n \rightarrow \pi^+\pi^-\pi^- \]

2785 EVENTS

\[ t = M^2(\pi^+\pi^-) \]

\[ s = M^2(\pi^+\pi^-) \quad (\text{GeV}/c)^2 \]

\[ N \]

\[ \bar{p}n \rightarrow (\pi^+\pi^-)\pi^- \]

\[ m^2_{\pi^+\pi^-} \]

\[ \bar{p}n \rightarrow (\pi^-\pi^-)\pi^+ \]

\[ m^2_{\pi^-\pi^-} \]
Fig. 4

Fig. 5
Fig. 9
\[ a_{\pi\pi} \text{ (mb)} \]

- \( \phi \): \( \sigma_{\pi\pi} (K^+p \rightarrow n^-\pi^+\Lambda) \)
- \( \psi \): \( \sigma_{\pi\pi} [K^+p \rightarrow \pi^-\pi^+(1385)] \)
- \( \Delta \): \( \sigma_{\pi\pi} [K^+p \rightarrow \pi^-\pi^+(1385)] \)
- \( \times \): computed predictions

\[ P_{LAB} \text{ (GeV/c)} \]
3.0 GeV/c
892 events

Fig. 12
Fig. 14
Fig. 20