I. Introduction

Accretion in the relativistic accretion disks - black hole physics, hydrodynamics – relativistic

Subject headings: accretion, accretion disks, black hole physics, hydrodynamics - relativistic

Accretion in the relativistic accretion disks

According to this model, we derive additional terms in the disk structure equations, and show
the matter of the disk in the accretion disks is comparable to that of the rest mass.
Accreting black holes are expected to acquire a large spin and then the relativistic effects are especially important. A special feature of an accretion disk around the rapidly rotating black hole is that the viscously dissipated power becomes comparable to $\dot{M} \alpha^2$, $\dot{M}$ being the accretion rate. As a result, the inertial mass associated with heat stored in the advective disk is approximately equal to the rest mass of the accreting gas. Lasota’s equations, being fully relativistic in all other respects, neglect the inertia of internal heat, and Peitz \\& Appl (1997) and Jaroszynski \\& Kuriwieski (1997) have partly included it in their calculations. In this note we show that accounting for the heat inertia is necessary to make the disk model consistent with the global energy conservation law, and derive the disk structure equations including this effect. In particular, an additional term appears in the radial Euler equation. We dub it “heat deceleration” as it describes the back reaction of energy release on the radial velocity of accreting matter.

2. Notation, assumptions and methods

Our approach and notation are very close to those in Abramowicz et al. (1996), except that we do not neglect the contribution of internal energy and pressure to the inertial mass of the flow. We denote by $r_g = 2G M/c^2$ the gravitational radius of the black hole with the mass $M$, and by $a = J/M c$ the Kerr parameter connected to the black hole angular momentum $J$. We use the Boyer-Lindquist coordinates $x^i = (t, r, \theta, \varphi)$, and the signature $(-+++)$. The metric tensor $g_{ij}$ is given e.g. by Misner, Thorne, \\& Wheeler (1973). The four-velocity of the accreting gas has components $u^i = (u^t, u^r, u^\theta, u^\varphi)$ in the Boyer-Lindquist coordinates. The disk is assumed to lie at the equatorial plane of the Kerr geometry. In the slim approximation, $u^\varphi$ is neglected and all the metric and connection coefficients are evaluated at the equatorial plane.

The gas Lorentz factor measured in the frame of local observers with zero angular momentum is connected to $u^t$ by $\gamma = u^t (-g^{tt})^{-1/2}$. The angular velocity of the gas rotation is defined as $\Omega = u^\varphi / u^t$. It is not equal in general to the Keplerian angular velocity,

$$\Omega_K = \pm \frac{c}{r (2r/r_g)^{1/2} \mp a}.$$

(Plus and minus signs in this formula correspond to co-rotating and counter-rotating orbits). The $r$-component of four-acceleration, $a_r$, is given by (Abramowicz et al., 1996)

$$a_r = \frac{1}{2} \frac{d}{dr} (u_r u^r) + \frac{1}{2} \frac{\partial g_{rr}}{\partial r} \gamma^2 (\Omega - \Omega_K^r) (\Omega - \Omega_K^r).$$

The dynamical equations are derived from the conservation laws $\nabla_i T^i_{\text{h}} = 0$, where $\nabla_i$ is the covariant derivative in the Kerr metric, and $T^i_{\text{h}}$ is the stress-energy tensor of the viscous gas flow (Misner et al. 1973),

$$T^i_{\text{h}} = (\rho + p) u^i u_k + p \delta^i_k = -2 c \eta \sigma^i_k + \frac{1}{c} (u^i q_k + u_k q^i),$$

where $\rho$ and $p$ are the total energy density and pressure in the comoving frame, $\sigma^i_k$ is the shear tensor, and $q^i$ is the energy flux vector assumed to be directed vertically in the disk. The dynamic viscosity $\eta$ is related to the kinematic viscosity $\nu$ by $\eta = \nu (\rho + p)/c^2$. (Note that this differs from the previously used relation that includes only the rest mass of the gas).

