Abstract : We propose a model for the geometry of a dynamical spherical shell in which the metric is asymptotically Schwarzschild, but deviates from Ricci-flatness in a finite neighbourhood of the shell. Hence, the geometry corresponds to a ‘hairy’ black hole, with the hair originating on the shell. The metric is regular for an infalling shell, but it bifurcates, leading to two disconnected Schwarzschild-like spacetime geometries. The shell is interpreted as either collapsing matter or as Hawking radiation, depending on whether or not the shell is infalling or outgoing. In this model, the Hawking radiation results from tunnelling between the two geometries. Using this model, the back reaction correction from Hawking radiation is calculated.
The back reaction of Hawking radiation on the black hole geometry has attracted considerable attention during recent years [1]. However, explicit calculation of the resulting corrections to the spectrum have only been performed in the semi-classical approximation using a thin shell model for the Hawking radiation field [1], wherein a “dressed shell” wave function was obtained in a reduced phase-space quantization formalism, with the background metric fixed to be Ricci-flat, and hence Schwarzschild, everywhere except at the location of the shell, assumed to be an imbedded 3-manifold. To lowest order, the shell traverses a geodesic in the ambient spacetime. Near the Schwarzschild radius (given in terms of the ADM mass $M_A$ by $r = 2M_A$), the affine parameter $t$ of the geodesic acquires a small imaginary part for an outgoing shell. The Hawking radiation is thereby generated in a narrow region near the Schwarzschild radius. In fact, the spectrum of Hawking radiation is completely determined by $\text{Im} t$, independently of the detailed form of the wave function.

In this paper we examine the dependence of the correction to the spectrum of Hawking radiation on the choice of spacetime metric used to model the back reaction. For this purpose, we introduce a more detailed model, so that the minimal modification to the Schwarzschild geometry reproduces the expected stress-energy tensor in the asymptotic region. The possible choices are severely constrained by this asymptotic condition as well as the boundary conditions at the shell. This is reviewed in Section 2. In Section 3, such a metric is displayed and its properties are discussed. In the absence of the back reaction, it reduces to the Schwarzschild metric, with the shell playing the role of collapsing matter, if it is infalling, and of the Hawking radiation, if it is outgoing. In the next approximation, the results of Kraus and Wilczek [1] are recovered, but with an additional correction of the same order of magnitude. This additional correction arises from the details of the dynamics. We also derive the corrections arising from a non-zero shell mass- i.e. from a massive scalar field. In Section 4, we examine the stress-energy tensor. In the last section, we conclude our discussion and anticipate a more complete quantum mechanical treatment of the problem.

Our model is based on a reparameterization, $r \to R(r; \alpha(\hat{r}), \beta(\hat{r}))$, where $\alpha(\hat{r}), \beta(\hat{r})$ are parameters depending on the shell position $\hat{r}$. The parameters $\alpha(\hat{r}), \beta(\hat{r})$ are uniquely determined by the boundary conditions at the shell. The ADM mass $M_A$ is then determined from the saddle point condition. The parameter $\beta(\hat{r})$ satisfies a cubic equation, the physical roots of which represent two disconnected spacetime geometries. For an outgoing shell, we
find that $\beta(\hat{r})$ acquires a small imaginary part in the narrow non-classical region around
the Schwarzschild radius. It is in this region that the Hawking radiation originates from
tunnelling between the two spacetime geometries. We obtain an expression for Im $t$ from
the solution of the geodesic equation- the lowest order approximation to the motion of the
shell. A notable feature of this approach is that the geodesics for an infalling particle, as a
function of Schwarzschild time $t$, are complete. This is due to the dynamical distortion of
the Schwarzschild metric- the back reaction. Hence, the affine parameter of these geodesics
can be used as the time variable in the calculation of the Hawking spectrum.

2 Review of Spherical Shells and Hawking Radiation

We will consider a spherically symmetric geometry in which there is a thin spherical shell
with mass $m$. In terms of spherical coordinates $(t, r, \theta, \phi)$, the metric is of the form

$$ds^2 = -N^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$  

(1)

where $N, L, N^r, R$ are functions of $(t, r)$ only. In fact, the dependence on the time coordinate
$t$ is only through $\hat{r}(t)$, where $r = \hat{r}(t)$ describes the position of the spherical shell.

