A Bayesian Estimate of the Primordial Helium Abundance

Craig J. Hogan\textsuperscript{1}, Keith A. Olive\textsuperscript{2} and Sean T. Scully\textsuperscript{2}

\textsuperscript{1}University of Washington
Astronomy and Physics Departments, Box 351580, Seattle, WA 98195-1580

\textsuperscript{2}School of Physics and Astronomy,
University of Minnesota, Minneapolis MN 55455 USA

Abstract

We introduce a new statistical method to estimate the primordial helium abundance $Y_p$ from observed abundances in a sample of galaxies which have experienced stellar helium enrichment. Rather than using linear regression on metal abundance we construct a likelihood function using a Bayesian prior, where the key assumption is that the true helium abundance must always exceed the primordial value. Using a sample of measurements compiled from the literature we find estimates of $Y_p$ between 0.221 and 0.236, depending on the specific subsample and prior adopted, consistent with previous estimates either from a linear extrapolation of the helium abundance with respect to metallicity, or from the helium abundance of the lowest metallicity H\textsuperscript{II} region, I Zw 18. We also find an upper limit which is insensitive to the specific subsample or prior, and estimate a model-independent bound $Y_p < 0.243$ at 95\% confidence, favoring a low cosmic baryon density and a high primordial deuterium abundance. The main uncertainty is not the model of stellar enrichment but possible common systematic biases in the estimate of $Y$ in each individual H\textsuperscript{II} region.
Big Bang Nucleosynthesis (BBN) makes a clean prediction for the primordial helium abundance $Y_p$, depending on only one parameter (the baryon-to-photon ratio $\eta$) and on that only weakly. A precise measurement of $Y_p$ is therefore necessary to test BBN even in the present situation where measurements of other predicted quantities constraining $\eta$, such as the primordial lithium abundance, deuterium abundance and the cosmic baryon density, are not yet so precise (e.g. Walker et al. 1991, Smith et al. 1993, Sarkar 1996, Fields et al. 1996, Hogan 1997ab).

Even though the bulk of the helium of the Universe originates in the Big Bang, the additional helium enrichment by stars cannot be ignored in estimating the primordial abundance from observations of present-day helium. Unlike the case of deuterium, we do not have the option of measuring directly the nearly primordial abundance of $^4\text{He}$ at high redshift. Lyman-\(\alpha\) absorption by He$^+$ at high redshift now gives a rough direct estimate of primordial abundance (Hogan, Anderson & Rugers 1997), but require uncertain ionization corrections to estimate $Y$. The high precision needed for strong cosmological tests can only be attained with high signal-to-noise measurements of nebular emission lines in well characterized nearby H\(\text{II}\) regions (Pagel et al. 1992, Izotov, Thuan, & Lipovetsky, 1994, 1997, Skillman et al. 1994, 1997), necessarily requiring measurements in gas with a certain amount of stellar helium superimposed on the primordial helium.

Under these circumstances, what is the best way to estimate the primordial helium abundance $Y_p$? It certainly helps to find H\(\text{II}\) regions with as little stellar helium as possible, as indicated both by their helium abundances and by the abundances of other heavier elements. The best studied example is I Zw 18 (Pagel et al. 1992, Skillman & Kennicutt 1993, Izotov, Thuan, & Lipovetsky, 1997), with an average estimated helium abundance from five independent measurements $Y = 0.230 \pm 0.004$ (Olive, Skillman, & Steigman 1997), and a metallicity 1/50th of solar. In the analysis below, we will use samples drawn from the complete sample of 62 individual H\(\text{II}\) regions (including two in I Zw 18) compiled by
Olive, Skillman, & Steigman (1997) from the data in Pagel et al. (1992), Izotov, Thuan, & Lipovetsky, (1994, 1996), and Skillman et al. (1994, 1997). This sample represents all published $Y$ measurements of $\text{H\ II}$ regions with $O/H \leq 1.5 \times 10^{-4}$ and $N/H \leq 1.0 \times 10^{-5}$ (for comparison $(O/H)_\odot = 8.5 \times 10^{-4}$ and $(N/H)_\odot = 1.1 \times 10^{-4}$).

