Boundaries of 11-Dimensional Membranes

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Abstract
The action for an 11-dimensional supermembrane contains a chiral Wess–Zumino–Witten model coupling to the E_8 super-Yang–Mills theory on the end-of-the-world 9-brane. It is demonstrated that this boundary string theory is dictated both by gauge invariance and by κ-symmetry.
Hořava and Witten [1] have described the heterotic and type I string theories as a compactification of M-theory [2] on $S^1/\mathbb{Z}_2$ (an interval). The cancellation of anomalies in the low-energy 11-dimensional supergravity on this space relies on the introduction of an extra $E_8$ super-Yang–Mills multiplet on each component of the boundary.

At each boundary 9-brane, only the field components survive that are even under reflections in the boundary. This projects the 11-dimensional supergravity [3] down to type I supergravity, to which the super-Yang–Mills fields on the boundary couple. The 3-form field strength $h$, which is what survives of the 4-form $H$ on the boundary, picks up an anomalous term in its Bianchi identity, $dh \sim \text{tr} F^2$, as is necessary in the coupling of super-Yang–Mills to type I supergravity [4]. This in turn implies boundary conditions on the $H$ field,

$$H|_{\partial\mathcal{M}} = -\frac{\kappa^2}{2\lambda^2} F^2$$

(1)

(the normalization of $H$ adopted here differs from ref. [1] by a factor $\sqrt{2}$, and is the natural one from a superspace point of view). It should be pointed out, that eq. (1), which so far has been seen as a boundary condition for the bosonic 4-form, now is promoted to a superspace identity, with the former as its leading component. Any transformation involving the super-Yang–Mills fields induces a transformation of the $H$ field. In ref. [1], this is used to verify anomaly cancellation in the entire model, including both the bulk and boundary fields (although some questions remain about the gravitational part). As a consequence, the gravity and gauge couplings must be related as $\lambda^3 = (2\pi)^{5/2} 2 \kappa^2$. They are both related to the string tension $(2\pi\alpha')^{-1}$ as $2\kappa^2 = (2\pi)^8 \alpha'^{9/2}$ and $\lambda^3 = (2\pi)^7 \alpha'^3$ [5].

In this note, the analogous mechanism for the supermembrane propagating in the 11-dimensional supergravity background will be studied, in the case where the membrane ends on the 10-dimensional boundary.

The ordinary action for the supermembrane [6] is

$$I_0 = -T \int_{\mathcal{M}} d^3\xi \sqrt{-g} + T \int_{\mathcal{M}} C.$$ 

(2)

where the membrane tension is $T = (2\pi)^{-2} \alpha'^{-3/2}$, and where all world-volume fields are constructed as pullbacks of superfields in 11-dimensional supergravity. The action (2) is invariant under $\kappa$-symmetry, as long as there are no boundaries present, thus ensuring that the number of physical fermions on the world-volume equals the number of transverse oscillations. When there is a boundary $\partial\mathcal{M}$, there must be additional terms to cancel both the gauge anomaly induced by (1) and the $\kappa$-symmetry anomaly. Both contributions originate from the Wess–Zumino term in the action.
After using the relations among the coupling constants, one finds that the potential $C$ close to the boundary behaves as a Chern–Simons form (modulo total derivatives):

$$C|\partial M = -\frac{1}{8\pi T} \Omega_{\text{CS}}(A) = -\frac{1}{8\pi T} \text{tr} \left( dA + \frac{2}{3} A^3 \right).$$  \hspace{1cm} (3)

The gauge anomaly of the membrane action is therefore

$$\delta_A I_0 = -\frac{1}{8\pi} \int_{\partial M} \text{tr} \Lambda dA .$$  \hspace{1cm} (4)

