Winding Solutions for the two Particle System
in 2+1 Gravity

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Abstract

Using a PASCAL program to follow the evolution of two gravitating particles in 2+1 dimensions we find solutions in which the particles wind around one another indefinitely. As their center of mass moves ‘tachyonic’ they form a Gott-pair. To avoid unphysical boundary conditions we consider a large but closed universe. After the particles have evolved for some time their momenta have grown very large. In this limit we quantize the model and find that both the relevant configuration variable and its conjugate momentum become discrete.

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1 Introduction

7 Years after the paper of Deser, Jackiw and ’t Hooft on 2+1 dimensional gravity [2], Gott published an article in which he constructed a space time containing closed time like curves (CTC) without introducing ‘exotic matter’ [4]. The basic idea was that two cosmic strings in 3+1 dimensional gravity, or particles in 2+1 dimensional gravity, approaching each other with high velocity and small impact parameter could produce these CTC’s. It was however recognized quickly thereafter that the CTC is not confined to a small region of space time but also exists at spatial infinity [3]. This fact implies that Gott’s space time violates ‘physical boundary conditions’ that should be imposed at infinity. Another way of saying it is that the center of mass (c.o.m.) of the two particles is ‘tachyonic’, although the strings themselves are particlelike. Tachyonic means that the energy momentum vector is spacelike. So physical reasonable boundary conditions mean that the total energy momentum of the universe is timelike. Next, Cutler showed that one can find a family of Cauchy surfaces, prior to the existence of the CTC’s, from which they evolve [6]. This implies that evolution of initial data from these Cauchy surfaces must run into a Cauchy Horizon. Another interesting question that arose was whether it was possible to start with a space time of timelike particles and accelerate two particles to high velocities in such a way that they form a Gott-pair. It was found in [5] that there is never enough energy in an open universe to achieve this. In a closed universe however the Gott-pair can be formed and initially it seemed possible to construct CTC’s again. But now the ‘chronology protection conjecture’ [23] was saved by a very different effect. ’t Hooft showed that a closed universe containing Gott-pairs will crunch before the CTC’s can be fully traversed. Once the Gott-pair has come into existence the cosmic strings start to spiral around one another with ever increasing velocity until they reach the ‘end of the universe’ causing it to crunch.

In this paper we reproduce this winding solution (in a slightly different way) using ’t Hooft’s Cauchy formulation. Despite the fact that the strings move subluminally, the c.o.m. moves ‘tachyonic’, i.e. faster than the speed of light. If we want to avoid the unphysical asymptotic boundary conditions we must close the universe, following the philosophy of ’t Hooft:

"A safe way to consider open universes is to view them as limiting cases of infinitely large closed spaces."

This means that we start with one of Cutler’s Cauchy surfaces and evolve the initial data from it until we approach the Cauchy Horizon. But in the philosophy explained above this really implies that the universe has crunched before we arrive at this horizon. This big crunch will happen in the very far future however if we choose our universe big enough.

In the last section we study the Hilbert-space of the two particles (treating the
rest of the universe at ‘infinity’ classically) and find that it is finite dimensional.

2 Two Particles described by Polygon Variables

A simple way to describe a particle with mass $M$ in 2+1 dimensional gravity was already discovered by Staruszkiewicz in 1963 [1]. Let’s suppose this particle is sitting at the location $a$. The effect of its gravitational field is described by cutting out a wedge from space time and identifying the boundaries. The angle of the excised region is $8\pi GM$. From now on we will set $8\pi G = 1$. The identification is done by a simple rotation:

$$x' = a + R(M)(x - a)$$

A moving particle is described by boosting this solution, using a boostmatrix $B(\xi) \in SO(2,1)$ (see figure 1). The velocity of the particle is $v = \tanh \xi$. Now we have:

$$x' = a + B(\xi) \; R(M) \; B^{-1}(\xi)(x - a)$$

Notice that if we choose the wedge behind (or in front) of the particle the identification rule has no time jump. By Lorentz contraction the excised angle is larger than is the case for a particle at rest. It turns out that this total angle represents the energy of the particle. It is given by:

$$\tan \frac{H}{2} = \cosh \xi \; \tan \frac{M}{2}$$

The fact that the energy grows as we enhance the velocity of the particle makes sense because we add kinetic energy to the particle. The next step is to describe two particles at positions $a_1$ and $a_2$ with arbitrary velocities $v_1$ and $v_2$. Now
cut out the wedges in front or behind the particles in such a way that they meet somewhere. The situation is pictured in figure (2). The point where the boundaries of the wedges meet is called a vertex. The idea is now that after this vertex we continue our cutting in such a way as if there were an effective center of mass (c.o.m.) particle with a certain speed and spin (which equals the total angular momentum of the system). Consider the point $b_1$ and $b_2$ in the figure. They are related by the identification rule:

\[ b_1 = B_1 R_1 B_1^{-1} \{ B_2 R_2 B_2^{-1} (b_2 - a_2) + a_2 \} - a_1 \]  

(4)

This can be written as:

\[ b_1 = B_T R_T B_T^{-1} (b_2 - d) + d + B_T \ell \]  

(5)

where $d$ is the location of the c.o.m. particle, $B_T$ is the total boost determined by the velocity of the c.o.m. particle, $R_T$ is a rotation over an angle representing the energy of the system in the c.o.m. frame and $\ell = (l, 0, 0)$ where $l$ is the total angular momentum of the system in the c.o.m. frame (see [20]). By comparing (4) and (5) we can express these quantities in the parameters of the individual particles. If we demand that there are no time jumps in (5), i.e. $t_1 = t_2$, we can calculate the location of the boundary of $L_3$. The angle of this wedge represents the total energy of this system $H_T$. The total angular momentum is represented as a spatial shift over the boundary. We mention that we can only avoid time jumps if we are not in the c.o.m. frame. We will introduce the polygon variables $\eta_i$ and $L_i$ in the following. $L_i$ is given by the length of the wedge behind (or in front) of particle $i$. $L_3$ is taken to be the length of the boundary of the total wedge. $\eta_i$ is the rapidity ($v_L = \tanh \eta$) of the boundary which moves perpendicular to itself (see figure (1)). We can express the total energy in terms of $\eta_i$ only. It is given by:

\[ H_T = H_1 + H_2 + 2 \pi - \alpha_1 - \alpha_2 - \alpha_3 \]  

(6)

where

\[ H_i = 2 \arccos \left( \frac{\cos \frac{M_i}{2}}{\cosh \eta_i} \right) \]  

(7)

and

\[ \cos \alpha_1 = \frac{\gamma_1 - \gamma_2 \gamma_3}{\sigma_2 \sigma_3} \]  

(8)

\[ \cos \alpha_2 = \frac{\gamma_2 - \gamma_1 \gamma_3}{\sigma_1 \sigma_3} \]  

(9)

\[ \cos \alpha_3 = \frac{\gamma_3 - \gamma_1 \gamma_2}{\sigma_1 \sigma_2} \]  

(10)

where we have defined:

\[ \gamma_i = \cosh(2 \eta_i) \quad \sigma_i = \sinh(2 \eta_i) \]  

(11)
Figure 2: Two particles which cut out a piece of space time.

The relation among the angles $\alpha_i$ and momenta $\eta_j$ can be calculated using the fact that there is no mass at the vertex, so space time should be flat there. We did however introduce a two dimensional curvature at the vertex as the angles $\alpha_i$ do in general not add up to $2\pi$. This then must be compensated by extrinsic curvature to produce a flat three dimensional space time. So if we move a Lorentz vector around the vertex it should not be Lorentz transformed after returning to its original position. This is expressed as follows:

$$I = R(\alpha_2) \ B(2\eta_3) \ R(\alpha_1) \ B(2\eta_2) \ R(\alpha_3) \ B(2\eta_1)$$  \hspace{1cm} (12)

From this we can deduce vertex relations \cite{8} which enable us to calculate for instance the angles $\alpha_i$ from the momenta $\eta_i$. If we use $H_T(\eta_i)$ we can verify that $L_i$ and $2\eta_i$ are canonically conjugate variables, i.e.:

$$\frac{d}{dt}L_i = \{H_T, L_i\} = \frac{1}{2} \frac{\partial H_T}{\partial \eta_i}$$  \hspace{1cm} (13)

$$\frac{d}{dt}\eta_i = \{H_T, \eta_i\} = -\frac{1}{2} \frac{\partial H_T}{\partial L_i} = 0$$  \hspace{1cm} (14)

