Abstract

It is well known that anomaly cancellations for $D_{16}$ Lie algebra are at the root of the first string revolution. For $E_8$ Lie algebra, cancellation of anomalies is the principal fact leading to the existence of heterotic string. They are in fact nothing but the 6th order cohomologies of corresponding Lie algebras. Beyond 6th order, the calculations seem to require special care and it could be that their study will be worthwhile in the light of developments of the second string revolution.

As we have shown in a recent article, for $A_N$ Lie algebras, there is a method which are based on the calculations of Casimir eigenvalues. This is extended to $E_8$ Lie algebra in the present article. In the generality of any irreducible representation of $E_8$ Lie algebra, we consider 8th and 12th order cohomologies while emphasizing the diversities between the two. It is seen that one can respectively define 2 and 8 basic invariant polynomials in terms of which 8th and 12th order Casimir eigenvalues are always expressed as linear superpositions. All these can be easily investigated because each one of these invariant polynomials gives us a linear equation to calculate $E_8$ weight multiplicities. Our results beyond order 12 are not included here because they get more complicated though share the same characteristic properties with 12th order calculations.
I. INTRODUCTION

It is a clear fact that anomaly cancellations play a unique role in the construction of the way of thinking and constructing models in high energy physics since the last two decades. The ones for $D_{10}$ Lie algebra [1] are principal for the first string revolution to begin. As it is also noted [2], the construction of heterotic string [3] is shortly thereafter. It is known [4] that the existence of a 10-dimensional string with a $E_8 \times E_8$ gauge symmetry relies heavily on $E_8$ anomaly cancellations.

On the other hand, these anomaly cancellations are in fact due to cohomology relations of corresponding Lie algebras. The cohomology for Lie algebras states non-linear relationships between elements of the center of their universal enveloping algebras [5]. The non-linearity comes from the fact that these relationships are between the elements of different orders and the non-linearly independent ones are determined by the Betti numbers [6]. A problem here is to determine the number of linearly independent elements of the same order. In two subsequent works [7], we studied this problem for $A_N$ Lie algebras and give a method which is based on explicit construction of Casimir eigenvalues. This will be extended here to $E_8$ Lie algebra.

$E_8$ is the biggest one of finite dimensional Lie algebras and besides its own mathematical interest it plays a striking role in high energy physics. It provides a natural laboratory to study the structure of $E_{10}$ hyperbolic Lie algebra [8] which is seen to play a key role in understanding the structure of infinite dimensional Lie algebras beyond affine Kac-Moody Lie algebras. There are so much works to show its significance in string theories and in the duality properties of supersymmetric gauge theories. This hence could give us some insight to calculate higher order cohomologies of $E_8$ Lie algebra. It will be seen in the following that this task is to be simplified to great extent when one uses a method based on explicit calculations of Casimir eigenvalues.

It is known that, beside degree 2, $E_8$ Betti numbers give us non-linearly independent Casimir elements for degrees 8,12,14,18,20,24,30. We must therefore calculate the Casimir eigenvalues for all these degrees. In the present state of work, to give only the results for 8th and 12th orders would be more instructive. This will be possible in terms of one of the maximal subalgebras of $E_8$, namely $A_8$. Although our method [7] for $A_N$ Lie algebras is previously presented, the calculations still need some special care for 8th and 12th orders. These are investigated in sections II and III. To this end, we especially emphasize our second permutational lemma to express the weights of an $E_8$ Weyl orbit and $A_8$ duality rules without which the calculations will be useless. In section IV, we show that the calculations find an end in the form of decompositions in terms of some properly chosen $A_8$ basis functions. The remarkable fact here is that the coefficients in these decompositions are constants and this shows us that the dependence on irreducible representations of $E_8$ Lie algebra are contained in these $A_8$ basis functions solely. For 12th order, the results of our calculations are given in three appendices because they are comparatively voluminous than 8th order calculations.

II. WEIGHT CLASSIFICATION OF $E_8$ WEYL ORBITS

We refer the excellent book of Humphreys [9] for technical aspects of this section though a brief account of our framework will also be given here. It is known that the weights of an irreducible representation $R(\Lambda^+)$ can be decomposed in the form of

$$R(\Lambda^+) = \Pi(\Lambda^+) + \sum m(\lambda^+ < \Lambda^+) \Pi(\lambda^+)$$ (II.1)

where $\Lambda^+$ is the principal dominant weight of the representation, $\lambda^+$’s are their sub-dominant weights and $m(\lambda^+ < \Lambda^+)$’s are multiplicities of weights $\lambda^+$ within the representation $R(\Lambda^+)$. Once a convenient definition of eigenvalues is assigned to $\Pi(\lambda^+)$, it is clear that this also means for the whole $R(\Lambda^+)$ via (II.1).

In the conventional formulation, it is natural to define Casimir eigenvalues for irreducible representations which are known to have matrix representations. In ref(7), we have shown that the eigenvalue concept can be conveniently extended to Weyl orbits of $A_N$ Lie algebras. The convenience comes from a permutational lemma governing $A_N$ Weyl orbits. This however could not be so clear for Lie algebras other than $A_N$. We therefore give in the following a second permutational lemma. To this end, it is useful to decompose $E_8$ Weyl orbits in the form of

$$\Pi(\lambda^+) \equiv \sum_{\sigma^+ \in \Sigma(\lambda^+)} \Pi(\sigma^+)$$ (II.2).
Σ(λ⁺) is the set of \( A_8 \) dominant weights participating within the same \( E_8 \) Weyl orbit \( \Pi(\lambda^+) \).

If one is able to determine the set \( \Sigma(\lambda^+) \) completely, the weights of each particular \( A_8 \) Weyl orbit \( \Pi(\sigma^+) \) and hence the whole \( \Pi(\lambda^+) \) are known. We thus extend the eigenvalue concept to \( E_8 \) Weyl orbits just as in the case of \( A_N \) Lie algebras.

