Bubble wall perturbations coupled with gravitational waves

Akihiro Ishibashi * and Hideki Ishihara †

Department of Physics, Tokyo Institute of Technology,
Oh-Okayama Meguro-ku, Tokyo 152, Japan

Abstract

We study a coupled system of gravitational waves and a domain wall which is the boundary of a vacuum bubble in de Sitter spacetime. To treat the system, we use the metric junction formalism of Israel. We show that the dynamical degree of the bubble wall is lost and the bubble wall can oscillate only while the gravitational waves go across it. It means that the gravitational backreaction on the motion of the bubble wall can not be ignored.

PACS number(s) 04.20.-q 04.40.-b 98.80.Cq 98.80.Hw

---

*E-mail: akihiro@th.phys.titech.ac.jp
†E-mail: ishihara@th.phys.titech.ac.jp
I. INTRODUCTION

Topological defects could be formed when symmetries are broken in the early universe. Domain walls, the simplest topological defect, emerge at the boundaries of true vacuum bubbles nucleating in a false vacuum at first-order phase transitions [1].

Motivated by recent observations which suggest that the present density parameter $\Omega_0$ is smaller than unity, a possible inflationary scenario which is compatible with an open universe is proposed [2]. The scenario requires two inflationary epochs. First, an old type inflation occurs and solves the horizon and the homogeneity problems. It is exited through a nucleation of a single vacuum bubble by quantum tunneling. If the tunneling process is represented by the Euclidean bounce solution with O(4) symmetry [3], the interior of the bubble has the O(3,1) invariance. This symmetry naturally induces negatively curved time-slices and an open Friedman-Robertson-Walker (FRW) universe is realized inside the bubble. Next, a slow rollover inflation occurs after that. The present value of $\Omega_0$ is determined by the form of the inflaton potential. This scenario is called “one-bubble open inflation”. The geometry of the scenario is schematically depicted in Fig. 1.

In the one-bubble open inflation scenario, the observable universe is surrounded by a bubble wall. A conclusive verification of the scenario is the detection of existence of the bubble wall. Though the homogeneous isotropic open time-slices do not intersect the bubble wall, it is causally visible from the observers inside the bubble. If the bubble wall is deformed in some reasons, we may observe the inhomogeneities of the universe produced by the deformation. Thus it is especially relevant to investigate the motion of the bubble wall in the one-bubble open inflation context.

In the thin wall approximation, the motion of a domain wall is described as a three-dimensional timelike hypersurface in the four-dimensional spacetime. The small deformation around background domain walls with planar or spherical symmetries are represented by a single scalar field on the background wall hypersurface [4]. The equation for the scalar field takes the form of the Klein-Gordon equation on the hypersurface. Thus, it is expected intuitively that the wave propagation of the scalar field represents the oscillation of the deformed wall. These analysis are based on the assumption that all influence of gravity on the wall motion is negligible.

In the thin wall approximation, the energy-momentum tensor of the wall is concentrated on the three-dimensional hypersurface. Thus, the spacetime should be singular at the hypersurface. In the general relativity, the spacetime containing such a singular hypersurface should be treated by the metric junction formalism [5]. In this method the wall is treated as a source of the gravitational field in which the wall moves. The motions of domain walls with spherical symmetry are analyzed by this formalism in Ref. [6,7].

In the metric junction formalism, Kodama, Ishihara, and Fujiwara studied the perturbations of a spherically symmetric domain wall system [8]. The interaction between the perturbed wall and gravitational waves is taken into account in their work. As a result, it turned out that the deformation of the domain wall occurs only while the gravitational wave goes through and does not propagate on the background hypersurface by itself.

The background configuration of the spacetime considered in Ref. [8] is constructed in the peculiar way: joining two copies of a portion of Minkowski spacetime with spherically symmetric boundary by identifying the boundary at the domain wall. So, there is a reflection symmetry with respect to the spherically symmetric domain wall. Thus, it is interesting to clarify whether the result is due to the symmetric configuration of the background, or to the domain wall’s feature itself. This also motivates us to investigate the motion of domain wall on the different background configuration: one-bubble open inflation configuration, in which there exists the vacuum energy deference between the inside and the outside of the bubble and no reflection symmetry.

In this paper, we investigate perturbations of a coupled system of gravity and a vacuum bubble in de Sitter spacetime. We treat the bubble wall by the thin wall approximation and apply the metric junction formalism to consider the gravitational back reaction on the bubble wall motion. We accomplish this in terms of the gauge-invariant perturbation method on spherically symmetric background spacetime established by Gerlach and Sengupta [9–11]. Though, in the main stream of our study, we consider a bubble wall whose inside is Minkowski and outside de Sitter spacetime as a background, we also examine the case that both sides of the bubble are de Sitter spacetimes with different cosmological constants. The issue is the extension of that in Ref. [8].
In the next section, we specify a background geometry which is used in the one-bubble open inflation scenario by using metric junction method. In section 3, we consider the metric perturbations in terms of gauge-invariant variables. We find master variables for two modes of gravitational waves and reduce the perturbed Einstein equations into a single master equation for them, and give the general solutions of it. In section 4, we write down the perturbed junction conditions as the boundary conditions on the master variable. We match the general solution along the bubble wall by using the perturbed junction conditions and obtain the global solutions. Section 5 is devoted to summary and discussion.

Throughout this paper, \( \kappa \) denotes \( 8\pi G \) and the unit \( c = 1 \) is taken.

II. BACKGROUND GEOMETRY

Let us consider a geometrical model in which a spherical vacuum bubble exists in de Sitter spacetime with a cosmological constant \( \Lambda \) as a background spacetime. The bubble wall divides the spacetime into two regions: Minkowski \((M_-, ds^2)\) inside and de Sitter spacetime \((M_+, ds^2_+)\) outside the bubble. Hereafter, we describe quantities in \( M_- \) with subscript \( "-" \) and in \( M_+ \), with \( "+" \).

We assume that the thickness of the bubble wall is small enough comparing to all other dimensions, that is, the thin wall approximation. Then the history of the bubble wall is described as a three-dimensional timelike hypersurface \( \Sigma \). The whole spacetime manifold containing the bubble wall is constructed as \( M := M_- \cup \Sigma \cup M_+ \) by matching the boundaries \( \Sigma_\pm := \partial M_\pm \) at \( \Sigma \). The spacetime metric \( ds^2 \) is required to be \( C^0 \) in \( M \). The geometries of the boundaries \( \Sigma_\pm \) are characterized by the induced intrinsic metrics \( (q_{ij}d\zeta^i d\zeta^j)_\pm \) and the extrinsic curvatures \( (K_{ij}d\zeta^i d\zeta^j)_\pm \), where \( \zeta^i_k \) (the indices \( i,j \) run over \( 1,2,3 \)) are three-dimensional coordinate systems on \( \Sigma_\pm \), respectively. The intrinsic metrics and the extrinsic curvatures are specified by the junction conditions.

In more realistic model of the one-bubble open inflationary universe, the interior of the bubble has a slow rollover inflationary phase and it should be treated locally as de Sitter spacetime. We will see later that the results for this case are essentially the same. For simplicity, we consider the time symmetric configuration as illustrated in Fig.2, instead of a bubble nucleation.

A. Intrinsic metric and extrinsic curvature

The background spacetimes have spherical symmetry. In general, the metric with spherical symmetry can be expressed in the form

\[
d s^2 = g_{\mu\nu}dx^\mu dx^\nu = \gamma_{ab}(y^c)dy^a dy^b + r^2(y^c)\Omega_{pq}dz^p dz^q, \tag{2.1}
\]

where \( \Omega_{pq}dz^p dz^q \) is the metric on a unit symmetric two-sphere with an angular coordinates \( z^p \), i.e.,

\[
\Omega_{pq}dz^p dz^q = d\theta^2 + \sin^2 \theta d\phi^2 = : d\Omega^2. \tag{2.2}
\]

The functions \( r(y^c) \) and \( \gamma_{ab}(y^c) \) are scalar and tensor fields on the two-dimensional orbit space \( ^2M \) spanned by the two-coordinates \( y^a \) and each point of \( ^2M \) represents a symmetric two-sphere. Hereafter, the indices on the orbit space \( a, b, c, \cdots \) run over \( 0,1 \) and the indices on the unit two-sphere \( p, q, r, \cdots \) run over \( 2,3 \).

