SELECTION RULES FOR SPLITTING STRINGS

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Abstract

It has been pointed out that Nielsen-Olesen vortices may be able to decay by pair production of black holes. We show that when the abelian Higgs model is embedded in a larger theory, the additional fields may lead to selection rules for this process - even in the absence of fermions - due to the failure of a charge quantization condition. We show that, when there is topology change, the criterion based on the charge quantization condition supplements the usual criterion based on $\pi_0(H)$. In particular, we find that, unless $2\sin^2\theta_W$ is a rational number, the thermal splitting of electroweak $Z$-strings by magnetically neutral black holes is impossible, even though $\pi_0(H)$ is trivial.

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Topological defects and solitons remain an active field of study in many different areas of physics. Generally, as is implied by the nomenclature, defects are argued to be stable from topological considerations of the vacuum structure of the particular theory in question. Recently, however, it was observed that supposedly stable vortices in the abelian-Higgs model may be able to decay by pair production of black holes [1–3]. The instanton describing this process is based on the C-metric [4], or the Israel-Khan (IK) metric [5] for thermal nucleation of uncharged black holes [6,7], where the conical singularity in the exact general relativistic solution is smoothed out by a real vortex with the same deficit angle. Briefly, the C-, and IK-metrics represent two black holes connected to infinity by conical singularities. For the C-metric these black holes are undergoing uniform acceleration away from each other and are not in causal contact. The IK-metric on the other hand is static, with the two black holes held in equilibrium by the conical deficits, whose tension exactly balances the gravitational attraction between the black holes. The spatial topology of these solutions is $S^2 \times S^1$ minus one (C) or two (IK) points. The euclideanization of these metrics gives the instanton which interpolates between the initial, infinite vortex, state and the final, split vortex, state. This process clearly involves a topology change in the spacetime, thus circumventing the usual stability arguments for the topological defect, which rely on spacetime being topologically trivial.

Nonetheless, topology change is not the only ingredient in this decay process; the decay of idealized strings (conical deficit sources) via purely gravitational processes has been commented upon in the literature some years ago [8]. Central to this particular construction of a field-theoretic vortex splitting is the demonstration [6,9] that an abelian-Higgs vortex [10] can emanate from a black hole, and the solution falls outside the scope of several well known “no hair” theorems for abelian Higgs fields [11]. While in flat spacetime there is no regular field configuration describing a topological string with ends, this is not true when there are black holes present. The existence of non-contractable spheres in spacetime (due to the black hole horizon) allows for the vortex to end in a black hole [1,9] in such a way that the field configuration is regular outside the horizon. In effect, the field configuration is
that of a magnetic monopole - the only difference being that the vortex lives in a phase with broken symmetry where electromagnetism is massive, causing the flux lines to be confined to the vortex core. It is well known that Dirac monopoles [12] obey a charge quantization condition

\[ eg = \hbar / 2 \]  

which is rather elegantly described in terms of the Wu-Yang construction [13]. The cosmic string also obeys a flux quantization condition coming from single-valuedness of the Higgs field which forms it, and the coincidence of these quantization conditions allows us to use a Wu-Yang construction to achieve a smooth field configuration for the instanton.

We can often embed the abelian Higgs model in a larger theory to obtain classically stable (topological or non-topological) strings whose energy momentum tensors are identical to those of the Nielsen-Olesen vortex, hence they can also avoid no-hair theorems and may in principle split by pair creation of black holes. This was somehow implicit in [1–3], where regularity of the instanton required a second U(1) gauge field (with respect to which the Higgs field was neutral). A less trivial example is the semilocal string [14] which occurs in a special limit of the Weinberg-Salam model where the SU(2) symmetry is global. Even though the string is not topological (because the vacuum manifold, \( S^3 \), is simply connected) it has been shown that whenever \( m_{\text{Higgs}} \leq m_{\text{gauge}} \), infinitely long, straight strings are stable both to small perturbations [14–16] and to semiclassical tunnelling in flat space [16,17]. When the \( SU(2) \) coupling is non-zero, electroweak Z-strings are also classically stable in a very small region of parameter space very close to that of stable semilocal strings [18,19], but they can decay by nucleation of magnetic monopoles [16,17,20]. The classical and semiclassical stability of more general embedded vortices in flat spacetime has been investigated in [21] and [16,17], respectively.