It is known, on the other hand, that elements of \( \Sigma(\lambda^+) \) have the same square length with the \( E_8 \) dominant weight \( \lambda^+ \). It is unfortunate that this remains insufficient to obtain the whole structure of the set \( \Sigma(\lambda^+) \). This exposes more severe problems especially for Lie algebras having Dynkin diagrams with higher degree automorphisms, for instance affine Kac-Moody algebras. To solve this non-trivial part of this problem, we introduce 9 fundamental weights \( \mu_I \) of \( A_8 \), via scalar products

\[
\kappa(\mu_I, \mu_J) \equiv \delta_{IJ} - \frac{1}{9}, \quad I, J = 1, 2, \ldots, 9
\]  

The existence of \( \kappa(., .) \) is known to be guaranteed by \( A_8 \) Cartan matrix. The fundamental dominant weights of \( A_8 \) are now expressed by

\[
\sigma_i = \sum_{j=1}^{i} \mu_j, \quad i = 1, 2, \ldots, 8
\]  

To prevent misconception, we list the main quantities which take place in the following discussions:

- \( \lambda^+ \), \( \Lambda^+ \) \( \rightarrow \) dominant weights of \( E_8 \)
- \( \lambda_i \) \( \rightarrow \) fundamental dominant weights of \( E_8 \), \( i = 1, 2, \ldots, 8 \)
- \( \sigma^+ \) \( \rightarrow \) dominant weights of \( A_8 \)
- \( \sigma_i \) \( \rightarrow \) fundamental dominant weights of \( A_8 \), \( i = 1, 2, \ldots, 8 \)
- \( \mu_I \) \( \rightarrow \) fundamental weights of \( A_8 \), \( I = 1, 2, \ldots, 9 \)

The correspondence \( E_8 \leftrightarrow A_8 \) is now provided by

\[
\begin{align*}
\lambda_1 &= \sigma_1 + \sigma_8 \\
\lambda_2 &= \sigma_2 + 2 \sigma_8 \\
\lambda_3 &= \sigma_3 + 3 \sigma_8 \\
\lambda_4 &= \sigma_4 + 4 \sigma_8 \\
\lambda_5 &= \sigma_5 + 5 \sigma_8 \\
\lambda_6 &= \sigma_6 + 3 \sigma_8 \\
\lambda_7 &= \sigma_7 + \sigma_8 \\
\lambda_8 &= 3 \sigma_8
\end{align*}
\]  

with

\[
\lambda^+ = \sum_{i=1}^{8} r_i \lambda_i, \quad r_i \in \mathbb{Z}^+.
\]  

\( Z^+ \) here is the set of positive integers including zero. It is clear that this last relation turns out to be

\[
\Lambda^+ = \sum_{i=1}^{8} q_i \sigma_i, \quad q_i \in \mathbb{Z}^+.
\]  

in view of (II.5) and hence \( E_8 \leftrightarrow A_8 \). By comparison between (II.6) and (II.7), note here that elements of \( \Sigma(\lambda_1) \) are dominant weights for \( A_8 \) but not for \( E_8 \).

It is clear that we only need here to know the weights of the sets \( \Sigma(\lambda_i) \) for \( i = 1, 2, \ldots, 8 \) explicitly. For instance,

\[
\Sigma(\lambda_1) = (\sigma_1 + \sigma_8, \sigma_3, \sigma_6)
\]
for which we have the decomposition
\[ \Pi(\lambda_1) = \Pi(\sigma_1 + \sigma_8) \oplus \Pi(\sigma_3) \oplus \Pi(\sigma_6) \] (II.8)
of 240 roots of \( E_8 \) Lie algebra. Due to permutational lemma given in ref(7), \( A_8 \) Weyl orbits here are known to have the weight structures
\[
\begin{align*}
\Pi(\sigma_1 + \sigma_8) &= (\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8) \\
\Pi(\sigma_3) &= (\mu_1 + \mu_2 + \mu_3) \\
\Pi(\sigma_6) &= (\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6)
\end{align*}
\] (II.9)
where all indices are permuted over the set \((1,2, .. 9)\) providing no two of them are equal. Note here by (II.4) that
\[
\begin{align*}
\sigma_1 + \sigma_8 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 \\
\sigma_3 &= \mu_1 + \mu_2 + \mu_3 \\
\sigma_6 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6
\end{align*}
\] (II.10).
The formal similarity between (II.9) and (II.10) is a resume of the first permutational lemma. Now, we are ready to state our second permutational lemma:

**For a dominant weight \( \lambda^+ \), the set \( \Sigma(\lambda^+) \) of \( A_8 \) dominant weights is specified by**
\[
\Sigma(\lambda^+) = \sum_{i=1}^{8} r_i \Sigma(\lambda_i).
\] (II.11)
**together with the equality of square lengths.**

In addition to \( \Sigma(\lambda_1) \) given above, the other 7 sets \( \Sigma(\lambda_i) \) have respectively 7,15,27,35,17,5 and 11 elements for \( i=2,3, .. 8 \) and they are given in appendix(1). It is therefore clear that the weight decomposition of any \( E_8 \) Weyl orbit is now completely known in terms of \( A_8 \) Weyl orbits in the presence of both of our lemmas.

### III. DUALITY RULES FOR \( A_8 \)

In this section, we present some rules which we call \( A_8 \) **Dualities** in calculating \( E_8 \) cohomology. They are however similarly generalized for Lie algebras other than \( A_8 \). It will be seen in the following that they are of crucial importance in calculating \( E_8 \) cohomology relations higher than degree 9.

