Canonical Quantization of Cylindrical Gravitational Waves with Two Polarizations

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Abstract

The canonical quantization of the essentially nonlinear midisuperspace model describing cylindrically symmetric gravitational waves with two polarizations is presented. A Fock space type representation is constructed. It is based on a complete set of quantum observables. Physical expectation values may be calculated in arbitrary excitations of the vacuum. Our approach provides a non-linear generalization of the quantization of the collinearly polarized Einstein-Rosen gravitational waves.

The quantization of dimensionally reduced models of 4d Einstein gravity serves as interesting testing ground for many issues of quantum gravity. The physical output of this approach to an understanding of characteristic features of the full theory however strongly depends on the complexity of the model under consideration.

The probably simplest and best understood examples are the mini-superspace models [1] which contain only a finite number of physical degrees of freedom and thus hide the field effects of quantum gravity. A more complicated example of steady interest is given by the midi-superspace model of cylindrically symmetric gravitational waves with one polarization [2, 3].

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This model already involves an infinite number of degrees of freedom. It becomes treatable with the methods of flat space quantum field theory, because the Einstein field equations essentially reduce to the axisymmetric 3d Laplace equation. This underlying linearity on the other hand may conceal typical nonlinear features of quantum gravity.

It is the purpose of this letter to generalize the results of [2, 3] to cylindrical gravitational waves with two polarizations, where the Einstein equations become truly nonlinear. We achieve a consistent canonical quantization in terms of a complete set of quantum observables. Creation and annihilation operators are identified in a kind of Fock space representation of these observables. The full Hilbert space of physical quantum states is build from excitations of the vacuum. The presented techniques allow to calculate all physical expectation values in arbitrary quantum states.

We start from a general space-time with cylindrical symmetry, i.e. assume existence of two commuting Killing vector fields, one of which has closed orbits. Choose coordinates such that the Killing vector fields are given by \( \partial_z \) and \( \partial_\varphi \) associated to the axis of symmetry \( z \) and the azimuthal angle \( \varphi \) respectively. Further gauge-fixing brings the metric into the standard form [4]

\[
ds^2 = e^{\Gamma(\rho, \tau)} (-d\tau^2 + d\rho^2) + \rho g_{ab}(\rho, \tau) dx^a dx^b \quad a, b = 2, 3 ,
\]

with \( x^2 \equiv z \), \( x^3 \equiv \varphi \), radial coordinate \( \rho \) and time \( \tau \). The symmetric \( 2 \times 2 \) matrix \( g \) is restricted by the condition \( \det g = 1 \).

The Einstein field equations consist of two parts: the Ernst equation for the matrix \( g \):

\[
\partial_\rho (\rho g^{-1} \partial_\rho g) + \partial_\tau (\rho g^{-1} \partial_\tau g) = 0 ,
\]

and the equation for the conformal factor \( \Gamma \):

\[
\Gamma(\rho, \tau) = \frac{1}{2} \int_0^\rho \rho' \, d\rho' \, \text{tr} \left( (g^{-1} \partial_{\rho'} g)^2 + (g^{-1} \partial_\tau g)^2 \right) .
\]

The conformal factor at spatial infinity \( \Gamma_\infty \equiv \Gamma(\rho=\infty) \) generates evolution with respect to the time coordinate \( \tau \). Its exponential measures the total energy per unit length in \( z \)-direction and the deficit angle in the asymptotic region

\[
H^t = \frac{1}{\pi G} \varphi_0 = \frac{2}{G} (1 - e^{-\Gamma_\infty / 2}) .
\]
effective Lagrangian density $L^{(2)}$ that comes from reduction via Killing symmetries and gauge fixing of the original Lagrangian $L^{(4)}_{EH} = (1/G)\sqrt{|g_{\mu\nu}|}R^{(4)}$:

$$L^{(2)}(\rho, \tau) = \frac{1}{2G} \rho \text{tr} \left( (g^{-1}\partial_{\rho}g)^2 - (g^{-1}\partial_{\tau}g)^2 \right).$$

In matrix components $g_{ab}$, the Poisson brackets read

$$\{g_{ab}(\rho), (g^{-1}\partial_{\tau}gg^{-1})_{cd}(\rho')\} = \frac{G}{\rho} \delta_{ad}\delta_{bc}\delta(\rho - \rho').$$

The restrictions of symmetry and unit determinant of $g$ require some additional technical effort and have been taken into account in the derivation of the following results.