The action can be written in ADM form [2, 1]:

$$S[\hat{r}, R, L, N, N^r] = \int dt \left[ \hat{p} \dot{\hat{r}} - M_A + \int dr \left( \Pi_R \dot{R} + \Pi_L \dot{L} - N \mathcal{H}_t - N^r \mathcal{H}_r \right) \right].$$  

(2)

In the above, $\hat{p}, \Pi_R, \Pi_L$ are the canonical momenta conjugate, respectively, to $\hat{r}, R, L$. The
term $M_A$ is a boundary term which is a constant of the motion and may be identified with
the ADM mass. We use units in which $G = c = 1$. The lapse $N$ and shift $N^r$ functions are
Lagrange multipliers enforcing the constraints:

$$\mathcal{H}_t := \frac{L \Pi_L^2}{2R^2} - \frac{\Pi_L \Pi_R}{R} + \frac{(R R')'}{L} - \frac{(R')^2}{2L} - \frac{L}{2} + \frac{\dot{E}}{L} \delta(r - \hat{r}(t)) \approx 0,$$  

(3)

$$\mathcal{H}_r := R' \Pi_R - L \Pi_L' - \hat{p} \delta(r - \hat{r}(t)) \approx 0.$$  

(4)

These are respectively the Hamiltonian and diffeomorphism constraints. Hatted quantities,
e.g. $\hat{L}$ are given by $\hat{L}(t) = L(r, t)|_{r = \hat{r}(t)}$. Also $\hat{L} = \partial L/\partial t, L' = \partial L/\partial r$, etc. Finally, the
quantity $\hat{E} := (\hat{p}^2 + m^2 L^2)^{\frac{1}{2}}$.

The constraints can be solved for the field momenta [2, 4]

$$\Pi_R = L \Pi_L / R', \text{if } r \neq \hat{r}(t);$$  

(5)
\[ \Pi_L = \begin{cases} R \left[ (R'/L)^2 - 1 + 2M_A/R \right]^{\frac{1}{2}}, & \text{if } r > \hat{r}(t) \\ R \left[ (R'/L)^2 - 1 + 2M/R \right]^{\frac{1}{2}}, & \text{if } r < \hat{r}(t) \end{cases} \]  

Moreover, the discontinuity at the shell gives the following boundary conditions:

\[ \Delta \Pi_L = \frac{-\hat{p}}{\hat{L}}; \quad \Delta R' = \frac{-\hat{E}}{\hat{R}}, \]

where

\[ \Delta \Pi_L := \lim_{\epsilon \to 0^+} (\Pi_L(\hat{r}(t) + \epsilon) - \Pi_L(\hat{r}(t) - \epsilon)), \]

etc. As usual in thin shell formulations, it is assumed that the metric is continuous, but the first derivatives normal to the shell are discontinuous at \( r = \hat{r}(t) \).

A semi-classical quantization of the system can be performed either in terms of the reduced phase space [1], or via Dirac quantization, wherein the constraints are imposed as operators which annihilate physical states [2, 4]. In either case, one ends up with a second- quantized wave function \( \psi(t, r) \) associated with the location of the shell. When expanded in modes the wave-function has the form:

\[ \psi(t, r) = \int \frac{d\omega}{2\pi} \left[ a_i(\omega)u_i(\omega, t, r) + a_i^\dagger(\omega)u_i^*(\omega, t, r) \right], \]  

where the index \( i = 1, 2 \) is not summed over, and denotes, respectively the quantities defined with respect to observers falling with the shell and in the asymptotic region. In particular, \( a_i(\omega), a_i^\dagger(\omega) \) are annihilation and creation operators, while the \( u_i \) and their complex conjugates \( u_i^* \) are the mode functions. The mode functions \( u_2 \), in the asymptotic region, are given by \( u_2 = u(\omega, r)e^{-i\omega t} \). The operators corresponding to the different observers are related by Bogoliubov transformations, with the coefficients \( A(\omega, \omega'), B(\omega, \omega') \) given by the Fourier transforms of the wave-function:

\[ A = \frac{1}{2\pi u(\omega, r)} \int_{-\infty}^{\infty} dt e^{i\omega t} \psi(t, r), \]

\[ B = \frac{1}{2\pi u^*(\omega, r)} \int_{-\infty}^{\infty} dt e^{i\omega t} \psi^*(t, r). \]