Let \( \nabla_\mu \), \( D_a \), and \( \hat{D}_p \) be the covariant derivatives with respect to \( g_{\mu\nu}, \gamma_{ab}, \) and \( \Omega_{pq} \), respectively. The covariant derivative \( \nabla_\mu \) is expressed by \( D_a \) and \( \hat{D}_p \), that is, the Christoffel symbol \( \Gamma^\lambda_{\mu\nu} \) associated with the spacetime metric \( g_{\mu\nu} \) is expressed as

\[
\begin{align*}
\Gamma^a_{bc} &= \hat{\Gamma}^a_{bc}, \quad \Gamma^r_{pq} = \hat{\Gamma}^r_{pq}, \\
\Gamma^a_{pq} &= \delta^p_q \left( \frac{D_a r}{r} \right), \quad \Gamma^a_{pq} = -r^2 \Omega_{pq} \left( \frac{D^a r}{r} \right),
\end{align*}
\tag{2.3}
\]
The orbit space metrics of $M_\pm$ can take the static form

$$\gamma_{ab} dy^a dy^b = -g(r_-) dt_+^2 + g(r_+)^{-1} dr_+^2,$$

where $H$ is defined by $H := \sqrt{\Lambda/3}$. The history of the spherically symmetric bubble wall is described by a timelike orbit in $^3M$.

Let $(n^\mu)_\pm$ be the unit normal vectors to the boundaries $\Sigma_\pm$, respectively, such that both $(n^\mu)_\pm$ direct from $M_-$ to $M_+$. Since the vectors $(n^\mu)_\pm$ have vanishing $z^p$ components, we can identify them with $(n^a)_\pm$. Let $(\tau^a)_\pm$ be the future-directed unit timelike tangential vectors to $\Sigma_\pm$ on the orbit space. Then the orbit space metric can be decomposed as

$$\gamma_{ab} = -(\tau_a)(\tau_b) + (n_a)(n_b).$$

The projection tensors $(q_{\mu\nu})_\pm$ onto $\Sigma_\pm$ are represented by

$$(q_{\mu\nu})_\pm := (g_{\mu\nu})_\pm - (n_\mu)_\pm(n_\nu)_\pm$$

$$= - (\tau_\mu)(\tau_\nu)_\pm + r_\pm^2 \Omega_{\mu\nu},$$

where $\tau^\mu$ and $\Omega_{\mu\nu}$ are the four-dimensional extensions of $\tau^a$ and $\Omega_{pq}$. The four-dimensional extensions of the extrinsic curvatures $(K_{\mu\nu})_\pm$ are defined as

$$(K_{\mu\nu})_\pm := -(q^\lambda_\mu q^\sigma_\nu \nabla_\lambda n_\sigma)_\pm.$$  

From Eqs. (2.7) and (2.8), the extrinsic curvatures $(K_{\mu\nu})_\pm$ can be represented by

$$(K_{ab})_\pm = -(\tau_a D^c n_c)_\pm,$$

$$(K_{ap})_\pm = 0,$$

$$(K_{pq})_\pm = -(D^r r)_\pm \Omega_{pq},$$

where

$$D^r := t^a D_a, \quad D_\perp := n^a D_a.$$  

B. Background bubble motion

By using the metric junction formalism [5], the motions of spherically symmetric thin walls are studied in Ref. [6]. To establish notation, we briefly review the metric junction formalism and determine the motion of a spherical vacuum bubble in de Sitter spacetime following Ref. [7].

For convenience, we introduce the Gaussian normal coordinate system $(x^\mu) = (\chi, \zeta^i)$ in the neighborhood of the hypersurface $\Sigma$ such that the spacetime metric takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = d\chi^2 + q_{ij} d\zeta^i d\zeta^j.$$  

The coordinate axis $\chi$ is normal to $\Sigma$ and can be set $\chi = 0$ on $\Sigma$.

Let $S_{\mu\nu}(\zeta^i)$ be the wall stress-energy tensor which is defined by the energy-momentum tensor $T_{\mu\nu}$ as

$$T_{\mu\nu} := 4\pi S_{\mu\nu}(\zeta^i).$$
\[
S_{\mu\nu}(\zeta^i) := \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} T_{\mu\nu} d\chi
\] (2.14)
in the thin wall approximation. Then the energy-momentum tensor can be described as
\[
T_{\mu\nu} = S_{\mu\nu}(\zeta^i) \delta(\chi) - \rho g_{\mu\nu} \theta(\chi),
\] (2.15)
where \(\rho = 3H^2/\kappa\) is the vacuum energy of de Sitter spacetime and \(\theta(\chi)\) is a step function. Provided that the spacetime metric \(g_{\mu\nu}\) is continuous at \(\chi = 0\), \((q_{ij})_+\) coincides with \((q_{ij})_-\) on \(\Sigma\). Hence,
\[
[q_{ij}] = 0,
\] (2.16)
where, for geometrical quantity \(Q\) on \(M_\pm\), \([Q]\) denotes
\[
([Q] := \lim_{\epsilon \to 0} \{Q|_{\chi=+\epsilon} - Q|_{\chi=-\epsilon}\}.
\] (2.17)

With the help of the Gauss-Codazzi equation, the \((i,j)\) components of the Einstein equations are written by
\[
(3) R_{ij} - K K_{ij} + 2 K_{il} K_l^j + \partial_\chi K_{ij} = \kappa (T_{ij} - \frac{1}{2} g_{ij} T),
\] (2.18)
where \((3) R_{ij}\) is the Ricci curvature on \(\Sigma\) with respect to \(q_{ij}\). Substituting Eq. (2.15) into this equation (2.18) and integrating it with respect to \(\chi\) in an infinitesimally small interval \((-\epsilon, +\epsilon)\), we obtain
\[
[K^i_j] = \kappa \left( S^i_j - \frac{1}{2} \delta^i_j S \right).
\] (2.19)

Similarly, from the \((\chi,i)\) and the \((\chi,\chi)\) components, we obtain
\[
(3) D_j S^{ij} = 0, \quad (3) D_i S^{ij} = \rho,
\] (2.20)
(2.21)
where, for \(Q\) on \(M_\pm\), \(\overline{Q}\) denotes the mean value
\[
\overline{Q} := \frac{1}{2} \lim_{\epsilon \to 0} \{Q|_{\chi=+\epsilon} + Q|_{\chi=-\epsilon}\},
\] (2.22)
and \((3) D_i\) is the covariant derivative with respect to \(q_{ij}\).

Assuming that a scalar field comprises the bubble wall, we can write the wall stress-energy tensor \(S_{\mu\nu}\) by a single scalar field \(\sigma\) [7] as
\[
S_{\mu\nu} = -\sigma q_{\mu\nu},
\] (2.23)
From the condition (2.21), \(\sigma\) should be constant. Then, the junction conditions (2.19) and (2.20) reduce to
\[
[K^i_j] = \frac{1}{2} \kappa \sigma \delta^i_j, \quad (3) D_j S^{ij} = 0,
\] (2.24)
(2.25)
The bubble wall motion is determined by the conditions (2.16) and (2.24). Eq. (2.25) is then satisfied automatically.

The unit tangential vectors to \(\Sigma_{\pm}\) are expressed as
\[
\tau^a_{\pm} := dy^a_{\pm} / d\tau = (\dot{t}, \dot{r})_{\pm}
\] (2.26)
\[ i_{\pm} = \epsilon \frac{1}{g(r_{\pm})} \sqrt{g(r_{\pm}) + \dot{r}_{\pm}^2}, \quad (\epsilon = \pm 1). \]  

(2.27)

From the orthogonality \( n_a \tau^a = 0 \) and the normalization \( n_a n^a = 1 \), the components \( (n^t, n^r)_{\pm} \) are written as

\[ (n^t)_{\pm} = \frac{\dot{r}_{\pm}}{g(r_{\pm})}, \quad (n^r)_{\pm} = \epsilon \sqrt{g(r_{\pm}) + \dot{r}_{\pm}^2} = g(r_{\pm}) \dot{t}_{\pm}. \]  

(2.28)

Then, from Eq. (2.11), we get the \((p, q)\) components of the extrinsic curvatures of \( \Sigma_{\pm} \) in the form

\[ (K^p_q)_{\pm} = -\epsilon \sqrt{g(r_{\pm}) + \dot{r}_{\pm}^2} \delta^p_q. \]  

(2.29)

The \((p, q)\) components of the continuity condition (2.16) require \( r_+ (\tau) = r_- (\tau) = r(\tau) \). Then, substituting Eq. (2.29) into the condition (2.24), we obtain

\[ 1 + \dot{r}^2 = \left( \frac{r}{\alpha} \right)^2, \]  

(2.30)

where \( \alpha \) is the constant defined by

\[ \alpha := \frac{4\kappa \sigma}{4H^2 + \kappa^2 \sigma^2}. \]  

(2.31)

The value of the constant \( \alpha \) gives the minimal radius of the bubble, which is the radius on the instance of the bubble nucleation.

