New Mass Relations for Heavy Quarkonia

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Abstract

By assuming the existence of (quasi)-linear Regge trajectories for heavy quarkonia in the low energy region, we derive a new, sixth power, meson mass relation which shows good agreement with experiment for both charmed and beauty mesons. This relation may be reduced to a quadratic Gell-Mann–Okubo

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type formula by fitting the values of the Regge slopes of these (quasi)-linear trajectories. For charmed mesons, such a formula holds with an accuracy of ~ 1%, and is in qualitative agreement with the relation obtained previously by the application of the linear mass spectrum to a meson hexadecuplet.

Key words: flavor symmetry, quark model, charmed mesons, Gell-Mann–Okubo, Regge phenomenology


The generalization of the standard $SU(3)$ Gell-Mann–Okubo mass formula [1] to higher symmetry groups, e.g., $SU(4)$ and $SU(5)$, became a natural subject of investigation after the discovery of the fourth and fifth quark flavors in the mid-70’s [2]. Attempts have been made in the literature to derive such a formula, either quadratic or linear in mass, by a) using group theoretical methods [3, 4, 5], b) generalizing the perturbative treatment of $U(3) \times U(3)$ chiral symmetry breaking and the corresponding Gell-Mann-Oakes-Renner relation [6] to $U(4) \times U(4)$ [7, 8], c) assuming the asymptotic realization of $SU(4)$ symmetry in the algebra $[A_\alpha, A_\beta] = i f_{\alpha\beta\gamma} V_\gamma$ (where $V_\alpha, A_\beta$ are vector and axial-vector charges, respectively) [9], d) extending the Weinberg spectral function sum rules [11] to accommodate the higher symmetry breaking effects [10], and e) applying alternative methods, such as the linear mass spectrum for meson multiplets\(^1\) [12, 13]. In the following\(^2\), $\eta, \eta_s, \eta_c, \eta_b, K, D, D_s, B, B_s, B_c$ stand for the masses of the $n\bar{n}$ ($n \equiv u$ or $d$), $s\bar{s}, c\bar{c}, b\bar{b}, s\bar{n}, c\bar{c}, c\bar{s}, b\bar{n}, b\bar{s}, b\bar{c}$ mesons, respectively\(^3\).

The linear mass relations

\[
\begin{align*}
D &= \frac{\eta + \eta_c}{2}, \\
D_s &= \frac{\eta_s + \eta_c}{2}, \\
B &= \frac{\eta + \eta_b}{2}, \\
B_s &= \frac{\eta_s + \eta_b}{2}, \\
B_c &= \frac{\eta_c + \eta_b}{2}
\end{align*}
\]

found in [5, 8], although perhaps justified for vector mesons, since a vector meson mass is given approximately by a sum of the corresponding constituent quark masses,

\[
m(i\bar{j}) \simeq m(i) + m(j)
\]

\(^1\)Here, we speak of linear spectrum over the multiplet quantum numbers, taking proper account of degeneracy, not (directly) make use of linear Regge trajectories.

\(^2\)Here $\eta$ stands for the masses of both isovector and isoscalar $n\bar{n}$ states which coincide on a naive quark model level.

\(^3\)Since these designations apply to all spin states, vector mesons will be confusingly labelled as $\eta$’s. We ask the reader to bear with us in this in the interest of minimizing notation.
(in fact, for vector mesons, the relations (1),(2) hold with an accuracy of up to \( \sim 4\% \)),
are expected to fail for other meson multiplets, as confirmed by direct comparison with experiment. Similarly, the quadratic mass relation

\[ D_s^2 - D^2 = K^2 - \eta^2 \]  

obtained in ref. [7] by generalizing the \( SU(3) \) Gell-Mann-Oakes-Renner relation [6]
to include the \( D \) and \( D_s \) mesons (here \( \pi \) stands for the mass of the \( \pi \), etc.),

\[ \frac{\pi^2}{2n} = \frac{K^2}{n + s} = \frac{D^2}{n + c} = \frac{D_s^2}{s + c}, \]  