In the saddle point limit one gets the result [1] \( M_A = M \pm \omega \), where the \( +(-) \) sign refers to an ingoing (outgoing) shell. This fixes \( M_A \) in the semi-classical approximation. [3]

By standard arguments [6, 5, 1], one may compute the number of outgoing particles in Hawking radiation with frequencies between \( \omega \) and \( \omega + d\omega \):

\[ n(\omega) \frac{d\omega}{2\pi} = \left[ |A/B|^2 - 1 \right]^{-1} \frac{d\omega}{2\pi}. \]
The quantity \( n(\omega) \) gives the spectrum if the contribution of those particles falling back into the black hole can be ignored. If back reaction is also neglected, \( n(\omega) \) is given by the well-known thermal spectrum \([5, 6]\)

\[
n(\omega) = \frac{1}{e^{8\pi M \omega} - 1}.
\]

(12)

3 The Back Reaction

In the absence of the shell, the one-parameter family of Schwarzschild metrics is the unique solution of the vacuum Einstein equations with spherical symmetry. The presence of the shell disrupts this uniqueness. In fact, classically the metric would be Schwarzschild everywhere except at the location of the shell itself. The metric has a jump discontinuity at the location of the shell, and the Schwarzschild metrics on either side of the shell can have unequal Schwarzschild mass. In a quantum mechanical treatment, classical geometry may not be completely established. In a semi-classical analysis, there is a transitory regime of duration \( \Delta \tau \) specified by the quantum uncertainty principle, during which the Schwarzschild mass, or total energy, \( M \) is uncertain by \( \Delta M \), where \( \Delta M \Delta \tau \sim \hbar \). We expect that in this approximation, the uncertainty in total energy is approximately given by the energy of the shell, \( \Delta M \sim \hbar \omega \).[3]

We can model the situation via the “semi-classical Einstein equations”

\[
G_{\mu\nu}(x, \omega) = 8\pi T_{\mu\nu}(x, \omega),
\]

(13)

where the stress-energy tensor \( T_{\mu\nu}(x, \omega) \) includes the Hawking radiation itself, with a concomitant distortion of the “background” Schwarzschild geometry. The program would begin with consideration of some matter field, say a scalar field governed classically by the Klein-Gordon equations, propagating in the distorted Schwarzschild background. However, in the absence of an exact classical solution which describes the back reaction effect, we will propose a physically reasonable metric as a “solution” of Eq.(13). For this purpose, we will rely on the constraints and the boundary conditions at the shell to limit the possibilities. In addition to this, the stress-energy tensor \( T_{\mu\nu} \) will be required to be consistent with well-known predictions for the flux of Hawking radiation in the asymptotic region, i.e., as \( \hat{r} \to \infty \). Moreover, Hawking radiation may be viewed as arising from a mismatch between the reference frames of an observer at infinity and that of an infalling observer \([6]\). This mismatch is represented by the Bogolubov coefficients given by Eq.(10). Therefore, in
our model, the distorted metric must approach that of an infalling observer as $r \to \hat{r}(t)$. We can satisfy this requirement by choosing coordinates such that in the “inner region” $r < \hat{r}(t)$, $\Pi_L = 0 = \Pi_R$. These conditions severely limit the choice of metric.

We have found the following metric satisfies the above requirements:

$$R = \begin{cases} R_+ := r [1 + \gamma(r, \hat{r}(t), \omega)], & \text{if } r > \hat{r}(t) \\ R_- := r [1 + \beta(\hat{r}(t), \omega)], & \text{if } r < \hat{r}(t) \end{cases}$$  \hspace{1cm} (14)$$

$$L = \begin{cases} L_+ := (1 - 2M/R_+)^{-\frac{1}{2}}, & \text{if } r > \hat{r}(t) \\ L_- := (1 + \beta(\hat{r}(t), \omega))(1 - 2M/R_-)^{-\frac{1}{2}}, & \text{if } r < \hat{r}(t) \end{cases}$$  \hspace{1cm} (15)$$

Furthermore, we will choose coordinates such that the lapse and shift functions are $N = 1/L, N^r = 0$.

The functional dependence of the parameters $\gamma(r, \hat{r}, \omega), \beta(\hat{r}, \omega)$ on $\hat{r}$ will be determined by the boundary conditions at the shell and in the asymptotic region.