The solution of Eq. (2.30) is obtained as

\[ r = \alpha \cosh \tau. \]  

(2.32)

Here and for the rest of this paper, \( \tau \) denotes the normalized propertime by \( \alpha \). Then from Eq. (2.7), the intrinsic metric on the wall takes the form

\[ q_{ij} d\zeta^i d\zeta^j = \alpha^2 (-d\tau^2 + \cosh^2 \tau d\Omega^2). \]  

(2.33)

This is the three-dimensional de Sitter spacetime metric with the O(3,1) symmetry of the bubble wall.

As a natural extension of the wall metric (2.33), we introduce the hyperbolic charts \((\tau_{\pm}, \chi_{\pm}, \theta_{\pm}, \phi_{\pm})\) defined by

\[ t_- =: a(\chi_-) \sinh \tau_-; \quad \sqrt{g(r_+)} \sinh Ht_+ =: a(\chi_+) \sinh \tau_+, \]  

\[ r_{\pm} =: a(\chi_{\pm}) \cosh \tau_{\pm}, \]  

(2.34)

where

\[ a(\chi_{\pm}) := \begin{cases} (1/H) \sin H\chi_+ & \text{for } \chi > 0 \quad \text{— de Sitter side}, \\ \chi^- & \text{for } \chi < 0 \quad \text{— Minkowski side}. \end{cases} \]  

(2.35)

This is a Gaussian normal coordinate system (2.13) of the considering model. The coordinates \( \chi_{\pm} \) are related to \( \chi \) as

\[ \chi = a(\chi_{\pm}) - \alpha, \]  

(2.36)

and hence at the bubble wall

\[ a(\chi_{\pm}) \big|_{\Sigma} = \alpha. \]  

(2.37)

In addition, on the wall, it can be set naturally as

\[ \tau_{\pm} = \tau, \quad \theta_{\pm} = \theta, \quad \phi_{\pm} = \phi. \]  

(2.38)
Then, in the hyperbolic charts, the spacetime metrics take the form

\[ (ds^2)_\pm = d\chi^2 + a^2(\chi\pm) \left\{ -d\tau^2 + \cosh^2 \tau d\Omega^2 \right\}. \]  

(2.39)

Since these coordinate systems, which are also called the spherical Rindler charts, do not cover the whole spacetime, an analytic extension is necessary to cover the interior of the forward light cone from the center of the bubble, which corresponds to the observable universe. By the coordinate transformations,

\[ \chi_- = i\eta, \quad \tau_- = \xi - i\frac{\pi}{2}, \]  

(2.40)

the metric takes the form

\[ ds_-^2 = -d\eta^2 + \eta^2 \left\{ d\xi^2 + \sinh^2 \xi d\Omega^2 \right\}. \]  

(2.41)

This is the Milne universe, which has open FRW time-slices.

In the hyperbolic charts, the extensions of the normal vectors \( (n^\mu)_\pm \) and the tangential vectors \( (\tau^\mu)_\pm \) to \( \chi = \) const. hypersurfaces in a neighborhood of \( \Sigma \) are expressed, respectively, as

\[ (n^\mu)_\pm = (\partial \chi_\pm)^\mu, \quad (\tau^\mu)_\pm = \frac{1}{a(\chi_\pm)}(\partial \tau)^\mu. \]  

(2.42)

The extrinsic curvatures of the \( \chi = \) const. hypersurfaces are calculated explicitly as

\[ (K_{ij})_\pm = -\partial_{\chi_\pm} \ln a(\chi_\pm)(q_{ij})_\pm, \]  

(2.43)

and evaluated on \( \Sigma_\pm \) as

\[ (K_{ij})_- = -\frac{1}{\alpha}(q_{ij})_-, \quad (K_{ij})_+ = -\frac{\sqrt{1-H^2\alpha^2}}{\alpha}(q_{ij})_+. \]  

(2.44)

### III. PERTURBATIONS

In this section, we consider the metric perturbations on the background geometry constructed in the previous section. We describe the metric perturbations by the tensor harmonics defined in Appendix A. We introduce gauge-invariant variables in accordance with Gerlach and Sengupta [9] and express the perturbed Einstein equations in terms of them. Furthermore, we show that the perturbed Einstein equations reduce to a single scalar master equation in \( ^2M \).

#### A. Gauge-invariant perturbation variables

Let us consider the metric perturbations \( (h_{\mu\nu}dx^\mu dx^\nu)_\pm \) on the background metrics \( (g_{\mu\nu}dx^\mu dx^\nu)_\pm \). Due to the spherical symmetry of the background geometry, we can expand the perturbations \( (h_{\mu\nu}dx^\mu dx^\nu)_\pm \) by using the tensor harmonics on the unit sphere (see Appendix A, for their definitions) as follows:

for odd modes,

\[ h^{(o)}_{\mu\nu}dx^\mu dx^\nu = \sum_{l,m} \left\{ r(f^{(o)1}lm)a(V^{(o)1}lm)p(dy^adz^b + dz^ady^b) + r^2(f^{(o)2}lm)(T^{(o)2}lm)pqdz^pdz^q \right\}, \]  

(3.1)

for even modes,

\[ h^{(e)}_{\mu\nu}dx^\mu dx^\nu = \sum_{l,m} \left\{ (f^{(e)1}lm)abY_{im}dy^ady^b + r(f^{(e)1}lm)a(V^{(e)1}lm)p(dy^adz^b + dz^ady^b)
+ r^2(f^{(e)0}lm)(T^{(e)0}lm)pq + r^2(f^{(e)2}lm)(T^{(e)2}lm)pq \right\} dz^pdz^q. \]  

(3.2)
The subscripts $e$ and $o$ denote the even and the odd modes, respectively. The expansion coefficients, $f(y^c), f_a(y^c), f_{ab}(y^c)$ are scalar, vector, and symmetric tensor fields on the orbit space $^2\mathbb{M}$, respectively. Hereafter we suppress the angular integers $l, m$ and summation $\sum_{l,m}$ for notational simplicity and discuss the modes $l \geq 2$, which are relevant to the gravitational waves.

The metric perturbations $h_{\mu\nu}dx^{\mu}dx^{\nu}$ have the gauge freedoms. Associated with an infinitesimal coordinate transformations, $x^\mu \to x^\mu + \xi^\mu$, the gauge transformed metric perturbations are

$$h_{\mu\nu}dx^{\mu}dx^{\nu} = h_{\mu\nu}dx^{\mu}dx^{\nu} - (\nabla_\mu \xi^\nu + \nabla_\nu \xi^\mu)dx^{\mu}dx^{\nu}.$$  (3.3)

The generator $\xi^\mu dx^\mu$ of the infinitesimal coordinate transformations is expanded by the tensor harmonics as

$$\xi^{(o)}_{\mu} dx^\mu = r\xi^{(o)}_{\mu} V^{(o)}_{\mu} dz^p,$$  (3.4)

$$\xi^{(e)}_{\mu} dx^\mu = \xi^{(e)}_{\mu} Y^a dy^a + r\xi^{(e)}_{\mu} V^{(e)}_{\mu} dz^p.$$  (3.5)

Then, from Eqs. (3.1), (3.2), and (3.3), the gauge transformed expansion coefficients are expressed as follows:

for odd modes,

$$\bar{\delta} f^{(o1)}_a = - rD_a \left( \xi^{(o)}_r \right),$$  (3.6)

$$\bar{\delta} f^{(o2)} = - \frac{2}{r} \xi^{(o)},$$  (3.7)

for even modes,

$$\bar{\delta} f^{(e)}_{ab} = - D_a \xi_b - D_b \xi_a,$$  (3.8)

$$\bar{\delta} f^{(e1)}_a = - rD_a \left( \xi^{(e)}_r \right) - \frac{1}{r} \xi_a,$$  (3.9)

$$\bar{\delta} f^{(e0)} = 2 \frac{l(l+1)}{r} \xi^{(e)} - \frac{4}{r} \xi^a D_a r,$$  (3.10)

$$\bar{\delta} f^{(e2)} = - \frac{2}{r} \xi^{(e)}.$$  (3.11)

According to Gerlach and Sengupta [9], let us introduce the gauge-invariant variables by combining the expansion coefficients of the metric perturbations.