**Collinear polarizations.** Among the simplest nontrivial metrics of this model are the collinearly polarized gravitational waves discovered by Einstein and Rosen [5]. They correspond to a diagonal form of the matrix $g \equiv \text{diag}(e^{\phi}, e^{-\phi})$, i.e. the number of degrees of freedom reduces to one. The Ernst equation (2) in this case reduces to the cylindrical Laplace equation

$$-\partial^2_{\tau} \phi + \rho^{-1}\partial_{\rho}\phi + \partial^2_{\rho} \phi = 0,$$

with general solution

$$\phi(\rho, \tau) = \int_{0}^{\infty} \left[ A_+(\lambda)J_0(\lambda \rho)e^{i\lambda \tau} + A_-(\lambda)J_0(\lambda \rho)e^{-i\lambda \tau} \right] d\lambda,$$

where $J_0$ denote Bessel functions of the first kind. The coefficients $A_+ = A_-^*$ build a complete set of observables with canonical Poisson brackets

$$\{ A_+(\lambda), A_-(\lambda') \} = G \delta(\lambda - \lambda').$$

Thus, quantization of this structure is straightforward [3] and gives rise to a representation in terms of creation and annihilation operators

$$A_-|0\rangle = 0 \quad \text{with} \quad A_+ = A_-^\dagger.$$  \hspace{1cm} (6)

In particular, coherent quantum states may be constructed in the same way as in flat space quantum field theory. Recent discussion however has shown, that these states do not provide coherence of all essential physical quantities [6].

As the first step towards the general case, we cast the truncated model of collinear polarization into a form that will allow proper generalization. Introduce new variables

$$T_\pm(w) \equiv \exp \int_{0}^{\infty} A_\pm(\lambda)e^{\pm iw\lambda}d\lambda,$$

which build an equivalent complete set of observables. In the Fock space representation (6) $T_-|0\rangle$ is represented as identity, whereas $T_+|0\rangle$ generates
the coherent state associated to a classical field that on the symmetry axis
\( \rho = 0 \) is peaked as a \( \delta \)-function at \( \tau_0 = w \). In terms of these new variables,
the Poisson structure (5) becomes
\[
\left\{ T_-(v), T_+(w) \right\} = -\frac{G}{v - w} T_-(v) T_+(w) .
\] (8)

We shall see in the sequel, that it is this quadratic form of Poisson brackets
which generically appears in the case of two polarizations. Linearization to
(5) is a special feature of the truncated model but not possible in the general
case.

Two polarizations. In general, the Ernst equation (2) does not admit
explicit solution. However, it is possible to construct the analogue of the
quantities \( T_\pm \) defined above. Inspired by the auxiliary linear system associ-
ated to the Ernst equation [7] we define for real \( w \)
\[
T_\pm(\tau, w) \equiv \lim_{\epsilon \to 0} \{ \mathcal{P} \exp \int_0^\infty d\rho 2 \left( \frac{\gamma_\pm^2 g^{-1} \partial_\rho g}{1 - \gamma_\pm^2} - \frac{\gamma_\pm^{-1} \partial_\tau g}{1 - \gamma_\pm^2} \right) \} ,
\] (9)
with
\[
\gamma_\pm = -\frac{1}{\rho} \left( w \pm i \epsilon - \tau + \sqrt{(w \pm i \epsilon - \tau)^2 - \rho^2} \right) .
\]
For diagonal \( g \), this definition indeed reduces to (7) above. The variables \( T_\pm \)
are still constants of motion, i.e. \( \partial_\tau T_\pm(\tau, w) = 0 \). They turn out to admit
holomorphic expansion into the upper and lower half of the complex plane
respectively. Definition (9) further implies \( \det T_\pm = 1 \) and \( T_+ = T_- \).

As another important result, the matrix product
\[
M(w) \equiv T_+(w) T_-^t(w) = g(\rho = 0, \tau = w)
\] (10)
In particular, it is symmetric and real:
\[
M(w) = M^t(w) \quad \text{and} \quad M(w) = M(w) .
\] (11)
Since the \( T_\pm \) contain the initial values of the metric on the symmetry axis,
they contain sufficient information to recover \( g \) everywhere by means of (2)
(note that \( \partial_\rho g(\rho = 0) = 0 \)). Thus, the set of \( T_\pm(w) \) builds a complete set of
observables for the Ernst equation.

