We examine one of the advantages of Ashtekar’s formulation of general relativity: a tractability of degenerate points from the point of view of following the dynamics of classical spacetime. Assuming that all dynamical variables are finite, we conclude that an essential trick for such a continuous evolution is in complexifying variables. In order to restrict the complex region locally, we propose some ‘reality recovering’ conditions on spacetime. Using a degenerate solution derived by pull-back technique, and integrating the dynamical equations numerically, we show that this idea works in an actual dynamical problem. We also discuss some features of these applications.

PACS No.: 04.20.Cv, 02.40.-k, 04.20.Fy, 04.25.Dm

I. INTRODUCTION

A decade has passed since the proposal of the new formulation of general relativity by Ashtekar [1]. By using this special pair of variables, the framework has many advantages in the treatment of gravity. The constraint equations which appear in the theory become low-order polynomials, and the theory has the correct form for gauge theoretical features. These suggest possibilities for treating a quantum description of gravity nonperturbatively.

Here we examine another advantage of the SO(3)-ADM (Ashtekar) formulation of general relativity: the ability to dynamically evolve a spacetime with a degenerate metric consistently. A ‘degenerate point’, here, is defined as a point in the spacetime where the density \( e \) of 3-space vanishes, as we denote in §II. This advantage comes from the fact that all equations in Ashtekar’s formulation do not contain the inverses of the variables.

There are several motivations for studying ‘degenerate points’. The first one comes from a quantum cosmological description of the very early history of the Universe. One scenario describes that our Universe was born and evolved in Euclidean spacetime, then emerged with Lorentzian metric through a quantum tunneling process. Kodama [2] found an exact solution for Bianchi type IX spacetime using Ashtekar’s framework, which has both Euclidean and Lorentzian sections connected by an analytic continuation. Apparently, such a signature changing process should pass through a degenerate point, but so far the dynamics of this process remains unclear.

A second motivation comes from the gauge theoretical point of view. We can set a finite gauge transformation so as to have a degenerate point of spacetime from a non-degenerate one naturally [3]. Therefore when we calculate the functional integrals, we have to include degenerate solutions due to the gauge invariance, and studies on degenerate points will be indispensable to construct the complete theory of quantum gravity in the near future.

A third interesting application exists in topological field theory. Since Ashtekar’s dynamical variables \( \tilde{E}^a_a \) and \( A^a_i \) are assigned positive mass dimension one, a short-distance limit (i.e., region of quantum gravity) must have degenerate phase: \( \tilde{E} = A = 0 \). Such a degenerate phase is called the topological (unbroken) phase of quantum gravity, in which diffeomorphism is unbroken [4]. Finding how to evolve from the degenerate unbroken phase to the non-degenerate phase is one of the important issues, and it is expected to be a key to construct quantum gravity in four dimensions.

Another possibility which makes degenerate points interesting is the evolution of classical (Lorentzian) spacetime. As already pointed out by some authors [5,6], the Ashtekar formulation is also attractive for numerical relativity. If this tractability of a degenerate point works also in this context, then we will be able to analyze focusing or shell-crossing features or (coordinate) singularities in spacetime.

In this paper, we study what problems appear and how to solve them when we treat the dynamics through a degenerate point. Here, the dynamics means the evolution of the three-hypersurfaces expressed by Ashtekar’s variables \( (\tilde{E}^a_a, A^a_i) \), fixing the gauge freedom of the lapse func-
In the last few years, several studies have been done on degenerate metrics. Some solutions have been found by connecting degenerate and non-degenerate metrics [7]-[9], and degenerate examples in terms of Ashtekar variables are shown in [10]. Some alternative approaches over the framework of Ashtekar were considered in [11] and [12]. However there are few comments on the dynamical evolution of a degenerate metric. Bombelli and Torrence [13] commented on the conditions for ‘passing’ degenerate points. They speculate the degeneracy of lapse function, condition for finite ‘passing’ coordinate-time, and possibility of divergence of variables at the degenerate point.

We develop their idea in §III A.

Assuming that both all dynamical variables and the coordinate time are finite (passing condition in §II), we seek the condition which enables us to pass degenerate points for the case of a vacuum spacetime with/without a cosmological constant. We take mainly two approaches. The first approach, we named intersecting slice approach, purposes that a foliation passes a degenerate point directly. This is a natural passing behavior which we expect from a context of Ashtekar’s formulation. However, we show that such a direct treatment does not work normally both in ADM and Ashtekar formulation, whether we impose the reality conditions or not. This is described in §III.

The second approach, we call deformed slice approach, takes a complex path around a real degenerate point. This approach is described in §IV, in which we propose some conditions for recovering the real manifold after a detour around a degenerate point by complexification.

In §V, we derive a degenerate solution by pull-back technique and show that the deformed slice approach works well for ‘passing’ a degenerate point using a numerical integration of the dynamical equations. We conclude that the essential trick for ‘passing’ a degenerate point in Ashtekar’s formulation exists in their complexification of the basic dynamical variables. We devote §VI to discussion.