The final equations are written for the vertically integrated thermodynamic quantities: surface rest mass density $\Sigma$, surface energy density $U$ (that includes both the rest mass energy $\Sigma c^2$, and internal energy $\Pi$),
and the vertically integrated pressure \( P \). \( F^+ \) denotes the surface rate of viscous heating, and \( F^- \) denotes the radiation flux radiated from both faces of the disk. All the thermodynamic quantities, and both fluxes \( F^- , F^+ \) are measured in the comoving frame. The dimensionless specific enthalpy is defined as,

\[
\mu = \frac{U + P}{\Sigma c^2} = 1 + \frac{\Pi + P}{\Sigma c^2}
\]

Neglecting the internal heat contribution to the inertia of the flow is the same as assuming that \( \mu = 1 \). In this paper we do not assume that, but keep in all equations \( \mu > 1 \). Consistently, we keep the viscous term in the radial Euler equation (see Section 4).

We skip all computational details, because our derivation goes along the same standard lines explained e.g. by Page & Thorne (1974), and used recently by Lasota (1994) and all the other above quoted subsequent authors. In the next Sections we give only the final results.

3. Conservation of energy and angular momentum

The conservation of energy and angular momentum are expressed by equations

\[
\frac{d}{dr} \left[ \mu \left( \frac{\dot{M} u_r}{2\pi} + 2\nu \Sigma r \sigma^r_r \right) \right] = \frac{F^-}{c^2} r u_r, \quad (1)
\]

\[
\frac{d}{dr} \left[ \mu \left( \frac{\dot{M} u_\phi}{2\pi} + 2\nu \Sigma r \sigma^\phi_\phi \right) \right] = \frac{F^-}{c^2} r u_\phi, \quad (2)
\]

Here \( \dot{M} \) is the accretion rate related to \( u^r \) and \( \Sigma \) by the barion conservation law,

\[
2\pi r u^r \Sigma = -\dot{M}, \quad (3)
\]

and \( \sigma^r_r, \sigma^\phi_\phi \) are the shear tensor components,

\[
\sigma^r_r = \frac{1}{2} g^{rr} g_{\phi\phi} \sqrt{-g^\gamma_\gamma} \frac{d\Omega}{dr}, \quad \sigma^\phi_\phi = -\Omega \sigma^r_r.
\]

These equations differ from those with neglected inertia of heat (see Abramowicz et al. 1996) just by the presence of the factor \( \mu > 1 \). They also differs from the equations in Peitz & Appl (1997) and Jaroszynski & Kurpiewski (1997) who take into account the contribution of heat to the specific orbital energy and angular momentum of the flow but neglect the similar term in the relation between the dynamic and kinematic viscosity.

In the standard thin disk \( \mu \approx 1 \) with high accuracy, and the rotation is very close to Keplerian. Then the equations (1,2) become the same as those derived by Page & Thorne (1974). For the advective disk, however, the factor \( \mu \) should not be neglected. This can be easily seen from the global energy conservation law. Integrating equation (1) from the transonic inner edge of the disk, \( r_{in} \), to infinity, and neglecting the viscous stress at \( r_{in} \), we get

\[
L = -\frac{2\pi}{c} \int_{r_{in}}^{\infty} u_r F^- r dr = \dot{M} c^2 \left( 1 + \mu_{in} \frac{u^r_{in}}{c} \right). \quad (4)
\]
It follows that the radiative efficiency of the disk equals

$$\epsilon = 1 - \mu \ln \left( 1 - \frac{b_0}{c^2} \right),$$

where \(b_0 = c^2 + u_0^2c\) is the specific binding energy at the inner edge. In the advective disks, most of the dissipated binding energy is stored as internal heat and the radiative losses are negligible, \(\epsilon \rightarrow 0\). From this we conclude that \(\mu \ln \rightarrow (1 - b_0/c^2)^{-1}\). In the particular case of the extremely rotating black hole, assuming e.g. that \(b_0/c^2\) is close to the corresponding standard value \((1 - 1/\sqrt{3}) \approx 0.42\), one has \(\mu \ln = \sqrt{3}\). Note that the usual assumption \(\mu = 1\) follows that \(b_0 = 0\) for any advective disk. In fact, the position and binding energy of the inner edge can be found only by the integration of the disk structure equations, and it is the difference \((\mu - 1)\) that adjusts to keep \(\epsilon \approx 0\). In this respect the heat inertia is essential in the case of non-rotating black hole as well as extremely rotating.