In the asymptotic region, given by $r \to \infty$, we require that $R_+ \to r$. Hence

$$\gamma(r, \hat{r}, \omega) \to 0,$$  \hspace{1cm} (16)$$

in the above limit. This ensures that the metric is asymptotically Schwarzschild.

The continuity of $R$ at the shell requires that $\gamma(\hat{r}, \hat{r}, \omega) = \beta(\hat{r}, \omega)$. Furthermore, the derivative of $\gamma(r, \hat{r}, \omega)$ with respect to its first argument, evaluated at $r = \hat{r}$ is determined from the junction conditions Eq.(7) and the constraints Eq.(5),Eq.(6). The result is:

$$\hat{r}\alpha = \frac{(2 + \beta)}{2(1 + \beta)} + \frac{(m/\hat{r})^2}{2\beta(1 + \beta - 2M/\hat{r})},$$  \hspace{1cm} (17)$$

where we have written:

$$\alpha(\hat{r}, \omega) := \frac{-1}{\beta(\hat{r}, \omega)} \frac{\partial}{\partial r} \gamma(r, \hat{r}, \omega)|_{r=\hat{r}}.$$  \hspace{1cm} (18)$$

We specify the function $\gamma(r, \hat{r}, \omega)$ only up to the correct limiting behaviour, as discussed above.

It remains to ensure that $L$ is continuous across $r = \hat{r}(t)$. This will determine the parameter $\beta$. The equation $L_+ = L_-$ at $r = \hat{r}(t)$ implies that

$$x^3 - (hf)x^2 - x + f = 0,$$  \hspace{1cm} (19)$$

where

$$x := 1 + \beta; \hspace{0.5cm} f := 2M/\hat{r}; \hspace{0.5cm} h := M_A/M.$$  \hspace{1cm} (20)$$
The three roots of this cubic are given by

\[ P_+ + P_- + \frac{hf}{3}; \quad -\frac{P_+ + P_-}{2} \pm i\sqrt{3} \left( \frac{P_+ - P_-}{2} \right) + \frac{hf}{3}, \]  

(21)

where

\[
\begin{align*}
a &= -1 - \frac{1}{3}(hf)^2; \\
b &= -\frac{2}{27}(hf)^3 - \frac{1}{3}(hf) + f; \\
c &= \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}; \\
P_\pm &= \left(-\frac{b}{2} \pm c\right)^{1/3}.
\end{align*}
\]

(22, 23, 24)

If \( c^2 > 0 \), there is one real and a complex conjugate pair of roots; if \( c^2 < 0 \), there are three real roots.

The nature of the roots depends on the value of \( h \). In fact, \( h = 1 \pm \omega/M \), with the (+) sign for ingoing, and the (−) sign for outgoing radiation. It is easy to see that all the roots are real for ingoing radiation \( (h > 1) \), but complex roots are possible for outgoing radiation. Figures 1a and 1b display the behaviour of the roots, for infalling and outgoing shells, respectively.[7]

The behaviour of the roots in the respective regions where \( \tilde{r} >> 2M \) or where \( \tilde{r} \approx 2M \) is most important. In the former, where the shell is far from the horizon, \( x \) is very close to 1. We call this root “physical”, since for it, the metric is asymptotically flat. We note that for the physical root, \( \beta = \pm \omega/\tilde{r} + O\left(\tilde{r}^{-2}\right) \) for the + (−) sign for ingoing (outgoing) shell.

In the other region, near the event horizon, we expand in powers of the two parameters \( \omega/M \) and \( \epsilon := \tilde{r} - 2M \). To lowest order, for the outgoing shell, the roots are

\[
-1 - \frac{9}{4} \left( \frac{\omega}{M} \right); \quad 1 - \frac{1}{8} \left( \frac{\omega}{M} \right) - \frac{1}{2} \left( \frac{\epsilon}{2M} \right) \pm i\Delta,
\]

(25)

where

\[
\Delta := \sqrt{\frac{\omega}{2M}} \left[ 1 - \frac{5}{8} \left( \frac{\omega}{M} \right) + \frac{1}{2} \left( \frac{\epsilon}{2M} \right) \right].
\]

(26)