When we apply the Ashtekar formalism in classical general relativity, we need to impose reality conditions. We treat these reality conditions based on work developed by two of us [5], and some essential points are attached in the Appendix along with a full description of the notations we use throughout this paper.

We use greek letters (µ,ν,ρ,...), which range over the four spacetime coordinates 0,...,3, while uppercase latin letters from the middle of the alphabet (I,J,K,...) range over the four internal SO(1,3) indices (0),..., (3). Lower case latin indices from the middle of the alphabet (i,j,k,...) range over the three spatial indices 1,...,3, while lower case latin indices from the beginning of the alphabet (a,b,c,...) range over the three internal SO(3) indices (1),..., (3). We use volume forms εabc; εabcεabc = 3!

II. DEFINITIONS OF ‘PASSING’

In the beginning, let us clarify some terminology. The basic variables and equations in Ashtekar’s formulation are summarized in the Appendix.

A ‘degenerate point’, in this paper, is the point in the spacetime where the density e of 3-space vanishes (not 4-space). In the Ashtekar formulation, the density is defined as $e = \sqrt{\det E^a_i}$, which corresponds to the same condition in the ADM formulation as $e^2 = \det \gamma_{ij}$, where $\gamma_{ij}$ is 3-metric. We ask readers to remember that such a degenerate point does not always mean a physical singularity. In the basic equations neither the constraints (A5)-(A7) nor the dynamical equations (A8)-(A9) include any inverse of the dynamical variables ($A^i_a$ and $E^i_a$), even if we include a cosmological constant. This fact suggests to us that we can ‘pass’ such a degenerate point. That is, we expect that all calculations can be continued even if we have a degenerate metric during the time evolution.

Note that, in the ADM formulation, this is impossible because the equations in the ADM include an inverse of the variables. We also remark that we can not transform the Ashtekar variables onto the ADM three-hypersurface $\Sigma$ at the degenerate point. This is because the 3-metric $\gamma_{ij}$ is given by $\gamma_{ij} = e_i^a e_j^a$, where $e_i^a = (E^a_i)^{-1}$ and $E^a_i = E^i_a / e$, and the last quantity diverges as $e \to 0$. Therefore, at the degenerate points, one can not recover the ADM variables from Ashtekar variables.

In order to say ‘pass’ degenerate points, we require the following four passing conditions:

(a) Ashtekar variables $E^a_i, A^i_a, N, N^i, A^a_b$ must remain finite throughout the calculation,
(b) the spatial derivatives of them must also be finite throughout the calculation (because they appear in the equations of motion),
(c) all the constraints (A5),(A6),(A7) and the equations of motion (A8),(A9) are satisfied, and
(d) some conditions for recovering the real manifold after a detour around a degenerate point.

1. We raise and lower µ,ν,ρ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (Lorentzian metric); I, J, K by $\eta^{IJK} = \text{diag}(-1,1,1,1)$ and $\eta_{ij}$; i, j, k by $\gamma_{ij}$ (3-metric).
2. The cosmological constant term in (A8) looks as if it might be divergent, but the relation $ee_i^a = (1/2)\epsilon^{abc} \gamma_{ijk} E^j_a E^k_b$ guarantees its finiteness.
3. From the assumptions (b) and (c), we have that the time differentiation of dynamical variables $E^a_i, A^i_a$ must also be finite.
(d) the calculation must be finished in finite coordinate time.

The problem is whether we can make dynamical foliation through a degenerate point under these passing conditions (a)-(d) above. In the following sections, we take two approaches. The first one, which we call the ‘intersecting slice approach’, attempts to pass a degenerate point directly, and the second one, ‘deformed slice approach’, takes a foliation in complex region.

III. INTERSECTING SLICE APPROACHES

In this section, we describe the possibility of an ‘intersecting slice approach’, which attempts to pass a degenerate point directly. Here ‘direct’ means that the dynamical Ashtekar variables run into the degenerate points, never detour the points. We assume that $N$ and $N^i$ are real as usual. To say our conclusion first, we face at least two problems, which we call “cusped lapse/density problem” and “divergence problem”, the latter requires severe conditions.

A. Cusped lapse/density problem

A troublesome variable in an evolution through a degenerate point is the (inverse) densitized lapse $N := N/e_{\Sigma}$, where $e_{\Sigma} = \sqrt{\det \gamma_{ij}}$. Since $N$ is held finite [condition (a)], the ADM lapse $N = N e_{\Sigma}$ vanishes at the degenerate point $e_{\Sigma} = 0$. Notice that the ratio of the proper time $\tau$ to the coordinate time $t$ is not $N$ but $N$. Thus we are afraid that the calculation exhausts an infinite amount of time, i.e., condition (d) is violated. Let us take $t$ and $\tau$ such that $\tau = t = 0$ at the degenerate point $e_{\Sigma} = 0$. Then condition (d) is denoted by

$$\delta t = \int_{-\tau_0}^{t_1} dt = \int_{-\tau_0}^{\tau_1} \frac{d\tau}{N(\tau)} < \infty. \quad (3.1)$$

Here the range of the integral is arbitrary but includes the zero point $\tau = t = 0$. Note that $N \geq 0$ since it is a lapse function. This condition originally appeared in [13].