Here we are interested in semiclassical decay by spacetime topology change – which can obviously affect both topological and non-topological defects, so we will concentrate on the former. The tunneling amplitude found in [1–3] for Nielsen-Olesen vortices was incredibly
small, of order $e^{-10^{12}}$ for GUT strings, however, it does provide the only possible decay channel for otherwise stable defects and it is interesting that this channel can close when the field theory model is extended. It is worth stressing that the arguments we will use make no reference to fermions. It is well known that the presence of fermions can jeopardize instanton-mediated processes due to incompatibility of spin structures [22]. However in a purely bosonic extension of the theory one would expect the decay mode to persist, since the instanton will still be a solution to the (Euclidean) field equations with all extra fields set to zero. In fact, in [23] it was shown that the presence of additional symmetries should typically enhance the decay rate of a false vacuum by bubble nucleation due to the extra zero modes. By contrast, the decay of vortices by black hole pair creation seems to suffer from the opposite effect: there may be selection rules for the process precisely due to the extra bosons.

Let us now consider the conditions for the splitting of a string. Recall that the abelian-Higgs vortex could split because it was possible to use a Wu-Yang construction to smoothly convert from the vacuum configuration exterior to the string ($\Phi = \eta e^{i\phi}; A_\mu = \frac{i}{e} \nabla_\mu \phi$) to the ‘true’ vacuum ($\Phi = \eta; A_\mu = 0$) using the gauge transition function $G(\phi) = e^{i\phi} \in U(1)$ on an equatorial region in which two distinct gauge patches overlap. Now consider what happens if we extend our original model by the addition of extra bosonic fields. If these extra fields couple to the original $U(1)$ gauge field, or indeed if there are additional gauge fields (e.g. if the $U(1)$ becomes a subgroup of a larger symmetry group), then they too will have to be defined separately on each gauge patch, and transformed by the gauge transition function according to the particular representation in which they lie. A necessary condition for the string to split would therefore seem to be that these field configurations can all be consistently defined. One can view this as a relic of the Dirac charge quantization conditions (DQC’s) in the unbroken theory. A string ending on a black hole represents confined magnetic flux from a magnetically charged black hole created prior to a phase transition. Such a charged black hole must be made up of particles in the spectrum of the theory, which are known to
obey quantization conditions analogous to eq. (1). Therefore a string can terminate on a black hole (or monopole) only if their combined (confined plus unconfined) flux corresponds to some magnetic monopole in the symmetric phase. (Note that this condition must hold over and above any other conditions necessary for the regularity of the instanton [1,2], which we assume are satisfied in what follows.)

Clearly this criterion also applies to the case of a non-topological vortex ending in a regular monopole, where it is customary to state the condition in terms of properties of the unbroken residual symmetry group, $H$. We shall do this now for the black hole case, making reference to two specific examples, discrete gauge symmetry and the electroweak string, in order to illustrate: a) that a commonly used relation between $\Pi_0(H)$ and Aharonov-Bohm phases is not strictly correct when there is gauge mixing, and b) that neither Aharonov-Bohm interactions nor $\pi_0(H)$ are enough to determine when a string can split and one should always look at the DQC’s as well.

Given a symmetry breaking $G \rightarrow H$ which admits localized vortices with finite energy per unit length, parallel transport of the Higgs field around the string implies

$$\mathbf{P} \exp \{ i \oint \mathbf{A}_H \} \Phi \equiv \mathbf{g} \Phi = \Phi$$

(2)

where $\mathbf{A}_H = e_{\text{Higgs}} A^a T^a_{\text{Higgs}}$; i.e. the group element $\mathbf{g}$ must sit in the unbroken group $H$. To derive a condition for the splitting of a string, we must consider what happens to $\mathbf{g}$ as we deform a loop encircling the string, slipping it off the end of the string and contracting it to a point.

As long as there is no patching involved, it is clear that infinitesimal transformations of the loop affect $\mathbf{g}$ continuously, and that when the loop is contracted to a point the final value of $\mathbf{g}$ must be the identity. In that case a necessary condition for the string to split is that $\mathbf{g}$ be continuously connected to the identity - the usual criterion for a (non-topological) string to end in a regular monopole. The only subtlety, when a string ends in a black hole, is how patching conditions may affect $\mathbf{g}$. If $H$ is abelian, $\mathbf{g}$ is unchanged; if $H$ is not abelian, $\mathbf{g}$ is not uniquely defined but depends on a reference point on the closed loop (all physically
measurable quantities are, of course, independent of such a reference point). Under a gauge patching, $g$ will be conjugated by the transition element at that point, which is well defined because the transition function must be single valued in $G$ (this is the DQC). Once in the “vacuum” patch we are back to the previous case: the endpoint value is the identity, and the conjugated $g$ must be continuously connected to it; but, since the identity is fixed under conjugation, and conjugation is a continuous function, this statement is also true in the other patch. We conclude that a necessary condition for a string to split, whether it is by black holes or monopoles, is that $g$ lie in the trivial component of $\pi_0(H)$. This condition was given in refs. [1,2] with an acknowledgment to a private communication by J. Preskill. However the condition is not sufficient, as we will now show.