Continuing the program of canonical quantization we next calculate their
Poisson algebra to subsequently quantize it. A direct but lengthy calculation
reveals a quadratic Poisson algebra for the matrix entries \( T_\pm^{ab}(w) \):
\[
\left\{ T_\pm^{ab}(v), T_\pm^{cd}(w) \right\} = \frac{G}{v - w} \left( T_\pm^{ad}(v) T_\pm^{cb}(w) - T_\pm^{cb}(v) T_\pm^{ad}(w) \right) ,
\] (12)
\[
\left\{ T_-^{ab}(v), T_+^{cd}(w) \right\} = \frac{G}{v - w} \left( T_-^{ab}(v) T_+^{cd}(w) - T_-^{cb}(v) T_+^{ad}(w) - \delta^{bd} T_-^{am}(v) T_+^{cm}(w) \right) .
\] (13)
which consistently encloses the scalar algebra (8) in the components $T_{\pm}^{11}(w)$. Quantization of this quadratic structure is rather more subtle than that of a linear algebra, since there appear obvious ambiguities on the r.h.s. due to different orderings of the quadratic expressions. Fortunately, the proper quantum analogue of the Poisson brackets (12) is known in the theory of integrable systems [8] as the so-called $\mathfrak{sl}(2)$-Yangian algebra

$$\left[ T_{\pm}^{ab}(v), T_{\pm}^{cd}(w) \right] = \frac{i\hbar G}{v-w} \left( T_{\pm}^{cb}(w)T_{\pm}^{ad}(v) - T_{\pm}^{cb}(v)T_{\pm}^{ad}(w) \right).$$

(14)

The proper quantization of (13) leads to a set of mixed relations

$$\left[ T_{\pm}^{-}(v), T_{\pm}^{+}(w) \right] = \frac{i\hbar G}{v-w + 2i\hbar G} \left( T_{\pm}^{ab}(v)T_{\pm}^{cd}(w) - T_{\pm}^{cb}(v)T_{\pm}^{ad}(w) \right)$$

$$- \frac{i\hbar G}{v-w} \delta^{bd} T_{\pm}^{cm}(w)T_{\pm}^{am}(v).$$

(15)

The shift of the denominator on the r.h.s. provides quantum corrections of (13) of higher order in $\hbar$ which are necessary for compatibility of these commutation relations with the quantum analogue of the symmetry (11):

$$T_{\pm}(w)T_{\pm}^{\dagger}(w) = \left( T_{\pm}(w)T_{\pm}^{\dagger}(w) \right)^{\dagger}.$$  

(16)

Again, the ordering of these quadratic expressions is now essential. Classically, $M(w)$ contains the essential physical objects according to (10). In the quantum model, the definition $M(w) = T_{\pm}(w)T_{\pm}^{\dagger}(w)$ ensures, that the commutation relations (14), (15) actually yield a closed commutator algebra of the matrix entries of $M(w)$. Moreover, these are hermitean operators, provided that

$$T_{\pm}^{ab}(w) = \left( T_{\pm}^{ab}(w) \right)^{\dagger}, \quad w \in \mathbb{R},$$

(17)

in accordance with the classical relations. Finally, the classical condition of unit determinant $\det T_{\pm}(w) = 1$ requires quantum corrections because of the nonlinear terms and is substituted by the “quantum determinant” [9]

$$T_{\pm}^{11}(w+i\hbar G)T_{\pm}^{22}(w) - T_{\pm}^{12}(w+i\hbar G)T_{\pm}^{21}(w) = 1,$$

(18)

which is indeed compatible with the relations (14), (15) and may as such be imposed as an operator identity.