From Eqs. (3.6) and (3.7), we take the odd mode gauge-invariant perturbation variables

$$\mathcal{F}_a := f^{(o1)}_a - \frac{1}{2} rD_a f^{(o2)}.$$  (3.12)

From Eqs. (3.8)–(3.11), we take the even mode gauge-invariant variables

$$\mathcal{F}_{ab} := f_{ab} - D_a \xi_b - D_b \xi_a + \frac{1}{2} \gamma_{ab} \left\{ f^{(e0)} + l(l+1) f^{(e2)} - \frac{4}{r} X^c D_c r \right\},$$  (3.13)

where the vector $X^a$ is defined by

$$X^a := r f^{(e1)a} - \frac{1}{2} r^2 D^a f^{(e2)}.$$  (3.14)

From Eqs. (3.9) and (3.11), we see that the vector $X^a$ transforms as

$$\bar{\delta} X^a = - \xi^a.$$  (3.15)
The perturbed Einstein equations on de Sitter background spacetime are described by

\[
2\delta G_{\mu\nu} \equiv -\Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h^{\lambda}_\lambda + g_{\mu\nu} \Box h^{\lambda}_\lambda + \nabla_\mu \nabla_\lambda h^{\lambda}_\nu + \nabla_\nu \nabla_\lambda h^{\lambda}_\mu - g_{\mu\nu} \nabla_\lambda \nabla_\sigma h^{\lambda\sigma} + H^2 g_{\mu\nu} h^{\lambda}_\lambda - 4H^2 h_{\mu\nu} = 0,
\]

(3.16)

where \( \Box := \nabla^\mu \nabla_\mu \). We express this perturbed Einstein equations by the gauge-invariant variables introduced in Eqs. (3.12) and (3.13). Further, we rewrite them in the form of the wave equation for a single scalar field on the orbit space.

For a while, we consider only the de Sitter side (\( M_+ , ds^2_+ \)) and suppress the suffix “+”. All geometrical quantities for Minkowski side (\( M_-, ds^2_- \)) are reproduced by taking the limit \( H \to 0 \). We shall discuss the odd and the even modes separately.

1. Odd modes

The odd mode perturbed Einstein equations are written by the odd mode gauge-invariant variables in the form

\[
D_c \left\{ r^4 D^c \left( \frac{F^a}{r} \right) - r^4 D^a \left( \frac{F^c}{r} \right) \right\} - (l - 1)(l + 2) r F^a = 0,
\]

(3.17)

\[
D^a (r F_a) = 0.
\]

(3.18)

From Eq. (3.18), we can find a scalar variable \( \Phi_{(o)} \) such that

\[
r F^a = \epsilon^{ab} D_b \Phi_{(o)},
\]

(3.19)

where \( \epsilon^{ab} := n^a \tau^b - \tau^a n^b \) is the antisymmetric two-tensor. Substituting Eq. (3.19) into Eq. (3.17), we obtain

\[
\left\{ r^2 \Box_2 - 2r (D^r D_c - (l - 1)(l + 2)) \right\} \Phi_{(o)} = 0,
\]

(3.20)

where \( \Box_2 := D^a D_a \). We can further rewrite this equation in the form

\[
\left( -\Box_2 + \frac{l(l + 1)}{r^2} \right) \left( \Phi_{(o)} \right) = 0.
\]

(3.21)

This is the odd mode master equation for \( \Phi_{(o)} \).

2. Even modes

The even mode perturbed Einstein equations are written by the even mode gauge-invariant variables as the following set of equations:

\[
D_b F^b_a = D_c F^c_b,
\]

(3.22)

\[
\frac{1}{2} \left\{ \Box_2 + 2 \left( \frac{D^r}{r} \right) D_c - (l - 1)(l + 2) \right\} F_{ab} - \left( \frac{D^r}{r} \right) \left( \frac{D^d}{r} \right) F_{cd\gamma ab} - \left( \frac{D^r}{r} \right) \left( \frac{D^b}{r} \right) F_{ab} + \left( \frac{D^r}{r} \right) \left( \frac{D^r}{r} \right) F_{d\gamma ab} - \left( \frac{D^r}{r} \right) \left( D_b F_{ac} + D_a F_{bc} \right) + \left( \frac{D^r}{r} \right) D_c F^c_b + \left( \frac{D^r}{r} \right) D_c F^c_a = 0.
\]

(3.23)
Eq. (3.22) implies that there exists a master scalar variable $\Phi(e)$ such that
\[
\mathcal{F}_{ab} = (D_a D_b + H^2 \gamma_{ab}) \Phi(e).
\] (3.24)

Substituting Eq. (3.24) into Eq. (3.23), we obtain
\[
\left( -\Box + \frac{l(l+1)}{r^2} \right) \left( \frac{\Phi(e)}{r} \right) = 0.
\] (3.25)

For the derivation of Eq. (3.25), see Appendix B. This is the even mode master equation and is the same form with the odd mode master equation (3.21).

**C. General solutions of the master equation**

Let us give general solutions of the odd and the even mode master equations. Here, we omit the subscripts $e$ and $o$, since the both master equations (3.21) and (3.25) take the same form. We express the master equations in the hyperbolic charts defined by Eq. (2.34) in the form
\[
\left( -a(\chi \pm) \partial_{\chi \pm}^2 - a(\chi \pm) (\partial_{\chi} a(\chi \pm)) \partial_{\chi \pm} + \partial_{\tau}^2 + \frac{l(l+1)}{\cosh^2 \tau} \right) \left( \frac{\Phi}{r} \right) \pm = 0.
\] (3.26)

Assuming that $(\Phi/r)\pm$ can be decomposed into $R(\chi \pm)T(\tau)$, we obtain the equations for $R(\chi \pm)$ and $T(\tau)$, respectively,
\[
\left( \frac{1}{a(\chi \pm)} \partial_{\chi \pm} a(\chi \pm) \partial_{\chi \pm} + \frac{k_\pm^2}{a(\chi \pm)} \right) R(\chi \pm) = 0,
\] (3.27)
\[
\left( \partial_{\tau}^2 + \frac{l(l+1)}{\cosh^2 \tau} + k_\pm^2 \right) T(\tau) = 0,
\] (3.28)

where $k_\pm$ are the separation constants. The solutions of Eq. (3.27) are given by
\[
R(\chi \pm) = e^{\pm ik_\pm \zeta(\chi \pm)},
\] (3.29)
where
\[
\zeta(\chi_-) := \ln \chi_-,
\] (3.30)
\[
\zeta(\chi_+) := - \ln \left\{ \tan \left( \frac{H \chi_+}{2} \right) \right\}.
\] (3.31)

The solutions of Eq. (3.28) are given by
\[
T(\tau) \propto P_l^{\pm ik_\pm} (- \tanh \tau)
\] (3.32)
\[
\propto G_l \left( 1, 1 \mp ik_\pm; \frac{1 + \tanh \tau}{2} \right) e^{\mp ik_\pm \tau},
\] (3.33)

where $P_l$ and $G_l$ are the Legendre function and the Jacobi polynomial.

Let us introduce the set of functions $\{ \psi_l(\tau, k) \}$ defined by
\[
\psi_l(\tau, k) := G_l \left( 1, 1 \mp ik; \frac{1 + \tanh \tau}{2} \right) e^{\mp ik \tau}.
\] (3.34)

This satisfies the orthogonality condition
\[
\int_{-\infty}^{\infty} d\tau \psi_l^* (\tau, k') \psi_l (\tau, k) = 2\pi \delta(k - k'),
\] (3.35)
and the set \( \{ \psi_l(\tau, k) \} (\infty < k < \infty) \) is complete in the \( L^2(\infty, \infty) \) space of functions of \( \tau \). Then we can express the solutions \( (\Phi/r)_\pm \) as
\[
\frac{\Phi}{r}_\pm = \int_{-\infty}^{\infty} dk \psi_l(\tau, k) \left\{ A_\pm(k)e^{ik\zeta(x_\pm)} + B_\pm(k)e^{-ik\zeta(x_\pm)} \right\}.
\] (3.36)

The mode expansion coefficients \( A_\pm \) represent the gravitational waves propagating from the past Rindler horizon \( \tau = -\infty \) toward the bubble wall and \( B_\pm \) the waves propagating from the bubble wall toward the future Rindler horizon \( \tau = \infty \), as depicted schematically in Fig. 3.

**IV. GLOBAL SOLUTIONS**

In this section, we first write down the perturbed junction conditions. We can express them in terms of the master variables. Then, by matching the solutions of the master equations in \( M_- \) and \( M_+ \) along the bubble wall, we construct the global solutions of the perturbed Einstein equations in the whole spacetime \( M \).