It will furthermore be important that the $\eta_i$ fulfill a triangle inequality:

$$|\eta_i| + |\eta_j| \geq |\eta_k| \text{ and cyclic permutations}$$  \hspace{1cm} (15)

Let us choose a global Lorentz frame, i.e. we fix the velocity of the c.o.m. frame. This amounts to fixing the value of $\eta_3$. What are now the possible values of $\eta_1$ and $\eta_2$ if we take the triangle inequalities into account properly. This is best seen in figure (3). The coloured region contains the allowed values for $\eta_i$. A negative sign for $\eta$ means that the edges are moving inward instead of outward. This is indicated with an arrow: if the arrow is pointing away from the vertex
the sign of \( \eta \) is positive and vice versa. We may use as a rule of thumb that the arrows considered as vectors must add up to zero. If all the signs of the \( \eta_i \) are the same (i.e. the same as the fixed sign of \( \eta_3 \)) then all angles \( \alpha_i \) are smaller then \( \pi \) and we are in region A. If one of the signs differs from its neighbours, the opposite angle \( \alpha \) is larger than \( \pi \) as is the case in region B, C and D. We chose a fixed orientation for particle 1 and 2 but the opposite orientation is of course also possible. If the system evolves, two kind of transitions can occur if one of the lengths \( L_i \) shrinks to zero. The transitions between C and B or D are easy and orientation preserving. We will call them type-1 transitions in the following. Let us take for instance the transition from C to B. In this case only the sign of \( \eta \) changes while absolute value remains unchanged. The values and signs of \( \eta_2 \) and \( \eta_3 \) remain fixed. This implies that \( \alpha_2 \to \alpha_2 + \pi \) and \( \alpha_3 \to \alpha_3 - \pi \) according to the 'sign rule' above. A more interesting transition occurs between region A and B (or D). We will call these transitions type-2 in the following. Let us consider a transition from A to B. Because this transition changes orientation we have to interchange particle 1 and 2 in region B. As particle 1 hits the edge it appears on the other side, but its speed is boosted according to:

\[
\gamma'_1 = \gamma_2 \gamma_3 + \sigma_2 \sigma_3 \cos(\alpha_2 + \alpha_3 - H_1)
\]

(16)

where \( \alpha_2, \alpha_3 \) and \( H_1 \) are given in terms of \( \eta_i \). The values and signs of \( \eta_2 \) and \( \eta_3 \) remain unchanged. It seems strange that particle 1 is boosted while the momentum of particle 2 stays the same. Where does the energy come from? The answer is that there is gravitational energy stored in the three-vertex which is transformed into kinetic energy for particle 1. (16) is one of the transition rules that can be derived from (12) [8]. After the new rapidities have been calculated we can use (8,9,10) to calculate the new angles.
Summarizing we can say that the system evolves for a while linearly (the $L_i$ change linear in time) after which a transition takes place. This process may repeat itself for some time. For low energy scattering we expect only a few transitions to occur. For high energy processes however we will find that the particles can wind around each other indefinitely.

### 3 The winding Solution

In order to follow the evolution of this two particle system we wrote a program in PASCAL. Typically one starts in region C of figure (3) where both particles move towards the vertex. Depending on whether particle 1 or particle 2 hits the vertex first we make a type-1 transition into region B or D respectively. After that we move into sector A by a type-2 transition. For low energies the particles, once they are in region A, move apart linearly in time, i.e. they are scattered. Because of the deficit angles these particles are deflected by some scattering angle. If we add more energy to the system we also find solutions that wind around one another for some time after which they break free and move apart. If we add so much energy to the system that the total energy exceeds $\pi$ (but is less than $2\pi$) we find that there are many solutions that wind around each other indefinitely. In other words, the particles are trapped in their own gravitational field.

Let us briefly describe how the computer program works. First it determines in what sector the particles are. It then calculates the values of $\dot{L}_1$ and $\dot{L}_2$. It is important that $\dot{L}_i$ has two contributions, one from the fact that the particles move through Minkowski space and one from the fact that the vertex moves. For $\dot{L}_1$ we find:

$$
\dot{L}_1 = \frac{1}{2} \frac{\partial H_T}{\partial \eta_1} = \frac{1}{2} \frac{\partial H_1}{\partial \eta_1} - \frac{1}{2} \frac{\partial}{\partial \eta_1} (\alpha_1 + \alpha_2 + \alpha_3) \\
= \frac{\cos(M_2^{\frac{1}{2}}) \tanh \eta_1}{\sqrt{\cosh^2 \eta_1 - \cos^2 M_2}} + \text{sign}(\sin \alpha_2) \sqrt{\frac{\gamma_1 - 1}{\gamma_1 + 1}} \frac{(1 + \gamma_1 - \gamma_2 - \gamma_3)}{\sqrt{1 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 + 2 \gamma_1 \gamma_2 \gamma_3}}
$$