To summarize, the physical root \( 1 + \beta \) of Eq.(19) is always real for an ingoing shell, but for an outgoing shell, it acquires a small imaginary part in a narrow region around \( \tilde{r} = 2M \). This is the region where Hawking radiation originates and escapes by way of tunnelling through the classical barrier. In this region the solution of the geodesic equation can become complex, as will be shown below.
At the classical level, each point on the spherical shell traverses a geodesic. The geodesic equation can be obtained directly from the boundary condition Eq.(7) and from the expression for the momentum canonically conjugate to $\hat{r}$,

$$\hat{p} = m\hat{L}^2\hat{r} \left[ \hat{N}^2 - \hat{L}^2 \hat{r}^2 \right]^{-\frac{1}{2}}. \quad (27)$$

The result is that

$$\dot{\hat{r}} = \pm \hat{N} \left( \Delta \Pi_{\hat{L}}/\hat{L} \right) / \Delta \hat{r}, \quad (28)$$

where the $+$ ($-$) sign applies to an outgoing (ingoing) geodesic. (If $m = 0$, it follows from Eq.(28) and from Eq.(7) that $\dot{\hat{r}} = \pm \hat{N}$, so that the points on the shell traverse null geodesics.)

The geodesic equation Eq.(28) is integrated to yield

$$t = - \left\{ \pm \int \frac{d\hat{r}}{\hat{R}} \frac{1}{2(1 + \beta)^{\frac{1}{2}} \sqrt{\left( \hat{R} \right)^2 - 1}} \left[ \beta(2 + \beta) + \frac{\hat{R}}{\hat{R} - 2M} \left( \frac{m}{\hat{r}} \right)^2 \right] \right\} + t_0, \quad (29)$$

where $\hat{R} = \hat{r}(1 + \beta)$ and $r_0 =$ constant. For $\beta = 0$ and $m = 0$, this reduces to the usual...
equation for a null geodesic in Schwarzschild spacetime:

\[ t = \pm \left[ \hat{r} + 2M \ln \left( \frac{\hat{r}}{2M} - 1 \right) \right] + t_0. \] (30)

However, if \( \beta \) is complex, then \( t \) becomes complex. In this case, the integral has a branch cut along the real axis in the complex \( \hat{r} \)-plane. The region in which \( \beta \) is complex is quite narrow, centered around \( \hat{r} = 2M_A \). In this region, \( \beta \approx \) constant, and the branch cut can be replaced with a simple pole. The imaginary part of \( t \) can then be evaluated as the residue at the pole, with the result:

\[ t = t_{\text{real}} + i4\pi M_A \left\{ 1 - \frac{3}{8} \left( \frac{\omega}{M_A} \right) \left[ 1 + \frac{1}{16} \left( \frac{m}{M_A} \right)^2 \right] \right\}. \] (31)

Here we have also assumed that \( m \ll \omega \). Only the lowest order corrections in \( \omega/M_A, m/M_A \) and \( m/\omega \) are retained Eq.(31). If \( m = 0 \), one obtains

\[ \text{Im } t = 4\pi M_A (1 - 3\omega/8M_A) = 4\pi (M_A - 3\omega/8). \] (32)

If the last term is ignored, we have the result of Ref. 1: \( \text{Im } t = 4\pi M_A = 4\pi (M - \omega) \). If all the corrections of order \( \omega/M \) are ignored, we have Hawking’s result: \( \text{Im } t = 4\pi M \), obtained in the Euclidean formulation.[5, 6]
The spectrum of Hawking radiation is given by Eq.(10) and Eq.(11). By virtue of Eq.(31) we obtain:

\[ n(\omega) = \frac{1}{\exp\left\{8\pi\omega \left( M_A - \frac{3}{8}\omega \left( 1 + \frac{m^2}{16M_A^2} \right) \right) \right\} - 1. \] (33)

For the case of a massless shell \((m = 0)\), we have:

\[ n(\omega) = \exp\left\{8\pi\omega (M_A - 3\omega/8) \right\} - 1. \] (34)

With \(M_A = M - \omega\), we recover the results of Ref. 1, if the additional correction of \(3\omega/8\) is ignored. The origin of this additional correction lies in the details of the dynamics, which are sensitive to the form of the metric. Our choice of metric, given by Eq.(14) and Eq.(15), reproduces the results of Ref. 1, if the details of the shell dynamics in the distorted background, represented by Eq.(29) are ignored. However, this does not seem to be justified since the additional correction in Eq.(33) is of the same order in \(\omega\) as the corrections in Ref. 1. It appears that the back reaction correction is sensitive to the details of the metric. The ambiguity in the metric can only be diminished by an exact semi-classical solution of the Einstein equations. In the absence of such solutions, we have to rely on the consistency of our choice of metric. We will confirm this consistency through a calculation of the stress-energy tensor \(T_{\mu\nu}\) in the next section.