The solutions of the perturbed Einstein equations (3.16) should satisfy the following perturbed junction conditions
\[
[\delta g_{ij}] = 0,
\] (4.1)
\[
[\delta K^i_j] = \kappa(\delta S^i_j - \frac{1}{2} \delta^i_j \delta S).
\] (4.2)

Since the wall stress-energy tensor \( S_{\mu\nu} \) is given by Eq. (2.23) with the constant \( \sigma \), the second condition reduces to
\[
[\delta K^i_j] = 0.
\] (4.3)

As in the case of the metric perturbations, we can expand the perturbed extrinsic curvature \( \delta K_{ij} \) by the tensor harmonics:

for odd modes,
\[
(\delta K^{(o)})_{ij}dx^idx^j = r(\delta K^{(o1)})_{ij}(V^{(o)})_p(dy^a_dz^p + dz^pdy^a) + r^2(\delta K^{(o2)})_{pq}dz^pdz^q,
\] (4.4)

for even modes,
\[
(\delta K^{(e)})_{ij}dx^idx^j = (\delta K^{(e1)})_{ab}Y_{ij}dy^adp + r(\delta K^{(e1)})_{ij}(V^{(e)})_p(dy^adz^p + dz^pdy^a)
\]
\[
+ r^2 \left( (\delta K^{(e0)})_{pq} + (\delta K^{(e2)})_{pq} \right) dz^pdz^q.
\] (4.5)

For convenience, let us define the decompositions of tensors on \( \tilde{M} \) by using \( \tau^a \) and \( n^a \). A vector \( V^a \) is decomposed as
\[
V^a = -V_\parallel \tau^a + V_\perp n^a,
\] (4.6)
where
\[
V_\parallel := \tau^a V_a, \quad V_\perp := n^a V_a.
\] (4.7)

Similarly, a symmetric tensor \( T_{ab} \) is decomposed as
\[
T_{ab} = T_\parallel \parallel \tau_a \tau_b + T_\perp \parallel n_a n_b - T_\parallel \perp (\tau_a n_b + n_a \tau_b),
\] (4.8)
where
\[
T_\parallel \parallel := \tau^a \tau^b T_{ab}, \quad T_\perp \parallel := n^a n^b T_{ab}, \quad T_\parallel \perp := \tau^a n^b T_{ab}.
\] (4.9)

We also define the projections of the perturbed extrinsic curvature

11
\((\delta K_{\beta})_a^\mu := q_\lambda^\mu \delta K_{\epsilon, \lambda}^a\),
\(\delta K_{\beta} f_{\gamma}^{(e) \alpha} = \frac{1}{2} \epsilon_{\alpha \beta \gamma} D_\alpha \left( \frac{\mathcal{F}_\alpha}{\tau} \right), \quad (4.11)
\)
\((\delta K_{\beta} f_{\gamma}^{(o) 2}) = \mathcal{F}_{\perp, \beta} / \tau. \quad (4.12)
\)
\[\begin{align}
(\delta K_{\beta} f_{\gamma}^{(o) 1})_a^\alpha & = -D_\beta \mathcal{F}_{\perp} + \frac{1}{2} D_\beta \left( \mathcal{F}_{\perp} - \frac{1}{2} \mathcal{F}_{\epsilon} \right) - \frac{1}{2} (K^a_{\beta}) \left( \mathcal{F}_{\perp} - \frac{1}{2} \mathcal{F}_{\epsilon} \right) \\
& + \left\{ -D_\beta^2 + (K^a_{\beta})^2 + H^2 \right\} X_{\perp} , \quad (4.13)
\end{align}\]
\[\begin{align}
(\delta K_{\beta} f_{\gamma}^{(e) 1})_a^\alpha & = \frac{1}{2} \mathcal{F}_{\perp} + r D_\beta \left( \frac{X_{\perp}}{\tau} \right), \quad (4.14)
\end{align}\]
\[\begin{align}
(\delta K_{\beta} f_{\gamma}^{(e) 0}) & = -\left( \frac{1}{2} (K^p_{\beta}) \left( \mathcal{F}_{\perp} - \frac{1}{2} \mathcal{F}_{\epsilon} \right) - 2 \left( \frac{D_{\beta r}}{\tau} \right) \mathcal{F}_{\perp} - \frac{1}{2} D_{\perp} \mathcal{F}_{\epsilon} \\
& + \left\{ -2 \left( \frac{D_{\beta r}}{\tau} \right) D_\beta - \frac{l(l+1)}{r^2} + \frac{1}{2} (K^p_{\beta})^2 + 2 H^2 \right\} X_{\perp} , \quad (4.15)
\end{align}\]
\[\begin{align}
(\delta K_{\beta} f_{\gamma}^{(e) 2}) & = \frac{X_{\perp}}{r^2} . \quad (4.16)
\end{align}\]

We note that these quantities contain the gauge dependent variable \(X_{\perp}\). In the perturbation theory, we treat the quantities on the perturbed geometry by embedding to the background geometry. In the present case, we treat the geometry constructed by gluing two different spacetimes: Minkowski spacetime and de Sitter spacetime. In addition, the geometry contains a singular hypersurface (the bubble wall) at the boundaries of the two spacetimes. We therefore must choose the embedding such that it maps the perturbed geometry of Minkowski (de Sitter) spacetime to the background Minkowski (de Sitter) spacetime and furthermore the perturbed bubble wall to that on the background. Thus, we consider the gauge transformations under the restriction such as \(\xi_{\perp} = 0\) on the bubble wall. The variable \(X_{\perp}\) is gauge-invariant just for the gauge transformations on the bubble wall and can be interpreted as the wall displacement variable [8]. This is discussed in a separate work [12].

Using Eqs. (4.11)–(4.16), we can rewrite the junction condition (4.3) in terms of the gauge-invariant variables and further in terms of the master variables \(\Phi_{(o)}\) and \(\Phi_{(e)}\). We consider the odd and the even modes separately in the following subsections.

### A. Odd modes

First, from Eq. (4.1), we see that \(f_{(o) 2}^{(o)}\) and \(f_{(o) 1}^{(o)}\) are continuous. Then,
\[\mathcal{F}_{\beta} = \left[ f_{\beta}^{(o) 1} - \frac{1}{2} D_{\beta} f_{(o) 2} \right] = 0. \quad (4.17)\]
From Eq. (3.19), it follows that
\[\left[ D_{\perp} \Phi_{(o)} \right] = 0. \quad (4.18)\]
This is rewritten as
\[\left[ D_{\perp} \left( \frac{\Phi_{(o)}}{\tau} \right) \right] = \frac{1}{2} \kappa \sigma \left( \frac{\Phi_{(o)}}{\tau} \right). \quad (4.19)\]
Second, from Eq. (4.11), we see that
\[ 0 = \left[ e^{ab} D_b \left( \mathcal{F}_a \right) \right] = \frac{1}{r^2} \left[ \left( \frac{2}{r} D^a r D_a - \square_2 \right) \Phi_{(o)} \right]. \quad (4.20) \]

Using the master equation (3.21), we rewrite Eq. (4.20) as
\[ \left[ \Phi_{(o)} \right] = 0. \quad (4.21) \]

Eqs. (4.19) and (4.21) should be imposed on the solutions of the master equation as the boundary conditions at the bubble wall. We obtain no new condition from Eq. (4.12).

Substituting the solutions (3.36) into the junction condition (4.21), we get
\[ \int_{-\infty}^{\infty} dk \psi_l(\tau, k) \left\{ A_+ - A_- + B_+ - B_- \right\} = 0, \quad (4.22) \]
where
\[ A_\pm := A_\pm e^{ik\pm \zeta_\pm |\Sigma|}, \quad B_\pm := B_\pm e^{-ik\pm \zeta_\pm |\Sigma|}. \quad (4.23) \]
Here, \( e^{\pm ik\pm \zeta_\pm |\Sigma|} \) are constant phase factors. Multiplying \( \int_{-\infty}^{\infty} d\tau \psi_l^* (\tau, k') \) and using the orthogonality condition (3.35), we obtain
\[ [A] + [B] = 0. \quad (4.24) \]

Similarly, from the junction condition (4.19), we obtain
\[ \left( k - i\kappa \alpha \sigma \right) \mathcal{A} - \left( k + i\kappa \alpha \sigma \right) \mathcal{B} = 0. \quad (4.25) \]

From the conditions (4.24) and (4.25), we get consequently the relations of the mode expansion coefficients,
\[ B_\pm = \frac{-i\mu}{k + i\mu} A_\pm + \frac{k}{k + i\mu} A_\mp, \quad (4.26) \]
where
\[ \mu := \frac{\kappa \alpha \sigma}{4}. \quad (4.27) \]

**B. Even modes**

From Eq. (4.1), the functions \( f_\parallel, f^{(e_1)}_\parallel, f^{(e_0)}_\parallel, \) and \( f^{(e_2)}_\parallel \) are continuous. Then
\[ [X_f] = \left[ r f^{(e_1)}_\parallel - \frac{1}{2} r^2 D_\parallel f^{(e_2)}_\parallel \right] = 0. \quad (4.28) \]

Similarly, from Eq. (4.16), we see that
\[ [X_\perp] = 0. \quad (4.29) \]