The formula for $\dot{L}_2$ can be obtained from (17) by interchanging all indices 1 and 2. The first term is clearly due to the velocity of the particle itself. Notice that for large values of $\eta_1$ this velocity is suppressed like $\dot{L}_1 \sim e^{-\eta_1}$. The second term is the contribution from the vertex. For large $\eta_{1,2}$ this term survives and its value depends on the difference $(\eta_1 - \eta_2)$ (see (26,27)). We find that for small values of $\eta_i$ only the first term is important and for large values the second term dominates.

After the program has calculated the velocities $\dot{L}_i$ it determines what kind of transition will occur. In the case of a type-1 transition it only changes the sign of the $\eta$ that is involved in the transition. Furthermore it changes two angles by $\pi$ as was explained in section 1. In case of a type-2 transition it uses equation (16) to calculate the new value for $\eta$ (or $\gamma$). Of course all angles $\alpha_i$ change in such a
way that the total energy is conserved. This change is calculated using (8,9,10).

As the kinetic energy of the particle is increased some energy from the vertex is
transferred to the particle that is boosted. After this it reevaluates the velocities
and determines whether they will move to infinity or wind yet another time.

Now let’s assume we are in sector A, \( \dot{L}_1 < 0 \) and \( \dot{L}_2 > 0 \). The next transition
will involve particle 1. After the transition we have two possibilities; either \( \dot{L}_1 > 0 \)
and \( \dot{L}_2 > 0 \) which implies that the particles break free, or \( \dot{L}_1 > 0 \) and \( \dot{L}_2 < 0 \)
which implies that they will wind another time. As it turns out, once we are
in region A and we choose \( \eta_{1,2} \) large enough and not equal, we have solutions
that wind indefinitely. We will now investigate this behaviour closer. First we
introduce new phase space variables in the following way:

\[
p_{\pm} = \eta_1 \pm \eta_2 \\
L_{\pm} = L_1 \pm L_2
\]

If we substitute these variables in the expression for the total Hamiltonian (6)
and take the limit for large \( \eta_{1,2} \), i.e. \( p_+ \gg 1 \), we find:

\[
H_T = 3\pi - \arccos\left(\frac{e^{2p} - \gamma_3}{\sigma_3}\right) - \arccos\left(\frac{e^{-2p} - \gamma_3}{\sigma_3}\right) + O(e^{-\frac{1}{2}p_-})
\]

or

\[
\cos \frac{H_T}{2} = -\frac{\cosh p_-}{\cosh \eta_3} + O(e^{-\frac{1}{2}p_-})
\]

First off all we notice that \( H_T \) is inevitably in the range \( H_T \in [\pi, 2\pi] \). Secondly,
the first term that contains \( p_+ \) is exponentially suppressed! It is also important
to keep in mind that \( p_- \) is restricted by the triangle inequalities to the range:

\[
p_- \in [-\eta_3, \eta_3]
\]

as can be seen in figure (3) by rotating it over 45 degrees. We also see that in
region A we have:

\[
p_+ \in [\eta_3, \infty)
\]

The fact that \( p_- \) is restricted to a ‘Brillouin zone’ will cause quantization of \( L_- \) in
the quantum theory. Let us now return to the transitions. In the case of type-2
we find the following result in the limit \( p_+ \gg 1 \):

\[
p'_- = -p_- + O(e^{-\frac{1}{2}p_-})
\]