One statistical technique to extract a primordial helium abundance from such a sample was introduced by Peimbert & Torres-Peimbert (1974) and is still widely used. For each galaxy in the sample, a metallicity $Z$ (either $O/H$ or $N/H$) and helium abundance $Y$ are measured; one then fits a linear relation between $Y$ and $Z$ and extrapolates to zero metallicity to find the primordial value. This technique offers insights into the nuclear evolution of these systems, and for relatively large enrichments the extrapolation is empirically well founded. In Olive, Steigman & Walker (1991) and in Olive & Steigman (1995), it was shown that these two quantities as measured in extragalactic $\text{H\ II}$ regions are in fact strongly correlated. The stability of the resulting fits was tested by a statistical bootstrap in Olive & Scully (1996) and in Olive, Skillman, & Steigman (1997) showing that the results were not particularly sensitive to any individual data point.

However, one of the limitations of this method is the need to assume a linear relation between $Y$ and $Z$, which is not well motivated. For example, helium and oxygen are not produced in the same stars; while helium is produced primarily in intermediate mass stars, oxygen is produced only in stars with masses $\gtrsim 8 \, M_{\odot}$. Physically realistic enrichment models often include quadratic terms, as well as stochastic variations in enrichment history, including different slopes $dY/dZ$. It is not clear that complicating the fits to include such effects results in a value for primordial $Y_p$ with any greater statistical significance or reliability. Moreover most of the information on the primordial abundance is contained in the lowest metallicity points, where the correlation is not very reliably established; this information is not being efficiently used in regression fits dominated by highly enriched regions.

Another approach has been to simply take the lowest, best measured points and use them as estimates of (or at least limits on) the primordial abundance. Indeed, it has been argued that it may be sufficient to determine the primordial $^4\text{He}$ abundance from even a
single well studied low metallicity H II region such as I Zw 18 (Kunth et al. 1994), although to avoid bias it should be chosen on the basis of low metallicity and not low helium abundance. Although these approaches yield results consistent with the linear fit to the data, caution is needed to avoid subtle biases if one starts off by selecting a sample by intentionally choosing the lowest points. Moreover in a sample of more than just one point we need a statistical method to combine the data, which recognizes the real spread in stellar helium enrichment and still recovers an unbiased estimate of the primordial value in spite of the obvious “bias” that the nonprimordial contributions are always additive.

We explore here a simple but systematic Bayesian approach to these issues, which we believe is the simplest recipe to extract an unbiased primordial abundance estimate from the data, free from detailed assumptions about metal enrichment. We aim to assume as little as possible—only that there is some universal primordial helium abundance, and that all subsequent evolution has increased the abundance. We encapsulate these assumptions mathematically with a Bayesian “prior” and derive statistical constraints on the primordial abundance which explicitly recognize the bias introduced by stellar enrichment. Inspiration for our approach and more background on the statistical methods is given by Press’ (1996) application of Bayesian arguments to estimates of the Hubble constant.

II. METHOD AND APPLICATION

Suppose we have a sample $S$ consisting of a series of abundance measurements $Y_i$ in a set of galaxies. In each case $i$ there is some true abundance $Y_{iT}$ which we do not measure, because of the measurement error. We seek to evaluate the relative probability (or likelihood) of obtaining the data in sample $S$, given a primordial abundance $Y_p$:

$$\mathcal{L}(Y_p) \equiv P(S|Y_p) = \prod_i P(Y_i|Y_p)$$

(1)

where $P(Y_i|Y_p)$ denotes the probability of obtaining the measurement $Y_i$ given the primordial value $Y_p$. For this in turn we write
\[ P(Y_i|Y_p) = \int dY_{iT} P(Y_i|Y_{iT})P(Y_{iT}|Y_p). \] (2)

The first term in the integrand is just the distribution of measurement errors in each case. For the second term we need to explicitly construct a Bayesian prior \( P(Y_{iT}|Y_p) \). This “enrichment probability function” encodes our assumptions about what the distribution of helium abundances in galaxies in our sample ought to be, given a primordial abundance.