We start by expressing an \( A_8 \) dominant weight \( \sigma^+ \) in the form
\[
\sigma^+ \equiv \sum_{i=1}^{8} k_i \mu_i \quad , \quad k_1 \geq k_2 \geq \ldots \geq k_8 \geq 0.
\] (III.1)

To prevent repetitions, we reproduce here the main definitions and formulas of ref(7) for \( A_8 \). The eigenvalues of a Casimir operator of degree \( M \) then are known to be defined by the aid of the formal definition
\[
ch_M(\sigma^+) \equiv \sum_{\mu \in \Pi(\sigma^+)} (\mu)^M
\] (III.2)
for a Weyl orbit \( \Pi(\sigma^+) \). Our way of calculation the right hand side of (III.2) is given in appendix(2). To this end, we need to define the following generators:
\[
\mu(M) \equiv \sum_{i=1}^{9} (\mu_1)^M.
\] (III.3)
It is also convenient to define the following ones which we would like to call \textbf{K-generators}:

\[ K(M) \equiv \sum_{i=1}^{8} (k_i)^M. \quad (III.4) \]

We remark here by definition that \( \mu(1) \equiv 0 \) and hence

\[ (\mu(1))^M \equiv 0, \quad M = 1, 2, \ldots, 9, 10, \ldots. \quad (III.5) \]

It can be readily seen that (III.5) is fulfilled for \( M = 2, 3, \ldots, 9 \) without any other restriction. It gives rise however to the fact that, for \( M \geq 10 \), all the generators \( \mu(M) \) are non-linearly depend on the ones for \( M = 2, 3, \ldots, 9 \). These non-linearities are clearly the reminiscents of \( A_8 \) cohomology. We therefore call them \( A_8 \) Dualities. It will be seen that the cohomology of \( E_8 \) Lie algebra will be provided by these \( A_8 \) dualities.

The first example is

\[
\mu(10) \equiv \frac{1}{8!} (25200 \mu(2) \mu(8) + 19200 \mu(3) \mu(7) + 16800 \mu(4) \mu(6) \\
- 8400 \mu(2)^2 \mu(6) - 13440 \mu(2) \mu(3) \mu(5) + 8064 \mu(5)^2 + 2100 \mu(2)^3 \mu(4) \\
- 5600 \mu(3)^2 \mu(4) - 6300 \mu(2) \mu(4)^2 + 2800 \mu(2)^2 \mu(3)^2 - 105 \mu(2)^5 ) \quad (III.6)
\]

It is seen that \( \mu(10) \) consists of \( p(10) = 11 \) monomials coming from the partitions of 10 into the set of numbers \( \{2, 3, 4, 5, 6, 7, 8, 9\} \). We also have \( p(8) = 7 \), \( p(9) = 8 \), \( p(11) = 13 \), \( p(12) = 19 \) and these are the maximum numbers of monomials for corresponding degrees. We thus obtain the following expressions:

\[
\mu(11) \equiv \frac{1}{362880} (-3465 \mu(2)^4 \mu(3) + 12320 \mu(2) \mu(3)^3 + 41580 \mu(2)^2 \mu(3) \mu(4) \\
- 41580 \mu(3)^2 \mu(4)^2 + 16632 \mu(2)^3 \mu(5) - 44352 \mu(3)^3 \mu(5) \\
- 99792 \mu(2) \mu(4) \mu(5) - 110880 \mu(2) \mu(3) \mu(6) + 133056 \mu(5) \mu(6) \\
- 71280 \mu(2)^2 \mu(7) + 142560 \mu(4) \mu(7) + 166320 \mu(3) \mu(8) + 221760 \mu(2) \mu(9) )
\]

and

\[
\mu(12) \equiv \frac{1}{725760} (322560 \mu(3) \mu(9) + 136080 \mu(2)^2 \mu(8) + 272160 \mu(4) \mu(8) + 248832 \mu(5) \mu(7) \\
- 60480 \mu(2)^3 \mu(6) - 80640 \mu(3)^2 \mu(6) + 129960 \mu(6)^2 - 72576 \mu(2)^2 \mu(3) \mu(5) \\
- 145152 \mu(3) \mu(4) \mu(5) + 17010 \mu(2)^4 \mu(4) - 34020 \mu(2)^2 \mu(4)^2 \\
- 22680 \mu(4)^3 + 20160 \mu(2)^3 \mu(3)^2 + 4480 \mu(3)^4 - 945 \mu(2)^6 )
\]

\textbf{IV. DECOMPOSITIONS OF INVARIANT POLYNOMIALS IN THE \( A_8 \) BASIS}

Let us start with the decomposition

\[ ch_8(\Lambda^+) \equiv \sum_{\alpha=1}^{7} Q_\alpha(\Lambda^+) T(\alpha) \quad (IV.1) \]

where 7 generators \( T(\alpha) \) signify monomials

\[ \mu(8), \mu(2)\mu(6), \mu(3)\mu(5), \mu(4)^2, \mu(4)\mu(2)^2, \mu(3)^2\mu(2), \mu(2)^4 \]

which are known to exist because \( p(8) = 7 \). One must stress in (IV.1) that coefficients \( Q_\alpha(\Lambda^+) \) are assumed to be defined by comparison of (IV.1) with (III.2). These 7 monomials play a prominent role in expressing
where $dimR$ is the allowance of obtaining the decompositions (IV.3) and (IV.4) with coefficients which are constant for all $E$. It is however seen that the cohomology of $P$ exists from the beginning. It is however seen that the cohomology of $P$ is reflected by $\Theta$ for basis functions $\rho$. What is really significant here is the Weyl vector of $E$ which come from the monomial $\mu(2)^4$: $P_1(8, \Lambda^+) \equiv 729 \, \Theta(8, \Lambda^+) - 71757069294212$. (IV.3)

Only the following one is obtained for all other monomials:

$$P_2(8, \Lambda^+) \equiv 68580 \, \Theta(8, \Lambda^+)$$
$$- 42672 \, \Theta(6, \Lambda^+) \, \Theta(2, \Lambda^+)$$
$$- 42672 \, \Theta(5, \Lambda^+) \, \Theta(3, \Lambda^+)$$
$$- 13335 \, \Theta(4, \Lambda^+)^2$$
$$+ 13335 \, \Theta(4, \Lambda^+) \, \Theta(2, \Lambda^+)^2$$
$$+ 17780 \, \Theta(3, \Lambda^+)^2 \, \Theta(2, \Lambda^+)$$
$$- 939 \, \Theta(2, \Lambda^+)^4$$
$$+ 385526887200$$ (IV.4)

The functions $\Theta(M, \Lambda^+)$ can be considered here as $A_8$ basis functions which are defined by

$$\Theta(M, \Lambda^+) \equiv \sum_{I=1}^{9} (\vartheta_I(\Lambda^+))^M \ , \ M = 1, 2, \ldots$$ (IV.5)

where

$$\vartheta_I = \kappa(\Lambda^+ + \rho_w, \mu_I) \ .$$ (IV.6)

$\rho_w$ here is the Weyl vector of $E_8$ Lie algebra. We notice that $A_8$ dualities are valid exactly in the same way for basis functions $\Theta(M, \Lambda^+)$ because $\Theta(1, \Lambda^+) = 0$. This highly facilitates the work by allowing us to decompose all invariant polynomials $P_\alpha(\Lambda^+)$ in terms of $\Theta(M, \Lambda^+)$’s but only for $M=2,3,\ldots,9$.