In terms of $\tau$, the request (3.1) is considerably restrictive because $N = 0$ at $\tau = 0$. For example, let us consider the form $N = |\tau|^s$ where $s$ is constant. The condition for (3.1) is $0 < s < 1$, e.g., $N = \sqrt{|\tau|}$ satisfies (3.1). We see $dN/d\tau$ does not exist at $\tau = 0$. This is the reason we call this the *cusped lapse problem*. We note that this is not a serious problem to execute, because the choice of lapse in such a cusped form is only within a freedom of gauge, although there remains its naturalness of foliation.

In terms of $t$, however, the request (3.1) is not restrictive as follows. When $N = d\tau/dt = \tau^s (0 < s < 1)$, we see $t = \tau^{1-s}/(1 - s)$, thus $N = (1-s)^{s/(1-s)}\tau^{s/(1-s)}$. The power $s/(1-s)$ can be taken arbitrary positive even if we require $0 < s < 1$. The only request here is $N = 0$ at $t = 0$. So the lapse is not necessarily cusped in terms of $t$.

Furthermore we will see that the density $|e|$ should also satisfy a condition which is similar to the cusped lapse condition. We call this problem the *cusped density problem*. We assume $e = 0$ at $t = \tau = 0$ and $|N| < M$ (bounded). Since $N \geq 0$, we have $e\tilde{N} = |e| \tilde{N}$.

Then we see

$$\infty > \int_{-\tau_0}^{\tau_1} \frac{d\tau}{N} = \int_{-\tau_0}^{\tau_1} \frac{d\tau}{e\tilde{N}} = \int_{-\tau_0}^{\tau_1} \frac{d\tau}{|e| \tilde{N}} > \frac{1}{M} \int_{-\tau_0}^{\tau_1} \frac{d\tau}{|e|}.$$

Similarly to the lapse, the density is restricted in terms of $\tau$ as $|e| = \tau^s (0 < s < 1)$, and is not restricted in terms of $t$.

B. Divergence problem

Let us consider the quantity $\omega^0_a$ which appears in the definition of $A^0_a$ [(A1) in Appendix]. We have

$$\omega^0_a = -K^i_a E^j = \frac{1}{e^2 N} \left( \gamma_{ij} - D_i N_j - D_j N_i \right) \tilde{e}^j_a.$$  

In order to examine this finiteness, we prepare

$$\omega^0_a \tilde{e}^i_a = \frac{1}{N} \left( \frac{\partial_i (e^2)}{e^2} - D_i (N^i) - D_j (N^j) \right)$$

$$= \frac{1}{N} \left[ \frac{\partial_i (e^2)}{e^2} - \frac{\partial_j (e^2)}{e^2} - \frac{\partial_s (e^2)}{e^2} N^j - 2\partial_s (N^i) \right]. \quad (3.2)$$

Now we assume the conditions (a)-(d). Then $N$ is finite, so the term $1/N$ is bounded below. When a parameter $s$ is taken such that $\partial_s = \{1, -N^i\}$ and $s = 0$ corresponds the degenerate point, the first and second terms in the right-hand side of (3.2) are rewritten as,

$$\frac{\partial_i (e^2)}{e^2} - \frac{\partial_j (e^2)}{e^2} N^j = \frac{\partial_i (e^2)}{e^2} = \partial_i \log (e^2),$$

which diverges at $s = 0$. The $\partial_i (N^i)$ term in (3.2) is finite since we assume condition (b). To sum up, we see (3.2) diverges at the degenerate point. If we assume the triad reality conditions (A16)-(A17), this fact tells us the

4Strictly, this is an improper Riemann integral.

5We see $\lim \sup_{\tau \to 0^+} dN/d\tau = \infty$ when $dN/d\tau$ exists for $0 < \tau$. 
passing is impossible. Here the triad reality conditions are useful and stronger than the usual metric reality conditions as in Appendix A.2. Since the triad is real, \( \omega_{i}^{a} \) is a real part of \( \mathcal{A}_0^a \), so it is finite. And, since \( \tilde{E}_0^a \) is finite, \( \omega_{i}^{a} \tilde{E}_0^a \) must be finite. The above conclusion contradicts this. Thus, unfortunately, we can not pass the degenerate point when the triad reality is assumed.

Even if we do not impose the triad reality conditions, i.e. we impose the metric reality conditions or do not impose any reality conditions, this problem still exists as follows. In order to enforce the finiteness of \( \mathcal{A}_0^a \), its second term, \( i\epsilon^{abc} \omega_{i}^{bc} \), must diverge such that it cancels out the divergence of \( \omega_{i}^{a} \).

At the present stage, we do not know any exact solutions which satisfy this condition. Thus we conclude that the passing conditions (a)-(d), which aimed to pass a degenerate point directly, are difficult to satisfy simultaneously due to this divergence of variable.