So once we are in region A the transition will not kick us out of the allowed range.
We can also deduce what \( p_+ \) does:

\[
p'_+ = p_+ + 2|p_-| + O(e^{-\frac{1}{2}p_+})
\]
As the $\eta_i$ grow, the actual speed of the particles increases until it reaches the speed of light: $v_i \simeq 1$. In the limit for $p_+ \gg 1$ we have for $\dot{L}_i$:

\begin{align}
\dot{L}_1 &= \frac{\sinh p_-}{\sqrt{\cosh^2 \eta_3 - \cosh^2 p_-}} + O(e^{-\frac{1}{2}p_+}) \\
\dot{L}_2 &= \frac{-\sinh p_-}{\sqrt{\cosh^2 \eta_3 - \cosh^2 p_-}} + O(e^{-\frac{1}{2}p_+}) \\
\dot{L}_3 &= \frac{-\cosh p_-}{\sqrt{\cosh^2 \eta_3 - \cosh^2 p_-}} + O(e^{-\frac{1}{2}p_+})
\end{align}

(26)

(27)

(28)

We notice that:

\[ \dot{L}_+ = 0 + O(e^{-\frac{1}{2}p_+}) \]  

(29)

This was to be expected as in this limit $H_T$ only depends on $p_-$. All dependence on $p_+$ is exponentially suppressed as $p_+$ grows. This implies that we may view $L_+$ as a fixed parameter in this limit. Its value depends on the initial conditions. The only configuration variable to survive is $L_-$ and we find:

\[ \dot{L}_- = \frac{2 \sinh p_-}{\sqrt{\cosh^2 \eta_3 - \cosh^2 p_-}} + O(e^{-\frac{1}{2}p_+}) \]  

(30)

Next we turn to construct a geometric picture of what is going on in this limit. Therefore we calculate all angles $H_i$ and $\alpha_i$:

\begin{align}
H_{1,2} &= \pi + O(e^{-\frac{1}{2}p_+}) \\
\alpha_3 &= \pi + O(e^{-p_+}) \\
\alpha_{1,2} &= \arccos\left(\frac{e^{2p_- - \gamma_3}}{\sigma_3}\right) + O(e^{-2p_+})
\end{align}

(31)

(32)

(33)

Putting this information together we construct a picture of (the finite part of) our universe: see figure (4). Firstly we notice that $H_T \in [\pi, 2\pi]$. As $H_{1,2}$ and $\alpha_3$ are equal to $\pi$ it follows that $L_1$ and $L_2$ form a straight line. The total length of this line is $2L_1 + 2L_2 = 2L_+$. As we have seen this must be a constant in this limit.

The distance between the particles is equal to $L_+$. We also know that $L_3$ moves perpendicular to itself with a velocity $v_3 = \tanh \eta_3$ and the particles move with the speed of light perpendicular to the above mentioned line ($\eta_{1,2} \to \infty$, $\xi_i \to \infty$).

Using this information we can construct the location of the $L_i$ and the particle’s positions at a time step later. The angles $\alpha_i$ remain constant because they depend only on the momenta $\eta_i$. At $t = t_2$ we see that the location of the * has moved closer to particle 2. This implies that $L_- = L_1 - L_2$ has grown according to (30). If we let the system evolve to $t = t_3$, $L_2$ has shrunk to zero and must make a transition of type-2 (24). We see from the picture that the particle simply reappears on the other side of the line. It is as if the particles were moving on
Figure 4: Two particle universe in the limit $p_+ \to \infty$. The shaded area is removed from space.

a circle. This will also become important in the quantum theory as it implies quantization of $p_-$!

It is important to notice that after each transition $p_+ \to p_+ + 2|p_-|$, so the approximation becomes better and better as the system evolves in time. Once we have entered the region in phase space for which $p_+ \gg 1$, evolution only drags us in further.

One can easily see from the figure that the c.o.m. moves tachyonic, or faster than the speed of light. As the individual particles move (almost) with the speed of light it follows that the vertex-points at $\alpha_1$ and $\alpha_2$ move superluminal along the vertical dashed lines. Because the angles themselves do not change this is the same speed as the tip of the cone (i.e. the c.o.m.-tachyon). The fact that this is possible while the particles move subluminal is caused by the instant jumps during a transition along the vertical dashed lines. One can verify this by a simple calculation:

$$v_{\text{com}} = \frac{\tanh \eta_3}{\sin \frac{H}{2}}$$

$$= \frac{\sinh \eta_3}{\sqrt{\cosh^2 \eta_3 - \cosh^2 p_-}} \geq 1 \quad \text{if} \quad p_- \in [-\eta_3, \eta_3]$$