Summarizing, we have formulated the consistent quantum model in terms of the operators $T_{\pm}^{ab}(w)$, subject to the commutation relations (14), (15), as well as to unit quantum determinant (18), hermiticity (17) and symmetry (16). We are now in position to introduce a Fock space type representation of this algebra, inspired by the scalar case (6). Let therefore $T_{-}(w)$ act trivially on the vacuum

$$T_{-}^{ab}(w)|0\rangle = \delta^{ab}|0\rangle,$$

(19)
and \( T_+ (w) \) generate the Hilbert space of physical states

\[
\mathcal{H} = \left\{ \left( \prod_{i=1}^{m} T_{+}^{a_i b_i} (w_i) \right) |0\rangle \mid m, a_i, b_i, w_i \right\},
\]

where all the excitations are not independent but obey the relations (14), (16) and (18) for \( T_+ \). The intuitive idea that the \( T^{ab} (w) \) generate the complete spectrum of states is not only supported by the exactly solved scalar case from above, but even stronger by the fact, that the conserved charges \( T_+ (w) \) canonically Poisson-generate the Geroch group [10] which as a symmetry group acts transitively among the classical solutions of the field equations [11].

It is straightforward to further extract all relevant physical information from the quantum model. The hermiticity relations (17) together with the commutation relations (15) allow to calculate the expectation values of arbitrary polynomials in the \( T^{ab} (w) \) in arbitrary excitations of the vacuum. Indeed, a closer look at (15) shows, that by means of these relations, the \( T_{-}^{ab} (w) \) may be shuffled through to the right in any sequence of operators, where they finally “annihilate” the vacuum according to (19). As an illustration we state the scalar product between excitations of the first level \( |T^{ab}_+(w)\rangle \equiv T^{ab}_+(w) |0\rangle \):

\[
\langle T^{ab}_+(v) | T^{cd}_+ (w) \rangle = \langle 0 | T^{ab}_+(v) T^{cd}_+ (w) |0\rangle = \delta^{ab} \delta^{cd} + \frac{i \hbar G}{v - w} \left( \delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd} \right).
\]

We can also derive expectation values of the conformal factor \( \Gamma_{\infty} \) and its exponential \( e^{\Gamma_{\infty}} \), related to energy, deficit angle and metric components at infinity (4). Namely, it may be shown that classically

\[
\{ \Gamma_{\infty}, T_{\pm} (w) \} = G \partial_w T_{\pm} (w).
\]

In the quantum theory, the conformal factor can thus be represented as derivation operator \( i \hbar G \partial / \partial w \), such that its exponential \( e^{\Gamma_{\infty}} \) becomes the shift operator \( w \mapsto w + i \hbar G \). It is then an elementary exercise to calculate its matrix elements between arbitrary states of \( \mathcal{H} \).

The presented quantum model provides the exact quantization of a midisuperspace model of quantum gravity with essential nonlinear characteristics. The complete set of quantum observables and the complete spectrum of physical quantum states are at hand. The techniques are sufficiently developed to start exploring the properties of the spectrum and relevant observables.

It would be of high interest to identify some kind of coherent states in this model, i.e. quantum states with certain semi-classical properties. Due to the nonlinear setting it is reasonable to suspect, that not all the standard properties of usual coherent states can be satisfied. The fact, that
the traditional framework of coherent states may be too restrictive for the description of quantized gravitational waves is actually supported by recent observations in the linear model [6].

Another exciting feature of this quantum model emerges from the quantum analogue of the determinant (18): In view of the physical interpretation of $M(w)$ (10), which supplies the spectral parameter $w$ with a space-time meaning, it is tempting to consider (18) as a sign of arising nonlocality of the quantum operators on the Planck scale.

Since the presented quantization mainly employs the group-theoretical properties of the model, it will allow natural generalization to other and more complicated models of dimensionally reduced gravity, including higher dimensional supergravity as well as Einstein-Maxwell systems. Similarly it should find application to the Gowdy model, where $\rho$ becomes a time-like variable [12]. The weak field limit of the nonlinear Poisson structure in this case is isomorphic to the isomonodromic Poisson structure quantized in [13]. With different norm of the reducing Killing vector fields, the whole scheme may furthermore be applied to stationary axisymmetric spacetimes, providing an exact quantization of the black hole solutions in a vast class of models.

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References


[10] D. Korotkin and H. Samtleben, Preprint DESY-96-245, gr-qc/9611061, *Class. Quantum Grav.* accepted for publication. To be precise, the classical Geroch group is generated by the operators \( T_{\pm}^{-1}(w) \, \text{ad}_{T_+(w)} \), which as a corollary of (12) build half of the affine algebra \( \tilde{sl}_2 \).