IV. DEFORMED SLICE APPROACHES

In the previous section, we showed that foliations through a degenerate point are difficult even within the Ashtekar framework. Therefore, in the second approach, we try to make a spacetime foliation avoid a degenerate point by breaking reality conditions. That is, we impose the foliations of 3-space which detour into complex spacetime only in the vicinity of such a degenerate point.

This means that locally we are out of Einstein’s framework. In this section, we examine what conditions are needed when we impose such a locally deforming time evolution.

As we mentioned in the introduction, our aim is to keep continuous time evolution in the presence of a degenerate point. We extend the spacetime using complex numbers, and impose reality on the coordinates \((t,x)\), so the foliation maintains 3+1 dimensions \( \mathbb{R}^3 \times \mathbb{R} \) in \( \mathbb{C}^4 \). Note that our proposal of real coordinate is different from the Wick rotation, rather it is close to the recent proposal of complex lapse function by Hayward [14]. We use a freedom of gauge function to foliate into the complex spacetime.

We assume a spacetime has a degenerate point and has also its future (a kind of coordinate singularity or focusing). In the following expression, we postulate that a single degenerate point exists at \((t,x) = (t_*,x_*)\), but extending to many degenerate points is straightforward.

In order to recover the Einstein spacetime, the foliations have to satisfy the following conditions, such as the ‘foliation recovering’ condition which ensures that the foliation locally deforms into the imaginary region only in the vicinity of the degenerate point.

A. Foliation recovering condition

We seek the foliation which starts from the real section, evolves in the complex region only in the vicinity of a degenerate point, and ends in the real section. We call this the ‘foliation recovering’. In the local deformation, the lapse function and the shift vector become complex valued to foliate in the complex region. Thus the ‘foliation recovering condition’ is expressed in terms of the gauge functions as:

\[
\int_{t_-}^{t_+} \Im \mathcal{N}(t,x) dt = 0, \quad (4.1)
\]

\[
\int_{t_-}^{t_+} \Im \mathcal{N}^i(t,x) dt = 0, \quad (4.2)
\]

where \( t_- < t_* \) and \( t_+ > t_* \) are defined in appropriate far time from a degenerate point at \( t = t_* \). The reason for not imposing the other gauge function, the triad lapse \( \mathcal{A}_0^a \), is that the actual spacetime foliation is determined only by the lapse and the shift vector, and the triad lapse does not contribute in physical (external spacetime) foliations.

B. Asymptotic reality condition

Although we are considering deforming Einstein spacetime by using a complex metric in the time evolution, we require that such deformations are as local as possible. This is because our proposal is just a technique for treating a degenerate point. Therefore we impose reality conditions both at the spatial far limit and time far limit from a degenerate point, in order to ensure that the metric becomes complex only in the vicinity of the degenerate points.

In the spatial far region from a degenerate point, the spacetime must be in real Einstein spacetime. In the time far limit, we also impose a condition to ensure that the metric becomes complex only in the time vicinity of the degenerate point. In other words, the initial value of \( \tilde{E}_0^a(t,x) \) is chosen to be real. Then we make time evolution within Einstein spacetime before we meet a degenerate point, and in the vicinity of a degenerate point we make time evolution into complex values, and return to the real values at the end.

Those are expressed by gauge functions and metric variables. The former are:

\[
\Im \mathcal{N}(t,x) \rightarrow 0 \quad (4.3)
\]

\[
\Im \mathcal{N}^i(t,x) \rightarrow 0 \quad (4.4)
\]

for all four limits \( x \rightarrow x_* \pm \Delta x, \ t \rightarrow t_* \pm \Delta t \) where \( \Delta x \) and \( \Delta t \) determine where we call the far region from a degenerate point, which depends on the particular problem. The latter is:
for all four limits \( x \to x_\pm \Delta x, \ t \to t_\pm \Delta t \). This expresses the primary reality condition (A11), of which general discussions are in Appendix.

As for the time-far limit of the metric reality condition (4.5), the expression can be written in an integrated form

\[
\int_{t_-}^{t_+} \frac{d}{dt} \left( 2\tilde{E}_{\alpha} \tilde{E}^{\alpha} \right) / \det \tilde{E}(t, \mathbf{x}) dt = 0. \tag{4.6}
\]

The integrand is calculated using the dynamical evolution equation (A9), so the explicit form of (4.6) is not given at an arbitrary time except in some simple analytic cases. Normally, the condition (4.6) becomes a boundary problem of the evolution both in initial and final time-slice, so that any general application of this method requires numerical iterations for finding gauge functions to satisfy all conditions (4.1)-(4.5).

In general, we cannot deny the possibility that two different gauges satisfying (4.1)-(4.5) exist and each yields a different real metric after deforming. However, there is no problem when we know the metric beyond the degenerate point. In the next section, we show an example of such a foliation, where we fix our background metric to be flat spacetime with one degenerate point.