As in the similar way with $A_8$ basis functions defined above, the two polynomials $P_1$ and $P_2$ can be considered as $E_8$ basis functions in the sense that for any 8th order Casimir operator of $E_8$ the eigenvalues can always be expressed as linear superpositions of these $E_8$ basis functions. What is really significant here is the allowance of obtaining the decompositions (IV.3) and (IV.4) with coefficients which are constant for all irreducible representations of $E_8$ Lie algebra. In other words, beside constant coefficients, $E_8$ characteristic is reflected by $A_8$ basis functions.

$E_8$ cohomology manifests itself here by the fact that we have 2 polynomials $P_1$ and $P_2$ as $E_8$ Basis functions in spite of the fact that we have 7 polynomials from the beginning. As will be summarized in appendix(3), the same considerations lead us for degree 12 to 19 different polynomials which are known to exist from the beginning. It is however seen that the cohomology of $E_8$ dictates only 8 invariant polynomials for degree 12.

Careful reader could now raise the question that is there a way for a direct comparison of our results in presenting the $E_8$ basis functions

$$P_\alpha(8, \Lambda^+) \text{ for } \alpha = 1, 2 \ ,$$
$$P_\alpha(12, \Lambda^+) \text{ for } \alpha = 1, 2, \ldots, 8 \ .$$
A simple and might be possible way for such an investigation is due to weight multiplicity formulas which can be obtained from these polynomials. The method has been presented in another work \[10\] for $A_N$ Lie algebras and it can be applied here just as in the same manner. This shows the correctness in our conclusion that any Casimir operator for $E_8$ can be expressed as linear superpositions of $E_8$ basis functions which are given in this work. An explicit comparison has been given in our previous works but only for 4th and 5th order Casimir operators of $A_N$ Lie algebras and beyond these this does not seem to be tractable in practice.

As the final remark, one can see that the method presented in this paper are to be extended in the same manner to cases $E_7$ and $G_2$ in terms of their sub-groups $A_7$ and $A_2$.

REFERENCES


[10] H.R.Karadayi ; Non-Recursive Multiplicity Formulas for $A_N$ Lie algebras, physics/9611008
APPENDIX 1

The Weyl orbits of $E_8$ fundamental dominant weights $\lambda_i$ (i=1,2, .. 8) are the unions of those of the following $A_8$ dominant weights:

$\Sigma(\lambda_2) \equiv (2\sigma_1 + \sigma_7, \sigma_1 + \sigma_3 + \sigma_8, \sigma_1 + \sigma_6 + \sigma_8, \sigma_2 + \sigma_4, \sigma_2 + 2\sigma_8, \sigma_3 + \sigma_6, \sigma_5 + \sigma_7)$

$\Sigma(\lambda_3) \equiv (\sigma_3 + 3\sigma_8, \sigma_2 + \sigma_3 + 2\sigma_8, \sigma_2 + \sigma_6 + 2\sigma_8, \sigma_1 + \sigma_5 + \sigma_7 + \sigma_8, 2\sigma_1 + \sigma_3 + \sigma_6 + \sigma_8, 2\sigma_1 + \sigma_3 + \sigma_7, \sigma_1 + \sigma_2 + \sigma_4 + \sigma_8, \sigma_1 + 2\sigma_4, 2\sigma_2 + \sigma_5, 2\sigma_5 + \sigma_8, \sigma_2 + \sigma_4 + \sigma_6, \sigma_3 + \sigma_5 + \sigma_7)$

$\Sigma(\lambda_4) \equiv (\sigma_4 + 4\sigma_8, 2\sigma_3 + 3\sigma_8, \sigma_3 + \sigma_6 + 3\sigma_8, \sigma_1 + \sigma_4 + 2\sigma_7 + 2\sigma_8, 2\sigma_3 + \sigma_6 + 3\sigma_8, \sigma_1 + \sigma_4 + 3\sigma_8, \sigma_2 + 4\sigma_7, \sigma_3 + \sigma_5 + \sigma_7 + 3\sigma_8, \sigma_1 + \sigma_3 + 3\sigma_7 + \sigma_8, 5\sigma_1 + \sigma_4, 2\sigma_3 + 3\sigma_7, 2\sigma_1 + \sigma_4 + 3\sigma_7, 3\sigma_2 + \sigma_5 + 2\sigma_8, \sigma_2 + 2\sigma_5 + 3\sigma_8, \sigma_1 + \sigma_2 + \sigma_3 + \sigma_6 + 2\sigma_7 + 2\sigma_8, 2\sigma_2 + \sigma_4 + \sigma_6 + \sigma_8, 2\sigma_4 + \sigma_6 + 2\sigma_8, 4\sigma_2 + \sigma_7, \sigma_2 + \sigma_3 + \sigma_5 + \sigma_7 + 2\sigma_8, 3\sigma_2 + \sigma_6, 3\sigma_1 + 2\sigma_4 + \sigma_7, 2\sigma_1 + 2\sigma_2 + \sigma_5 + \sigma_7, \sigma_1 + \sigma_2 + \sigma_3 + \sigma_7 + \sigma_8, 4\sigma_1 + \sigma_3 + \sigma_5, 2\sigma_1 + \sigma_2 + \sigma_4 + \sigma_6 + \sigma_7, \sigma_1 + \sigma_3 + 2\sigma_5 + 2\sigma_8, 3\sigma_1 + \sigma_2 + \sigma_4 + \sigma_6, 2\sigma_2 + \sigma_4 + 2\sigma_7, 2\sigma_4 + \sigma_6 + \sigma_8, \sigma_1 + 2\sigma_2 + \sigma_5 + \sigma_6 + \sigma_8, 3\sigma_4 + \sigma_6, \sigma_1 + \sigma_2 + \sigma_4 + \sigma_5 + \sigma_7, 2\sigma_2 + 2\sigma_5 + \sigma_7, \sigma_2 + \sigma_4 + 2\sigma_5 + \sigma_8, \sigma_3 + 3\sigma_5)$