In physical terms, an element $g$ not equal to the identity simply indicates a nonzero magnetic flux emanating from the monopole or black hole. If one demands that the string split by nucleating neutral black holes (as required by the Israel-Khan instanton or in a theory with no residual electromagnetic symmetry) then the condition $g=1$ is not only necessary but sufficient. Indeed, the vortex corresponds to an element of $\pi_1(G/H)$; if $g=1$, the loop in $G/H$ is closed in $G$ and one can simply use the ‘inverse loop’ as a transition function to make the string disappear into a Schwarzschild black hole. Therefore, if $g \neq 1$, thermal splitting of a string by nucleation of magnetically neutral black holes will not take place even if $g$ is continuously connected with the identity.

Let us illustrate this rule with two specific examples. The first one, also mentioned in [1], is the coupling of gravity to an abelian Higgs model with an extra, fractionally charged field. This model has been extensively studied in connection with (discrete) quantum hair for black holes [24]. The matter part of the action is:

$$S = \int d^4x \sqrt{g} \left[ |D_\mu \Phi|^2 + |D_\mu \chi|^2 - \frac{1}{4} F_{\mu\nu}^2 - V(\Phi, \chi) \right]$$  (3)

where $D_\mu \Phi = (\nabla_\mu - i NeA_\mu) \Phi$, $D_\mu \chi = (\nabla_\mu - ieA_\mu) \chi$ and $V(\Phi, \chi)$ contains a mexican hat potential for $\Phi$. Under a U(1) gauge transformation,

$$\Phi \rightarrow e^{iNe\alpha} \Phi, \quad \chi \rightarrow e^{ie\alpha} \chi, \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha,$$  (4)
thus $\alpha$ is to be identified with a period of $2\pi/e$. If the Higgs field $\Phi$ condenses, this results in a symmetry breaking $U(1) \to \mathbb{Z}_N$, topological vortices can form and although they are classified by $\Pi_1(G/H) = \mathbb{Z}$, they also can be labelled by $\pi_0(H) = \mathbb{Z}_N$, which characterises their Aharonov-Bohm interactions with the fractionally charged $\chi$ quanta. Applying the $\pi_0(H)$ criterion, it is obvious that strings whose winding number is not a multiple of $N$ cannot break by nucleating black holes, because $g \neq 1$ for such strings and $H$ is discrete. It is interesting that for the string itself, the putative gauge transition function $G(\phi) = e^{i\phi}$ or $\alpha(\phi) = \phi/Ne$, would ‘unwind’ the vortex, but it is not legal since it is not closed in $G$ (as can be seen by its non-single valuedness acting on $\chi$). This is clearly equivalent to the DQC in the symmetric phase, since the presence of particles of fundamental electric charge $N$ times smaller than that of $\Phi$, gives a minimal magnetic monopole charge $N$ times larger. Thus, both criteria are equivalent in this case.

Our second example is a ‘pruned’ electroweak string. Consider a symmetry breaking $U(1) \times U(1)/\mathbb{Z}_2 \to U(1)$, realised by two Higgs fields $\phi_1, \phi_2$ which have equal charge $g/2$ with respect to the first $U(1)$, but opposite charge $\pm g'/2$ with respect to the second $U(1)$. Writing $\Phi = (\phi_1, \phi_2)$ for brevity, the gauge covariant derivative is

\[ D_\mu \Phi = \left[ \nabla_\mu - \frac{1}{2}ig'B'_\mu \sigma^3 - \frac{1}{2}igB_\mu \right] \Phi. \tag{5} \]

We suppose that $\phi_2$ condenses at some high energy scale, $\eta$, whereas other interactions occur at a much lower scale (and will be ignored). This model has stable topological strings classified by the winding number of the second Higgs field. The reason for calling this a pruned electroweak string should now be apparent. Although the $\phi_2$-condensation ensures that there is no $SU(2)$ symmetry, the solution for a static straight string of winding number $N$ in cylindrical polars,

\[ \Phi = \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 \\ X(\rho)e^{iN\phi} \end{pmatrix}, \tag{6} \]

\[ Z_\mu = \cos \theta_\mu B'_\mu - \sin \theta_\mu B_\mu = \frac{2N}{\alpha}(1 - P(\rho))\partial_\mu \phi, \tag{7} \]

\[ A_\mu = \sin \theta_\mu B'_\mu + \cos \theta_\mu B_\mu = 0, \tag{8} \]
(where $X$ and $P$ are the Nielsen-Olesen profiles, and $\sin \theta_w = g/\alpha$, $\alpha^2 = g^2 + g'^2$) is in fact identical to that of the electroweak $Z$-string (with $g$, $g'$ interchanged) [20,18]. However, unlike the electroweak string, this vortex is topologically stable.