It therefore turns out that the vector \( X_a \) is continuous quantity. Then, from Eqs. (2.44), (2.31), and the definition (3.13) of \( \mathcal{F}_{ab} \), we get
\[ [\mathcal{F}_{\parallel \parallel}] = \left[ f_{\parallel \parallel} - \frac{2}{r} D_{\parallel} (r X_{\parallel}) - 4(K_a^u) X_{\perp} - \frac{1}{2} f^{(e0)} - \frac{1}{2} l(l + 1) f^{(e2)} \right] \]
\[ = -4[K_a^u] X_{\perp}, \quad (4.30) \]
\[ [\mathcal{F}_c] = \left[ f^{(e0)} + l(l + 1) f^{(e2)} + 4 \frac{D_{\parallel r} r X_{\parallel}}{r} + 4(K_a^u) X_{\perp} \right] \]
\[ = 4[K_a^u] X_{\perp}, \quad (4.31) \]

Hence,
\[ [\mathcal{F}_{\parallel \parallel}] + [\mathcal{F}_c] = 0, \quad (4.32) \]
and equivalently
\[ [\mathcal{F}_{\perp \perp}] = 0. \quad (4.33) \]

It follows from Eq. (4.14) that
\[ [\mathcal{F}_{\parallel \perp}] = 0. \quad (4.34) \]

With the help of Eq. (4.34) and the relation
\[ \frac{1}{2}(K_p^p)_\perp^2 + 2H^2 = \frac{1}{2}(K_p^p)_\perp^2, \quad (4.35) \]
Eq. (4.15) yields
\[ [D_{\perp} \mathcal{F}_c] + \left[ (K_p^p) \left( \mathcal{F}_{\perp \perp} - \frac{1}{2} \mathcal{F}_c \right) \right] = 0. \quad (4.36) \]

By using Eqs. (2.44), (2.31), and (4.33), we can rewrite Eq. (4.36) as
\[ [D_{\perp} \mathcal{F}_c] = -\frac{H^2}{\kappa \sigma} [\mathcal{F}_c] - \kappa \sigma \left( \mathcal{F}_{\perp \perp} - \frac{1}{2} \mathcal{F}_c \right). \quad (4.37) \]

Using Eq. (3.22), we see that no new constraint is obtained from the continuity of \((\delta K_{\parallel}^{(e)})_a^u\).

Now let us express the junction conditions (4.32), (4.34), and (4.37) in terms of the master variable \(\Phi_{(e)}\). It is useful to rewrite Eq. (3.24) in the form
\[ \mathcal{F}_{ab} = r D_a D_b \left( \frac{\Phi_{(e)}}{r} \right) + r \left( D_a r \right) D_b \left( \frac{\Phi_{(e)}}{r} \right) + r \left( D_b r \right) D_a \left( \frac{\Phi_{(e)}}{r} \right). \quad (4.38) \]

First, by using the even mode master equation (3.25), we see that the jump of the trace part of Eq. (4.38) becomes
\[ [\mathcal{F}_c] = 2 \cosh \tau \left\{ \sqrt{1 - H^2 \alpha^2} \left( \partial_\chi \frac{\Phi_{(e)}}{r} \right)_+ - \left( \partial_\chi \frac{\Phi_{(e)}}{r} \right)_- \right\} - \frac{1}{\alpha} \left\{ 2 \sinh \tau \partial_\tau - \frac{l(l + 1)}{2 \cosh \tau} \right\} \left[ \frac{\Phi_{(e)}}{r} \right], \quad (4.39) \]
and the jump of \(\mathcal{F}_{\parallel \parallel}\),
\[ [\mathcal{F}_{\parallel \parallel}] = \frac{2}{\alpha} \left\{ \cosh \tau \partial_\tau^2 + \sinh \tau \partial_\tau + \frac{l(l + 1)}{2 \cosh \tau} \right\} \left[ \frac{\Phi_{(e)}}{r} \right]. \quad (4.40) \]

Substituting Eqs. (4.39) and (4.40) into the condition (4.32), we obtain
\[ \left\{ \partial_\tau^2 + \frac{l(l + 1)}{\cosh^2 \tau} \right\} \left[ \frac{\Phi_{(e)}}{r} \right] = -\alpha \left\{ \sqrt{1 - H^2 \alpha^2} \left( \partial_\chi \frac{\Phi_{(e)}}{r} \right)_+ - \left( \partial_\chi \frac{\Phi_{(e)}}{r} \right)_- \right\}. \quad (4.41) \]
Next, from Eqs. (4.38) and (4.34), we obtain
\[ \partial_\tau \left( \cosh \tau \left( \frac{\Phi(e)}{r} \right) \right) = 0. \quad (4.42) \]

With the help of Eq. (3.25), from Eq. (4.38), we get
\[ [D \cdot F_c]^e = \frac{2}{\alpha^2} \left\{ \cosh \tau \partial_\tau^2 + \sinh \tau \partial_\tau + \frac{l(l+1)}{2 \cosh \tau} \left( \sqrt{1 - H^2 \alpha^2} \left( \frac{\Phi(e)}{r} \right) + \left( \frac{\Phi(e)}{r} \right) \right) \right\} \]
\[ - \frac{2}{\alpha} \left\{ \tanh \tau \partial_\tau \left( \cosh \tau \left( \frac{\Phi(e)}{r} \right) \right) - \frac{(l-1)(l+2)}{2 \cosh \tau} \left( \frac{\Phi(e)}{r} \right) \right\}. \quad (4.43) \]

Substituting Eqs. (4.39), (4.42), (4.43), and (4.44) into the condition (4.37), we obtain
\[ \partial_\chi \left( \Phi(e) \right) = 0. \quad (4.45) \]

Eqs. (4.41) and (4.45) are the boundary conditions at the bubble wall for the even mode master variable \( \Phi(e) \).

Now, let us translate the conditions (4.41) and (4.45) into the conditions on the mode expansion coefficients. Substituting the solutions (3.36) into the conditions (4.41) and (4.45), we obtain, for \( k \neq 0 \),
\[ ik \left( [A] + [B] \right) = \sqrt{1 - H^2 \alpha^2} (A_+ - B_+) + A_- - B_- , \]
\[ A - B = 0. \quad (4.47) \]

Then, using the equation
\[ 1 - \sqrt{1 - H^2 \alpha^2} = 2 \mu, \]
we finally obtain the relations of the even mode expansion coefficients,
\[ B_\pm = \frac{i \mu}{k + i \mu} A_\pm + \frac{k}{k + i \mu} A_\mp. \quad (4.49) \]

Eq. (4.49) are the same form as the odd mode case Eq. (4.26), apart from the difference of the sign of the first term. They gives the manner of reflection and transmission of incident gravitational waves by the bubble wall.

One expect that the poles of Eqs. (4.26) and (4.49) might correspond to the characteristic modes of the gravitational waves emitted by the bubble wall. Both the poles, in the present cases, are pure imaginary: \( k = -i \mu \). The behavior of the pole modes should be examined in the Milne universe \( t_- > r_- \) in accordance with the one-bubble open inflation context. By the analytic extension of \( \Phi \) with respect to the coordinates transformation (2.40), the metric perturbations \( F_{ab} \) and \( F_a \) in the Milne universe are obtained. Then, one can easily observe that \( F_{ab} \) and \( F_a \) corresponding to the pole modes diverge at the center \( r_- = 0 \). Hence, the solution \( \Phi \) of the pole modes should be excluded from the global solutions by the regularity condition at the center. For the same reason, \( k = 0 \) even mode should also be excluded. As a result, there is no characteristic mode.

Since, for odd mode perturbations, there is no variable which describes physical deformation of the bubble wall, the result for the odd modes is naturally understood. However, for even modes, \( X_- \) expresses the physical deformation of the bubble wall [12]. Its behavior is described by the even mode master variables \( \Phi(e) \) and hence the bubble wall is not a rigid surface. In fact, from Eqs. (4.30) and (4.40), it is obtained that
\[ X_- = -\frac{1}{\kappa \alpha \sigma} \left\{ \cosh \tau \partial_\tau^2 + \sinh \tau \partial_\tau + \frac{l(l+1)}{2 \cosh \tau} \right\} \left( \frac{\Phi(e)}{r} \right). \quad (4.50) \]

Thus, the result for the even modes is nontrivial. It is concluded that the behavior of the bubble wall is completely accompanied with the metric perturbations and the bubble wall has no its own dynamical degree of freedom.
It is straightforward to calculate the case that the interior of the bubble is also de Sitter spacetime which corresponds to a slow rollover inflation phase in the one-bubble open inflation scenario. In this case we can also obtain the same results, Eqs. (4.26) and (4.49), only apart from the value of $\alpha$. In this case

$$\alpha = \frac{4\kappa\sigma}{\sqrt{16(H_+^2 - H_-^2)^2 + 8\kappa^2\sigma^2(H_+^2 + H_-^2) + \kappa^4\sigma^4}}$$

where $\Lambda_\pm := 3H_\pm^2$ ($H_+^2 > H_-^2$) are the cosmological constants of the outside and the inside of the bubble (see Appendix C).

