Abstract

After a summary of a recently proposed new type of instant form of dynamics (the Wigner-covariant rest-frame instant form), the reduced Hamilton equations in the covariant rest-frame Coulomb gauge for the isolated system of N scalar particles with pseudoclassical Grassmann-valued electric charges plus the electromagnetic field are studied. The Lienard-Wiechert potentials of the particles are evaluated and it is shown how the causality problems of the Abraham-Lorentz-Dirac equation are solved at the pseudoclassical level.
Then, the covariant rest-frame description of scalar electrodynamics is given. Applying to it the Feshbach-Villars formalism, the connection with the particle plus electromagnetic field system is found.

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I. INTRODUCTION

In a recent paper [1] a new type of instant form of dynamics [2] was introduced with special Wigner-covariance properties and was named “covariant rest-frame instant form”. The twofold motivations which led to its discovery were the problem of understanding the role of relative times in the description of N relativistic scalar particles [whose phase space coordinates are $x_i^\mu(\tau), p_i^\mu(\tau), i=1,\ldots,N$ with first class constraints $p_i^2 - m_i^2 \approx 0$ [see Ref. [1] and the references quoted there for the solution of this problem] and the need of a formulation already adapted to the transition from special to general relativity.

The N particle system (starting with N free particle for the sake of simplicity) was reformulated on spacelike hypersurfaces $\Sigma_\tau$, all diffeomorphic to a given one $\Sigma$, foliating Minkowski spacetime ($\tau$ is a scalar parameter labelling the leaves of the foliation) following Dirac’s approach to parametrized field theory [3], subsequently extended to curved spacetimes by Kuchar [4]. In this way one adds an infinite number of configuration variables, the coordinates $z^\mu(\tau, \vec{\sigma})$ of the points of $\Sigma_\tau$ [$\vec{\sigma}$ are Lorentz-scalar curvilinear coordinates on the abstract $\Sigma$ embedded in Minkowski spacetime as $\Sigma_\tau$]. The position in $\Sigma_\tau$ of a particle traveling on a timelike worldline $\gamma_i$ is determined only by the 3 Lorentz-scalar numbers $\vec{\sigma} = \vec{\eta}_i(\tau)$ determining the intersection of $\gamma_i$ with $\Sigma_\tau$: $x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$. But this means that the constraints $p_i^2 - m_i^2 \approx 0$ have been solved in this description, so that $p_i^\mu \approx \pm \sqrt{m_i^2 + \vec{p}_i^2}$ and all the particle time variables $x_i^\alpha(\tau)$ have been replaced by the $\tau$-value identifying $\Sigma_\tau$ [namely all relative times are put equal to zero from the beginning in a covariant way]. Therefore, in this 1-time description every particle has a well defined sign $\eta_i = \pm 1$ of the energy and we cannot describe simultaneously both times like in the N-times description.

The standard Lagrangian of N free particles was rewritten in terms of $\vec{\eta}_i(\tau), \vec{\eta}_i'(\tau) = \frac{d}{d\tau} \vec{\eta}_i(\tau)$ and of the metric induced on $\Sigma_\tau$ by the Minkowski metric $\eta_{\mu\nu}$, namely of $g_{\hat{A}\hat{B}}(\tau, \vec{\sigma}) = z^\mu_{\hat{A}}(\tau, \vec{\sigma})\eta_{\mu\nu}z^\nu_{\hat{B}}(\tau, \vec{\sigma})$, where $z^\mu_{\hat{A}}(\tau, \vec{\sigma}) = \frac{\partial}{\partial \sigma^\hat{A}} z^\mu(\tau, \vec{\sigma})$ [$\sigma^{\hat{A}} = (\sigma^0 = \tau; \vec{\sigma} = \{\sigma^r\})$]. It turns out that there are 4 first class constraints determining the momentum $\rho_{\mu}(\tau, \vec{\sigma})$, conjugate to $z^\mu(\tau, \vec{\sigma})$, in terms of the momenta $\vec{\kappa}_i(\tau)$ conjugate to the $\vec{\eta}_i(\tau)$’s and of the vectors
normal and tangent to $\Sigma_\tau$ [all functions of the $z_\mu^\nu(\tau, \vec{\sigma})$]: the component of $\rho_\mu(\tau, \vec{\sigma})$ along the normal is the particle energy-density on $\Sigma_\tau$, while the components tangent to $\Sigma_\tau$ are the components of the particle momentum density on $\Sigma_\tau$. These 4 first class constraints say that the description of the N particles is independent from the choice of the foliation. Therefore, in special relativity we can get a simpler description by restricting ourselves to foliations whose leaves are spacelike hyperplanes of Minkowski spacetime. Finally, if we select all particle configurations whose total 4-momentum is timelike (they are dense in the space of all possible configurations), we can restrict ourselves to the special foliation whose hyperplanes are orthogonal to the total 4-momentum: these hyperplanes, named Wigner hyperplanes $\Sigma_{w,\tau}$, are intrinsically determined by the physical system itself (in this case N free scalar particles). It is shown in Ref. [1], that at this stage all the degrees of freedom $z_\mu^\nu(\tau, \vec{\sigma}), \rho_\mu(\tau, \vec{\sigma})$, have disappeared except for the canonical coordinates $\tilde{x}_s^\mu(\tau), p_s^\mu$, of a point. While $p_s^\mu$ is a timelike 4-vector (playing the role of the total 4-momentum) orthogonal to $\Sigma_{w,\tau}$ with $p_s^2 \approx$ squared invariant mass of the system due to the constraints, $\tilde{x}_s^\mu(\tau)$ is not a 4-vector. It describes the canonical relativistic center of mass of the system: it is the classical analogue of the Newton-Wigner position operator and, like it, it is covariant only under the little group of timelike Poincaré orbits (the Euclidean group).

As shown in Ref. [1], the restriction to Wigner hyperplanes forces the Lorentz-scalar 3-vectors $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)$, to become Wigner spin-1 3-vectors [they transform under induced Wigner rotations when one rotates the Wigner hyperplanes with Lorentz boosts in Minkowski spacetime], since in the gauge-fixing procedure use is made of the Wigner standard boost for timelike Poincaré orbits. Therefore, tensors on the Wigner hyperplane have Wigner covariance and the Wigner hyperplanes are intrisically Euclidean: an 1-time Wigner-covariant relativistic statistical mechanics can be developed on them as shown in Ref. [1].

Only 4 first class constraints remain on Wigner hyperplanes: i) one determines $\sqrt{p_s^2}$ in terms of the particle-system invariant mass; ii) the other 3 are $\vec{p}_W = \sum_{i=1}^{N} \vec{\kappa}_i(\tau) \approx 0$ [$\vec{p}_W$ must not be confused with the space part of $p_s^\mu$, which is arbitrary, being connected
with the chosen frame of reference of Minkowski spacetime in which Wigner hyperplanes are embedded], saying that the Wigner hyperplanes are the intrinsic rest frames after the separation of the center of mass motion in Minkowski spacetime. Since the Lorentz-scalar Minkowski-rest-frame time $T_s = p_s \cdot \tilde{x}_s / \sqrt{p_s^2}$ is the variable conjugate to $\sqrt{p_s^2}$, one can add the gauge-fixing $T_s - \tau \approx 0$ and obtain a description of the evolution in $T_s$ of the system by using the invariant mass as Hamiltonian [the other 6 degrees of freedom in $\tilde{x}_\mu, p^\mu_s$, are the 6 canonical coordinates of the free noncovariant center of mass]. If one adds the 3 gauge-fixings $\sum_{i=1}^{N} \eta_i(\tau) \approx 0$, one can reduce the 6N degrees of freedom $\bar{\eta}_i(\tau), \bar{\kappa}_i(\tau)$, to 3(N-1) pairs of relative variables [for general systems usually one does not know the form of the needed 3 gauge-fixings to be added]. In this way one gets the rest-frame instant form of dynamics on the Wigner hyperplanes and has the relativistic generalization of the Newtonian separation of the center-of-mass motion in phase space.

On the Wigner hypersurface one can introduce any kind of instantaneous action-at-a-distance interactions (without the complications of the N-times formalism) and to treat the problem of cluster decomposition in Newtonian terms.

Then in Ref. [1] the isolated system of N scalar particles with Grassmann-valued electric charges plus the electromagnetic field was studied in this way till the level of Wigner hyperplanes. One also found the Dirac observables with respect to the gauge transformations of the whole system and obtained the Wigner-covariant rest-frame version of the Coulomb gauge. In particular, this allows to extract from field theory the interparticle instantaneous Coulomb potential (which appears in the invariant mass as an additive term to the particle relativistic kinetic energies) and to regularize the Coulomb self-interactions due to the Grassmann character of the electric charges $Q_i$, which implies $Q_i^2 = 0$. However, the Hamilton equations of motion and their implications in the rest-frame instant form were not given. Moreover, even if there were some comments about the relation of charged particles with the Feshbach-Villars [5] formalism for charged Klein-Gordon fields with external electromagnetic fields, no clear connection was established.

In this paper we shall study the Hamilton equations on the Wigner hyperplane of the
isolated system of N charged scalar particles plus the electromagnetic field in the pseudoclassical case of Grassmann-valued electric charges of the particles. We shall find the rest-frame formulation of the Lienard-Wiechert potentials and the pseudoclassical regularization of the causality problems of the Abraham-Lorentz-Dirac equation. Each particle has its Lienard-Wiechert potential, but it does not directly produce radiation because $Q_i^2 = 0$; however, one gets a Larmor formula for the radiated energy containing terms $Q_i Q_j$, $i \neq j$, from the interference of the various Lienard-Wiechert potentials in wave zone and this is macroscopically satisfactory.

Then, we reformulate scalar electrodynamics, namely the isolated system of a complex Klein-Gordon field coupled to the electromagnetic field, on spacelike hypersurfaces and we obtain its rest-frame description on Wigner hyperplanes. Following Ref. [6], we give the Dirac observables with respect to the gauge transformations of the theory. Then, we evaluate the rest-frame reduced Hamilton equations and we apply to them the Feshbach-Villars formalism. Finally, we show that one can recover the previous constraints of charge particles plus the electromagnetic field from this treatment of scalar electrodynamics, if one makes a strong eikonal approximation which eliminates the mixing of positive and negative energy solutions of the Klein-Gordon theory, which is induced by effects that, in second quantization, are interpreted as pair production from vacuum polarization.

In Section II we remind some results of Ref. [1]. In Section III we evaluate the reduced Hamilton equations in the rest-frame instant form. In Section IV we find the Lienard-Wiechert potentials of the particles and we study their equations of motion, showing which is the pseudoclassical way out from the causality problems of the Abraham-Lorentz-Dirac equation. In Section V we study scalar electrodynamics on spacelike hypersurfaces and we find its rest-frame formulation; then we recast it in the Feshbach-Villars formalism and look for connections with the previous theory. Some final comments are put in the Conclusions.
II. PRELIMINARIES

In this Section we will introduce the background material from Ref. [1] needed in the description of physical systems on spacelike hypersurfaces, integrating it with the definitions needed to describe the isolated system of N scalar particles with pseudoclassical Grassmann-valued electric charges plus the electromagnetic field [1].