4 The Stress-Energy Tensor

We can calculate \(T_{\mu\nu}\) using the standard thin shell formalism [8]. We use Gaussian normal coordinates \((\eta, x^i, i = 1, 2, 3)\) so that the metric is of the form:

\[ ds^2 = d\eta^2 + h_{ij}dx^i dx^j, \] (35)

where \(\eta(x^i) = 0\) is the equation of the shell and \(h_{ij}\) is the induced metric on the hypersurface \(\Sigma_r\) swept out by the trajectory of the shell. In our case, \(\eta = r - \hat{r}(t)\) for a spherical shell. The stress-energy tensor takes the form [8]

\[ T_{\mu\nu} = \delta(\eta) S_{\mu\nu} + \theta(\eta) T_{\mu\nu}^+ + \theta(-\eta) T_{\mu\nu}^-, \] (36)

where \(\theta\) is the unit step-function. The tensor \(S_{\mu\nu}\) is the surface stress-energy carried by the shell, while \(T_{\mu\nu}^\pm\) denote the regular background contributions on the two sides of the shell. Hence, \(T_{\mu\nu}^\pm\) describe the black hole hair originating on the shell. Both fall off rapidly with distance from the shell. Therefore Hawking radiation contributes to \(S_{\mu\nu}\), but not
directly to $T_{\mu\nu}^{\pm}$. The latter represent the transient stress-energy originating in the quantum fluctuations of the geometry.

The surface stress-energy tensor $S_{\mu\nu}$ is determined by the extrinsic curvature (second fundamental form) $K_{\mu\nu}$ of $\Sigma_r$ by

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n h_{\mu\nu}; \quad h_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu, \quad (37)$$

where $n_\mu$ is the unit normal to $\Sigma_r$. The Einstein equations now take the form:

$$S_{ij} = -(1/8\pi) (\kappa_{ij} - \kappa h_{ij}), \quad (38)$$

$$D_j S^{ij} = -T^i_\eta, \quad (39)$$

$$\frac{1}{2} \left( \hat{K}^+_{ij} + \hat{K}^-_{ij} \right) S^{ij} = T^\eta_\eta, \quad (40)$$

where $\kappa_{ij} := \hat{K}^+_{ij} - \hat{K}^-_{ij}$, $\kappa := h^{ij} \kappa_{ij}$ and $D_j$ is the covariant derivative with respect to the metric $h_{ij}$.

We are primarily interested in the component $S_{tt}$ for an outgoing shell, which determines the surface energy per unit area of the shell. Other components of the surface stress-energy tensor determine the pressure and tension. From Eq.(37)–Eq.(38) we obtain for a massless shell:

$$S_{tt} = -\hat{N}^3 \Delta \hat{R}' / 4\pi \hat{R}, \quad (41)$$

$$\rightarrow -\frac{\omega}{4\pi \hat{r}^2}, \quad (42)$$

as $\hat{r} \rightarrow \infty$. Therefore, energy outflow of Hawking radiation from a sphere of radius $\hat{r}$ at infinity is $\hbar \omega$, as expected. As a result, Eq.(14)–Eq.(15) are consistent with the expected asymptotic behaviour of the stress-energy tensor. Moreover, it can be shown that the $T_{\mu\nu}^{\pm}$ fall off rapidly away from $\Sigma_r$ and therefore do not contribute to the Hawking radiation.

The total flux of Hawking radiation can now be calculated using Eq.(12). The total flux is

$$\int_0^\infty (2\pi)^{-1} d\omega n(\omega) \omega = 1/768\pi M^2,$$

which recovers a well-known result [5].