The $\kappa$-symmetry, on the other hand, transforms $C$ as $\delta'_\kappa C = \mathcal{L}_\kappa C = (i_\kappa d + di_\kappa)C$ (the prime is only to distinguish gauge and $\kappa$-transformations), and the presence of a boundary generates the anomaly

$$\delta'_\kappa I_0 = -\frac{1}{8\pi} \int_{\partial M} i_\kappa C = -\frac{1}{8\pi} \int_{\partial M} i_\kappa \Omega_{\text{CS}}(A) .$$  \hspace{1cm} (5)

It is convenient to consider instead the modified $\kappa$-transformation $\Delta_\kappa = \delta'_\kappa - \delta_{i_\kappa} A$, under which the anomaly becomes

$$\Delta_\kappa I_0 = -\frac{1}{8\pi} \int_{\partial M} \text{tr} i_\kappa FA .$$  \hspace{1cm} (6)

We will demonstrate that both these anomalies are cancelled by contributions from a chiral level one Wess–Zumino–Witten model living on the boundary string of the membrane.

The coupling of a Wess–Zumino–Witten model [7] to external gauge fields can be accomplished by replacing the left-invariant Maurer–Cartan 1-form $\omega = g^{-1} dg$ by $g^{-1} Dg = \omega - A$, $D$ being the covariant derivative for the right action of the gauge group. Gauge invariance under

$$\delta_A A = DA ,$$
$$\delta_A g = gA ,$$  \hspace{1cm} (7)

also requires the inclusion of a Wess–Zumino term and a Chern–Simons term for $A$, and the invariant action reads

$$I_A = \frac{1}{8\pi} \int_{\partial M} \text{tr} \left( \frac{1}{2} (\omega - A)^*(\omega - A) - \omega A \right) + \frac{1}{8\pi} \int_{M} \left( \Omega_{\text{CS}}(\omega) - \Omega_{\text{CS}}(A) \right)$$
$$\equiv I_1 - \frac{1}{8\pi} \int_M \Omega_{\text{CS}}(A) .$$  \hspace{1cm} (8)

The last term is of course impossible in the present context, where $A$ lives only on the boundary, but it plays exactly the same rôle as the Wess–Zumino term in the membrane action, so by leaving
it out we have verified gauge invariance of the full membrane action $I = I_0 + I_1$:

$$I = - T \int d^3 \xi \sqrt{-g} + \int d^3 \xi \left( TC + \frac{1}{8\pi} \Omega_{CS}(\omega) \right) + \frac{1}{8\pi} \int d^3 \xi \, \text{tr} \left( \frac{1}{2} (\omega - A) \ast (\omega - A) \right). \quad (9)$$

It may be in order to comment on chirality of the Wess–Zumino–Witten model. The ordinary Wess–Zumino–Witten action, obtained from (8) by setting $A$ to zero, leads to the equations of motion for $g$: $\partial_- j_+ = 0 = \partial_+ j_-$, where $j_+ = \partial_+ g^{-1}$, $j_- = g^{-1} \partial_- g$, and where a 1-form $\beta$ is decomposed in selfdual and anti-selfdual components as $\beta_\pm d\xi = \frac{1}{2} (\beta \pm \ast \beta)$. The two equations are equivalent. The solution is $g(\xi^+, \xi^-) = g_L(\xi^+) g_R(\xi^-)$, and it may consistently be truncated by setting (e.g.) $j_+ = 0$. When the $A$ field is present, the two equivalent versions of the equations of motion for $g$ become $\partial_- J_+ = 0 = D_+ J_+ + F_{+-}$, where $J_+ = D_+ g^{-1}$, $J_- = g^{-1} D_- g$. Now, the only possible chirality constraint is $J_+ = 0$, since $J_-$ couples to the background fields. This is the model under consideration here. The mechanism is completely analogous to the one at work for free chiral bosons or selfdual forms in $4n+2$ dimensions [8]. The chirality constraint is invariant under all symmetries considered.