Let \( \{ \Sigma_\tau \} \) be a one-parameter family of spacelike hypersurfaces foliating Minkowski spacetime \( M^4 \) and giving a 3+1 decomposition of it. At fixed \( \tau \), let \( z^\mu(\tau, \vec{\sigma}) \) be the coordinates of the points on \( \Sigma_\tau \) in \( M^4 \), \( \{ \vec{\sigma} \} \) a system of coordinates on \( \Sigma_\tau \). If \( \sigma^\hat{A} = (\sigma^\tau, \vec{\sigma} = \{ \sigma^\hat{r} \}) \) [the notation \( \hat{A} = (\tau, \hat{r}) \) with \( \hat{r} = 1, 2, 3 \) will be used; note that \( \hat{A} = \tau \) and \( \hat{A} = \hat{r} = 1, 2, 3 \) are Lorentz-scalar indices] and \( \partial_\hat{A} = \partial/\partial\sigma^\hat{A} \), one can define the vierbeins

\[
z^\mu_\hat{A}(\tau, \vec{\sigma}) = \partial_\hat{A}z^\mu(\tau, \vec{\sigma}), \quad \partial_\hat{B}z^\mu_\hat{A} - \partial_\hat{A}z^\mu_\hat{B} = 0, \tag{1}
\]

so that the metric on \( \Sigma_\tau \) is

\[
g_{\hat{A}\hat{B}}(\tau, \vec{\sigma}) = z^\mu_\hat{A}(\tau, \vec{\sigma})\eta_{\mu\nu}z^\nu_\hat{B}(\tau, \vec{\sigma}), \quad g_{\tau\tau}(\tau, \vec{\sigma}) > 0,
\]

\[
g(\tau, \vec{\sigma}) = -\det | g_{\hat{A}\hat{B}}(\tau, \vec{\sigma}) | = (\det | z^\mu_\hat{A}(\tau, \vec{\sigma}) |)^2,
\]

\[
\gamma(\tau, \vec{\sigma}) = -\det | g_{\hat{r}\hat{s}}(\tau, \vec{\sigma}) |. \tag{2}
\]

If \( \gamma^{\hat{r}\hat{s}}(\tau, \vec{\sigma}) \) is the inverse of the 3-metric \( g_{\hat{r}\hat{s}}(\tau, \vec{\sigma}) \) \( [\gamma^{\hat{r}\hat{a}}(\tau, \vec{\sigma})g_{\hat{a}\hat{s}}(\tau, \vec{\sigma}) = \delta^\hat{r}_\hat{s}] \), the inverse \( g^{\hat{A}\hat{B}}(\tau, \vec{\sigma}) \) of \( g_{\hat{A}\hat{B}}(\tau, \vec{\sigma}) \) \( [g^{\hat{A}\hat{C}}(\tau, \vec{\sigma})g_{\hat{b}\hat{d}}(\tau, \vec{\sigma}) = \delta^\hat{A}_\hat{B}] \) is given by

\[
g^{\tau\tau}(\tau, \vec{\sigma}) = \gamma(\tau, \vec{\sigma})/g(\tau, \vec{\sigma}), \]

\[
g^{\tau\hat{r}}(\tau, \vec{\sigma}) = -[\gamma_{\hat{r}\hat{a}}\gamma^{\hat{a}\hat{r}}]/g(\tau, \vec{\sigma}), \]

\[
g^{\hat{r}\hat{s}}(\tau, \vec{\sigma}) = \gamma^{\hat{r}\hat{s}}(\tau, \vec{\sigma}) + \gamma_{\hat{r}\hat{a}}g_{\hat{a}\hat{b}}\gamma^{\hat{a}\hat{b}}\gamma^{\hat{b}\hat{s}}(\tau, \vec{\sigma}), \tag{3}
\]

so that \( 1 = g^{\tau\hat{C}}(\tau, \vec{\sigma})g_{\hat{C}\tau}(\tau, \vec{\sigma}) \) is equivalent to

\[
g(\tau, \vec{\sigma})/\gamma(\tau, \vec{\sigma}) = g_{\tau\tau}(\tau, \vec{\sigma}) - \gamma^{\hat{r}\hat{s}}(\tau, \vec{\sigma})g_{\tau\hat{r}}(\tau, \vec{\sigma})g_{\tau\hat{s}}(\tau, \vec{\sigma}). \tag{4}
\]

We have
\[ z^\mu_\nu(\tau, \sigma) = (\sqrt{\frac{g}{\gamma}} l^\mu + g_{\tau \tau} \gamma^{s s} z^\mu_\nu(\tau, \sigma)), \] (5)

and

\[ \eta^{\mu \nu} = z^\mu_A(\tau, \sigma)g^{AB}(\tau, \sigma)z^\nu_B(\tau, \sigma) = (l^\mu l^\nu + z^\mu_\tau \gamma^{s s} z^\nu_\tau)(\tau, \sigma), \] (6)

where

\[ l^\mu(\tau, \sigma) = \left( \frac{1}{\sqrt{\gamma}} \epsilon^\mu_{\alpha \beta \gamma} z^\alpha_\tau z^\beta_\tau z^\gamma_\tau \right)(\tau, \sigma), \]
\[ l^2(\tau, \sigma) = 1, \quad l^\mu(\tau, \sigma)z^\mu_\nu(\tau, \sigma) = 0, \] (7)

is the unit (future pointing) normal to \( \Sigma_\tau \) at \( z^\mu(\tau, \sigma) \).

For the volume element in Minkowski spacetime we have

\[ d^4z = z^\mu_\nu(\tau, \sigma)d\tau d^3\Sigma_\mu = \sqrt{\gamma(\tau, \sigma)d^3\sigma} \]
\[ = \sqrt{g(\tau, \sigma)d\tau d^3\sigma}. \] (8)

Let us remark that according to the geometrical approach of Ref. [4], one can use Eq.(5) in the form \( z^\mu_\nu(\tau, \sigma) = N(\tau, \sigma)l^\mu(\tau, \sigma) + N^\tau(\tau, \sigma)z^\mu_\tau(\tau, \sigma) \), where \( N = \sqrt{g/\gamma} = \sqrt{g_{\tau \tau} - \gamma^{s s}g_{\tau s}g_{\tau s}} \) and \( N^\tau = g_{s \tau} \gamma^{s s} \) are the standard lapse and shift functions, so that

\[ g_{\tau \tau} = N^2 + g_{s \tau} N^s N^s, g_{\tau s} = g_{s \tau} N^s, g_{s s} = N^{-2}, g^{s s} = \gamma^{s s} + \frac{N^s N^s}{N^2}, \]
\[ \frac{\partial}{\partial \sigma_\tau} = l^\mu \frac{\partial}{\partial \Sigma_\mu} + z^\mu_\tau \gamma^{s s} \frac{\partial}{\partial \Sigma_\tau}, \quad d^4z = N \sqrt{\gamma}d\tau d^3\sigma. \]

The rest frame form of a timelike fourvector \( p^\mu \) is \( \hat{p}^\mu = \eta \sqrt{\hat{p}^2(1; \hat{0})} = \eta^{\mu \nu} \eta \sqrt{\hat{p}^2}, \hat{p}^2 = p^2 \), where \( \eta = \text{sign} p^\mu \). The standard Wigner boost transforming \( \hat{p}^\mu \) into \( p^\mu \) is

\[ L^\mu_\nu(p, \hat{p}) = e^\mu_\nu(u(p)) = \eta^{\mu \nu} + 2P^\mu P^\nu \frac{(P^\mu + \hat{p}^\mu)(P^\nu + \hat{p}^\nu)}{P^\mu P^\nu + p^2} = \eta^{\mu \nu} + 2u^\mu(p)u^\nu(\hat{p}) - \frac{(u^\mu(p) + u^\mu(\hat{p}))(u^\nu(p) + u^\nu(\hat{p}))}{1 + u^\nu(p)}, \]

\[ \nu = 0 \quad \epsilon^\mu_0(u(p)) = u^\mu(p) = p^\mu / \eta \sqrt{p^2}, \]
\[ \nu = r \quad \epsilon^\mu_r(u(p)) = (-u_r(p); \delta^\mu_r - \frac{u^\nu(p)u_i(p)}{1 + u^\nu(p)}). \] (9)
The inverse of \( L_{\nu}(p, \hat{p}) \) is \( L_{\nu}(\hat{p}, p) \), the standard boost to the rest frame, defined by
\[
L_{\nu}(\hat{p}, p) = L_{\nu}(p, \hat{p}) = L_{\nu}(p, \hat{p})|_{\vec{p} \to -\vec{p}}.
\]
(10)

Therefore, we can define the following vierbeins [the \( \epsilon_{\mu}^{\nu}(u(p)) \)'s are also called polarization vectors; the indices r, s will be used for A=1,2,3 and \( \bar{\sigma} \) for A=0]

\[
\epsilon_{A}^{\mu}(u(p)) = L_{\mu}^{A}(p, \hat{p}),
\]
\[
\epsilon^{A}_{\mu}(u(p)) = L_{\mu}^{A}(\hat{p}, p) = \eta^{AB} \eta_{\mu \nu} \epsilon_{B}^{\nu}(u(p)),
\]
\[
\epsilon^{\sigma}_{\mu}(u(p)) = \eta_{\mu \nu} \epsilon_{\nu}^{\sigma}(u(p)) = u_{\mu}(p),
\]
\[
\epsilon_{\mu}^{r}(u(p)) = -\delta^{rs} \eta_{\mu \nu} \epsilon_{\nu}^{s}(u(p)) = (\delta^{rs} u_{s}(p); \delta^{r}_{j} - \delta^{rs} \delta^{j}_{s} \frac{u^{h}(p) u_{s}(p)}{1 + u^{o}(p)}),
\]
\[
\epsilon^{A}_{\sigma}(u(p)) = u_{A}(p),
\]
(11)

which satisfy

\[
\epsilon_{\mu}^{A}(u(p)) \epsilon_{A}^{\nu}(u(p)) = \eta_{\mu \nu},
\]
\[
\epsilon^{A}_{\nu}(u(p)) \epsilon^{A}_{\mu}(u(p)) = \eta^{A}_{B},
\]
\[
\eta^{\mu \nu} = \epsilon_{A}^{\mu}(u(p)) \eta^{AB} \epsilon_{B}^{\nu}(u(p)) = u_{\mu}(p) u_{\nu}(p) - \sum_{r=1}^{3} \epsilon_{r}^{\mu}(u(p)) \epsilon_{r}^{\nu}(u(p)),
\]
\[
\eta_{AB} = \epsilon_{A}^{\mu}(u(p)) \eta_{\mu \nu} \epsilon_{B}^{\nu}(u(p)),
\]
\[
p_{\alpha} \frac{\partial}{\partial p_{\alpha}} \epsilon_{A}^{\mu}(u(p)) = p_{\alpha} \frac{\partial}{\partial p_{\alpha}} \epsilon^{A}_{\mu}(u(p)) = 0.
\]
(12)