$\Sigma(\lambda_5) \equiv (\sigma_6 + 3\sigma_8, \sigma_4 + \sigma_7 + 2\sigma_8, 3\sigma_7, \sigma_2 + 2\sigma_7 + \sigma_8, \sigma_1 + \sigma_3 + 2\sigma_7, 2\sigma_2 + \sigma_6 + \sigma_8, \sigma_1 + 2\sigma_2 + \sigma_7, \sigma_2 + \sigma_4 + \sigma_7 + \sigma_8, 3\sigma_2, 3\sigma_1 + \sigma_3, \sigma_3 + \sigma_5 + 2\sigma_8, 2\sigma_1 + \sigma_2 + \sigma_5, \sigma_1 + \sigma_4 + \sigma_5 + \sigma_8, 2\sigma_1 + \sigma_4 + \sigma_6, 2\sigma_4 + \sigma_7, \sigma_1 + \sigma_2 + \sigma_5 + \sigma_7, \sigma_2 + 2\sigma_5)$

$\Sigma(\lambda_6) \equiv (\sigma_7 + \sigma_8, \sigma_2 + \sigma_7, \sigma_1 + \sigma_2, \sigma_4 + \sigma_8, \sigma_1 + \sigma_5)$

$\Sigma(\lambda_7) \equiv (3\sigma_8, \sigma_3 + \sigma_7 + \sigma_8, 3\sigma_1, 2\sigma_2 + \sigma_8, \sigma_5 + 2\sigma_8, \sigma_1 + 2\sigma_7, \sigma_1 + \sigma_2 + \sigma_6, \sigma_1 + \sigma_4 + \sigma_7, 2\sigma_1 + \sigma_4, \sigma_2 + \sigma_5 + \sigma_8, \sigma_4 + \sigma_5)$
As an example of (II.11), let us construct the set \( \Sigma(\lambda_1 + \lambda_7) \) from \( \Sigma(\lambda_1) \) and \( \Sigma(\lambda_7) \) in view of our second lemma. The lemma states that elements \( \sigma \in \Sigma(\lambda_1 + \lambda_7) \) are to be chosen from 15 elements of \( \Sigma(\lambda_1) \oplus \Sigma(\lambda_7) \) providing the conditions

\[
\kappa(\sigma, \sigma) = \kappa(\lambda_1 + \lambda_7, \lambda_1 + \lambda_7)
\]

In result, one has only the following 13 elements:

\[
\begin{align*}
\Sigma(\lambda_1 + \lambda_7) &\equiv (\sigma_1 + \sigma_7 + 2\sigma_8, \\
&\quad \sigma_1 + \sigma_2 + \sigma_7 + \sigma_8, \\
&\quad 2\sigma_1 + \sigma_2 + \sigma_8, \\
&\quad \sigma_1 + \sigma_4 + 2\sigma_8, \\
&\quad \sigma_6 + \sigma_7 + \sigma_8, \\
&\quad 2\sigma_1 + \sigma_5 + \sigma_8, \\
&\quad \sigma_2 + \sigma_3 + \sigma_7, \\
&\quad \sigma_1 + \sigma_2 + \sigma_3, \\
&\quad \sigma_3 + \sigma_4 + \sigma_8, \\
&\quad \sigma_2 + \sigma_6 + \sigma_7, \\
&\quad \sigma_4 + \sigma_6 + \sigma_8, \\
&\quad \sigma_1 + \sigma_3 + \sigma_5, \\
&\quad \sigma_1 + \sigma_5 + \sigma_6).
\end{align*}
\]

**APPENDIX.2**

Let us first borrow the following quantities from ref(7):