(and therefore \( D_s^2 - D^2 = K^2 - \pi^2 \propto (s - n) \), also found in refs. [4, 10, 9]), does not agree with experiment. For pseudoscalar mesons, for example, one has (in GeV^2)
0.388 for the l.h.s. of (3) vs. 0.226 for the r.h.s. For vector mesons, the corresponding
quantities are 0.424 vs. 0.199, with about 100% discrepancy. The reason that the
relation (3) does not hold is apparently due to the impossibility of perturbative treat-
m of \( U(4) \times U(4) \) symmetry breaking, as a generalization of that of \( U(3) \times U(3) \),
due to very large bare mass of the \( c \)-quark as compared to those of the \( u \)-, \( d \)- and \( s \)-quarks. In ref. [13] by the application of the linear spectrum to \( SU(4) \) meson
hexadecuplet, the following relation was obtained,

\[ 12D^2 = 7\eta_0^2 + 5\eta_c^2, \]  

where \( \bar{D} \) is the average mass of the \( D \) and \( D_s \) states (which are mass degenerate
when \( SU(4) \) flavor symmetry is broken to \( SU(3) \)), and \( \eta_0 \) is the mass average of
the corresponding \( SU(3) \) nonet. As shown in ref. [13], this relation holds with an
accuracy of up to \( \sim 5\% \) for all well established meson hexadecuplets.

It is well known that the hadrons composed of light \( (u,d,s) \) quarks populate linear
Regge trajectories; i.e., the square of the mass of a state with orbital momentum \( \ell \) is
proportional to \( \ell : M^2(\ell) = \ell/\alpha' + \text{const} \), where the slope \( \alpha' \) depends weakly on the
flavor content of the states lying on the corresponding trajectory,

\[ \alpha'_{n\bar{n}} \simeq 0.88 \text{ GeV}^{-2}, \quad \alpha'_{s\bar{n}} \simeq 0.84 \text{ GeV}^{-2}, \quad \alpha'_{s\bar{s}} \simeq 0.80 \text{ GeV}^{-2}. \]  

In contrast, the data on the properties of Regge trajectories of hadrons containing
heavy quarks are almost nonexistent at the present time, although it is established
[14] that the slope of the trajectories decreases with increasing quark mass (as seen
in Eq. (6)) in the mass region of the lowest excitations. This is due to an increasing
(with mass) contribution of the color Coulomb interaction, leading to a curvature of
the trajectory near the ground state. However, as the analyses show [14, 15, 16], in
the asymptotic regime of the highest excitations, the trajectories of both light and
heavy quarkonia are linear and have the same slope \( \alpha' \simeq 0.9 \text{ GeV}^{-2} \), in agreement
with natural expectations from the string model.
Knowledge of Regge trajectories in the scattering region, i.e., at \( t < 0 \), and of the intercepts \( a(0) \) and slopes \( \alpha' \) is also useful for many non-spectral purposes, for example, in the recombination [17] and fragmentation [18] models. Therefore, as pointed out in ref. [14], the slopes and intercepts of the Regge trajectories are the fundamental constants of hadron dynamics, perhaps generally more important than the mass of any particular state. Thus, not only the derivation of a mass relation but also the determination of the parameters \( a(0) \) and \( \alpha' \) of heavy quarkonia is of great importance, since they afford opportunities for better understanding of the dynamics of the strong interactions in the processes of production of charmed and beauty hadrons at high energies.

Here we apply Regge phenomenology for the derivation of a mass formula for the SU(4) meson hexadecuplet, by assuming the (quasi)-linear form of Regge trajectories for heavy quarkonia with slopes which are generally different from (less than) the standard one, \( \alpha' \approx 0.9 \text{ GeV}^{-2} \). We show that for the formula to avoid depending on the values of the slopes, it must be of sixth power in meson masses. It may be reduced to a quadratic Gell-Mann–Okubo type relation, by fitting the values of the slopes, which is in qualitative agreement with Eq. (5).

Let us assume the (quasi)-linear form of Regge trajectories for hadrons with identical \( J^{PC} \) quantum numbers (i.e., belonging to a common multiplet). Then for the states with orbital momentum \( \ell \) one has

\[
\ell = \alpha'_{ii} m_{ii}^2 + a_{ii}(0),
\]

\[
\ell = \alpha'_{jj} m_{jj}^2 + a_{jj}(0),
\]

\[
\ell = \alpha'_{jj} m_{jj}^2 + a_{jj}(0).
\]

Using now the relation among the intercepts [19, 20],

\[
a_{ii}(0) + a_{jj}(0) = 2a_{ji}(0),
\]

one obtains from the above relations

\[
\alpha'_{ii} m_{ii}^2 + \alpha'_{jj} m_{jj}^2 = 2\alpha'_{ji} m_{ji}^2.
\]