The Wigner rotation corresponding to the Lorentz transformation \( \Lambda \) is

\[
R_{\mu \nu}(\Lambda, p) = [L(\hat{p}, p) \Lambda^{-1} L(\Lambda p, \hat{p})]_{\mu}^{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & R_{i j}(\Lambda, p) \end{pmatrix},
\]

\[
R_{i j}(\Lambda, p) = (\Lambda^{-1})^{i}_{j} - \frac{(\Lambda^{-1})^{i}_{\rho} p_{\rho}(\Lambda^{-1})^{\beta}_{\jmath} \eta^{\rho \jmath}}{p^{\beta} + \eta \sqrt{p^{2}}} - \frac{p^{i}}{p^{\rho} + \eta \sqrt{p^{2}}} [(\Lambda^{-1})^{\alpha}_{\rho} \eta^{\beta}_{\jmath} p_{\rho}(\Lambda^{-1})^{\beta}_{\jmath} - (\Lambda^{-1})^{\alpha}_{\rho} - 1] p_{\rho}(\Lambda^{-1})^{\beta}_{\jmath} \eta^{\rho \jmath}.
\]
(13)

The polarization vectors transform under the Poincaré transformations \((a, \Lambda)\) in the following way
\[ e^\nu_r(u(Ap)) = (R^{-1})_{\nu}^\mu \Lambda^\mu_\nu e^\nu_s(u(p)). \] (14)

In Ref. [1], the system of N charged scalar particles was considered. As said in the Introduction, on the hypersurface \( \Sigma \), the particles are described by variables \( \vec{\eta}_i(\tau) \) such that \( x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau)) \). The electric charge of each particle is described in a pseudoclassical way by means of a pair of complex conjugate Grassmann variables [8] \( \theta_i(\tau), \theta_i^*(\tau) \) satisfying

\[ [I_i = I_i^* = \theta_i^* \theta_i \text{ is the generator of the } U_{em}(1) \text{ group of particle } i] \]

\[
\theta_i^2 = \theta_i^{*2} = 0, \quad \theta_i \theta_i^* + \theta_i^* \theta_i = 0,
\]

\[
\theta_i \theta_j = \theta_j \theta_i, \quad \theta_i \theta_j^* = \theta_j^* \theta_i, \quad \theta_i^* \theta_j^* = \theta_j^* \theta_i^*, \quad i \neq j. \quad (15)
\]

This amounts to assume that the electric charges \( Q_i = e_i \theta_i^* \theta_i \) are quantized with levels 0 and \( e_i \) [8].

On the hypersurface \( \Sigma \), we describe the electromagnetic potential and field strength with Lorentz-scalar variables \( A_A(\tau, \vec{\sigma}) \) and \( F_{AB}(\tau, \vec{\sigma}) \) respectively, defined by

\[
A_A(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma}) A_\mu(z(\tau, \vec{\sigma})),
\]

\[
F_{AB}(\tau, \vec{\sigma}) = \partial_A A_B - \partial_B A_A = z^\mu_A(\tau, \vec{\sigma}) z^\nu_B(\tau, \vec{\sigma}) F_{\mu\nu}(z(\tau, \vec{\sigma})). \quad (16)
\]

The system is described by the action [1]

\[
S = \int dt d^3 \sigma L(\tau, \vec{\sigma}) = \int dt L(\tau),
\]

\[
L(\tau) = \int d^3 \sigma \mathcal{L}(\tau, \vec{\sigma}),
\]

\[
\mathcal{L}(\tau, \vec{\sigma}) = \frac{i}{2} \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))[\theta_i^* (\tau) \dot{\theta}_i (\tau) - \dot{\theta}_i^* (\tau) \theta_i (\tau)] -
\]

\[
- \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) [\eta_i m_i \sqrt{g_{\tau\tau}(\tau, \vec{\sigma}) + 2 g_{\tau\vec{r}}(\tau, \vec{\sigma}) \eta_i^\mu(\tau) + g_{\vec{r}\vec{r}}(\tau, \vec{\sigma}) \eta_i^\mu(\tau) \eta_i^\mu(\tau) +
\]

\[
+ e_i \theta_i^* (\tau) \theta_i (\tau) (A_{\tau}(\tau, \vec{\sigma}) + A_{\vec{r}}(\tau, \vec{\sigma}) \eta_i^\mu(\tau))] -
\]

\[
- \frac{1}{4} \sqrt{g(\tau, \vec{\sigma}) g_{A\bar{C}}(\tau, \vec{\sigma}) g_{B\bar{D}}(\tau, \vec{\sigma}) F_{AB}(\tau, \vec{\sigma}) F_{CD}(\tau, \vec{\sigma})}, \quad (17)
\]

where the configuration variables are \( z^\mu(\tau, \vec{\sigma}), A_A(\tau, \vec{\sigma}), \eta_i(\tau), \theta_i(\tau) \) and \( \theta_i^* (\tau), i=1,..,N \). We have
\[
-\frac{1}{4}\sqrt{g}g^{AC}g^{BD}\sum a F_{\alpha\beta} F_{\alpha\beta} = \\
-\frac{1}{4}\sqrt{g} \sum a[2(g^{r\tau} g^{s\tilde{\tau}} - g^{r\tilde{s}} g^{s\tau}) F_{\alpha r\tau} F_{\alpha s\tilde{\tau}} + 4g^{r\tilde{s}} g^{s\tau} F_{\alpha r\tilde{\tau}} F_{\alpha s\tau} + g^{r\tilde{\tau}} g^{s\tilde{\tau}} F_{\alpha r\tilde{\tau}} F_{\alpha s\tilde{\tau}}] = \\
-\sqrt{g} \sum a[\frac{1}{2} \sqrt{g} F_{\alpha r\tau} \gamma^{r\tilde{s}} F_{\alpha s\tilde{\tau}} - \sqrt{g} g_{r\tau\tilde{\tau}} \gamma^{r\tilde{s}} F_{\alpha r\tilde{\tau}} F_{\alpha s\tilde{\tau}} + \frac{1}{4} \sqrt{g} \gamma^{r\tilde{s}} F_{\alpha r\tilde{\tau}} F_{\alpha s\tilde{\tau}} (\gamma^{r\tilde{s}} + 2\frac{\sqrt{g}}{2}(F_{r\tau} - N^\alpha F_{\alpha r\tau})(F_{r\tilde{\tau}} - N^\alpha F_{\alpha r\tilde{\tau}}) - \frac{N}{4} \frac{\sqrt{g}}{2}(F_{r\tilde{\tau}} - N^\alpha F_{\alpha r\tau})(F_{r\tilde{\tau}} - N^\alpha F_{\alpha r\tilde{\tau}}) - \frac{N}{4} \frac{\sqrt{g}}{2}(F_{r\tilde{\tau}} - N^\alpha F_{\alpha r\tau})(F_{r\tilde{\tau}} - N^\alpha F_{\alpha r\tilde{\tau}})].
\]

The action is invariant under separate \(\tau\)- and \(\tilde{\sigma}\)-reparametrizations, since \(A_\tau(\tau, \tilde{\sigma})\) transforms as a \(\tau\)-derivative; moreover, it is invariant under the odd phase transformations \(\delta \theta_i \rightarrow i\alpha \theta_i\), generated by the \(I_i\)'s.

The canonical momenta are \([E_\tau = F_{r\tau}\) and \(B_\tau = \frac{1}{2}\epsilon_{\tau \tilde{\sigma} \tau} F_{\tau \tilde{\sigma}} (\epsilon_{\tau \tilde{\sigma} \tau} = \epsilon_{\tilde{\sigma} \tau \tau})\) are the electric and magnetic fields respectively; for \(g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}\) one gets \(\pi^\tau = -E_\tau = E^{\tau}\)

\[
\rho_\mu(\tau, \tilde{\sigma}) = -\frac{\partial L(\tau, \tilde{\sigma})}{\partial z_\mu(\tau, \tilde{\sigma})} = \sum_{i=1}^{N} \delta^3(\tilde{\sigma} - \bar{\eta}_i(\tau))\eta_i m_i \\
\]

\[
\frac{z_{\tau \mu}(\tau, \tilde{\sigma}) + z_{\tilde{\tau} \mu}(\tau, \tilde{\sigma})\eta_i(\tau)}{\sqrt{g(\tau, \tilde{\sigma})}} + \frac{\sqrt{g(\tau, \tilde{\sigma})}}{4} [g^{\tau \tau} z_{\tau \mu}(\tau, \tilde{\sigma}) + g^{\tilde{\tau} \tilde{\tau}} z_{\tilde{\tau} \mu}(\tau, \tilde{\sigma})] g^{C C}(\tau, \tilde{\sigma}) g^{B D}(\tau, \tilde{\sigma}) F_{\alpha \beta}(\tau, \tilde{\sigma}) F_{\bar{C} \bar{D}}(\tau, \tilde{\sigma}) - \\
-2[z_{\tau \mu}(\tau, \tilde{\sigma}) g^{\bar{A} C} g^{D D} + g^{A C} g^{B D}(\tau, \tilde{\sigma})] F_{\alpha \beta}(\tau, \tilde{\sigma}) F_{\bar{C} \bar{D}}(\tau, \tilde{\sigma}) = \\
= [(\rho^\nu z_\nu) l_\mu + (\rho^\nu z_\nu) \gamma^{r\tilde{s}} z_{\beta \mu}] (\tau, \tilde{\sigma}),
\]

\[
\pi^\tau(\tau, \tilde{\sigma}) = \frac{\partial L}{\partial \dot{\tau}_A(\tau, \tilde{\sigma})} = 0, \\
\pi^\tilde{\tau}(\tau, \tilde{\sigma}) = \frac{\partial L}{\partial \dot{\tilde{\tau}}_A(\tau, \tilde{\sigma})} = -\frac{\gamma(\tau, \tilde{\sigma})}{g(\tau, \tilde{\sigma})} \gamma^{\tilde{r}\tilde{s}}(\tau, \tilde{\sigma}) (F_{r\tilde{s}} - g_{r\tau} \gamma^{\tilde{r}\tilde{u}} F_{\tilde{u} \tilde{s}})(\tau, \tilde{\sigma}) = \\
= \frac{\gamma(\tau, \tilde{\sigma})}{g(\tau, \tilde{\sigma})} \gamma^{\tilde{r}\tilde{s}}(\tau, \tilde{\sigma}) (E_{\tilde{s}}(\tau, \tilde{\sigma}) + g_{r\nu}(\tau, \tilde{\sigma}) \gamma^{\tilde{r}\tilde{u}}(\tau, \tilde{\sigma}) \epsilon_{\tilde{u} \tilde{s} \tilde{\sigma}} B_\tau(\tau, \tilde{\sigma}),
\]

\[
\kappa_{\nu \tau}(\tau) = -\frac{\partial L(\tau)}{\partial \dot{\eta}_i(\tau)} = \\
= \eta_i m_i \frac{g_{\tau\tau}(\tau, \bar{\eta}_i(\tau)) + g_{r\tilde{\tau}}(\tau, \bar{\eta}_i(\tau))\tilde{\eta}_i(\tau)}{\sqrt{g_{\tau\tau}(\tau, \bar{\eta}_i(\tau)) + 2g_{r\tau}(\tau, \bar{\eta}_i(\tau))\tilde{\eta}_i(\tau) + g_{\tilde{\tau}\tilde{\tau}}(\tau, \bar{\eta}_i(\tau))\tilde{\eta}_i(\tau)\tilde{\eta}_i(\tau)} + \epsilon_i \theta_i^\tau(\tau) \theta_i(\tau) A_\tau(\tau, \bar{\eta}_i(\tau)),
\]
\[
\pi_{\theta_i}(\tau) = \frac{\partial L(\tau)}{\partial \dot{\theta}_i(\tau)} = -\frac{i}{2} \theta^*_i(\tau)
\]
\[
\pi_{\theta^*_i}(\tau) = \frac{\partial L(\tau)}{\partial \dot{\theta}^*_i(\tau)} = -\frac{i}{2} \theta_i(\tau),
\]
and the following Poisson brackets are assumed
\[
\{z^\mu(\tau, \bar{\sigma}), \rho_\nu(\tau, \bar{\sigma}')\} = -\eta^\mu_\nu \delta^3(\bar{\sigma} - \bar{\sigma}')
\]
\[
\{A_A(\tau, \bar{\sigma}), \pi^B(\tau, \bar{\sigma}')\} = \eta^B_A \delta^3(\bar{\sigma} - \bar{\sigma}')
\]
\[
\{\eta^i_\bar{\nu}(\tau), \kappa_{j\bar{s}}(\tau)\} = -\delta_{ij} \delta^\bar{s} \bar{\nu}
\]
\[
\{\theta_i(\tau), \pi_{\theta^*_j}(\tau)\} = -\delta_{ij},
\]
\[
\{\theta^*_i(\tau), \pi_{\theta^*_j}(\tau)\} = -\delta_{ij}.
\]