The $\kappa$-transformation of $A$ is given as $\delta'_\kappa A = \mathcal{L}_\kappa A$, $A$ being the pullback of the background superfield. The transformation of $g$ has to be guessed, but there are not many possibilities. If $\delta'_\kappa g = gi_\kappa A$, the modified $\kappa$-transformations act as

$$\Delta_\kappa A = i_\kappa F,$$
$$\Delta_\kappa g = 0 \quad (10)$$

(since $g$ is defined in the bulk and $A$ only on the boundary, it is actually necessary to use the modified transformation $\Delta_\kappa$ for the $\kappa$-transformations to be well defined). The transformation of the new terms in the membrane action is then

$$\Delta_\kappa I_1 = \frac{1}{8\pi} \int d^3 \xi \, \text{tr} \left( (i_\kappa F + \ast i_\kappa F) \omega - \ast i_\kappa FA \right), \quad (11)$$

which obviously cancels against eq. (6) if $\ast i_\kappa F = -i_\kappa F$. This is the last condition needed, and it follows from the equations of motion for the background superfields (these are of course expected to be needed for $\kappa$-symmetry) as follows.

In the 10-dimensional type I superspace associated with the boundary, $F$ is a superspace 2-form. There is not any known off-shell superspace formulation of 10-dimensional super-Yang–Mills, but by imposing a set of constraints on $F$, the field content is reduced down to the correct one and the theory is put on-shell. With $a, b, \ldots$ and $\alpha, \beta, \ldots$ denoting inertial frame vector and spinor indices, respectively, the relevant constraint is $F_{\alpha\beta} = 0$. Once it is imposed, the Bianchi identity
\[(DF)_{\alpha\beta\gamma} \sim (\gamma_a)_{(\alpha\beta} F^{a\gamma)} = 0\] may be solved as \(F_{aa} = (\gamma_a)_{\alpha}^{\beta} \bar{\psi}_\beta\). The parameter \(\kappa\) is constrained to obey
\[
\kappa = \frac{1}{2}(1 + \Gamma)\kappa, \quad \Gamma = -\frac{1}{2\sqrt{-g}} \varepsilon^{ij} \gamma_{ij}, \tag{12}
\]
on the boundary, and this implies that
\[
(i\kappa F)_i = (\bar{\psi}_i \gamma_i \kappa) = -\frac{1}{\sqrt{-g}} g_{ij} \varepsilon^{jk} (\bar{\psi}_j \gamma_k \kappa) = -(*_{i\kappa} F)_i, \tag{13}
\]
which completes the proof of \(\kappa\)-symmetry.

We have found that the incorporation of a chiral level 1 \(E_8\) Wess–Zumino–Witten model on the boundary \(\partial \mathcal{M}\) of the world-volume \(\mathcal{M}\) of an 11-dimensional supermembrane ending on an end-of-the-world 9-brane of M-theory cancels the apparent gauge and \(\kappa\)-symmetry anomalies of the bulk theory of the membrane. The cancellations rely on the same relation between coupling constants that was used for the anomaly cancellation in the background theory. The boundary conformal field theory is the membrane analog of string endpoint Chan–Paton factors. As in the context of the heterotic string, there are of course other possible formulations of the boundary theory; the chiral current may e.g. be fermionized. It is however striking that the 3-manifold with the string as a boundary, introduced in ref. [7], from being a mathematical construction, now obtains a concrete reality as the membrane world-volume. One aesthetic feature is that instead of introducing a separate \(E_8\) for each boundary string, one naturally gets a single \(E_8\) on the union of the boundaries.

There are a number of questions that remain unanswered. One is the microscopic M-theory calculation verifying the exact form of the Lorentz anomalies that was left out in ref. [1] (although specific predictions were made). Another one is the question to which extent this modified supermembrane theory can be seen as a microscopic origin of the boundary super-Yang–Mills. It is conceivable that the boundary string theory, due to its bulk interaction becomes “critical”, and that its massless spectrum contains exactly the 10-dimensional super-Yang–Mills fields. The analogous question could be raised for supermembranes ending on M-theory 5-branes [9], where the massless selfdual tensor multiplet is believed to be related to a 6-dimensional string theory without gravitational coupling. There seems to be a deep connection between M-theory and selfduality in various dimensions.
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