What do we know about \( P(Y_{iT}|Y_p) \)? The most important property, supported by all models of nuclear evolution, is that there is no net destruction of helium: stars can only increase the primordial abundance. Thus nowhere can the true abundance be less than \( Y_p \): \( P(Y_{iT}|Y_p) = 0 \) for \( Y_{iT} \leq Y_p \). At a more detailed level, the shape of this prior depends on what we think about the chemical histories of the galaxies in the sample under study; there will be some range of \( Y_p \) of width \( w \) for which \( P(Y_{iT}|Y_p) \neq 0 \). Fortunately as we shall demonstrate, within reasonable limits the detailed shape for \( Y_{iT} \geq Y_p \) does not much affect the statistical limits on \( Y_p \); the most important thing is its asymmetry about \( Y_{iT} = Y_p \).

Within this framework we can now explore estimates of \( P(S|Y_p) \), and consequent statistical constraints on \( Y_p \). To start with, (1) assume that the observational errors are normally distributed with variances \( \sigma_i^2 \), and (2) assume the simplest form for \( P(Y_{iT}|Y_p) \), a uniform distribution or top-hat function of width \( w \), \( P(Y_{iT}|Y_p) = c(\Theta(Y_{iT} - Y_p) - \Theta(Y_{iT} - (Y_p + w))) \).

The likelihood function is then given by

\[ \mathcal{L}(Y_p) = \prod_{i=1}^{N} \int_{0}^{1} dY_{iT} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(Y_i - Y_{iT})^2/2\sigma_i^2} P(Y_{iT}|Y_p), \] (3)

where the product is over the \( N \) sample data points. The integral in (3) can be performed analytically, resulting in

\[ \mathcal{L}(Y_p) = \prod_{i=1}^{N} P(Y_i|Y_p) = \prod_{i=1}^{N} \frac{c_i}{2} \left\{ \text{erf} \left( \frac{Y_p + w - Y_i}{\sqrt{2}\sigma_i} \right) - \text{erf} \left( \frac{Y_p - Y_i}{\sqrt{2}\sigma_i} \right) \right\} \] (4)

The normalization constants \( c_i \) are chosen so that each of the individual functions \( P(Y_i|Y_p) \), when integrated over \( Y_p \), yield unity. To a very good approximation, \( c = 1/w \). For sufficiently large \( w \), the first error function is approximately unity.
We choose to leave $w$ as a free parameter and maximize the likelihood (4) in both $Y_p$ and $w$—that is, we estimate $w$ from the sample itself. For each choice of $(Y_p, w)$ we compute the likelihood $\mathcal{L}(Y_p, w)$; the maximum of the likelihood gives the best values of these parameters. The relative likelihood of other choices then allow us to derive constraints on $Y_p$ and $w$. This can be illustrated in contour plots showing the equal likelihood contours representing 1, 2 and 3 standard deviations. These are determined by comparing the log of the likelihood to its peak value

$$\ln \mathcal{L} = \ln \mathcal{L}_{\text{peak}} - s^2/2$$

for $s$ standard deviations. The contours in the figures represent 1, 2, and 3 standard deviations from the peak value as determined by (5). If a large number of points contribute, the $s = 2$ contour translates in the usual way to approximately 95% confidence.

As an illustrative exercise it is useful first to consider a sample of just one galaxy, the lowest metallicity galaxy I Zw 18 where $Y$ has been measured independently in two distinct H$_{\text{II}}$ regions, yielding an average helium abundance $^1 \left< Y_1, Y_2 \right> = 0.230 \pm 0.004$. Because the measured values of $Y_p$ for the two H$_{\text{II}}$ regions in I Zw 18 are consistent with each other, the likelihood function (4) is peaked at $w = 0$ and prefers a value $Y_p = 0.230 \pm 0.004$, equal to the average of the two points (see figure 1a).