The Grassmann momenta give rise to the second class constraints \(\pi_{\theta_i} + \frac{i}{2} \theta^*_i \approx 0, \pi_{\theta^*_i} + \frac{i}{2} \theta_i \approx 0\) \([\{\pi_{\theta_i} + \frac{i}{2} \theta^*_i, \pi_{\theta^*_j} + \frac{i}{2} \theta_j\} = -i \delta_{ij}; \pi_{\theta_i} \text{ and } \pi_{\theta^*_i} \text{ are then eliminated with the help of Dirac brackets}
\]
\[
\{A, B\}^* = \{A, B\} - i[\{A, \pi_{\theta_i} + \frac{i}{2} \theta^*_i\}\{\pi_{\theta^*_i} + \frac{i}{2} \theta_i, B\} + \{A, \pi_{\theta^*_i} + \frac{i}{2} \theta_i\}\{\pi_{\theta_i} + \frac{i}{2} \theta^*_i, B\}]\]
so that the remaining Grassmann variables have the fundamental Dirac brackets [which we will still denote \(\{., .\}\) for the sake of simplicity]
\[
\{\theta_i(\tau), \theta_j(\tau)\} = \{\theta^*_i(\tau), \theta^*_j(\tau)\} = 0,
\]
\[
\{\theta_i(\tau), \theta^*_j(\tau)\} = -i \delta_{ij}.
\]

We obtain four primary constraints
\[
\mathcal{H}_\mu(\tau, \bar{\sigma}) = \rho_\mu(\tau, \bar{\sigma}) - l_\mu(\tau, \bar{\sigma})[T_{\tau\bar{\tau}}(\tau, \bar{\sigma}) + \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) \times \eta_i \sqrt{m_i^2 - \gamma^{i\bar{s}}(\tau, \bar{\sigma})[\kappa_{i\bar{s}}(\tau) - e_i \theta^*_i(\tau) \theta_i(\tau) A_{i\bar{s}}(\tau, \bar{\sigma})][\kappa_{i\bar{s}}(\tau) - e_i \theta^*_i(\tau) \theta_i(\tau) A_{i\bar{s}}(\tau, \bar{\sigma})]} - z_{i\mu}(\tau, \bar{\sigma}) \gamma^{i\bar{s}}(\tau, \bar{\sigma})\{-T_{\tau\bar{s}}(\tau, \bar{\sigma}) + \sum_{i=1}^{N} \delta^3(\bar{\sigma} - \bar{\eta}_i(\tau)) [\kappa_{i\bar{s}} - e_i \theta^*_i(\tau) \theta_i(\tau) A_{i\bar{s}}(\tau, \bar{\sigma})]\} \approx 0,
\]
\]
\[
12
\]
where
\[
T_{\tau\tau}(\tau, \vec{\sigma}) = -\frac{1}{2} \left( \frac{1}{\sqrt{\gamma}} \pi^{\hat{\tau}} g_{\hat{\tau}\hat{s}} \pi^{\hat{s}} - \frac{\sqrt{\gamma}}{2} \gamma^{\hat{\tau}\hat{s}} \gamma^{\hat{u}\hat{v}} F_{\hat{r\hat{u}}} F_{\hat{s\hat{v}}} \right)(\tau, \vec{\sigma}),
\]
\[
T_{\tau\hat{s}}(\tau, \vec{\sigma}) = -F_{\hat{s\hat{t}}} (\pi^{\hat{t}}(\tau, \vec{\sigma}) = -\epsilon_{\hat{s\hat{t}}\hat{u}} \pi^{\hat{u}}(\tau, \vec{\sigma}) B_{\hat{u}}(\tau, \vec{\sigma}) =
\frac{1}{2} \frac{\sqrt{\gamma}}{\gamma^{\hat{u}\hat{v}}} F_{\hat{r\hat{u}}} F_{\hat{s\hat{v}}} \right)(\tau, \vec{\sigma}),
\]
are the energy density and the Poynting vector respectively. We use the notation \((\vec{\pi} \times \vec{B})_{\hat{s}}\) because it is consistent with \(\epsilon_{\hat{s\hat{t}}\hat{u}} \pi^{\hat{u}} B_{\hat{u}}\) in the flat metric limit \(g_{\hat{A}\hat{B}} \rightarrow \eta_{\hat{A}\hat{B}}\); in this limit \(T_{\tau\tau} \rightarrow \frac{1}{2} (\vec{E}^2 + \vec{B}^2).\)

These constraints are first class \([1]\) and their existence implies that the description of the system is independent from the choice of the foliation.

Since the canonical Hamiltonian is (we assume boundary conditions for the electromagnetic potential such that all the surface terms can be neglected; see Ref. \([9]\))
\[
H_c = -\sum_{i=1}^{N} \kappa_{i\hat{t}}(\tau) \bar{\eta}_i^{\hat{t}}(\tau) + \int d^3\sigma [\pi^{\hat{A}}(\tau, \vec{\sigma}) \partial_\tau A^{\hat{A}}(\tau, \vec{\sigma}) - \rho_{\mu}(\tau, \vec{\sigma}) z^\mu(\tau, \vec{\sigma}) - L(\tau, \vec{\sigma})] =
\int d^3\sigma [\partial_\tau (\pi^{\hat{t}}(\tau, \vec{\sigma}) A_\tau(\tau, \vec{\sigma})) - A_\tau(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma})] = -\int d^3\sigma A_\tau(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma}),
\]
with
\[
\Gamma(\tau, \vec{\sigma}) = \partial_\tau \pi^{\hat{t}}(\tau, \vec{\sigma}) - \sum_{i=1}^{N} \epsilon_i \theta_i^{\hat{t}}(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)),
\]
we have the Dirac Hamiltonian \((\lambda^\mu(\tau, \vec{\sigma}) and \lambda_\tau(\tau, \vec{\sigma}) are Dirac’s multipliers))
\[
H_D = \int d^3\sigma \lambda^\mu(\tau, \vec{\sigma}) H^\mu(\tau, \vec{\sigma}) + \lambda_\tau(\tau, \vec{\sigma}) \pi^{\hat{t}}(\tau, \vec{\sigma}) - A_\tau(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma})].
\]

The Lorentz scalar constraint \(\pi^\tau(\tau, \vec{\sigma}) \approx 0\) is generated by the gauge invariance of \(S\); its time constancy will produce the only secondary constraint (Gauss law)
\[
\Gamma(\tau, \vec{\sigma}) \approx 0.
\]

The ten conserved Poincaré generators are
\[
P^\mu = p^\mu_s = \int d^3\sigma p^\mu(\tau, \vec{\sigma}),
\]
\[
J^{\mu\nu} = J^{\mu\nu}_s = \int d^3\sigma (z^\mu \rho^\nu - z^\nu \rho^\mu)(\tau, \vec{\sigma}).
\]
As shown in Ref. [1], one can restrict himself to foliations whose leaves are space-like hyperplanes with constant timelike normal \(l^\mu\), by adding the gauge-fixings \(z^\mu(\tau, \vec{\sigma}) \approx x_s^\mu(\tau) + b_s^\mu(\tau)\sigma^s\) [\(b_s^\mu = \{b_s^\mu = e^\mu_{\alpha\beta\gamma}b_1^\mu(\tau)b_2^\beta(\tau)b_3^\gamma(\tau); b_s^\mu(\tau)\} (\partial_\tau l^\mu = 0)\) is an orthonormal tetrad] and by going to Dirac brackets. In this way the hypersurface degrees of freedom \(z^\mu(\tau, \vec{\sigma}), \rho_{\mu}(\tau, \vec{\sigma})\), are reduced to 20 ones: i) 8 are \(x_s^\mu(\tau), p_s^\mu\); ii) 12 are the 6 independent pairs of canonical variables hidden in \(b_s^\mu\) and \(S_s^{\mu\nu} = J_s^{\mu\nu} - (x_s^\mu p_s^\nu - x_s^\nu p_s^\mu)\). The constraints \(\mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0\) are reduced to only 10 constraints: \(\tilde{\mathcal{H}}(\tau) = \int d^3\sigma \mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0\), \(\tilde{\mathcal{H}} = b_s^\mu(\tau) \int d^3\sigma \sigma^s\mathcal{H}^\mu(\tau, \vec{\sigma}) - b_s^\mu(\tau) \int d^3\sigma \sigma^s\mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0\).

Then, if one restricts himself to configurations with \(p_s^2 > 0\), one can make a further canonical reduction to the special foliation whose hyperplanes (the Wigner hyperplanes \(\Sigma_{w\tau}\) are orthogonal to \(p_s^\mu\) [namely \(l^\mu = p_s^\mu/\sqrt{p_s^2}\)]. This is achieved in two steps: i) firstly, one boosts at rest the variables \(b_s^\mu, S_s^{\mu\nu}\), with the standard Wigner boost \(L^\mu{}_{\nu}(p_s, 0)\) for timelike Poincaré orbits; ii) then, one adds the gauge-fixings \(b_s^\mu - L^\mu{}_{\nu}(p_s, 0)\) and goes to Dirac brackets. Only 4 pairs, \(\tilde{x}_s^\mu(\tau), \tilde{p}_s^\mu\), of canonical variables are associated with the Wigner hyperplane \(\Sigma_{w\tau}\) and the final constraints can be put in the form

\[
\mathcal{H}(\tau) = \frac{\eta_s\sqrt{p_s^2}}{\sum_{i=1}^N \eta_i\sqrt{m_i^2}} + \left[\bar{\kappa}_i(\tau) - \varepsilon_i\theta_i^*(\tau)\theta_i(\tau)\bar{A}(\tau, \vec{\eta}_i(\tau))\right]^2 + \\
- \frac{1}{2} \int d^3\sigma [\bar{\pi}^2(\tau, \vec{\sigma}) + \bar{B}^2(\tau, \vec{\sigma})] \approx 0,
\]

\[
\tilde{\mathcal{H}}(\tau) = \sum_{i=1}^N [\bar{\kappa}_i(\tau) - \varepsilon_i\theta_i^*(\tau)\theta_i(\tau)\bar{A}(\tau, \vec{\eta}_i(\tau)) + \\
+ \int d^3\sigma \bar{\pi}(\tau, \vec{\sigma}) \times \bar{B}(\tau, \vec{\sigma}) \approx 0,
\]

\[
\pi^r(\tau, \vec{\sigma}) \approx 0,
\]

\[
\Gamma(\tau, \vec{\sigma}) \approx 0.
\]

(29)

As said in the Introduction, \(\tilde{x}_s^\mu(\tau)\) is not a 4-vector [see Ref. [1] for its definition], \(A^\tau(\tau, \vec{\sigma})\) and \(\pi^r(\tau, \vec{\sigma})\) are Lorentz scalars, while \(\bar{A}(\tau, \vec{\sigma})\) and \(\bar{\pi}(\tau, \vec{\sigma})\) are Wigner spin-1 3-vectors.