So we must conclude that we formed a Gott-pair! Because we used a Cauchy formulation we do not have to worry about CTC’s. As Cutler showed, a family of Cauchy surfaces exists before the appearance of the CTC’s. But then we do
have to worry about the fact that our Cauchy formulation runs into a Cauchy Horizon and we cannot integrate past this point\(^2\). At this Cauchy Horizon the first closed lightlike curve comes into existence. In that case we have chosen an ‘unphysical’ universe. This happens for instance in an open universe with a Gott-pair. Carroll, Farhi and Guth have shown that there is simply not enough energy in the (open) universe to create such a pair from regular initial conditions. So then one must conclude that tachyons have created the pair and as we don’t observe these tachyons we exclude this possibility. Another way of stating this is that the identification at infinity is boostlike instead of rotationlike. This in turn implies that the total energy momentum vector is spacelike. This boundary condition was characterized as unphysical in [3]. A possible way to avoid the problem of unphysical boundary conditions is to close our universe ‘at infinity’ by adding some ‘spectator particles’. It was shown that in the literature [5] that the Gott-pair can be formed safely in a closed universe. The fact that no CTC’s appear in ’t Hooft’s description is due to the fact that before they can be traversed the Gott-pair has reached the spectator particles at ‘infinity’. ’t Hooft showed that the universe then necessarily crunches [9]. So the Cauchy Horizon is screened off by the big crunch. But we can postpone this dramatic ending of the universe by simply locating the observer-particles very far away. Before the crunch the spectator particles have plenty of time to study the two particles. This will be the philosophy in the next section. The particles that close the universe at infinity are treated classically. They have clocks that define time in the universe. They study the Hilbert-space of the two particles that approach them superluminally. This will however be a difficult job because the light that reaches them has necessarily undergone lots of transitions themselves (light that travels in straight lines will be overtaken by the c.o.m.-tachyon). Notice that a local observer also necessarily makes a lot of transitions and is boosted to very high velocities. Due to time dilatation he may observe the two particles very differently. An important lesson is that these states cannot exist in open universes with physical boundary conditions (which makes them unimportant for scattering calculations), but only exist in closed universes (although they may be very large). The final crunch in this universe is however unavoidable as it must screen off the potential CTC’s after the Cauchy Horizon.

4 Quantum Theory

In this section we will consider the quantum theory of our model in the limit \( p_+ \to \infty \). In this limit our particles essentially become massless. With this we mean that any reference to the mass disappears from our model (because the vertex contributions are dominant) and the velocity of the particles is the speed of light. \( L_+ \), being the conjugate variable to \( p_+ \), becomes a fixed constant in this

\(^2\)I thank H.J. Matschull for pointing this out to me
limit. It determines the relative distance between the particles. $L_+$ can be any positive number and is not quantized in our model. Let us remind the reader that $p_-$ was restricted to the range:

$$p_- \in [-\eta_3, \eta_3]$$  

$\eta_3$ determines the velocity of the c.o.m. and is treated classically in the following. It is chosen to be a fixed constant. Analogously to condensed matter physics we may view this as a ‘Brillouin zone’. We know what the effect of a Brillouin zone in momentum space is. It implies that the conjugate configuration variable is quantized in the following way:

$$L_n^- = \frac{n\pi}{\eta_3} \quad n \in \mathbb{Z}$$  

Notice that the distance between the lattice points depends on the center of mass momentum $\eta_3$. This is a peculiar feature of our model. There is however still a difficulty that we want to get rid of. Each time a transition occurs, $p_-$ reverses sign (and orientation). Let us therefore define the orientation $\zeta$ as follows: $\zeta = +1$ if particle 1 is to the right of particle 2 in figure (4). This is equivalent to saying that we encounter $L_i$ in the order 1-2-3 if we traverse around the vertex counter clockwise. If particle 1 is to the left of particle 2, $\zeta = -1$. If we define the quantity:

$$P = \zeta p_-$$  

then this new momentum will not change sign during a transition. It is actually a constant of motion. The sign of $P$ determines the ‘sense’ in which the particles wind around each other: clockwise or anti-clockwise. We can now distinguish two situations. First we will consider the case when the particles are identical. Two states that only differ by orientation are then indistinguishable and should be considered the same. If the particles have some extra quantum number that makes them non-identical the orientation does matter however. First we will describe the case of identical particles.