V. EXAMPLES

In this section, we discuss dynamical evolution through a degenerate point, using an analytic solution with the pull-back technique and a numerical demonstration.

A. Exact solution with degenerate point

One method to transform an exact solution is by the pull-back technique. When pull-back includes a singularity, it is not a diffeomorphism, and the coordinate transformation from the non-degenerate metric to the degenerate metric is singular. This technique was also used by Horowitz [3], when he discussed a possibility of topology change of the spacetime. Here, we derive a flat Minkowski spacetime which has one degenerate point at the origin by using the pull-back method for later numerical demonstration of deforming slices. From the flat Minkowski metric, the following degenerate metric is constructed:

\[ ds^2 = \left[ -1 - f(t) h(x) \right] dt^2 + \left[ -2f(t) h(x) \left( 1 - f(t) h'(x) \right) \right] dx^2 + \left( 1 - f(t) h'(x) \right)^2 dz^2 + dy^2 + dz^2, \tag{5.1} \]

where \( f(t), h(x) \) are an arbitrary functions which satisfy

\[ f(\pm \infty) = f'(\pm \infty) = 0, \quad -1 < f'(t) < 1, \quad -1 < f(t) \leq 1, \tag{5.2} \]

where the equality in the last equation is satisfied only when \( t = 0 \), and

\[ h(\pm \infty) = h'(\pm \infty) = 0, \quad -1 < h(x) < 1, \quad -1 < h'(x) \leq 1, \tag{5.3} \]

where the equality in the last equation is satisfied only when \( x = 0 \).

We will see that this metric has the following properties: (1°) flat, (2°) asymptotic Minkowski, (3°) degenerate at \( t = x = 0 \), and (4°) Lorentzian (t-time, x, y, z-space) at elsewhere.

**Proof**

(1°) Let us put \( a = t, \ b = x - f(t) h(x) \). Then we see \( ds^2 = -da^2 + b^2 + dy^2 + dz^2. \)

(2°) Since \( f(t) \to 0 \) and \( f'(t) \to 0 \) when \( t \to \pm \infty \), \( h(x) \to 0 \) and \( h'(x) \to 0 \) when \( x \to \pm \infty \), we see that the metric becomes flat for all four limits \( t \to \pm \infty, x \to \pm \infty \).

(3°) The determinant of this metric is det \( g_{\mu \nu} = -1 - f(t) h'(x)^2 \). Noting \( -1 < f(t) \leq 1 \) (equality is satisfied only when \( t = 0 \)) and \( -1 < h'(x) \leq 1 \) (equality is satisfied only when \( x = 0 \)), we obtain det \( g_{\mu \nu} \leq 0 \) (equality is satisfied only when \( t = x = 0 \)).

(4°) We have \( g_{00} = -1 - f'(t) h(x)^2 \). Since \( |f'(t)| < 1, \ |h(x)| < 1 \), we obtain \( g_{00} < 0 \). We easily see \( g_{11} \geq 0 \). And we have det \( g_{\mu \nu} \leq 0 \). Thus we see the metric is Lorentzian (t-time, x, y, z-space) except for \( t = x = 0 \).

For example, \( f(t) = e^{-t^2}, h(x) = xe^{-x^2} \) satisfy conditions (5.2) and (5.3). We start from the metric given by the substitution of \( f(t) = e^{-t^2}, h(x) = xe^{-x^2} \) into (5.1),

\[
\begin{align*}
ds^2 & = \left[ -1 - (2txe^{-t^2-x^2})^2 \right] dt^2 \\
& \quad + \left[ 4txe^{-t^2-x^2}\left[ 1 - (1 - 2x^2)e^{-t^2-x^2} \right] \right] dx^2 \\
& \quad + \left[ 1 - (1 - 2x^2)e^{-t^2-x^2} \right] dx^2 + dy^2 + dz^2.
\end{align*}
\tag{5.4}
\]

Hereafter, we call this the dM solution (a degenerate expression of Minkowski spacetime). This metric is plane symmetric so that a degenerate point \( t = x = 0 \) is not a point in spacetime. But this metric will be sufficient to show our proposal works in the next demonstration.

B. Numerical demonstration of two approaches

Using the dM solution (5.4) as a background metric, we here show some numerical demonstration of our discussions in the previous sections. The following computations were made by conventional finite difference

\[ ^6 \]Of course the reality conditions at an arbitrary time include this asymptotic condition. But reality conditions at an arbitrary time are too restrictive to solve in the deformed slice approach.
scheme using independent programs written in Fortran and Mathematica, both of which passed the appropriate convergence tests of grid size using the final imaginary part of density \( e \).

Since the metric (5.4) has a degenerate point at \( x = t = 0 \), it is clear that we can not evolve the solution beyond a degenerate point because the inverse 3-metric diverges at the degenerate point within the ADM formulation. As we denoted in §III, even if we take the Ashtekar formulation in time evolutions, some variables diverge at a degenerate point because of the ‘divergence problem’. Thus we may say it is hard to ‘pass’ such a point.