It would be instructive to compare the angular momentum equation (2) with its commonly used standard version (Novikov & Thorne 1973; Page & Thorne 1974). While \(\mu = 1\) can be safely assumed in the standard disk, the radiative losses of angular momentum represented by the right hand side of equation (2) become important when the accreting black hole is rapidly rotating (e.g. Lamb 1996). Contrary to this situation, in the advection dominated disk the radiative losses are always small, but the deviation of \(\mu\) from unity becomes significant. In a sense, the angular momentum which would be radiated away by the standard disk remains now in place, being carried by the increased inertia of internal energy.

4. Radial motion and viscous heating

To clarify the role of heat inertia in the equation of radial motion we derive this equation in two ways. Firstly, we perform vertical integration of the \(r\)-component of the equation \(\nabla_i T^i_k = 0\), to get

$$u^r u_r \left[ \frac{d}{dr}(\Pi + P) - \xi \frac{\Pi + P}{\Sigma} \frac{d\Sigma}{dr} \right] + a_r (U + P) + \frac{dP}{dr} + u_r \frac{F^-}{c} = 0,$$

where \(\xi \approx 1\) is a numerical factor accounting for non-homogeneity of the disk in vertical direction. In this equation we have taken into account that the divergency of the vertical energy flux \(q^{\perp}\) becomes equal to \(F^-\) after the vertical integration.

Secondly, we project the equation \(\nabla_i T^i_k = 0\) onto the hypersurface orthogonal to \(u^r\) to get the relativistic Euler equation (c.f. Lightman et al. 1979, problem 5.31). Then the flux term vanishes, but the viscous term arises in the radial equation, and we get after the vertical integration

$$a_r (U + P) + \frac{dP}{dr} (1 + u^r u_r) + \frac{F^-}{c} u_r = 0,$$

where

$$F^+ = 2\nu \Sigma \mu \sigma^2 c^2, \quad \sigma^2 = \frac{1}{2} g^{rr} g_{\varphi \varphi} \left( -g^{tt} \right) \xi^4 \left( \frac{d\Omega}{dr} \right)^2.$$  

Keeping \(F^{\pm}\) terms in the equations (6,7) allows to get the first law of thermodynamics for a vertically integrated accretion disk as a consequence of these two equations,

$$F^+ - F^- = c u^r \left( \frac{d\Pi}{dr} - \xi \frac{\Pi + P}{\Sigma} \frac{d\Sigma}{dr} \right).$$
Finally, substituting $a_r$ to (7), we write down the equation of radial motion with $u_r^2 u_r \ll 1$,

$$\frac{1}{2} \frac{d}{dr} (u_r u_r) = -\frac{1}{2} \frac{\partial g_{rr}}{\partial r} g^{rr} \gamma^2 \left( \Omega - \Omega_K \right) \left( \Omega - \Omega_K \right) - \frac{1}{c^2\Sigma\mu} \frac{dP}{dr} - \frac{F^+ u_r}{c^2\Sigma\mu}. \quad (10)$$

This equation differs from the radial equation in Abramowicz et al. (1996) by the presence of the factor $\mu$, and by the additional term proportional to $F^+$. Without this term, equation (10) would have a simple meaning: radial acceleration is a combined result of the deviation from Keplerian rotation and the radial pressure gradient. The viscous term $\propto F^+$ represents the heat deceleration effect which can be interpreted in the following way. The mass of the accreting gas measured in its local rest frame increases due to the stored heat. Thus, from the local observer point of view, the dissipation of orbital binding energy is an external source of mass-energy, and the matter inflow proceeds like motion of a body with changing mass. In this case, in addition to acting forces there is a contribution to acceleration due to the change of mass. The mass of the accreting gas increases with rate proportional to $F^+$ and this tends to decelerate the matter inflow. The “deceleration by heating” is a purely relativistic effect as it accounts for the mass-energy relation. It is proportional to the ratio of the released power to $\dot{M}c^2$ and can be essential near a rapidly rotating black hole. Then the additional term must influence the structure of the advective disk in the innermost region. There it becomes of the same order as the left hand side of the equation (10).

Note that the deceleration term is independent of the radiative losses, $F^-$, and remains the same even when $F^+ \sim F^+$. In this case the bulk of the dissipated energy is radiated away. However, this energy first appears as heat decelerating the radial inflow. Only then it is radiated away. The net momentum taken away by the radiation flux vanishes in the rest frame of the accreting gas, and does not contribute to the velocity change.

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