In order to eliminate the Regge slopes from this formula, we need a relation among the slopes. Two such relations have been proposed in the literature,

\[
\alpha'_{ii} \cdot \alpha'_{jj} = \left(\alpha'_{ji}\right)^2,
\]

which follows from the factorization of residues of the \( t \)-channel poles [21, 22], and

\[
\frac{1}{\alpha'_{ii}} + \frac{1}{\alpha'_{jj}} = \frac{2}{\alpha'_{ji}},
\]

based on topological expansion and the \( q\bar{q} \)-string picture of hadrons [20].
For light quarkonia (and small differences in the $\alpha'$ values), there is no essential difference between these two relations; viz., for $\alpha'_{ji} = \alpha'_{ii}/(1 + x)$, $x \ll 1$, Eq. (10) gives $\alpha'_{jj} = \alpha'_{ii}/(1 + 2x)$, whereas Eq. (9) gives $\alpha'_{jj} = \alpha'_{ii}/(1 + x)^2 \approx \alpha'/(1 + 2x)$, i.e., essentially the same result to order $x^2$. However, for heavy quarkonia (and expected large differences from the $\alpha'$ values for the light quarkonia) these relations are incompatible; e.g., for $\alpha'_{ji} = \alpha'_{ii}/2$, Eq. (9) will give $\alpha'_{jj} = \alpha'_{ii}/4$, whereas Eq. (10) $\alpha'_{jj} = \alpha'_{ii}/3$. One has therefore to choose one of these relations in order to proceed further. Here we use Eq. (10), since it is much more consistent with (8) than is Eq. (9), which we tested by using measured quarkonia masses in Eq. (8). We shall justify this choice in more detail in a separate publication [23].

Since we are interested in $SU(4)$ breaking, and since it simplifies the discussion, we take average slope in the light quark sector:

$$\alpha'_{nh} \cong \alpha'_{sh} \cong \alpha'_{sh} \cong \alpha' \simeq 0.85 \text{ GeV}^{-2}. \tag{11}$$

It then follows from the relations based on (8),

$$\alpha' \eta^2 = \alpha' \eta^2 + \alpha'_{\bar{c}c} \eta^2_c = 2\alpha'_{\bar{c}c} D_s^2, \tag{12}$$

$$\alpha' \eta^2_s + \alpha' \eta^2_c = 2\alpha'_{\bar{c}c} D_s^2, \tag{13}$$

that

$$\alpha' \cong \alpha'_{\bar{c}c} = \alpha' \frac{(\eta^2_s - \eta^2) D_s^2 - \eta^2_s}{(D_s^2 - D_s^2) \eta^2_c - \eta^2_s}. \tag{14}$$

Using these values of the slopes in Eq. (10) with $i = n, j = c$, we obtain

$$\left(\eta^2_s D_s^2 - \eta^2_s D_s^2\right) \left(\eta^2_c - \eta^2_s\right) + \eta^2_c \left(D_s^2 - D_s^2\right) \left(\eta^2_s - \eta^2\right) = 4 \left(\eta^2_s D_s^2 - \eta^2 D_s^2\right) \left(D_s^2 - D_s^2\right), \tag{16}$$

which is a new mass relation for the $SU(4)$ meson hexadecuplet. To test (16), we use the four well-established hexadecuplets\footnote{We use the value $\eta_s = 0.686$ GeV for pseudoscalar nonet, as follows from $\eta^2_s = 2K^2 - \pi^2$. We also use the simple average of the masses of the isovector and isoscalar states as the value of $\eta$ in Eqs. (16),(17) for the remaining three multiplets.} [24]:

1) $1 \; ^1 S_0 \; J^{PC} = 0^{-+}$, $m(\pi) = 0.138$ GeV, $m(\eta_s) = 0.686$ GeV, $m(D) = 1.868$ GeV, $m(D_1) = 1.969$ GeV, $m(\eta_c) = 2.980$ GeV,

$1 \; ^3 S_1 \; J^{PC} = 1^{+-}$, $m(\rho) = 0.775$ GeV, $m(\phi) = 1.019$ GeV, $m(D^*) = 2.009$ GeV, $m(D_{1*}) = 2.112$ GeV, $m(J/\psi) = 3.097$ GeV,