\[
\Omega_8(\sigma^+) \equiv \begin{array}{c}
40320 \ K(8) \ \mu(8) + \\
20160 \ ( \\
35 \ K(4,4) \ \mu(4,4) + 14 \ K(5,3) \ \mu(5,3) + 7 \ K(6,2) \ \mu(6,2) + 2 \ K(7,1) \ \mu(7,1) ) + \\
40320 \ ( \\
20 \ K(3,3,2) \ \mu(3,3,2) + 15 \ K(4,2,2) \ \mu(4,2,2) + \\
5 \ K(4,3,1) \ \mu(4,3,1) + 3 \ K(5,2,1) \ \mu(5,2,1) + 2 \ K(6,1,1) \ \mu(6,1,1) ) + \\
13440 \ ( \\
540 \ K(2,2,2,2) \ \mu(2,2,2,2) + 30 \ K(3,2,2,1) \ \mu(3,2,2,1) + \\
40 \ K(3,3,1,1) \ \mu(3,3,1,1) + 15 \ K(4,2,1,1) \ \mu(4,2,1,1) + 18 \ K(5,1,1,1) \ \mu(5,1,1,1) ) + \\
483840 \ ( \\
3 \ K(2,2,2,1,1) \ \mu(2,2,2,1,1) + K(3,2,2,1,1) \ \mu(3,2,2,1,1) + 2 \ K(4,1,1,1,1) \ \mu(4,1,1,1,1) ) + \\
967680 \ ( \\
3 \ K(2,2,1,1,1,1) \ \mu(2,2,1,1,1,1) + 5 \ K(3,1,1,1,1,1) \ \mu(3,1,1,1,1,1) ) + \\
29030400 \ K(2,1,1,1,1,1,1) \ \mu(2,1,1,1,1,1,1) + \\
1625702400 \ K(1,1,1,1,1,1,1) \ \mu(1,1,1,1,1,1,1)
\end{array}
\]
\[ \Omega_{12}(\Lambda^+) \equiv 40320 \; K(12) \; \mu(12) + \\
5040 \left( \\
1848 \; K(6, 6) \; \mu(6, 6) + 792 \; K(7, 5) \; \mu(7, 5) + 495 \; K(8, 4) \; \mu(8, 4) + \\
220 \; K(9, 3) \; \mu(9, 3) + 66 \; K(10, 2) \; \mu(10, 2) + 12 \; K(11, 1) \; \mu(11, 1) \right) + \\
95040 \left( \\
1575 \; K(4, 4, 4) \; \mu(4, 4, 4) + 210 \; K(5, 4, 3) \; \mu(5, 4, 3) + 252 \; K(5, 5, 2) \; \mu(5, 5, 2) + \\
280 \; K(6, 3, 3) \; \mu(6, 3, 3) + 105 \; K(6, 4, 2) \; \mu(6, 4, 2) + 42 \; K(6, 5, 1) \; \mu(6, 5, 1) + \\
60 \; K(7, 3, 2) \; \mu(7, 3, 2) + 30 \; K(7, 4, 1) \; \mu(7, 4, 1) + 45 \; K(8, 2, 2) \; \mu(8, 2, 2) + \\
15 \; K(8, 3, 1) \; \mu(8, 3, 1) + 5 \; K(9, 2, 1) \; \mu(9, 2, 1) + 2 \; K(10, 1, 1) \; \mu(10, 1, 1) \right) + \\
95040 \left( \\
11200 \; K(3, 3, 3, 3) \; \mu(3, 3, 3, 3) + \\
700 \; K(4, 3, 3, 2) \; \mu(4, 3, 3, 2) + 1050 \; K(4, 4, 2, 2) \; \mu(4, 4, 2, 2) + \\
350 \; K(4, 4, 3, 1) \; \mu(4, 4, 3, 1) + 420 \; K(5, 3, 2, 2) \; \mu(5, 3, 2, 2) + \\
280 \; K(5, 3, 3, 1) \; \mu(5, 3, 3, 1) + 105 \; K(5, 4, 2, 1) \; \mu(5, 4, 2, 1) + \\
168 \; K(5, 5, 1, 1) \; \mu(5, 5, 1, 1) + 630 \; K(6, 2, 2, 2) \; \mu(6, 2, 2, 2) + \\
70 \; K(6, 3, 2, 1) \; \mu(6, 3, 2, 1) + 70 \; K(6, 4, 1, 1) \; \mu(6, 4, 1, 1) + \\
60 \; K(7, 2, 2, 1) \; \mu(7, 2, 2, 1) + 40 \; K(7, 3, 1, 1) \; \mu(7, 3, 1, 1) + \\
15 \; K(8, 2, 1, 1) \; \mu(8, 2, 1, 1) + 10 \; K(9, 1, 1, 1) \; \mu(9, 1, 1, 1) \right) + \\
380160 \left( \\
1260 \; K(3, 3, 2, 2, 2) \; \mu(3, 3, 2, 2, 2) + \\
420 \; K(3, 3, 3, 2) \; \mu(3, 3, 3, 2, 1) + 1890 \; K(4, 2, 2, 2, 2) \; \mu(4, 2, 2, 2, 2) + \\
105 \; K(4, 3, 2, 2) \; \mu(4, 3, 2, 2, 1) + 140 \; K(4, 3, 3, 1) \; \mu(4, 3, 3, 1, 1) + \\
105 \; K(4, 4, 2, 1) \; \mu(4, 4, 2, 1, 1) + 189 \; K(5, 2, 2, 2) \; \mu(5, 2, 2, 2, 1) + \\
42 \; K(5, 3, 2, 1) \; \mu(5, 3, 2, 1, 1) + 63 \; K(5, 4, 1, 1) \; \mu(5, 4, 1, 1, 1) + \\
42 \; K(6, 2, 2, 1, 1) \; \mu(6, 2, 2, 1, 1) + 42 \; K(6, 3, 1, 1, 1) \; \mu(6, 3, 1, 1, 1) + \\
18 \; K(7, 2, 1, 1, 1) \; \mu(7, 2, 1, 1, 1) + 18 \; K(8, 1, 1, 1, 1) \; \mu(8, 1, 1, 1, 1) \right) + \\
570240 \left( \\
56700 \; K(2, 2, 2, 2, 2) \; \mu(2, 2, 2, 2) + \\
1260 \; K(3, 2, 2, 2, 2) \; \mu(3, 2, 2, 2, 2) + 280 \; K(3, 3, 2, 2, 1, 1) \; \mu(3, 3, 2, 2, 1, 1) + \\
840 \; K(3, 3, 3, 1, 1, 1) \; \mu(3, 3, 3, 1, 1, 1) + 315 \; K(4, 2, 2, 2, 1, 1) \; \mu(4, 2, 2, 2, 1, 1) + \\
105 \; K(4, 3, 2, 1, 1, 1) \; \mu(4, 3, 2, 1, 1, 1) + 420 \; K(4, 4, 1, 1, 1, 1) \; \mu(4, 4, 1, 1, 1, 1) + \\
126 \; K(5, 2, 2, 1, 1, 1) \; \mu(5, 2, 2, 1, 1, 1) + 168 \; K(5, 3, 1, 1, 1, 1) \; \mu(5, 3, 1, 1, 1, 1) + \\
84 \; K(6, 2, 1, 1, 1, 1) \; \mu(6, 2, 1, 1, 1, 1) + 120 \; K(7, 1, 1, 1, 1, 1) \; \mu(7, 1, 1, 1, 1, 1) \right) + \\
79833600 \left( \\
90 \; K(2, 2, 2, 2, 2, 1) \; \mu(2, 2, 2, 2, 2, 1) + \\
9 \; K(3, 2, 2, 2, 1, 1, 1) \; \mu(3, 2, 2, 2, 1, 1, 1) + 8 \; K(3, 3, 2, 1, 1, 1, 1) \; \mu(3, 3, 2, 1, 1, 1, 1) + \\
6 \; K(4, 2, 2, 1, 1, 1, 1) \; \mu(4, 2, 2, 1, 1, 1, 1) + 10 \; K(4, 3, 1, 1, 1, 1, 1) \; \mu(4, 3, 1, 1, 1, 1, 1) + \\
6 \; K(5, 2, 1, 1, 1, 1, 1) \; \mu(5, 2, 1, 1, 1, 1, 1) + 12 \; K(6, 1, 1, 1, 1, 1, 1) \; \mu(6, 1, 1, 1, 1, 1, 1) \right) + \\
479001600 \left( \\
36 \; K(2, 2, 2, 1, 1, 1, 1) \; \mu(2, 2, 2, 1, 1, 1, 1) + \\
10 \; K(3, 2, 1, 1, 1, 1, 1) \; \mu(3, 2, 1, 1, 1, 1, 1) + 40 \; K(3, 3, 1, 1, 1, 1, 1) \; \mu(3, 3, 1, 1, 1, 1, 1) + \\
15 \; K(4, 2, 1, 1, 1, 1, 1) \; \mu(4, 2, 1, 1, 1, 1, 1) + 42 \; K(5, 1, 1, 1, 1, 1, 1) \; \mu(5, 1, 1, 1, 1, 1, 1) \right)
In all these expressions, the so-called K-generators are to be reduced to the ones defined by (III.3) for which the parameters $k_i$ are determined via (III.1) for a dominant weight $\sigma^+$ which we prefer to suppress from K-generators. The reduction rules can be deduced from definitions given also in ref(7). The left-hand side of (III.2) can thus be calculated from