Fig. 1 shows such a feature. The figure is given by a time evolution of a solution (5.4) using Ashtekar’s dynamical equation, and the real part of density \( e \) is plotted versus spatial and time comoving coordinates \( x \) and \( t \). We set the gauge conditions (slicing conditions) the same as the exact solution (5.4), i.e., the lapse \( N = 1 \), the shift vector \( N_i = 2txe^{-t^2-x^2}[1 - (1 - 2x^2)e^{-t^2-x^2}] \), and the triad lapse \( A^a_0 = 0 \). Initial data are constructed from the exact solution (5.4) taking triad \( e^a_i \) as the diagonal matrix \((\sqrt{\gamma_{xx}}, 1, 1)\). We set that the time evolutions do not work properly after intersecting \( t = x = 0 \), because of the ‘divergence problem’.

Next, we show that a ‘deformed slicing’ approach works well. In order to make a deformed foliation, we allow the shift vector to be complex. The imaginary parts of \( N_i \) are arbitrary but should satisfy conditions (4.2) and (4.4). As an example, we choose

\[
\Im N_i = ate^{-b(t^2+x^2)}, \tag{5.5}
\]

where \( a \) and \( b \) are arbitrary constants. We take the lapse \( N = 1 \) and the triad lapse \( A^a_0 = 0 \), the simplest ones as before.

In order to judge whether an evolution recovers asymptotic flatness and satisfies asymptotic reality conditions, we checked two values at the final slice of our evolution;

\[
F(t_{\text{final}}) := \max_x |\Re(e(t = t_{\text{final}}, x) - 1)| \tag{5.6}
\]
\[
R(t_{\text{final}}) := \max_x |\Im(e(t = t_{\text{final}}, x))| \tag{5.7}
\]

where these indicate the time asymptotic reality condition (4.5). (\( F \) and \( R \) were used in convergence tests of the programs.)

By changing a deformation of slice (two parameters \( a \) and \( b \)), we searched for a solution which minimize the
flattening $F$ and asymptotic reality $R$ at the final time slice. Fig.2 shows a successful example of a local violation of reality, in which we used $a = 0.003982$ and $b = 2.5$ in the imaginary part of shift vector (5.5), and set $t_{\text{initial}} = -4, t_{\text{final}} = +4$ in coordinate time. In the case of Fig.2, the flatness $F$ and asymptotic reality $R$ are $F < O(10^{-2}), R < O(10^{-2})$ respectively when we take $\Delta t = 0.01$. We also should mention that this example satisfies asymptotic reality conditions at spatial far limit and foliation recovering conditions in the previous section. In the figure, density $e$ is plotted versus coordinates $x$ and $t$. We see that the time evolution does work properly in the sense that the real part recovers the analytic evolutions and the imaginary part vanished asymptotically.

We found such a solution appears discretely in the two-parameter $a$-$b$ plane. This eigensystem-like behavior is what we expected, since our requirements are over the freedom of this dynamical system as we discussed in §IV. This suggests that the spacetime dynamics in the complexified manifold will not necessary develop (or converge) into the real section of the manifold, without imposing some reality recovering conditions. And note that a solution which satisfies asymptotic reality also satisfies asymptotic flatness.

We also tried with other choices of shift vector, e.g., $\Re N_i = atxe^{-b(t^2+x^2)}, \Im N_i = 2atxe^{-(t^2+x^2)}\{1 - 2(1 - 2x^2)e^{-b(t^2+x^2)}\}$ and other forms where $a$ and $b$ are arbitrary constants. We found that the general behavior of the discrete appearance of solutions is quite similar to the above example. The existence of solutions in the case of small $b$ suggests to us that our deformed slicing approach is applicable not only for degenerate ‘point’ but also for degenerate ‘region’.

VI. DISCUSSION

We have studied one of the advantages of Ashtekar’s formulation: a tractability of degenerate points. We showed that a direct passing of degenerate point (which we call the intersecting slice approach) is hard to work in general if we impose some natural assumptions on dynamical variables to express Lorentzian spacetime evolution. This means we can not make an evolution of hypersurface directly through a degenerate point even if we use Ashtekar’s variables.

Therefore we propose a deformed slice approach, which violates reality conditions locally near a degenerate point, and showed that this proposal works in an actual dynamical problem. Therefore we conclude that a trick for dynamically passing a degenerate point exists in using complexified spacetime. This deformed slice approach is a natural foliation within Ashtekar’s formulation since variables are originally defined as complex variables.

Readers may think that if the essential trick of passing a degenerate point exists in complexifying variables, then we might find an example even within the complexified ADM formulation. We found also that this statement is true, and found similar discrete behavior in the deformed parameters.

This result suggests to us that there is a new possibility for dynamical problems in classical spacetime such as focusing or shell-crossing of coordinate points and also for singularities. Although this foliation requires parameter tuning to satisfy reality recovering conditions, the fact that a real part of the solution always expresses the expected analytic solution indicates this dynamical evolution technique works for more general problems.