As shown in Ref. [1] and as it is shown in Section V for scalar electrodynamics, one can eliminate the electromagnetic gauge degrees of freedom and reexpress everything in terms of the Dirac observables: i) \(\bar{A}_\perp(\tau, \vec{\sigma}), \bar{\pi}_\perp(\tau, \vec{\sigma}), \{\bar{A}_\perp^* (\tau, \vec{\sigma}), \bar{\pi}_\perp^* (\tau, \vec{\sigma})\} = -P_{\perp}^r(\vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}')\)
$P^r_s(\vec{\sigma}) = \delta^{rs} + \frac{\partial^r \partial^s}{\Delta}$, $\Delta = -\vec{\sigma}^2$] for the electromagnetic field; ii) $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau) = \vec{\kappa}_i(\tau) + Q_i \vec{A}_{\perp}(\tau, \vec{\sigma})$. $\vec{A}(\tau, \vec{\sigma})$ for the particles [they become dressed with a Coulomb cloud]; iii) $\vec{\theta}_i(\tau), \vec{\tilde{\theta}}_i(\tau)$, such that $Q_i = e_i \theta_i \theta_i = e_i \tilde{\theta}_i \tilde{\theta}_i$. This is the Wigner-covariant rest-frame Coulomb gauge. The reduced form of the 4 remaining constraints is

$$H(\tau) = \epsilon_s - \left\{ \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + (\vec{\kappa}_i(\tau) - Q_i \vec{A}_{\perp}(\tau, \vec{\eta}_i(\tau)))^2} + \right.$$

$$+ \sum_{i \neq j} \frac{Q_i Q_j}{4\pi | \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) |} + \int d^3 \sigma \frac{1}{2} [\vec{\tau}_{\perp}^2(\tau, \vec{\sigma}) + \vec{\tilde{B}}^2(\tau, \vec{\sigma})] \right\} \approx 0,$$

$$\bar{H}_p(\tau) = \vec{\kappa}_{\perp}(\tau) + \int d^3 \sigma [\vec{\pi}_{\perp} \times \vec{\tilde{B}}](\tau, \vec{\sigma}) \approx 0,$$

(30)

Note that in this way one has extracted the Coulomb potential from field theory and that the pseudoclassical property $Q_i^2 = 0$ regularizes the Coulomb self-energies.
III. REDUCED HAMILTON EQUATIONS FOR THE ELECTROMAGNETIC FIELD AND THE PARTICLES.

Let us add some more results regarding $N$ scalar electrically charged particles plus the electromagnetic field in the rest-frame instant form of the dynamics [1].

Let us first consider $N$ scalar free particles, which are described by the following 4 first class constraints on the Wigner hyperplane [on it the independent Hamiltonian variables are: $(T_s, \epsilon_s)$, $\vec{z}_s$, $(\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau))$, $i=1,...,N$; the other basis for the particles is $(\vec{\eta}_a(\tau), \vec{\kappa}_a(\tau))$, $a=1,..,N-1$]

$$\mathcal{H}(\tau) = \epsilon_s - \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + \kappa_i^2(\tau)} = \epsilon_s - H_{rel} \approx 0,$$

$$\mathcal{H}_p(\tau) = \sum_i \vec{\kappa}_i(\tau) \approx 0,$$

$$H_D = \lambda(\tau)\mathcal{H}(\tau) - \vec{\lambda}(\tau) \cdot \mathcal{H}_p(\tau). \quad (31)$$

If we add the gauge-fixing

$$\chi = T_s - \tau \approx 0, \quad (32)$$

its conservation in $\tau$ will imply $\lambda(\tau) = -1$. Going to Dirac brackets, we can eliminate the pair $(T_s, \epsilon_s)$. The Hamiltonian giving the evolution in the rest-frame time $\tau = T_s$ will be

$$H_R = H_{rel} - \vec{\lambda}(\tau) \cdot \mathcal{H}_p(\tau),$$

$$H_{rel} = \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + \kappa_i^2(\tau)}. \quad (33)$$

The associated Hamilton-Dirac equations are

$$\dot{\vec{\eta}}_i(\tau) \overset{\circ}{=} \frac{\vec{\kappa}_i(\tau)}{\eta_i \sqrt{m_i^2 + \kappa_i^2(\tau)}} - \vec{\lambda}(\tau)$$

$$\dot{\vec{\kappa}}_i(\tau) \overset{\circ}{=} 0,$$

$$\sum_{i=1}^{N} \vec{\kappa}_i(\tau) \overset{\circ}{=} 0. \quad (34)$$
The first line in invertible to give the momenta
\[ \vec{\kappa}_i(\tau) = \eta_i m_i \frac{\dot{\eta}_i(\tau) + \dot{\lambda}(\tau)}{\sqrt{1 - (\dot{\eta}_i(\tau) + \dot{\lambda}(\tau))^2}}, \] (35)
so that the associated Lagrangian is [\(\dot{\lambda}(\tau)\) are now Lagrange multipliers]
\[ L_R = \vec{\kappa}_i(\tau) \cdot \dot{\eta}_i(\tau) - H_R = -\sum_{i=1}^{N} \eta_i m_i \sqrt{1 - (\dot{\eta}_i(\tau) + \dot{\lambda}(\tau))^2} \] (36)
The Euler-Lagrange equations are
\[ \frac{d}{d\tau} \frac{\partial L_R}{\partial \dot{\eta}_i} = \frac{d}{d\tau} \left( \eta_i m_i \frac{\dot{\eta}_i(\tau) + \dot{\lambda}(\tau)}{\sqrt{1 - (\dot{\eta}_i(\tau) + \dot{\lambda}(\tau))^2}} \right) \overset{\circ}{=} 0 \]
\[ \frac{\partial L_R}{\partial \lambda} = \sum_i \eta_i m_i \frac{\dot{\eta}_i(\tau) + \dot{\lambda}(\tau)}{\sqrt{1 - (\dot{\eta}_i(\tau) + \dot{\lambda}(\tau))^2}} \overset{\circ}{=} 0. \] (37)
Let us remark that there is no equation of motion for the variables (\(\vec{z}_s, \vec{\kappa}_s\)) [the only left variables pertaining to the Wigner hyperplane]: this means that they are Jacobi data independent from \(\tau = T_s\).

In the free case one knows that by adding the gauge-fixings [its conservation imply \(\ddot{\lambda}(\tau) = 0\)]
\[ \ddot{\eta}_+(\tau) \approx 0, \] (38)
one can eliminate, by going to Dirac brackets, the variables (\(\vec{\eta}_+, \vec{\kappa}_+\)). Now the N particles are descibed by N-1 pairs of variables (\(\vec{\rho}_a, \vec{\pi}_a\)) relative to the center of mass and there is no constraint left. The Hamiltonian for the evolution in the rest-frame time \(\tau = T_s\) is
\[ H_{rel} = \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + (N \sum_{a=1}^{N-1} \dot{\gamma}_{ai} \dot{\vec{\pi}}_a(\tau))^2}, \] (39)
and the Hamilton equations are
\[ \dot{\vec{\rho}}_a(\tau) \overset{\circ}{=} \sum_{i=1}^{N} \sum_{b=1}^{N-1} \eta_i \sqrt{m_i^2 + (N \sum_{c=1}^{N-1} \dot{\gamma}_{ci} \dot{\vec{\pi}}_c(\tau))^2} \dot{\vec{\pi}}_b, \]
\[ \dot{\vec{\pi}}_a(\tau) \overset{\circ}{=} 0. \] (40)
However, this completely reduced description has the drawback that it is algebraically impossible to get explicitly the associated Lagrangian. Only in the case $N=2$ with $m_1 = m_2 = m$ one gets

\[ L_{rel} = -m \sqrt{4 - \dot{\rho}^2 (\tau)}, \quad \Rightarrow |\dot{\rho}(\tau)| \leq 2. \quad (41) \]

This result shows that there are kinematical restrictions on the relative velocities, which, not being absolute velocities, can exceed the velocity of light without violation of Einstein causality.

Let us now study the case of $N$ charged scalar particles plus the electromagnetic field on the Wigner hyperplane after the addition of the gauge-fixing $T_s - \tau \approx 0$. The Hamiltonian is [1]

\[ H_R = H_{rel} - \bar{\lambda}(\tau) \cdot \bar{\mathcal{H}}_p(\tau), \]

\[ H_{rel} = \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + (\tilde{k}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)))^2} + \]

\[ + \sum_{i \neq j} \frac{Q_i Q_j}{4 \pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)|} + \int d^3 \sigma \left[ \frac{1}{2} \tilde{\pi}_\perp^2(\tau, \sigma) + \tilde{B}^2(\tau, \sigma) \right], \]

\[ \bar{\mathcal{H}}_p(\tau) = \tilde{k}_+(\tau) + \int d^3 \sigma [\tilde{\pi}_\perp \times \tilde{\mathbf{B}}](\tau, \sigma) \approx 0, \quad (42) \]

where $\tilde{k}_i, \tilde{A}_\perp, \tilde{\pi}_\perp$ are the electromagnetic Dirac observables. The Grassmann-valued (Dirac observable) electric charges of the particles are $Q_i = e_i \tilde{\theta}_i^* \tilde{\theta}_i$, $Q_i^2 = 0$.