$$\text{ch}_M(\sigma^+) \equiv \frac{1}{9!} \text{dim} II(\sigma^+) \Omega_M(\sigma^+)$$

with which we obtain $E_8$ Weyl orbit characters. The dimension of a Weyl orbit $\Pi(\sigma^+)$ is the number of its elements and we show this number by $\text{dim} II(\sigma^+)$. Once again, we stress that both explicit forms and also the number of these weights are known due to permutational lemma given in ref(7).

**APPENDIX 3**

We now give the results of our 12th degree calculations. Explicit dependences on $\Lambda^+$ will be suppressed here. It will be useful to introduce the following auxiliary functions in terms of which the formal definitions of $E_8$ basis functions will be highly simplified:

$$W_1(8) \equiv 68580 \ \Theta(8) - 42672 \ \Theta(2) \ \Theta(6) -$$
$$42672 \ \Theta(3) \ \Theta(5) - 13335 \ \Theta(4)^2 +$$
$$13335 \ \Theta(2)^2 \ \Theta(4) + 17780 \ \Theta(2) \ \Theta(3)^2 - 939 \ \Theta(2)^4$$

$$W_2(8) \equiv 76765890960 \ \Theta(8) - 47741514624 \ \Theta(2) \ \Theta(6) -$$
$$4756928416 \ \Theta(3) \ \Theta(5) - 14950629660 \ \Theta(4)^2 +$$
$$14921466630 \ \Theta(2)^2 \ \Theta(4) + 19832476160 \ \Theta(2) \ \Theta(3)^2 - 1050561847 \ \Theta(2)^4$$

$$W_1(12) \equiv 302400 \ \Theta(3) \ \Theta(9) - 56700 \ \Theta(4) \ \Theta(8) -$$
$$51840 \ \Theta(5) \ \Theta(7) - 158400 \ \Theta(2) \ \Theta(3) \ \Theta(7) + 30240 \ \Theta(6)^2 -$$
$$168000 \ \Theta(3)^2 \ \Theta(6) + 33264 \ \Theta(2) \ \Theta(5)^2 - 80640 \ \Theta(3) \ \Theta(4) \ \Theta(5) +$$
$$16275 \ \Theta(4)^3 + 92400 \ \Theta(2) \ \Theta(3)^2 \ \Theta(4) + 19600 \ \Theta(3)^4$$

$$W_2(12) \equiv 42338419200 \ \Theta(3) \ \Theta(9) - 7938453600 \ \Theta(4) \ \Theta(8) -$$
$$250343238600 \ \Theta(2)^2 \ \Theta(8) - 7258014720 \ \Theta(5) \ \Theta(7) -$$
$$22177267200 \ \Theta(2) \ \Theta(3) \ \Theta(7) + 4233841920 \ \Theta(6)^2 -$$
$$23521344000 \ \Theta(3)^2 \ \Theta(6) + 156357159840 \ \Theta(2)^3 \ \Theta(6) +$$
$$4657226112 \ \Theta(2) \ \Theta(5)^2 - 11290245120 \ \Theta(3) \ \Theta(4) \ \Theta(5) +$$
$$160591001760 \ \Theta(2)^2 \ \Theta(3) \ \Theta(5) + 2278630200 \ \Theta(4)^3 +$$
$$48089818350 \ \Theta(2)^2 \ \Theta(4)^2 + 12936739200 \ \Theta(2) \ \Theta(3)^2 \ \Theta(4) -$$
$$48806484300 \ \Theta(2)^4 \ \Theta(4) + 2741456800 \ \Theta(3)^4 -$$
$$66618900600 \ \Theta(2)^3 \ \Theta(3)^2 + 3440480295 \ \Theta(2)^6$$