There are some points which remain unclear when we break reality conditions locally. One big problem is a causality of this spacetime. Matschull [15] commented on causality after a system passes a degenerate point. He classified the degenerate tetrad by rank, and considers causal structure of each degenerate space-time in a real manifold. However we do not know whether such a usual causal structure can be extended to the complex manifold.

If general relativity is extended to allow degenerate metrics, the topology of spacetime has a possibility to change even within a classical picture [3]. If we allow breakings of reality conditions locally, then our classical path may connect the spaces with different topology dynamically. We showed only the case that a spacetime will recover its original metric, but in general initial and final metrics are allowed to be different. Further analysis will broaden our research in dynamics which includes a degenerate point.

A discrete appearance of solutions suggests the existence of something like ‘topological charge’ of this system. We are now seeking such a charge, and plan to connect our discussions with a classically topology changing scenario and/or signature changing scenario in the early history of the Universe.

We believe that this work is the first example to display some classical dynamical behavior of Ashtekar’s variables for numerical evolution of spacetime. We expect our approach will become the first step towards understanding dynamics of the signature changing process, topology changing process and causal structure in a complex manifold.

ACKNOWLEDGMENTS

We thank Hitoshi Ikemori for useful discussions. We thank Paul Haines for his careful reading of our manuscript. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science, Sports and Culture No. 07854014, by NSF 96PHYS-00507, by the Grant-in-Aid for JSPS Fellow (AN) and by a Waseda University Grant for Special Research Projects 96A-153 and 96A-280.
APPENDIX A: ASHTEKAR’S FORMULATION

Here, we summarize the basic variables and equations in Ashtekar’s formulation of general relativity, used in this paper.

1. The Ashtekar variables

The key feature of Ashtekar’s formulation of general relativity [1] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad, $e^i_\mu$, and define its inverse, $E^i_\mu$, by $g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}$ and $E^{\mu i} := e^i_\mu g^{\mu\nu}\eta_{ij}$. We define a $SO(3, C)$ self-dual connection

$$\pm A_\mu^a := \omega^0_\mu + \frac{i}{2} e^a_{bc} \omega^{bc}. \quad (A1)$$

where $\omega^0_\mu$ is a spin connection 1-form (Ricci connection), $\omega^0_\mu := e^i_\nu \nabla_\mu e^\nu_i$. Ashtekar’s plan is to use only a self-dual part of the connection $\pm A_\mu^a$ and to use its spatial part $\pm A_\mu^a$ as a dynamical variable. Hereinafter, we simply denote $\pm A_\mu^a$ as $A_\mu^a$.

Note that the extrinsic curvature, $K_{ij} = - (\delta_i^k + n_i n_j^k) \nabla_i n_j$ in the ADM formalism, where $\nabla$ is a covariant derivative on $\Sigma$, satisfies the relation $-K_{ij} E^{ja} = \omega^{0a}_\mu$, when the gauge condition $E^a_0 = 0$ is fixed. So $A_\mu^a$ is also expressed by

$$A_\mu^a = - K_{ij} E^{ja} - \frac{i}{2} e^a_{bc} \omega^{bc}. \quad (A2)$$

The lapse function, $N$, and shift vector, $N^j$, are expressed as $E^0_a = (1/N, -N^j/N)$. This allows us to think of $E^0_a$ as a normal vector field to $\Sigma$ spanned by the condition $t = x^0 = \text{const}$, which plays the same role as that of ADM. Ashtekar treated the set $(A_\mu^a, E^a_i)$ as basic dynamical variables, where $\tilde{E}^a_i$ is an inverse of the densitized triad defined by

$$\tilde{E}^a_i := e E^a_i, \quad (A3)$$

where $e := \det e^a_i$ is a density. This pair forms the canonical set.

In the case of pure gravitational spacetime, the Hilbert action takes the form

$$S = \int d^4x [\tilde{A}^a_\mu \tilde{E}^{ja} + \frac{i}{2} N \tilde{E}^a_i \tilde{E}^b_j F^a_{ij} e^b_c - \Lambda_N \det \tilde{E} - N^i F^i_{jk} \tilde{E}^a_j + \Lambda_\Lambda \tilde{E}^a_\mu D_\mu \tilde{E}^a_\mu], \quad (A4)$$

where $\Lambda_N := e^{-1} N$, $\Lambda$ is the cosmological constant $^7$, $D_\mu \tilde{E}^a_i := \partial_\mu \tilde{E}^a_i - i e^a_{bc} A^b_\mu \tilde{E}^c_i$, and $\det \tilde{E}$ is defined to be

$$\det \tilde{E} = \frac{1}{6} \epsilon^{abc} \xi_{ijk} \tilde{E}^a \tilde{E}^b \tilde{E}^c, \text{ where } \epsilon_{ijk} := \epsilon^{abc} \epsilon^b_j \epsilon^c_k \text{ and } \xi_{ijk} := e^{-1} \epsilon_{ijk} 8.$$

Varying the action with respect to the non-dynamical variables $N, N^i$ and $A_0^a$ yields the constraint equations,

$$C_H = \frac{i}{2} e^{ab} c E^4_a E^4_b F^i_\mu - \Lambda \det \tilde{E} \approx 0, \quad (A5)$$
$$C_{Mi} = - F^a_\mu \tilde{E}^i_a \approx 0, \quad (A6)$$
$$C_{G\alpha} = D_i \tilde{E}^a_i \approx 0, \quad (A7)$$

where $F^a_\mu := (dA^a)_{\mu\nu} - \frac{i}{2} e^{bc} (A^b \wedge A^c)_{\mu\nu}$ is the curvature 2-form.