V. CONCLUSIONS AND DISCUSSION

We have analyzed the coupled system of perturbations of gravity and the bubble nucleating in de Sitter spacetime. We have adopted the thin wall approximation to treat the bubble wall. We have solved the perturbed Einstein equations on both sides of the bubble and connecting the solutions along the bubble wall by using the metric junction formalism. Thus we have obtained the global solutions of the perturbed Einstein equations in the whole spacetime. We concluded that the perturbative motion of the bubble wall is completely accompanied with the gravitational perturbations; the bubble wall oscillates only while incident gravitational waves go across. The result is essentially the same as that in Ref. [8].

In the context of the one-bubble open inflation scenario, bubble wall perturbations are investigated by several authors [13]. Recently, Tanaka and Sasaki have shown that, once the gravitational perturbations are taken into account, the bubble wall fluctuation modes disappear [14]. Our result is consistent with their work.

The behavior of the wall clarified in the present work is different from the intuitive expectation such that deformed bubble wall oscillates by its own dynamics and the oscillation damps gradually by emission of gravitational waves. As shown in Ref. [4], the perturbations on the wall can be described by a single scalar field $\phi$ on the hypersurface which represents the background motion of the wall: three-dimensional de Sitter spacetime in the present case. When we ignore the gravitational back reaction on the motion of the bubble wall, we obtain the perturbed equation of motion for the bubble wall as the Klein-Gordon wave equation on the background hypersurface,

$$\left(\Box_3 - m^2\right) \phi = 0,$$

where $\Box_3$ is the d’Alembertian on the three-dimensional background hypersurface and the mass term $m^2$ is expressed by $H^2$ and $\sigma$. This equation has oscillatory solutions and the non-spherical oscillations of the wall become a source of gravitational waves. Thus, if the influence of the gravity might be negligible, the solution of Eq. (5.1) have described the behavior of the wall well. In the present case, however, we see the qualitatively different behavior of the wall motion, taking the back reaction of the gravitational perturbations into account. Our results suggest that the back reaction of the gravitational perturbations on the motion of a domain wall should not be ignored.

The difference of our results and usual analysis may appear in the equation of motion for the bubble wall. By using the junction condition, we can easily obtain the equation of perturbative motion of the bubble wall coupled with the gravitational perturbations as the equation for $X_\perp$ on the orbit space. From the junction condition (2.25), we get

$$\delta K = 0.$$

Then substituting Eqs. (4.13), (4.15), and (4.16) into this condition, we obtain the perturbed equation of motion for the wall as

$$\left\{ -D_\perp^2 - 2\left(\frac{D_\perp C}{r}\right) D_\perp - \frac{l(l+1)}{r^2} + \frac{3}{2}H^2 + 3\left(\frac{H^2}{\kappa\sigma}\right)^2 + \frac{3\kappa^2\sigma^2}{16} \right\} X_\perp + \frac{3H^2}{2\kappa\sigma^2} \left(\mathcal{F}_\perp - \frac{1}{2}\mathcal{F}_c\right) - \frac{1}{2r^4}D_\perp \left(\frac{r^4}{2}\mathcal{F}_\perp\right) - \frac{1}{4}D_\perp \mathcal{F}_c = 0.$$
Comparing this equation with Eq. (5.1), we notice that there are additional terms described by the metric perturbations. The task which we should do is solving the coupled system of the perturbed Einstein equations and the equation of motion for the bubble wall (5.3) simultaneously. This work has been accomplished by using the metric junction formalism in the present paper. Consequently, the quantities of the second line in Eq. (5.3) are comparable to the quantities of the first line, i.e., the back reaction of the gravitational waves on the bubble wall motion is not negligible.

The equation of motion (5.3) is obtained from the junction condition which is derived from the Einstein equations. It is natural because the Einstein equations contain the equation of motion for matters. We can also derive the perturbed equation of motion (5.3) directly from the Nambu-Goto action for the domain wall. It will appear in Ref. [12].

It is interesting to study the quantum fluctuations of the bubble wall in the early universe. Garriga and Vilenkin investigated the quantization of the scalar field satisfying Eq. (5.1) [15]. However, if we consider a domain wall coupled with gravitational waves, the dynamical freedom of the wall is lost. Thus, another quantization scheme for the coupled system may be applied.

One of the reasons for the result obtained in the present paper may be the fact that the stress-energy tensor is proportional to the intrinsic metric of the wall hypersurface (see Eq. (2.23)). More general cases are under investigation.

ACKNOWLEDGMENTS

We are grateful to Dr. T. Tanaka for informative comments and fruitful discussion. We would like to thank Dr. K. Nakamura for helpful and useful discussion. We would also like to thank Professor A. Hosoya for his continuous encouragement.

APPENDIX A: TENSOR HARMONICS

Here we give the definitions and the basic properties of the tensor harmonics. We denote the metric of the unit two-sphere by

$$\Omega_{pq}dz^p dz^q = d\theta^2 + \sin^2 \theta d\phi^2,$$

(A1)

and the Laplace-Beltrami operator by $\hat{\Delta}_2 := \Omega^{pq} \hat{D}_p \hat{D}_q$.

The bases of the tensor harmonics are chosen so that each has a definite parity with respect to the transformation $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$ which maps each point on the unit sphere to its antipodal point. Under this choice the harmonics $T$ which transform as

$$T \longrightarrow (-1)^l T$$

(A2)

are called even, and those which transform as

$$T \longrightarrow (-1)^{l+1} T$$

(A3)

are called odd.

1. Scalar harmonics

The scalar harmonics are the spherical harmonic functions $Y_{lm}(z^p)$ which satisfy the equations

$$\left\{ \hat{\Delta}_2 + l(l + 1) \right\} Y_{lm} = 0,$$

(A4)

$$\partial_\phi Y_{lm} = im Y_{lm}.$$  

(A5)
They are expressed in terms of the Legendre functions as

\[
Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}} P^m_l(\cos \theta)e^{im\phi}.
\]  

These are all even.

2. Vector harmonics

The vector harmonics on the unit sphere are defined by

\[
(V^{(e)}_{lm})_p := \hat{D}_p Y_{lm} \quad \text{for even modes,}
\]
\[
(V^{(o)}_{lm})_p := \epsilon_{pq} \hat{D}_q Y_{lm} \quad \text{for odd modes,
}\]

where \(\epsilon_{pq}\) is the two-dimensional Levi-Civita antisymmetric tensor. Hereafter, the subscripts \((e)\) and \((o)\) express the even and the odd parity, respectively. Any vector on the unit sphere can be expanded by these vector harmonics. By definition, the vector harmonics satisfy

\[
\hat{D}_p (V^{(e)}_{lm})_p = -l(l + 1)Y_{lm},
\]
\[
\hat{D}_p (V^{(o)}_{lm})_p = 0,
\]
\[
\hat{D}_p (V^{(e)}_{lm})_q = 0,
\]
\[
\hat{D}_p (V^{(o)}_{lm})_q = \frac{1}{2} l(l + 1)\epsilon_{pq} Y_{lm},
\]
\[
\left(\hat{\Delta}^2 + l^2 + l - 1\right) (V_{lm})_p = 0 \quad \text{for both the even and the odd modes,}
\]

The vector harmonics vanish for \(l = 0\).