3) $1 \; ^1 P_1 \; J^{PC} = 1^{++}$, $m(b_1) = 1.201$ GeV, $m(h'_1) = 1.380$ GeV, $m(D_1) = 2.422$ GeV, $m(D_{1*}) = 2.535$ GeV, $m(h_{1*}(1P)) = 3.526$ GeV,

4) $1 \; ^3 P_2 \; J^{PC} = 2^{++}$, $m(a_2) = 1.297$ GeV, $m(f'_2) = 1.525$ GeV, $m(D_{2*}) = 2.459$ GeV, $m(D_{2*}) = 2.573$ GeV, $m(\chi_{2}(1P)) = 3.556$ GeV,
and rewrite (16) in the form

\[ \eta_c^2 = \frac{(\eta_s^2 D^2 - \eta^2 D_s^2) (4(D_s^2 - D^2) - (\eta_s^2 - \eta^2))}{(\eta_s^2 - \eta^2)(D_s^2 - D^2)}, \]  

(17)

We shall test the relation by comparing the values of \( \eta_c \) given by Eq. (17) with those established by experiment, using the known masses of the remaining states, for the four multiplets.

1) \( 1^1S_0 \) \( J^{PC} = 0^{-+} \). One obtains 3.137 GeV for the value of \( m(\eta_c) \) vs. experimentally established value 2.980 GeV; in this case the accuracy of Eq. (16) is \( \sim 5\% \).

2) \( 1^3S_1 \) \( J^{PC} = 1^{--} \). In this case one obtains 3.202 GeV vs. 3.097 GeV, as the value of \( m(J/\psi) \); the accuracy is \( \sim 3.5\% \).

3) \( 1^1P_1 \) \( J^{PC} = 1^{+-} \). Now one obtains 3.615 GeV vs. 3.526 GeV, as the value of \( m(h_c(1P)) \); the accuracy is \( \sim 2.5\% \).

4) \( 1^3P_2 \) \( J^{PC} = 2^{++} \). In this case one has 3.618 GeV vs. 3.556 GeV, as the value of \( m(\chi_{c2}(1P)) \); the accuracy is \( \sim 1.5\% \).

One sees that the formula (16) holds with a high accuracy for all four well established meson multiplets. The major contribution to the discrepancy between our formula result and experiment is the approximation (11). For higher excited states the trajectories become more accurately parallel, and the approximation (11) and subsequent relation \( \alpha'_c = \alpha'_s \) (Eq. (13)) become more exact. As shown above, the formula (16) does hold with improving accuracy as one proceeds to higher spin multiplets.

A possible additional reason for the discrepancy is the curvature of the \( \eta_c \) trajectory near \( \ell = 0 \) since the mass of the \( \eta_c \) is lower than expected from a linear extrapolation. Similarly, if one tries, apart from its Goldstone nature, to fit the pion to the trajectory on which the \( b_1(1231) \) and \( \pi_2(1670) \) lie \( [\ell = 0.85 \ M^2(\ell) - 0.30] \), extrapolation down to \( \ell = 0 \) gives \( m(\pi) \approx 0.6 \) GeV, much higher than the physical value \( m(\pi) = 0.138 \) GeV.

We note that a relation based on Eq. (9) for the slopes which is also of sixth power in meson masses,

\[ 4D^2 \left(D_s^2 - D^2\right) \left(\eta_s^2 - \eta^2\right) = \eta_c^2 \left(\eta_s^2 - \eta^2\right)^2 + 4\eta^2 \left(D_s^2 - D^2\right)^2, \]

holds for the four multiplets with an accuracy of 15-20\%. The reason for such a large discrepancy with experiment is a lower value for the slope of the charmonia trajectory given by Eq. (9), as compared to that given by (10), leading to higher values for the charmonia masses.

We emphasize that the formula (16) does not depend on the values of the Regge slopes, but only on the relation between them, Eq. (10), which justifies its use in both the low energy region where the slopes are different and the high energy
region where all the slopes coincide. In the latter case, as follows from (12),(13),
\[ \eta_s^2 - \eta^2 = 2(D_s^2 - D^2), \]
and Eq. (17) reduces to
\[ \eta^2 + \eta_c^2 = 2D^2, \tag{18} \]
consistent with Eq. (12) in this limit. One may also find from Eqs. (12),(13) with equal slopes, and the standard \( SU(3) \) Gell-Mann–Okubo relation,
\[ \eta^2 + \eta_s^2 = 2K^2, \tag{19} \]
that Eq. (3) also holds in this limit.