The equations of motion for the dynamical variables $(A_\mu^a, E^a_i)$ are

$$\tilde{A}^a_i = - i e^{ab} N \tilde{E}^b_j F^a_\mu j + N^j F^a_\mu j + D_\mu A_\mu^a + e \Lambda N e^a_i, \quad (A8)$$
$$\tilde{E}^a_i = - i D_j (e^{cb} N \tilde{E}^c_j \tilde{E}^b_a) + 2 D_j (N^j \tilde{E}^a_i) + i A^b_\mu \epsilon^{bc} \tilde{E}^c_i, \quad (A9)$$

where $D_j T^a_{ij} := \partial_j T^a_{ij} - i e^a_{bc} A^b_\mu T^c_\mu j$, for $T^a_{ij} + T^a_{ji} = 0$.

In order to construct metric variables from the variables $(A_\mu^a, E^a_i, N, N^i)$, we first prepare tetrad $E^a_i$ as $E^a_0 = (1/e_N, -N^j/e_N)$ and $E^a_i = (0, \tilde{E}^a_i/e)$. Using them, we obtain metric $g^{\mu\nu}$ such that

$$g^{\mu\nu} := E^i_\mu E^j_\nu \partial^l e^l. \quad (A10)$$

2. Reality conditions

Notice that in general the metric $(A10)$ is not real. In order to recover the real metric, we must impose the reality conditions.

To ensure the metric is real-valued, we need to impose two conditions; the primary is that the doubly densitized contravariant metric $\tilde{\gamma}^{ij} := \epsilon^{2} \tilde{E}^{ij}$ is real,

$$\Im(\tilde{E}^a_i \tilde{E}^a_j) = 0, \quad (A11)$$

and the secondary condition is that the time derivative of $\tilde{\gamma}^{ij}$ is real,

$$\Im(\partial_\tau (\tilde{E}^a_i \tilde{E}^a_j)) = 0. \quad (A12)$$

Using the equations of motion for $\tilde{E}^a_i$ (A9), the gauge constraint (A7) and the primary reality condition (A11), we can replace the secondary condition (A12) with a different constraint

$$W^{ij} := \Re(e^{abc} \tilde{E}^b_i \tilde{E}^c_j D_k \tilde{E}^k_c) \approx 0, \quad (A13)$$

$^7$We changed a factor in front of cosmological constant from [5].

$^8\epsilon_{xyz} = e, \epsilon_{xyz} = 1, e^{xyz} = e^{-1}, \tilde{e}^{xyz} = 1.$
which fixes six components of $\mathcal{A}_a^i$ and $\hat{E}_a^i$. Moreover, in order to recover the original lapse function $N := Ne$, we demand $\Im(N/e) = 0$, i.e. the density $e$ be real and positive. This requires that $e^2$ be positive, i.e.

$$\det\hat{E} > 0.$$  \hspace{1cm} (A14)

The secondary condition of (A14),

$$\Im[\partial_t(\det\hat{E})] = 0,$$  \hspace{1cm} (A15)

is automatically satisfied (see [5]). Therefore, in order to ensure that $e$ is real, we only require (A14).

Rather stronger reality conditions are sometimes useful in Ashtekar’s formalism for recovering the real 3-metric and extrinsic curvature. These conditions are

$$\Im(\hat{E}_i^a) = 0$$  \hspace{1cm} (A16)

and

$$\Im(\dot{\hat{E}}_i^a) = 0,$$  \hspace{1cm} (A17)

and we call them the “primary triad reality condition” and the “secondary triad reality condition”, respectively. Using the equations of motion of $\hat{E}_a^i$, the gauge constraint (A7), the metric reality conditions (A11), (A12) and the primary condition (A16), we see that (A17) is equivalent to [5]

$$\Re(\mathcal{A}_a^i) = \partial_i(N)\hat{E}_{ia} + \frac{1}{2} e^{-1} e_b^i N \hat{E}_{ja} \partial_j \hat{E}_{ib} + N^i \Re(\mathcal{A}_a^i).$$  \hspace{1cm} (A18)

From this expression we see that the second triad reality condition restricts the three components of “triad lapse” vector $\mathcal{A}_a^i$. Therefore (A18) is not a restriction on the dynamical variables ($\mathcal{A}_a^i$ and $\hat{E}_a^i$) but on the slicing, which we should impose on each hypersurface. Thus the second triad reality condition does not restrict the dynamical variables any further than the second metric condition does.

---