The full power of the method becomes clearer when we employ a larger statistical sample. In figures 1b-1e, we show the effect of progressively adding more $^4\text{He}$ data to our sample. The data is ordered by metallicity so that for example, the 11 point set includes the 11 extragalactic H$_{\text{II}}$ regions with the lowest values of O/H. Except for sample selection, the oxygen abundance is not explicitly utilized. We show the likelihood distributions up to what was described Olive, Skillman, and Steigman (1997) as set C corresponding to 32 data

$^1$Recently, the smallness of the error bars associated with one of the two H$_{\text{II}}$ regions in I Zw 18 has been questioned due to underlying stellar absorption (Skillman, Terlevich & Terlevich 1997). Of course, too much weight should never be given to any one single observation.
points. In this case, the linear extrapolation with respect to O/H gave $Y_p = 0.230 \pm 0.003$. The cutoffs for the various data sets considered here correspond to O/H <35, 45, 60, and $85 \times 10^{-6}$ for the 7, 11, 18, and 32 point sets respectively.\(^2\)

As soon as one goes beyond a 2-point data set, the dispersion in the data (normally associated with enrichment correlated with metallicity) produces a likelihood distribution which is peaked at a non-zero value for $w$— yielding an estimate of the stellar helium enrichment of the sample. Consider for example the 11-point data set. The peak of the distribution occurs at $w = 0.017$ and at that width, $Y_p = 0.228 \pm 0.003$. The individual helium abundances in this sample range from 0.225 to 0.251, with a weighted mean 0.237 ± 0.002. The value for $Y_p$ determined by the likelihood distribution is significantly below the mean value of the data. This bias is familiar from studies of the Malmquist effect: the best estimate of the distribution from which points are drawn is in general quite different from the distribution of the values for the points themselves. In the present context it is the asymmetry of the prior which leads to the apparent bias.

It is interesting to note that at the $2\sigma$ level, the data are consistent with a single primordial value with no subsequent enrichment ($w = 0$). This corresponds to a conservative $2\sigma$ upper limit, $Y_p \leq 0.240$. As the number of data points in the sample increase, the peak of the likelihood distribution shifts to higher values of $Y_p$. The $2\sigma$ upper limit however remains at $w = 0$, and its value is relatively insensitive to the data set. These results are summarized in table 1.

The most important arbitrary step in the above analysis is in the choice of prior, and

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\(^2\)The “full” data set of 62 points is not used here. Indeed, when one tries to use this data set, with O/H as high as $145 \times 10^{-6}$, the correlation between $Y$ and O/H cannot be neglected. Even in the 32 point set considered here, the correlation is statistically significant. Notice in table 1 that the likely value of $w$ increases from the 18 to the 32 point set; larger samples contain more information on enrichment but obscure the information on $Y_p$.  

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to calibrate its effect we consider a range of physically plausible possibilities. In fact the top-hat function is already quite conservative towards allowing the largest possible upper limits on $Y_p$—that is, since very low $Y$ systems in the present Universe, even in the current samples of low metal galaxies, are quite rare, it appears that stellar helium enrichment is seldom very small. We could however try to bias the result by choosing a different shape biased \textit{a priori} towards smaller (or larger) nonprimordial enrichments. Consider for example a sawtooth function prior, $P(Y_{iT}|Y_p) = (Y_p + w - Y_{iT})/w$ for $Y_{iT}$ between $Y_p$ and $Y_p + w$ and 0 otherwise. This stipulates \textit{a priori} that it is most likely that there has been very little helium enrichment, and no enrichment greater than $w$, thus $P$ rises abruptly to some value at $Y_{iT} = Y_p$, linearly decreasing for $Y_p \leq Y_{iT} \leq Y_p + w$, and zero for $Y_{iT} \geq Y_p + w$. We will refer to this prior as the negative bias (named for the slope with respect to $Y_{iT}$). Alternatively, we might choose $P(Y_{iT}|Y_p) = (Y_{iT} - Y_p)/w$ for $Y_{iT}$ between $Y_p$ and $Y_p + w$ and 0 otherwise. This assigns a greater probability for larger helium enrichment but still no enrichment greater than $w$; thus $P$ rises linearly from $Y_p$ to some value at $Y_{iT} = Y_p + w$. This is indeed the case for typical galaxy samples, where low enrichment systems are hard to find, so this is perhaps the more realistic prior. We refer to this prior as the positive bias. Taken together, these priors span the range of reasonable possibilities.