The Hamilton-Dirac equations are

\[ \dot{\tilde{\eta}}_i(\tau) \overset{\circ}{=} \frac{\tilde{k}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau))}{\eta_i \sqrt{m_i^2 + (\tilde{k}_i(\tau) - Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)))^2}} - \bar{\lambda}(\tau) \]

\[ \dot{\tilde{k}}_i(\tau) \overset{\circ}{=} - \sum_{k \neq i} \frac{Q_i Q_k (\tilde{\eta}_k(\tau) - \tilde{\eta}_i(\tau))}{4 \pi |\tilde{\eta}_k(\tau) - \tilde{\eta}_i(\tau)|^3} + \]

\[ + Q_i (\tilde{\eta}_i^u(\tau) + \lambda_u(\tau)) \frac{\partial}{\partial \tilde{\eta}_i} \tilde{A}_\perp^u(\tau, \tilde{\eta}_i(\tau))), \]

\[ \tilde{k}_+(\tau) + \int d^3 \sigma [\tilde{\pi}_\perp \times \tilde{\mathbf{B}}](\tau, \sigma) \approx 0. \quad (43) \]

In the second equation we have already used the following inversion of the first equation

\[ \tilde{k}_i(\tau) = \eta_i m_i \frac{\tilde{\eta}_i(\tau) + \bar{\lambda}(\tau)}{\sqrt{1 - (\tilde{\eta}_i(\tau) + \bar{\lambda}(\tau))^2}} + Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)). \quad (44) \]
The Hamilton-Dirac equations for the fields are

\[ \dot{A}_\perp(\tau, \vec{\sigma}) = -\pi_\perp(\tau, \vec{\sigma}) - [\vec{\lambda}(\tau) \cdot \vec{\partial}] A_\perp(\tau, \vec{\sigma}), \]

\[ \dot{\pi}_\perp(\tau, \vec{\sigma}) = \Delta A^r(\tau, \vec{\sigma}) - [\vec{\lambda}(\tau) \cdot \vec{\partial}] \pi_\perp(\tau, \vec{\sigma}) + \sum_i Q_i P^r_i(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)). \] (45)

The associated Lagrangian is

\[ L_R(\tau) = \frac{d}{d\tau} \left[ \frac{\dot{\eta}_i(\tau) + \vec{\lambda}(\tau)}{\sqrt{1 - (\dot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}} + Q_i A_\perp(\tau, \vec{\eta}_i(\tau)) \right] \]

\[ = \sum_{i=1}^N \frac{Q_i Q_j}{4\pi | \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) |} + \frac{1}{2} \sum_{i \neq j}^2 \frac{Q_i Q_j}{4\pi | \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) |} + \int d^3 \sigma \left[ \frac{(\dot{A}_\perp + [\vec{\lambda}(\tau) \cdot \vec{\partial}] \tilde{A}_\perp)^2}{2} - \frac{\tilde{B}^2}{2} \right](\tau, \vec{\sigma}) \] (46)

and its Euler-Lagrange equations are

\[ \frac{d}{d\tau} \left[ \frac{\dot{\eta}_i(\tau) + \vec{\lambda}(\tau)}{\sqrt{1 - (\dot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}} + Q_i A_\perp(\tau, \vec{\eta}_i(\tau)) \right] = \]

\[ = - \sum_{k \neq i} Q_i Q_k (\dot{\eta}_k(\tau) - \vec{\eta}_k(\tau)) + \frac{4\pi}{\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau) |} + Q_i (\dot{\eta}_i^u(\tau) + \vec{\lambda}_u(\tau) \frac{\partial}{\partial \vec{\eta}_i} A^u(\tau, \vec{\eta}_i(\tau))), \] (47)

\[ - \tilde{A}_\perp(\tau, \vec{\sigma}) - \frac{d}{d\tau} \left[ [\vec{\lambda}(\tau) \cdot \vec{\partial}] A^r(\tau, \vec{\sigma}) \right] = \]

\[ = \Delta \tilde{A}_\perp^r(\tau, \vec{\sigma}) + [\vec{\lambda}(\tau) \cdot \vec{\partial}] \{ \tilde{A}_\perp(\tau, \vec{\sigma}) + [\vec{\lambda}(\tau) \cdot \vec{\partial}] \tilde{A}_\perp^r(\tau, \vec{\sigma}) \} + \]

\[ - \sum_{i=1}^N Q_i P^r_i(\vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)), \] (48)

\[ \sum_{i=1}^N \left[ \frac{\dot{\eta}_i(\tau) + \vec{\lambda}(\tau)}{\sqrt{1 - (\dot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}} + Q_i \tilde{A}_\perp(\tau, \vec{\eta}_i(\tau)) \right] + \]

\[ + \int d^3 \sigma \sum_{r} \left[ (\vec{\partial} \tilde{A}^r_\perp) (\tilde{A}_\perp + [\vec{\lambda}(\tau) \cdot \vec{\partial}] \tilde{A}^r_\perp) \right](\tau, \vec{\sigma}) = 0. \] (49)

The Lagrangian expression for the invariant mass \( H_{rel} \) is

\[ E_{rel} = \sum_{i=1}^N \frac{\eta_i m_i}{\sqrt{1 - (\dot{\eta}_i(\tau) + \vec{\lambda}(\tau))^2}} + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi | \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) |} + \]

\[ + \int d^3 \sigma \frac{1}{2} [\tilde{E}^2_\perp(\tau, \vec{\sigma}) + \tilde{B}^2(\tau, \vec{\sigma})] = \text{const}. \] (50)
Eq. (47) may be rewritten as

\[
\frac{d}{d\tau} \left( \eta_i m_i \frac{\dot{\eta}_i(\tau) + \dot{\lambda}(\tau)}{\sqrt{1 + (\dot{\eta}_i(\tau) + \dot{\lambda}(\tau))^2}} \right) = -\sum_{k \neq i} \frac{Q_i Q_k (\dot{\eta}_k(\tau) - \dot{\eta}_k(\tau))}{4\pi |\eta_k(\tau) - \eta_k(\tau)|^3} + \\
+ Q_i \left[ \dot{E}_\bot(\tau, \dot{\eta}_i(\tau)) + (\dot{\eta}_i(\tau) + \dot{\lambda}(\tau)) \times \ddot{B}(\tau, \dot{\eta}_i(\tau)) \right],
\]

where the notation \( \ddot{E}_\bot^r = -\dot{A}_\bot^r - [\dot{\lambda}(\tau) \cdot \partial] \dot{A}_\bot^r = \ddot{\pi}_\bot^r \) has been introduced.

Let us remark that Eqs. (51) and (48) are the rest-frame analogues of the usual equations for charged particles in an external electromagnetic field and of the electromagnetic field with external particle sources; however, now, both particles and electromagnetic field are dynamical. Eq. (49) defines the rest frame by using the total (Wigner spin 1) 3-momentum of the isolated system formed by the particles plus the electromagnetic field. Eq. (50) gives the constant invariant mass of the isolated system: the electromagnetic self-energy of the particles has been regularized by the Grassmann-valued electric charges \( Q_i^2 = 0 \) so that the invariant mass is finite.

Let us now consider the gauge \( \ddot{\lambda}(\tau) = 0 \) [this result would be consistent with the addition of the unknown gauge-fixings for the 3 first class constraints \( \mathcal{H}_\rho(\tau) \approx 0 \)]. In this case, by eliminating the momenta, the equations of motion for the particles and for the electromagnetic field become

\[
\frac{d}{d\tau} \left( \eta_i m_i \frac{\dot{\eta}_i(\tau)}{\sqrt{1 - \eta_i^2(\tau)}} \right) = -\sum_{k \neq i} \frac{Q_i Q_k (\dot{\eta}_k(\tau) - \dot{\eta}_k(\tau))}{4\pi |\eta_k(\tau) - \eta_k(\tau)|^3} + \\
+ Q_i \left[ \dot{E}_\bot(\tau, \dot{\eta}_i(\tau)) + \dot{\eta}_i(\tau) \times \ddot{B}(\tau, \dot{\eta}_i(\tau)) \right],
\]

\[
\Box \dot{A}_\bot^r(\tau, \dot{\sigma}) = \ddot{A}_\bot^r(\tau, \dot{\sigma}) + \Delta \dot{A}_\bot^r(\tau, \dot{\sigma}) = J_\bot^r(\tau, \dot{\sigma}) = \\
= \sum_{i=1}^{N} Q_i P_{\bot}^{rs}(\dot{\sigma}) \dot{\eta}^s(\tau) \delta^3(\dot{\sigma} - \dot{\eta}_i(\tau)) = \\
= \sum_{i=1}^{N} Q_i \dot{\eta}^s(\tau) \left( \delta^r \partial^s + \frac{\partial^r \partial^s}{\Delta} \right) \delta^3(\dot{\sigma} - \dot{\eta}_i(\tau)) = \\
= \sum_{i=1}^{N} Q_i \dot{\eta}^s(\tau) \left[ \delta^3(\dot{\sigma} - \dot{\eta}_i(\tau)) +
\]

20
\[ + \int d^3\sigma' \frac{\pi^{rs}(\bar{\sigma} - \bar{\sigma}')}{|\bar{\sigma} - \bar{\sigma}'|^3} \delta^3(\bar{\sigma}' - \bar{n}_i(\tau)), \quad (53) \]

with

\[ \pi^{rs}(\bar{\sigma} - \bar{\sigma}') = \delta^{rs} - 3(\sigma^r - \sigma'^r)(\sigma^s - \sigma'^s)/(\bar{\sigma} - \bar{\sigma}')^2. \quad (54) \]

Due to the projector \( P_{rs}^{rs}(\bar{\sigma}) \) required by the rest-frame Coulomb gauge, the sources of the transverse (Wigner spin 1) vector potential are no more local and one has a system of integrodifferential equations (like with the equations generated by Fokker-Tetrode actions) with the open problem of how to define an initial value problem.

The equations identifying the rest frame become

\[
\sum_{i=1}^{N} \eta_i m_i \frac{\ddot{\eta}_i(\tau)}{\sqrt{1 - \dot{\eta}_i^2(\tau)}} + Q_i \tilde{A}_\perp(\tau, \tilde{\eta}_i(\tau)) + \\
+ \int d^3\sigma \sum_r [(\ddot{\tilde{A}}^r_\perp)(\tilde{A}^r_\perp)(\tau, \tilde{\sigma}) = 0. \quad (55) \]

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21
IV. ELECTROMAGNETIC LIENARD-WIECHERT POTENTIALS.

Eqs.(53) can be resolved by using the retarded Green function

\[ G_{\text{ret}}(\tau; \sigma) = (1/2\pi)\theta(\tau)\delta[\tau^2 - \sigma^2], \]

\[ \Rightarrow \Box G_{\text{ret}}(\tau, \sigma) = \delta(\tau)\delta^3(\sigma), \quad (56) \]

and one obtains

\[
\tilde{A}_{\perp \text{RET}}^r(\tau, \sigma) \equiv \tilde{A}_{\perp \text{IN}}^r(\tau, \sigma) + \sum_{i=1}^{N} \frac{Q_i}{2\pi}P_{\perp}^{rs}(\sigma) \int d\tau' d^3\sigma' \\
\theta(\tau - \tau')\delta[(\tau - \tau')^2 - (\sigma - \sigma')^2]\tilde{\eta}_i^r(\tau')\delta^3(\sigma' - \tilde{\eta}_i(\tau')) = \\
= \tilde{A}_{\perp \text{IN}}^r(\tau, \sigma) + \sum_{i=1}^{N} \frac{Q_i}{2\pi}P_{\perp}^{rs}(\sigma) \int d\tau' \\
\theta(\tau - \tau')\delta[(\tau - \tau')^2 - (\sigma - \tilde{\eta}_i^r(\tau'))^2]\tilde{\eta}_i^r(\tau'), \quad (57)
\]

where \( \Box A_{\perp \text{IN}}(\tau, \sigma) = 0 \) is a homogeneous solution describing arbitrary incoming radiation.

Let \( (\tau, \sigma) \) be the coordinates of a point \( z^\mu(\tau, \sigma) \) of Minkowski spacetime lying on the Wigner hyperplane \( \Sigma_W(\tau) \), on which the locations of the particles are \( (\tau, \tilde{\eta}_i(\tau)) \) [i.e. \( x_i^\mu(\tau) = z^\mu(\tau, \tilde{\eta}_i(\tau)) \)]. The rest-frame distance between \( z^\mu(\tau, \sigma) \) and \( x_i^\mu(\tau) \) is \( \tilde{r}_i(\tau, \sigma) = \sigma - \tilde{\eta}_i(\tau) \); let \( \tilde{r}_i(\tau, \sigma) = (\sigma - \tilde{\eta}_i(\tau))/|\sigma - \tilde{\eta}_i(\tau)| \) be the associated unit vector, \( \tilde{r}_i^2(\tau, \sigma) = 1 \).