$$W_3(12) \equiv 1976486400 \ \Theta(3) \ \Theta(9) - 370591200 \ \Theta(4) \ \Theta(8) +$$
$$63622800 \ \Theta(2)^2 \ \Theta(8) - 338826240 \ \Theta(5) \ \Theta(7) -$$
$$1035302400 \ \Theta(2) \ \Theta(3) \ \Theta(7) + 197648640 \ \Theta(6)^2 -$$
$$1098048000 \ \Theta(3)^2 \ \Theta(6) - 12136320 \ \Theta(2)^3 \ \Theta(6) +$$
$$217413504 \ \Theta(2) \ \Theta(5)^2 - 527063040 \ \Theta(3) \ \Theta(4) \ \Theta(5) +$$
$$185512320 \ \Theta(2)^2 \ \Theta(3) \ \Theta(5) + 106373400 \ \Theta(4)^3 -$$
$$39822300 \ \Theta(2)^2 \ \Theta(4)^2 + 603926400 \ \Theta(2) \ \Theta(3)^2 \ \Theta(4) +$$
$$6366150 \ \Theta(2)^4 \ \Theta(4) + 128105600 \ \Theta(3)^4 -$$
$$63571200 \ \Theta(2)^3 \ \Theta(3)^2 - 274935 \ \Theta(2)^6$$
\( \mathcal{W}_4(12) \equiv -1501985020838400 \Theta(3) \Theta(9) + 192772901311200 \Theta(4) \Theta(8) + \\
240792770302000 \Theta(2)^2 \Theta(8) - 13295642434560 \Theta(5) \Theta(7) + \\
760428342950400 \Theta(2) \Theta(3) \Theta(7) - 156516673824000 \Theta(6)^2 + \\
33565287369600 \Theta(2) \Theta(4) \Theta(6) + 88357745844800 \Theta(3)^2 \Theta(6) - \\
1515778400455200 \Theta(2)^3 \Theta(6) - 53070803904384 \Theta(2)^2 \Theta(5)^2 + \\
579544204861440 \Theta(3) \Theta(4) \Theta(5) - 1696086939738240 \Theta(2)^2 \Theta(3) \Theta(5) - \\
47654628701400 \Theta(4)^3 - 461057612469300 \Theta(2)^2 \Theta(4)^2 - \\
463327486742400 \Theta(2)^3 \Theta(4) + 472701971331450 \Theta(2)^4 \Theta(4) - \\
111245008649600 \Theta(3)^4 + 684206487048000 \Theta(2)^3 \Theta(3)^2 - 33351005297925 \Theta(2)^6 \)

It is first seen that the expression (IV.4) can be cast in the form

\[ P_2(8) \equiv \mathcal{W}_1(8) + 385526887200. \]

Let us further define

\[ \Delta_{12} \equiv (\Theta(2) - 620)(-105 \Theta(2)^5 + 341250 \Theta(2)^4 - 443780280 \Theta(2)^3 + \\
288672359200 \Theta(2)^2 - 93922348435072 \Theta(2) + 1222805588033560) \]

with the remark that the square length of \( E_8 \) Weyl vector is 620. At last, 8 basis functions of \( E_8 \) will be expressed as in the following:

\[ P_1(12) \equiv \mathcal{W}_1(12) + \\
\frac{105}{1392517035128} ( \\
\frac{1}{2327783} \mathcal{W}_2(8) \Theta(2)^2 + \\
\frac{1}{1641651348800} \mathcal{W}_1(8) \Theta(2)^2 + \\
\frac{1}{1853819288565353101504512} \Theta(2)^2 - \\
\frac{1}{5646385058438400} \mathcal{W}_1(8) \Theta(2) - \\
\frac{1}{245771496590103630881280000} \Theta(2) + \\
\frac{1}{187821352583849376} \mathcal{W}_1(8) + \\
\frac{1}{474462162108792} P_1(0) + \\
\frac{1}{8144998046410042555589009600} ) \]

\[ P_2(12) \equiv \mathcal{W}_1(12) + \\
\frac{105}{6580376} ( \\
\frac{1}{11} \mathcal{W}_2(8) \Theta(2)^2 - \\
\frac{1}{13946976} \mathcal{W}_1(8) \Theta(2)^2 - \\
\frac{1}{717386789108493504} \Theta(2)^2 + \\
\frac{1}{2185025000} \mathcal{W}_1(8) \Theta(2) + \\
\frac{1}{95108970408987600000} \Theta(2) - \\
\frac{1}{726826815792} \mathcal{W}_1(8) - \\
\frac{1}{31504388959739569446400} ) \]
\[
P_3(12) \equiv -\frac{10742925608415}{467309767} \Delta_{12} + \frac{6983349}{10867669} P_1(12) + \frac{3884320}{10867669} P_2(12)
\]
\[
P_4(12) \equiv -\frac{289884687985}{487195289} \Delta_{12} + \frac{39572311}{237932583} P_1(12) + \frac{198360272}{237932583} P_2(12)
\]
\[
P_5(12) \equiv -\frac{511567886115}{1063875427} \Delta_{12} + \frac{2327783}{173189023} P_1(12) + \frac{170861240}{173189023} P_2(12)
\]
\[
P_6(12) \equiv -\frac{2557839430575}{69599327} \Delta_{12} + \frac{11638915}{11330123} P_1(12) - \frac{308792}{11330123} P_2(12)
\]
\[
P_7(12) \equiv -\frac{17904876014025}{1362158257} \Delta_{12} + \frac{11638915}{31678099} P_1(12) + \frac{20039184}{31678099} P_2(12)
\]
\[
P_8(12) \equiv -\frac{26924625585}{9942761} \Delta_{12} + \frac{2327783}{30753191} P_1(12) + \frac{28425408}{30753191} P_2(12)
\]

It is seen that the generator \(\Delta_{12}\) plays the role of a kind of cohomology operators in the sense that 6 generators \(P_\alpha(12)\) (for \(\alpha = 3, 4, \ldots, 8\)) will depend linearly on the first 2 generators \(P_1(12)\) and \(P_2(12)\) modulo \(\Delta_{12}\). It is therefore easy to conclude that all our 8 generators \(P_\alpha(12)\) (for \(\alpha = 1, 2, \ldots, 8\)) are linearly independent due to the fact that there is no a linear relationship among the generators \(P_1(12)\) and \(P_2(12)\) modulo \(\Delta_{12}\).