Let’s define the variable $X$ which is conjugate to $P$:

$$X = \zeta L_-$$  

How does this variable evolve in time? Consider figure (4) and place particle 1 to the far right of the line on which they live. This implies that $L_1 = 0$, $L_2 = L_+$ and $\zeta = +1$. Together this gives: $X = -L_+$. Next consider an intermediate state where $L_1 = L_2$, we now have $X = 0$. If particle 2 has moved to the far left we find $X = +L_+$. Now a transition takes place and particle 2 reappears at the far right side of the line and the orientation is reversed. This implies that $X$ jumps from $+L_+$ to $-L_+$. Although the orientation is reversed, quantum mechanically this state is identical to the one we started with and we should identify them. This
implies that $X$ really lives on a circle with circumference $2L_+$. For distinguishable particles these states are different and the circumference becomes twice as large as we will see later. But if $X$ lives on a circle we should quantize $P$ in the quantum theory. So we find:

$$P_m = \frac{m\pi}{A} \in [-\eta_3, \eta_3] \quad m \in \mathbb{Z} \quad (40)$$

$$X_n = \frac{n\pi}{\eta_3} \in [-L_+, L_+] \quad n \in \mathbb{Z} \quad (41)$$

We will now define $A$ in the above formula. Call $M$ the maximum of $m$ and $N$ the maximum of $n$. We have:

$$M = N = \text{Maxint}\left(\frac{\eta_3 L_+}{\pi}\right) \quad (42)$$

$\text{Maxint}$ is the maximum integer contained in its argument. This implies that $L_+$ determines the upper bound for $m$ (which is $M$) and thus the dimension of our Hilbert-space. We will define $A$ to be:

$$A \equiv \frac{M\pi}{\eta_3} \in (L_+ - \frac{\pi}{\eta_3}, L_+] \quad (43)$$

The dimension of our Hilbert-space is $2M$. Notice that the first and the last point in our configuration space must be identified so we will consider the points: $-M+1, \ldots, M$ as being independent. A suitable basis in this Hilbert-space is:

$$\phi_m(n) = \sqrt{\frac{1}{2M}} e^{i\pi mn/M} \quad m = -M + 1, \ldots, M \quad (44)$$

One can check orthonormality$^3$: 

$$\sum_{m=-M+1}^{M} \phi_m(n) \phi_{m'}^*(n) = \sum_{n=-M+1}^{M} \frac{1}{2M} e^{i\pi (m-m')/M} = \delta_{m,m'} \quad (46)$$

Moreover, the basis set is complete:

$$\sum_{m=-M+1}^{M} \phi_m(n) \phi_{m}^*(n') = \delta_{n,n'} \quad (49)$$

$^3$To evaluate this sum one can use the formula

$$\sum_{n=-M+1}^{M} z^n = z^{-M+1} \frac{1 - z^{2M}}{1 - z} \quad (45)$$
It is now of course straightforward to define a Fourier transform as follows:

\[
\psi(m) = \sqrt{\frac{1}{2M}} \sum_{n=-M+1}^{M} e^{-i\pi mn/M} \psi(n) \tag{50}
\]

\[
\psi(n) = \sqrt{\frac{1}{2M}} \sum_{m=-M+1}^{M} e^{i\pi mn/M} \psi(m) \tag{51}
\]

Next we will turn our attention to distinguishable particles. The difference with the previous case is that two states with equal \(X\) values but different orientation \(\zeta\) are truly different. Let us define a new configuration variable conjugate to \(P\) for this case:

\[
Y = \zeta(L_+ - L_-) \tag{52}
\]

How does \(Y\) evolve classically? Start again with the case where particle 1 is located at the far right of the line \(L_1/L_2\). We have \(L_1 = 0, L_2 = L_+, \zeta = +1\), so \(Y = -2L_+\). Then the state where \(L_1 = L_2\) gives \(Y = -L_+\). If particle 2 is at the far left of the line we have \(L_1 = L_+, L_2 = 0, \zeta = +1\), so \(Y = 0\). Then the transition takes place and the orientation is reversed (\(Y\) stays 0). The next step \(L_1 = L_2\) again with \(\zeta = -1\), giving \(Y = L_+\). Next particle 1 is to the far right which implies \(Y = 2L_+\). Finally after the transition we are back at our starting point.