Likelihood contours for these two additional choices for priors are shown in figures 2 and 3 for the same subsamples discussed above. For all three priors, and for all the samples considered, the goodness of fit of the model— the effective reduced $\chi^2$—is less than unity, reflecting that the model is an adequate description of the data (and that the random observational errors have typically been overestimated.) The differences in the parameter estimates from the top-hat case are small, reflecting the fact that the most critical assumption affecting the results is the one we are most confident in, namely the assignment of $P(Y_{iT}|Y_p) = 0$ for $Y_{iT} \leq Y_p$. As is evident from the figures, the likelihood contours are shifted to the right for the negative bias prior, yielding a higher value for $Y_p$, and to the left for the positive bias, favoring a lower $Y_p$. However, the 2$\sigma$ upper limit is found to be nearly independent of our choice of prior. These results are summarized in tables 2 and 3.
III. CONCLUSIONS

The significant detection of nonzero $w$ reveals that even in these samples of metal-poor galaxies, there is evidence for a spread in $Y$ produced by stars. The best fit to the data is for values of the primordial abundance in the range $0.221 \leq Y_p \leq 0.236$, depending on the specific model and subsample. A $2\sigma$ upper limit, $Y_p \leq 0.243$, holds for all cases; this bound is virtually independent of the stellar helium enrichment model. There are enough data points entering so that even if individual errors are nongaussian, this limit corresponds approximately to a 95% confidence level.

The statistical evidence from this sample indicates that the internally estimated errors are if anything overly conservative, so the quoted limits are also generously estimated. The only reasonable way to reconcile the data with a higher value of $Y_p$ would be some systematic error in common among all the points— that is, an in-common mistake biasing the results of all the measurements. Of course, errors of this kind cannot be corrected by any purely statistical technique.

Even with quite simple and conservative assumptions, our analysis yields a final limit which is sufficiently precise to overconstrain the Big Bang picture when combined with other data. Our limit of $Y_p \leq 0.243$ corresponds to a limit on the baryon/photon ratio $\eta \leq 3.5 \times 10^{-10}$ and a predicted deuterium abundance $D/H \geq 6.2 \times 10^{-5}$, in conflict with some recent claims (e.g. Tytler et al. 1996) but in accord with others (e.g. Songaila et al. 1996). It is also consistent with BBN predictions of the Li abundance (Fields & Olive 1996, Fields et al. 1996). For the observed microwave background temperature, the baryon density is predicted to be $\Omega_b h^2 \leq 0.013$, which begins to conflict with some recent estimates based on models of quasar Lyman-α absorption (e.g., Rauch et al. 1997, Weinberg et al. 1997), but accords with other estimates of baryon density (e.g., Hogan 1997ab). An overall concordance of the Big Bang picture is certainly possible but will depend on which of these datasets “gives way”. It is clear that the reliability of $Y_p$ estimates would improve significantly with comprehensive studies of even one more region similar to I Zw 18.
We would like to thank E. Skillman and G. Steigman for helpful conversations. We are grateful to the Institute for Nuclear Theory, Seattle, funded by DOE, for sponsoring the 1996 workshop on Nucleosynthesis in the Big Bang, Stars, and Supernovae, where this work was started. This work was also supported at the University of Washington by NASA and NSF, and at the University of Minnesota in part by DOE grant DE-FG02-94ER-40823.
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Tables

Table 1: Likeliest values and limits for the top-hat prior

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<thead>
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<th># Regions</th>
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<th>$Y_p$</th>
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Table 2: Likeliest values and limits for the negative bias

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Table 3: Likeliest values and limits for the positive bias

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<td>32</td>
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Figure 1: Likelihood function showing 1σ, 2σ and 3σ contours in the \((Y_p, w)\) plane, for a top-hat prior of width \(w\). The ′+′s indicate the peaks of the likelihood functions. Results for different subsamples are shown in panels 1a-1e, starting with the 2 points of I Zw 18 and ending with the 32 lowest metal points.

Figure 2: Same as Figure 1, for the negative bias prior.

Figure 3: Same as Figure 1, for the positive bias prior.