The entire analysis may, of course, be repeated with \( B, B_s, \eta_b \) in place of \( D, D_s, \eta_c \), respectively, and will lead (assuming (10)) to a relation similar to (16):
\[ \left( \eta_s^2 B^2 - \eta^2 B_s^2 \right) \left( \eta_s^2 - \eta^2 \right) + \eta_b^2 \left( B_s^2 - B^2 \right) \left( \eta_s^2 - \eta^2 \right) = 4 \left( \eta_s^2 B^2 - \eta^2 B_s^2 \right) \left( B_s^2 - B^2 \right), \tag{20} \]
which is a new mass relation for the \( SU(5) \) meson 25-plet. We can test this relation for vector mesons since the masses of all of the beauty states involved are established experimentally only for vector mesons [24]: \( m(B^*) = 5.325 \) GeV, \( m(B_s^*) = 5.416 \) GeV, \( m(\Upsilon(1S)) = 9.460 \) GeV. The formula (20) yields \( m(\Upsilon(1S)) = 9.796 \) GeV, within \( \sim 3.5\% \) of the physical value, and the same accuracy as for charmed vector mesons.

Let us now discuss the question of the generalization of the standard \( SU(3) \) Gell-Mann–Okubo mass formula which is quadratic in mass to the case of heavier quarkonia. We shall continue to assume the validity of Eq. (11) and introduce \( x > 0 \) through the relation
\[ \alpha_{c\bar{n}}' = \alpha_{c\bar{s}}' = \frac{\alpha'}{1 + x}. \tag{21} \]
It then follows from (10) that
\[ \alpha_{c\bar{c}}' = \frac{\alpha'}{1 + 2x}, \tag{22} \]
and one obtains from (12),(13),
\[ (1 + x) \left( \eta^2 + \eta_s^2 \right) + \frac{2(1 + x)}{1 + 2x} \eta_c^2 = 2 \left( D_s^2 + D^2 \right). \tag{23} \]
Results of the calculations of the Regge slopes of heavy quarkonia in refs. [22]: \( \alpha_{c\bar{n}}'/\alpha' \simeq 0.73, \alpha_{c\bar{c}}'/\alpha' \simeq 0.58 \), and [20, 25]: \( \alpha_{c\bar{c}}' \simeq 0.5 \) GeV\(^{-2} \), support the value
\[ x \cong 0.355. \tag{24} \]
With this \( x \), it follows from (23) and the standard \( SU(3) \) Gell-Mann–Okubo formula (19) that
\[ 8.13 \; K^2 + 4.75 \; \eta_c^2 = 6 \left( D_s^2 + D^2 \right). \tag{25} \]
Thus, the new sixth order mass relations may be accurately reduced to quadratic ones by use of specific values for the Regge slopes.

For the four well-established meson hexadecuplets, the formula (23) gives in its l.h.s. and r.h.s., respectively\(^5\) (in GeV\(^2\)):

1) \(1^1S_0 \ J^{PC} = 0^{++}\), 44.17 vs. 44.20
2) \(1^3S_1 \ J^{PC} = 1^{--}\), 52.03 vs. 50.98
3) \(1^1P_1 \ J^{PC} = 1^{+-}\), 73.63 vs. 73.75
4) \(1^3P_2 \ J^{PC} = 2^{++}\), 76.67 vs. 76.00

One sees that the formula (25) holds at a 1% level for all four well established meson multiplets, thus confirming the assumption on the (quasi)-linearity of the Regge trajectories of heavy quarkonia in the low energy region. The formula (25) is in qualitative agreement with the relation (5) obtained by two of the present authors in ref. [13] by the application of the linear mass spectrum to a meson hexadecuplet.

Finally, we note that the derived Regge slopes in the charm sector are

\[\alpha'_{c\bar{n}} \simeq \alpha'_{c\bar{s}} \simeq 0.63 \text{ GeV}^{-2}, \quad \alpha'_{c\bar{c}} \simeq 0.50 \text{ GeV}^{-2}.\]  

(26)

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\(^5\)For the pseudovector nonet, we use the value \(K_{1B} = 1.339 \text{ GeV}\) which follows from the assumption on a 45° mixing between axial-vector and pseudovector nonets in the isodoublet channel [26].


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