3. Tensor harmonics

Any second rank tensor field on the unit sphere can be expanded by the tensor harmonics which are defined by

\[
(T^{(e0)}_{lm})_{pq} := \frac{1}{2} \Omega_{pq} Y_{lm},
\]
\[
(T^{(e2)}_{lm})_{pq} := \left\{ \hat{D}_p \hat{D}_q + \frac{1}{2} l(l + 1)\Omega_{pq} \right\} Y_{lm},
\]
\[
(T^{(o2)}_{lm})_{pq} := \frac{1}{2} (\epsilon_{qr} \hat{D}_p + \epsilon_{pr} \hat{D}_q) \hat{D}^r Y_{lm}.
\]

The harmonic tensor \((T^{(e0)}_{lm})_{pq}\) is essentially of scalar type and satisfies

\[
\left\{ \hat{\Delta}^2 + l(l + 1) \right\} (T^{(e0)}_{lm})_{pq} = 0,
\]
\[
(T^{(e0)}_{lm})_{pp} = Y_{lm},
\]
\[
\hat{D}_q (T^{(e0)}_{lm})^{qp} = \frac{1}{2} (V^{(e)}_{lm})^p.
\]

The harmonics \((T^{(e2)}_{lm})_{pq}\) and \((T^{(o2)}_{lm})_{pq}\) are purely tensorial and satisfy

\[
\left\{ \hat{\Delta}^2 + l^2 + l - 4 \right\} (T^{(e2,o2)}_{lm})_{pq} = 0,
\]
\[
(T^{(e2,o2)}_{lm})_{pp} = 0,
\]
\[
\hat{D}_q (T^{(e2,o2)}_{lm})^{qp} = -\frac{1}{2} (l - 1)(l + 2)(V^{(e,o)}_{lm})^p.
\]
We show that the even mode perturbed Einstein equations (3.22) and (3.23) reduce to the master equation (3.25). Substituting Eq. (3.24) into Eq. (3.23), we get

\[ (D_a D_b + H^2 \gamma_{ab}) L = 0, \]  

(B1)

where

\[ L := \left\{ -r^2 \Box_2 + 2r (D^r r) D_c + (l - 1)(l + 2) \right\} \Phi_{(e)}. \]  

(B2)

Regarding Eq. (B1) as a second order differential equation for \( L \), we obtain the general solution of Eq. (B1), denoted by \( L_s \), as

\[ L_s = C_1 r + \sqrt{1 - H^2 r^2} \left( C_2 \sinh H t + C_3 \cosh H t \right), \]  

(B3)

where \( C_1, C_2, \) and \( C_3 \) are arbitrary constants. Thus, \( \Phi_{(e)} \) should satisfy

\[ \left\{ -r^2 \Box_2 + 2r (D^r r) D_c + (l - 1)(l + 2) \right\} \Phi_{(e)} = L_s. \]  

(B4)

By the transformation of \( \Phi_{(e)} \) such as

\[ \Phi_{(e)} \rightarrow \Phi'_{(e)} := \Phi_{(e)} - \frac{L_s}{(l - 1)(l + 2)} + \frac{2C_1 r}{(l - 1)(l + 1)(l + 2)}, \]  

(B5)

we obtain

\[ L' := \left\{ -r^2 \Box_2 + 2r (D^r r) D_c + (l - 1)(l + 2) \right\} \Phi'_{(e)} = 0. \]  

(B6)

The transformation (B5) does not change \( F_{ab} \). Thus, rewriting \( \Phi'_{(e)} \) by \( \Phi_{(e)} \), we can reduce Eq. (B6) to the equation for \( \Phi_{(e)} \) of the form

\[ \left( -\Box_2 + \frac{l(l + 1)}{r^2} \right) \left( \frac{\Phi_{(e)}}{r} \right) = 0. \]  

(B7)

APPENDIX C: DE SITTER - DE SITTER CASE

Let us consider the case that the inside of the bubble is also de Sitter spacetime, where the cosmological constants of the outside (inside) are \( \Lambda_{+(-)} \equiv 3H_{+(-)}^2 \) \( (H_{+}^2 < H_{-}^2) \). From Eq. (2.11), the \( (p, q) \) components of the extrinsic curvatures are

\[ (K^q_p)_{p \pm} = - \sqrt{\frac{1 - H_{\pm}^2 r_{\pm}^2 + r_{\pm}^2}{r_{\pm}}} \delta^q_p. \]  

(C1)

Substituting Eq. (C1) into the junction condition (2.24), we obtain the equation of the bubble wall motion

\[ 1 + r^2 = \left( \frac{r}{\alpha} \right)^2, \]  

(C2)

where

\[ \alpha = \frac{4\kappa \sigma}{\sqrt{16(H_{+}^2 - H_{-}^2)^2 + 8\kappa^2 \sigma^2 (H_{+}^2 + H_{-}^2) + \kappa^4 \sigma^4}}. \]  

(C3)

The solution is
\[ r = \alpha \cosh \tau, \]  
\[ \text{(C4)} \]

where \( \tau \) is normalized by \( \alpha \). This conforms to the O(3, 1) symmetry of the bubble wall.

As in the Minkowski–de Sitter case, we can consider the metric perturbations of the geometry in terms of the gauge-invariant variables introduced in the section 3. In the hyperbolic charts defined by Eq. (2.39) with

\[ a(\chi_\pm) := \frac{1}{H_\pm} \sin H_\pm \chi_\pm, \quad a(\chi_\pm)|_\Sigma = \alpha, \]  
\[ \text{(C5)} \]

we obtain the master equations (3.21) and (3.25) for the master scalar \( \Phi \) and solutions

\[ \left( \frac{\Phi}{r} \right)_\pm = \int_{-\infty}^{\infty} dk \psi_l(\tau, k) \left\{ A_\pm(k)e^{ik\zeta_\pm} + B_\pm(k)e^{-ik\zeta_\pm} \right\}, \]  
\[ \text{(C6)} \]

where

\[ \zeta(\chi_\pm) := \mp \ln \left\{ \tan \left( \frac{H_\pm \chi_\pm}{2} \right) \right\}. \]  
\[ \text{(C7)} \]

The odd mode junction conditions for \( \Phi^{(o)} \) are described as

\[ \left[ D_\perp \left( \frac{\Phi^{(o)}}{r} \right) \right] = \frac{1}{2} \kappa \sigma \left( \frac{\Phi^{(o)}}{r} \right), \]  
\[ \text{(C8)} \]

\[ \left[ \frac{\Phi^{(o)}}{r} \right] = 0. \]  
\[ \text{(C9)} \]

These are the same forms as the Minkowski–de Sitter case. In terms of the expansion coefficients \( A_\pm \) and \( B_\pm \) of the solution, these conditions reduce to

\[ B_\pm = -\frac{i\mu}{k + i\mu} A_\pm + \frac{k}{k + i\mu} A_\mp. \]  
\[ \text{(C10)} \]

For even modes, the junction conditions of the metric perturbation expansion coefficients are

\[ [\mathcal{F}_c] = 2\kappa \sigma X_\perp, \]  
\[ \text{(C11)} \]

\[ [\mathcal{F}_\parallel] = -2\kappa \sigma X_\perp, \]  
\[ \text{(C12)} \]

\[ [\mathcal{F}_{\perp\perp}] = 0, \]  
\[ \text{(C13)} \]

\[ [\mathcal{F}_\perp] = 0, \]  
\[ \text{(C14)} \]

\[ [D_\perp \mathcal{F}_c] = -\frac{H_\perp^2 - H_\parallel^2}{\kappa \sigma} [\mathcal{F}_c] - \kappa \sigma \left( \mathcal{F}_{\perp\perp} - \frac{1}{2} \mathcal{F}_c \right). \]  
\[ \text{(C15)} \]

Rewriting these conditions in terms of the even mode master variable \( \Phi^{(e)} \) and combining them, we obtain the following constraints for \( \Phi^{(e)} \),

\[ \left\{ \partial_\tau^2 + \frac{l(l+1)}{\cosh^2 \tau} \right\} \left[ \frac{\Phi^{(e)}}{r} \right] = -\alpha \left\{ \sqrt{1 - H_\perp^2 \alpha^2} \left( \partial_\chi \frac{\Phi^{(e)}}{r} \right)_+ - \sqrt{1 - H_\parallel^2 \alpha^2} \left( \partial_\chi \frac{\Phi^{(e)}}{r} \right)_- \right\} = 0, \]  
\[ \text{(C16)} \]

\[ (l - 1)(l + 2) \left[ \partial_\chi \frac{\Phi^{(e)}}{r} \right] = 0. \]  
\[ \text{(C17)} \]

These constraints yield the following relations of the mode expansion coefficients

\[ B_\pm = \frac{i\mu}{k + i\mu} A_\pm + \frac{k}{k + i\mu} A_\mp. \]  
\[ \text{(C18)} \]

Eqs. (C10) and (C18) are the same forms as Eqs. (4.26) and (4.49), respectively.


FIG. 1. The Penrose diagram of the one-bubble open inflation universe.

FIG. 2. The Penrose diagram of the background geometry. Minkowski spacetime “$M_-$” and de Sitter spacetime “$M_+$” are connected at the bubble wall $\Sigma$ (thick line). The timelike dashed lines denote $\chi = \text{const.}$ hypersurfaces and the spacelike dashed lines in $M_-, \eta = \text{const.}$ hypersurfaces, which correspond to open FRW time-slices.

$M_+ : \text{de Sitter side}$

$M_- : \text{Minkowski side}$
FIG. 3. The block enclosed by dashed lines, the Rindler horizons, is covered by the hyperbolic (the spherical Rindler) charts. The thick line represents the bubble wall $\Sigma$. The mode expansion coefficient $A_+ (A_-)$ represents the wave propagating from the past Rindler horizon in $M_+ (M_-)$ toward the bubble wall $\Sigma$. Similarly, $B_+ (B_-)$ represents the wave from $\Sigma$ toward the future Rindler horizon in $M_+ (M_-)$. 