Let \( \tau_{i+}(\tau, \sigma) \) [the retarded times] denote the retarded solutions of the equations

\[ (\tau - \tau_{i+})^2 = (\sigma - \tilde{\eta}_i(\tau_{i+}))^2, \quad i = 1, ..., N. \quad (58) \]

The point \( z^\mu(\tau, \sigma) \) lies on the lightcones emanating from the particle worldlines at their points \( x_i^\mu(\tau_{i+}(\tau, \sigma)) = z^\mu(\tau_{i+}(\tau, \sigma), \tilde{\eta}_i(\tau_{i+}(\tau, \sigma))) \), lying on the Wigner hyperplanes \( \Sigma_W(\tau_{i+}(\tau, \sigma)) \) respectively. The point \( z^\mu(\tau, \sigma) \) on \( \Sigma_W(\tau) \) will define points \( z^\mu(\tau_{i+}(\tau, \sigma), \tilde{\sigma}) \) on the Wigner hyperplanes \( \Sigma_W(\tau_{i+}(\tau, \sigma)) \) by orthogonal projection [since \( (z^\mu(\tau, \sigma) - x_i^\mu(\tau_{i+}(\tau, \sigma)))^2 = 0 \), we have \( R_{i+}(\tau, \sigma) = \sqrt{(z(\tau, \sigma) - z(\tau_{i+}(\tau, \sigma), \tilde{\sigma}))^2} = \sqrt{-(z(\tau_{i+}(\tau, \sigma)))^2} - x_i(\tau_{i+}(\tau, \sigma))^2 \) and \( z^\mu(\tau, \sigma) - x_i^\mu(\tau_{i+}(\tau, \sigma)) = R_{i+}(\tau, \sigma)(t_{i+}^\mu(\tau, \sigma) + s_{i+}^\mu(\tau, \sigma)) \) with \( t_{i+}^\mu(\tau, \sigma) \) and \( s_{i+}^\mu(\tau, \sigma) \) being the timelike and spacelike unit vectors associated with \( z^\mu(\tau, \sigma) - z^\mu(\tau_{i+}(\tau, \sigma), \tilde{\sigma}) \) and...
\[ z^\mu(\tau_i, \sigma_i, \bar{\sigma}) - x^\mu_i(\tau_i, \sigma_i) \text{ respectively; } R_i(\tau, \bar{\sigma}) \text{ is the Minkowski retarded distance between } z^\mu(\tau, \bar{\sigma}) \text{ and } x^\mu_i(\tau, \sigma_i). \]

Let \( \hat{r}_i(\tau_i, \sigma_i, \bar{\sigma}) = \bar{\sigma} - \vec{\eta}_i(\tau_i, \sigma_i) \) denote the rest-frame retarded distance between the points \( z^\mu(\tau_i, \sigma_i, \bar{\sigma}) \) and the points \( x^\mu_i(\tau_i, \sigma_i) \) of the worldlines belonging to \( \Sigma_i(\tau_i, \sigma_i) \) [with \( \hat{r}_i(\tau_i, \sigma_i, \bar{\sigma}) \) being the unit vector, \( \hat{r}_i^2 = 1 \)]. Let us denote the length of the vectors \( \hat{r}_i(\tau_i, \sigma_i, \bar{\sigma}) \) with

\[ r_i(\tau_i, \sigma_i, \bar{\sigma}) = |\hat{r}_i(\tau_i, \sigma_i, \bar{\sigma})| = |\bar{\sigma} - \vec{\eta}_i(\tau_i, \sigma_i)| = \tau - \tau_i > 0. \] (59)

Then, we have

\[ \theta(\tau - \tau') \delta[(\tau - \tau')^2 - (\bar{\sigma} - \vec{\eta}_i(\tau'))^2] = \frac{\delta(\tau - \tau_i(\tau, \sigma))}{2\rho_i(\tau, \sigma_i)} \]

\[ \rho_i(\tau_i, \sigma_i, \bar{\sigma}) = \tau - \tau_i(\tau, \sigma) - \vec{\eta}_i(\tau_i, \sigma_i) \cdot [\bar{\sigma} - \vec{\eta}_i(\tau_i, \sigma_i)] = r_i(\tau_i, \sigma_i)[1 - \vec{\eta}_i(\tau_i, \sigma_i) \cdot \hat{r}_i(\tau_i, \sigma_i, \bar{\sigma})]. \] (60)

Eq.(57) can be rewritten as

\[ \tilde{A}_{\perp, \text{RET}}(\tau, \sigma) = \tilde{A}_{\perp, \text{IN}}(\tau, \sigma) + \sum_{i=1}^{N} Q_i \frac{P^{r_s}_i(\sigma_i)}{\rho_i(\tau_i, \sigma_i)} \frac{\vec{\eta}_i^s(\tau_i, \sigma_i)}{\theta_i(\tau_i, \sigma_i, \bar{\sigma})} = \]

\[ = \tilde{A}_{\perp, \text{IN}}(\tau, \sigma) + \sum_{i=1}^{N} Q_i \frac{\vec{\eta}_i^s(\tau_i, \sigma_i)}{\theta_i(\tau_i, \sigma_i, \bar{\sigma})} + \]

\[ + \int d^3\sigma' \pi^{r_s}(\sigma - \sigma') \frac{\vec{\eta}_i^s(\tau_i, \sigma')}{|\sigma - \sigma'|^3} \theta_i(\tau_i, \sigma', \sigma') = \]

\[ = \tilde{A}_{\perp, \text{IN}}(\tau, \sigma) + \sum_{i=1}^{N} \tilde{A}_{\perp, \text{IN}}(\tau, \sigma_i, \sigma) = \]

\[ = \tilde{A}_{\perp, \text{IN}}(\tau, \sigma) + \sum_{i=1}^{N} Q_i \tilde{A}_{\perp, \text{IN}}(\tau_i, \sigma_i, \sigma_i), \] (61)

where \( \tilde{A}_{\perp, \text{IN}}(\tau_i, \sigma, \bar{\sigma}) \) is the rest-frame form of the Lienard-Wiechert retarded potential produced by particle i [its Minkowski analogue, i.e. the relativistic generalization of the Coulomb potential, is \( A^\mu_{(i+)}(z) = \frac{Q_i}{4\pi z_i(\tau_i) - [z - x_i(\tau_i)]} = \frac{Q_i}{4\pi R_i + [1 - R_i + x_i(\tau_i)]} \)]. Since we are
in the rest-frame Coulomb gauge with only transverse Wigner-covariant vector potentials, 
\( \tilde{A}_{\perp(i+)}(\tau_i, \sigma, \tilde{\sigma}) \) has a first standard term generated at the retarded time \( \tau_i + (\tau, \tilde{\sigma}^\prime) \) at \( x_i^\mu(\tau_i, \tilde{\sigma}) \), which is, however, accompanied by a nonlocal term receiving contributions from all the retarded times \(-\infty < \tau_i + (\tau, \tilde{\sigma}) \leq \tau_i \), which is due to the elimination of the electromagnetic gauge degrees of freedom [this is the origin of the transverse projector]. If we put the \( \tilde{A}_{\perp(i+)}(\tau, \tilde{\sigma}) \)'s in the particle equations (52), with \( \tilde{A}_{\perp, L N}(\tau, \tilde{\sigma}) = 0 \), then the equations of motion become integro-differential equations like the ones generated by a Fokker action.

To evaluate the electric \( [\tilde{E}_\perp = -\tilde{A}_\perp] \) and magnetic \( [\tilde{B} = -\partial \times \tilde{A}_\perp] \) fields produced by \( \tilde{A}_{\perp(i+)}(\tau, \tilde{\sigma}) \), we need the rule of derivation of ‘retarded’ functions \( g(\tau, \tilde{\sigma}; \tau_i + (\tau, \tilde{\sigma})) \). From Eq.(58) we get \( (\tau - \tau_i + (d\tau - d\tau_i +) = r_{i+}(d\tau - d\tau_i +) = [\tilde{\sigma} - \tilde{\eta}_i(\tau_i +)] \cdot [d\tilde{\sigma} - \tilde{\eta}_i(\tau_i +)d\tau_i +] = r_i^\perp \cdot [d\tilde{\sigma} - \tilde{\eta}_i(\tau_i +)d\tau_i +] \). Therefore, by introducing the notation

\[
\tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma}) = \frac{\tilde{r}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})}{\tilde{g}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})} = \frac{\tilde{r}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})}{1 - \tilde{\eta}_i(\tau_i + (\tau, \tilde{\sigma})) \cdot \tilde{r}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})},
\]

\[
\tilde{r}_i^\perp = \frac{\tilde{v}_i^\perp}{|\tilde{v}_i^\perp|},
\]

\[
|\tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})| = \frac{1}{1 - \tilde{\eta}_i(\tau_i + (\tau, \tilde{\sigma})) \cdot \tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})} = \frac{\tau - \tau_i + (\tau, \tilde{\sigma})}{\tilde{g}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})},
\]

we get

\[
\frac{\partial \tau_i + (\tau, \tilde{\sigma})}{\partial \tau} = |\tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})|,
\]

\[
\frac{\partial \tau_i + (\tau, \tilde{\sigma})}{\partial \sigma^s} = \tilde{r}_i + s(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})|\tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})| = v_i + s(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma}),
\]

\[
\frac{\partial g(\tau, \tilde{\sigma}; \tau_i + (\tau, \tilde{\sigma}))}{\partial \tau} = \left[ \left( \frac{\partial}{\partial \tau} \right) |\tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})| \frac{\partial g(\tau, \tilde{\sigma}; \tau^\prime)}{\partial \tau} \right]|_{\tau^\prime = \tau_i + (\tau, \tilde{\sigma})},
\]

\[
\frac{\partial g(\tau, \tilde{\sigma}; \tau_i + (\tau, \tilde{\sigma}))}{\partial \sigma^s} = \left[ \left( \frac{\partial}{\partial \sigma^s} \right) |v_i + s(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})| \frac{\partial g(\tau, \tilde{\sigma}; \tau^\prime)}{\partial \sigma^s} \right]|_{\tau^\prime = \tau_i + (\tau, \tilde{\sigma})},
\]

so that [using \( \partial \tilde{r}_i^\perp(\tau_i +, \tilde{\sigma})/\partial \tau_i + = -\tilde{\eta}_i(\tau_i +), \partial r_i + (\tau_i +, \tilde{\sigma})/\partial \tau_i + = -\tilde{\eta}_i(\tau_i +) \cdot \tilde{r}_i + (\tau_i +, \tilde{\sigma}), \partial \rho_i + (\tau_i +, \tilde{\sigma})/\partial \tau_i + = \tilde{\eta}_i^2(\tau_i +) - (\tilde{\eta}_i(\tau_i +) + r_i + (\tau_i +, \tilde{\sigma})\tilde{\eta}_i(\tau_i +)) \cdot \tilde{r}_i + (\tau_i +, \tilde{\sigma}), \partial \rho_i + (\tau_i +, \tilde{\sigma})/\partial \sigma^s |_{\tau_i +, \rho_i + (\tau_i +, \tilde{\sigma})} = \delta^s_{\tau_i +}, \partial \rho_i + (\tau_i +, \tilde{\sigma})/\partial \sigma^s |_{\tau_i +, \rho_i + (\tau_i +, \tilde{\sigma})} = -\tilde{\eta}_i^s(\tau_i +) + \tilde{r}_i^s(\tau_i +, \tilde{\sigma}) \) ] we get

\[
E_{\perp, L E T}^r(\tau, \tilde{\sigma}) = -\frac{\partial}{\partial \tau} \tilde{A}_{\perp, L E T}^r(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma}) \equiv E_{\perp, L N}^r(\tau, \tilde{\sigma}) - P_{\perp, s}^s(\tilde{\sigma}) \sum_{i=1}^N \left[ \frac{Q_i}{4\pi} |\tilde{v}_i^\perp(\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})| \cdot \frac{-\tilde{\eta}_i^s(\tau_i + (\tau, \tilde{\sigma}))}{\rho_i + (\tau_i + (\tau, \tilde{\sigma}), \tilde{\sigma})} \right] \]