As in the Weinberg-Salam model, define $e = gg'/\alpha$. The generators of the broken ($\chi$) and unbroken ($\psi$) $U(1)$'s in terms of the original $U(1)$ generators ($\theta$, $\theta'$) are $\chi = \sin \theta_w \theta - \cos \theta_w \theta'$ and $\psi = \sin \theta_w \theta' + \cos \theta_w \theta$, thus

$$\phi_1 \rightarrow e^{i\frac{e}{2}(\psi - \frac{\alpha}{2} \cos 2\theta_w \chi)} \phi_1, \quad \phi_2 \rightarrow e^{\frac{i}{2} \chi} \phi_2 \quad (9)$$

Now consider $g$ given in ($\chi, \psi$) coordinates by $\left(\frac{\pi}{2}, 0\right)$, or in ($\theta, \theta'$) coordinates by $\left(\frac{\pi}{2}g', -\frac{\pi}{2}g'\right)$. Clearly, since $H = U(1)$, $g$ is connected to the identity, and indeed the rotation $\Delta \psi = \frac{\pi}{2} \tan \theta_w$ will return $g$ to the identity. Hence in principle these strings can split by nucleating charged black holes. To decide what charge the monopole must have, we can appeal to the DQC for the symmetric phase (that is, before the phase transition). The $U(1)_g$ magnetic charge must be a multiple of $1/g$, $q = n/g$, and the $U(1)_{g'}$ magnetic charge a multiple of $1/g'$, $q' = n'/g'$. Since we know that the $Z$-flux is $4\pi N/\alpha$, this implies that the $Z$-projection of magnetic charge, $q_Z = q' \cos \theta_w - q \sin \theta_w$, must be $N/\alpha$, i.e. $n' - n = N$.

This gives an $A$-magnetic charge of

$$q_A = \frac{n + n'}{2e} + \frac{N(g^2 - g'^2)}{2e\alpha^2} \quad (10)$$

consisting of a piece satisfying the DQC (with respect to the $A$-electric charge $e$) plus a non-trivial remaining part.

Note that if $\frac{2e}{\alpha} \tan \theta_w = \sin 2\theta_w \tan \theta_w \notin \mathbb{Z}$, then $\cos 2\theta_w \notin \mathbb{Z}$ hence $g\phi_1 \neq \phi_1$ and we have Aharonov-Bohm interactions of the $\phi_1$ quanta with the $\phi_2$-string (despite $H = U(1)$ being connected). The reason such a string can break is precisely that the monopoles nucleated do not satisfy the electromagnetic DQC and hence are naturally associated with ‘visible’ (non) Dirac strings. The Aharonov-Bohm phase simply gets “transferred” from the string to the monopoles.

Suppose instead we want to split the string by uncharged black holes. This we can only do if $g = 1$ i.e. if $\Delta \psi = 2\pi n/e$ for some $n \in \mathbb{Z}$, i.e. if $\tan \theta_w = \frac{n\pi}{2e}$. This requires $g = 0,$
g' = 0, or g = g'. Note again that, for all g, g', a G(ϕ) can be defined which would unwind the string, but is not closed in G.

It is obvious that these results generalize to electroweak strings: they can split by nucleating regular monopoles, which may then collapse to form magnetically charged black holes; but they can only split by neutral black holes when their Z flux satisfies the DQC, that is, if \(2N\sin^2\theta_W\) is an integer, where N is the winding number of the vortex (which is not a topological invariant in this case).

Now let us return to the original U(1) × U(1) → U(1) C-metric instanton of references [1–3]. The magnetic charge of the black hole in the C-metric is, in principle, arbitrary. However, it is interesting to note that adding a particle which is neutral under the broken U(1) (thereby having no Aharonov-Bohm interaction with the string), and charged under the unbroken U(1), has the effect of singling out a discrete set of allowed values of the black hole charge – those that satisfy the DQC – even though \(\pi_0(H)\) is unaffected by the presence of the new particle, or by the value of its charge. Once again we conclude that the topological \(\pi_0(H)\) criterion has to be supplemented by a careful analysis of the DQCs.

Finally, we would like to address an interesting point raised by these examples. Generally, when one talks of a tunnelling process, one is usually performing a semiclassical approximation, using a classical euclidean solution - the instanton - as a saddle point in the functional integral. In each case considered here, we can still define a euclidean solution by simply embedding the abelian Higgs instanton in the larger theory. However, if the DQC’s of the larger theory are not satisfied, the putative instanton is ‘isolated’ in field configuration space because small perturbations in certain directions cannot be defined. In that case it is no longer a viable saddle point and perturbation theory has broken down. Even so, it is reassuring (although perhaps misleading) that the path integral calculation can still yield the correct result – if one interprets the cancellation of the tunnelling amplitude as coming from the prefactor by virtue of the fact that the only allowable perturbations around the putative instanton form a set of measure zero. Nevertheless, the fact that the euclidean solution still appears to exist is deceptive, and does not indicate the presence of an approximate solution.
to the full quantum theory.

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