In the quantum theory we substitute again \(A'\) for \(L_+\) in (52) and we find:

\[
P_m = \frac{m\pi}{A'} \in [-\eta_3, \eta_3] \tag{53}
\]

\[
X_n = \frac{n\pi}{\eta_3} \in [-L_+, L_+] \tag{54}
\]

where we define again:

\[
A' \equiv \frac{M' \pi}{2\eta_3} \in (L_+ - \frac{\pi}{2\eta_3}, L_+) \tag{55}
\]

We notice that the number of modes (or the dimension of the Hilbert-space) has doubled:

\[
d = 2M' = \frac{4A'\eta_3}{\pi} \tag{56}
\]

Analoguesly to (44,50,51) we can define a complete set of orthonormal basis functions and a Fourier transform. In all these formula’s we should only replace \(M\) with \(M'\).

The Hamiltonian in terms of \(P\) can be obtained from:

\[
\cos \frac{H_T}{2} = -\frac{\cosh P}{\cosh \eta_3} \tag{57}
\]
Because \( H_T \in [\pi, 2\pi] \) the sign of \( \sin \frac{H_T}{2} \) is positive, so we have:

\[
\sin \frac{H_T}{2} = \frac{1}{\cosh \eta_3} \sqrt{\cosh^2 \eta_3 - \cosh^2 P} \quad (58)
\]

Combining these to equations we have:

\[
e^{-i\frac{H_T}{2}} = \frac{-1}{\cosh \eta_3} (\cosh P_m + i\sqrt{\cosh^2 \eta_3 - \cosh^2 P_m}) \quad (59)
\]

We now use the fact that the Hamiltonian is an angle to impose that time is quantized \([9]\). We find that all relations only involve cosines and sines of half the total Hamiltonian\([10]\). The Schrödinger equation is thus a difference equation that relates a state at time \( t \) to a state at a time \( t + 2 \):

\[
\psi(n, t + 2) = \frac{-1}{\cosh \eta_3} \sum_{n'=-M+1}^{M} \sum_{m=-M+1}^{M} (\cosh P_m + i\sqrt{\cosh^2 \eta_3 - \cosh^2 P_m}) e^{i\pi (n-n')/M} \psi(n', t) \quad (60)
\]

As the value of \( \psi(n, t + 2) \) depends on the values of \( \psi(n, t) \) at all the points \( X_n \), this Schrödinger equation is non local.

The spectrum of the relative motion of the particles is thus found to be discrete. If we add the c.o.m. motion to the Hilbert-space this is probably not the case anymore. The total Hilbert-space is however not a simple direct product of the relative motion and the c.o.m. motion because the range of the relative momentum depends on the c.o.m. momentum.

## 5 Discussion

In this paper we described the evolution of two gravitating particles in 2+1 dimensions. We used a ‘cut and identify’ procedure used by ’t Hooft in \([8]\). A simple computer program helped us follow the particles as they evolved in time. At low energy the model describes the bending of the particles due to their mutual gravitational interaction. At higher energy solutions emerged that showed the particles winding around one another a couple of times after which they broke free and moved towards infinity. For energies between \( \pi \) and \( 2\pi \), and relatively large, non equal values for \( \eta_{1,2} \) we found solutions that wind indefinitely. After each transition the value of \( p_+ = \eta_1 + \eta_2 \) grows, pulling the system into a region of phase space where \( p_+ \gg 1 \). In that case the center of mass of the particles moves faster then the speed of light, i.e. the particles form a Gott-pair. As we use a

\[\text{Notice that } 't \text{ Hooft chose units in such a way that his Hamiltonian for the one particle case is half the Hamiltonian that we use. Accordingly his time was quantized in units of one while our time is quantized in units of 2.}\]
Cauchy formulation it is impossible to encounter CTC’s. To avoid an unphysical boundary condition we add classical observers at infinity in such a way that the universe closes. Before any CTC can form (at the Cauchy Horizon) the universe will have destroyed itself in a big crunch.

Finally we quantized the two particles in the limit $p_+ \to \infty$ and found that both configuration space and momentum space live on a lattice. The number of lattice points (or the number of modes) and the lattice distance are determined by the two fixed parameters $\eta_3$ and $L_+$. The first parameter is determined by the c.o.m. motion. The second parameter $L_+$ is the conjugate to $p_+$ and is a constant in the limit $p_+ \to \infty$. It can acquire any value depending on the initial conditions of the particles. In the limit it determines the distance between the particles. The fact that the lattice distance is depending on these quantities is a peculiar feature of our model.

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References