5 Analysis of the Geometry

As we have seen, the spacetime geometry is determined by the properties of the function $\beta(\hat{r}, \omega)$. This function must satisfy the cubic equation Eq.(19). The properties of these roots are displayed in the graphs of the ‘optical scalar’ $\hat{R}$ vs. $\hat{r}$, Figure 2a and Figure 2b.
We may conclude from these that the root for which \( \hat{R} < 0 \) for all \( \hat{r} > 0 \) is unphysical, while the other two roots of the cubic represent a bifurcation of the spacetime geometry.

The collapsing shell, which has \( M_A = M + \omega \), for which the optical scalar corresponds to a Schwarzschild geometry asymptotically, i.e. which behaves as \( R \sim r \) for large \( r \), never reaches the location \( \hat{R} = 2M_A \) except in the limit as \( \hat{r} \to 0 \). The latter is a singular event in the geometry. The events at \( \hat{R} = 2M_A \) and at \( \hat{R} = 2M \) constitute apparent horizons. The two geometries are separated by a gap \( 2M \leq R \leq 2M_A \). A collapsing shell inside the \( \hat{R} = 2M \) horizon does fall into the singularity in a finite time. However, it cannot cross the apparent horizon \( \hat{R} = 2M \) in a finite time. Classically these two configurations are distinct solutions, and represent different spacetime geometries. It appears that the shell will only collapse into a black hole via quantum fluctuations of the spacetime geometry. Similarly, a shell in the inner region geometry can enter the outer region only via quantum tunnelling. Since the magnitude of the distortion of the outer geometry from that of Schwarzschild is small in the region where \( \hat{r} \) is near the value \( 2M_A \); and since, moreover, \( t \) is continuous at \( \hat{r} = 2M_A \), \( t \) can be used as an affine parameter.
When $M_A = M - \omega$, i.e., the shell is outgoing and the metric is complex in a small region near where the shell is close to the horizon.[3] A classical spacetime geometry does not exist in this region, and the Hawking radiation from classically forbidden configurations takes place through quantum tunnelling.

To summarize, in our model the fixed Schwarzschild geometry is split by the back reaction into two distinct spacetime geometries, both of which contain apparent horizons. The exterior geometry approaches the schwarzschild limit as $\hat{r} \to \infty$, while the interior geometry is Schwarzschildean in the limit $\hat{r} \to 0$.

6 Conclusion

We have calculated the back reaction correction for a massless as well as a massive thin shell in the semi-classical approximation. We have found an additional correction due to the back reaction for a massless shell, so that the total correction to $M$ in the Hawking spectrum is $-11\omega/8$ as opposed to $-\omega$, obtained in Ref. 1. The correction appears to be sensitive to the details of the dynamics, and, therefore, to the choice of metric distorted from that of
Schwarzschild by the presence of the shell. Our choice satisfies the asymptotic conditions as well as other consistency conditions at the shell. However, the ambiguity in the choice of metric will remain until an exact solution of the semi-classical Einstein equations can be found. [9]

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References


[3] The relation $M_A = M - \omega$ represents the absorption of negative energy by the black hole, associated perhaps with quantum fluctuations. This is one of the more well-known explanations of Hawking radiation [5]. Actually, the negative energy shell approaches $\hat{r} = 0$; but this is accompanied by an “outgoing shell” of positive energy Hawking radiation.


[7] Figures 1a, 1b, 2a and 2b are all MAPLE plots. The units are such that $M = 1$.


[9] In a recent preprint [10], it is claimed that the back-reaction correction due to Hawking radiation, which, in the semi-classical regime, consists of replacing $M$ by $M - \omega$ in expressions for the spectrum, is independent of the details of the metric. This is in seeming contradiction to our results, e.g. Eq.(34),
wherein the back-reaction correction is given by the replacement of $M$ by $M - 11\omega/8$. We note that there is no contradiction since the existence of a horizon at some value of $r$ is required by the family of metrics allowed in [10]. In such a case, $\dot{R} = \dot{r}$ in Eq.(29), and we obtain $\text{Im}t = 4\pi M_A = 4\pi (M - \omega)$, as noted after Eq.(32). In contrast, our allowed class of metrics given by Eq.(14),Eq.(15) can deviate significantly from a Schwarzschild geometry near the shell, when the latter is near the (putative) horizon. In fact, the geometry bifurcates and the usual horizon does not exist. All these effects result from the reparameterization $r \to R(r)$, which leads to a more general class of metrics. For this class, we find that the back reaction is sensitive to the details of the motion of the shell.