24
\[
\begin{aligned}
- \frac{\dot{\eta}_i^2}{\rho_i^2(\tau, \vec{\sigma})} (\eta_i^2 \dot{\eta}_i(\tau, \vec{\sigma})) &= \\
- (\dot{\eta}_i(\tau, \vec{\sigma})) + r_{i+}(\tau, \vec{\sigma}) \dot{\eta}_i(\tau, \vec{\sigma}) \cdot \vec{r}_{i+}(\tau, \vec{\sigma})) &= \\
\dot{E}_{\perp IN}(\tau, \vec{\sigma}) + \sum_{i=1}^{N} \dot{E}_{\perp(i+)}^r(\tau, \vec{\sigma}) &= \\
\dot{E}_{\perp IN}(\tau, \vec{\sigma}) + \sum_{i=1}^{N} Q_i \dot{E}_{\perp(i+)}^r(\tau, \vec{\sigma}),
\end{aligned}
\]

\[
\dot{B}_{\text{RET}}^r(\tau, \vec{\sigma}) = -\epsilon^{rsu}(\vec{\sigma}^s \dot{A}_{\text{RET}}^u(\tau, \vec{\sigma})) = \dot{B}_{IN}^r(\tau, \vec{\sigma}) + \\
+ \sum_{i=1}^{N} \frac{Q_i}{4\pi} \epsilon^{rsu} P_{\perp}^{uv}(\vec{\sigma}) \left( \frac{\partial}{\partial \vec{\sigma}^u} \big|_{\tau_{i+}} + v_{i+}(\tau, \vec{\sigma}, \vec{\sigma}) \right) \dot{\eta}_i^u(\tau, \vec{\sigma}))
\]

The particle equations of motion (51), the definition of the rest frame (49) and the conserved relative energy (50) have now the following form

\[
\frac{d}{d\tau} \left[ \eta_i m_i \right] = \sqrt{1 - \dot{\eta}_i^2(\tau)}
\]

\[
\begin{aligned}
&= - \sum_{k \neq i} \frac{Q_i Q_k}{4\pi} \left[ \dot{\eta}_k(\tau) - \dot{\eta}_k(\tau) \right] + \\
&+ \sum_{i=1}^{N} \left[ \eta_i m_i \frac{\eta_i m_i}{\sqrt{1 - \dot{\eta}_i^2(\tau)}} + Q_i \dot{\Lambda}_{\perp IN}(\tau, \vec{\sigma}_i) \right] + \\
&+ \sum_{i \neq j} Q_i Q_j \dot{\Lambda}_{\perp(j+)}(\tau_{j+}(\tau, \vec{\sigma}_i, \vec{\sigma}_j)) + \\
&+ \int d^3\sigma \left[ \dot{\Lambda}_{\perp IN} \times \vec{B}_{IN} + \sum_{i=1}^{N} Q_i (\dot{\Lambda}_{\perp IN} \times \vec{B}_{\perp(i+)}(\tau_{i+}, \vec{\sigma})) + 
\end{aligned}
\]

25
By using the equations of motion, it is verified that
follows. Besides the divergent Coulomb self-interaction it eliminates other divergent terms.

\[ E_{rel} = \sum_i \frac{\eta_i m_i}{\sqrt{1 - \beta_i^2(\tau)}} + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\vec{n}_i(\tau) - \vec{n}_j(\tau)|} + \]

\[ + \int d^3\sigma \left[ \frac{\vec{E}_{\perp IN}^2 + \vec{B}_{\perp IN}^2}{2} + \sum_{i=1}^N Q_i (\vec{E}_{\perp IN} \cdot \vec{E}_{\perp(i+)}(\tau_{i+}, \sigma) + \vec{B}_{\perp IN} \cdot \vec{B}_{(i+)}(\tau_{i+}, \sigma)) + \]

\[ + \sum_{i,j} Q_i Q_j (\vec{E}_{\perp(i+)}(\tau_{i+}, \sigma) \cdot \vec{E}_{\perp(j+)}(\tau_{j+}, \sigma) + \vec{B}_{(i+)}(\tau_{i+}, \sigma) \cdot \vec{B}_{(j+)}(\tau_{j+}, \sigma))] (\tau, \sigma) = \text{const.} \] (66)

The property \( Q_i^2 = 0 \) has been used in these equations and it will be used also in what follows. Besides the divergent Coulomb self-interaction it eliminates other divergent terms.

By using the equations of motion, it is verified that \( E_{rel} \) is a constant of the motion

\[ \frac{d}{d\tau} E_{rel} = \frac{d}{d\tau} \left[ \sum_{i=1}^N \frac{\eta_i m_i}{\sqrt{1 - \beta_i^2(\tau)}} - \sum_{i > j} \frac{Q_i Q_j}{4\pi |\vec{n}_i(\tau) - \vec{n}_j(\tau)|} + \right. \]

\[ \left. - \sum_{i=1}^N Q_i \dot{\vec{n}}_i(\tau) \cdot \vec{E}_{\perp IN}(\tau, \vec{n}_i(\tau)) - \sum_{i \neq j} Q_i \dot{\vec{n}}_i(\tau) \cdot \vec{E}_{\perp(j+)}(\tau_{j+}(\tau, \vec{n}_i(\tau)), \vec{n}_i(\tau)) + \right. \]

\[ \left. - \int_{S_{\infty}} d\Sigma \vec{n} \cdot \vec{E}_{\perp IN} \times \vec{B}_{\perp IN} + \right. \]

\[ + \sum_{i=1}^N Q_i (\vec{E}_{\perp IN} \cdot \vec{B}_{(i+)}(\tau_{i+}, \sigma) + \vec{E}_{\perp(i+)}(\tau_{i+}, \sigma) \times \vec{B}_{\perp IN} + \]

\[ + \sum_{i \neq j} Q_i Q_j (\vec{E}_{\perp(i+)}(\tau_{i+}, \sigma) \times \vec{B}_{(j+)}(\tau_{j+}, \sigma))] (\tau, \sigma) \triangleq 0, \] (67)

where we used the formula [\( V \) is a sphere in the Wigner hyperplane and \( S = \partial V \) its boundary with outer normal \( \vec{n} \); in Eq.(67), \( S_{\infty} \) is the limit of \( S \) when the radius of the sphere goes to infinity]

\[ \frac{dE}{d\tau} = \frac{d}{d\tau} \int_V d^3\sigma \left[ \frac{\vec{E}_{\perp}^2 + \vec{B}_{\perp}^2}{2} \right](\tau, \sigma) = - \int_S d\Sigma \vec{n} \cdot (\vec{E}_{\perp} \land \vec{B})(\tau, \sigma). \] (68)

In the nonrelativistic limit \( |\dot{\vec{n}}_i(\tau)| \ll 1 \) and in wave zone \( [\tau_{i+}(\tau, \sigma)) \rightarrow \tau, \)

26
\[ \rho_+(\tau_+, \sigma, \bar{\sigma}) \rightarrow r(\tau, \sigma) \approx |\bar{\sigma}| \rightarrow \infty \] with \( \tilde{A}_{\perp IN}(\tau, \bar{\sigma}) = 0 \), the asymptotic limit of the retarded fields is

\[
\tilde{E}_{\perp \text{RET,AS}}(\tau, \bar{\sigma}) \approx -P_{\perp}(\bar{\sigma}) \sum_{i=1}^{N} \frac{Q_i \tilde{\eta}_i(\tau)}{4\pi |\bar{\sigma}|}, \\
\tilde{B}_{\perp \text{RET,AS}}(\tau, \bar{\sigma}) \approx -P_{\perp}(\bar{\sigma}) \sum_{i=1}^{N} \frac{Q_i [\bar{\sigma} \times \tilde{\eta}_i(\tau)]^s}{|\bar{\sigma}|},
\]

(69)

so that the “Larmor formula” for the radiated energy become

\[
\frac{dE}{d\tau} \approx \int_{S} d\Sigma \cdot \tilde{E}_{\perp \text{RET,AS}} \times \tilde{B}_{\text{RET,AS}}(\tau, \bar{\sigma}) = \\
= \sum_{i \neq j} \frac{Q_i Q_j}{(4\pi)^2} \int d\Omega \cdot \tilde{\eta}_i(\tau) \times [\tilde{\eta}_i(\tau) \times [\tilde{\eta}_i(\tau)]] = \\
= \sum_{i \neq j} \frac{Q_i Q_j}{(4\pi)^2} \int d\Omega (\tilde{\eta}_i(\tau) \cdot (\tilde{\eta}_i(\tau) \times (\tilde{\eta}_i(\tau) \times (\tilde{\eta}_i(\tau)))) = \\
= \frac{2}{3} \sum_{i \neq j} \frac{Q_i Q_j}{(4\pi)^2} \tilde{\eta}_i(\tau) \cdot \tilde{\eta}_j(\tau). 
\]

(70)

The usual terms \( \frac{Q_i^2}{4\pi^2} \frac{2}{3} \tilde{\eta}_i(\tau) \) are absent due to the pseudoclassical conditions \( Q_i = 0 \). Therefore, at the pseudoclassical level, there is no radiation coming from single charges, but only interference radiation due to terms \( Q_i Q_j \) with \( i \neq j \). Since it is not possible to control whether the source is a single elementary charged particle (only macroscopic sources are testable), this result is in accord with macroscopic experimental facts.

For a single particle, \( N=1 \), the pseudoclassical equations (66) and (67) become

\[
\frac{d}{d\tau}(\eta m \frac{\tilde{\eta}(\tau)}{\sqrt{1 - \tilde{\eta}^2(\tau)}}) = Q[\tilde{E}_{\perp \text{IN}}(\tau, \bar{\eta}(\tau)) + \tilde{\eta}(\tau) \times \tilde{B}_{\text{IN}}(\tau, \bar{\eta})],
\]

\[
\eta m \frac{\tilde{\eta}(\tau)}{\sqrt{1 - \tilde{\eta}^2(\tau)}} + Q\tilde{A}_{\perp \text{IN}}(\tau, \bar{\eta}(\tau)) + \\
+ \int d^3\sigma [\tilde{E}_{\perp \text{IN}} \times \tilde{B}_{\text{IN}}](\tau, \bar{\sigma}) + Q(\tilde{E}_{\perp \text{IN}}(\tau, \bar{\sigma}) \times \tilde{B}_{\text{IN}}(\tau_+(\tau, \bar{\sigma}), \bar{\sigma}) + \\
+ \tilde{E}_{\perp (+)}(\tau_+(\tau, \bar{\sigma}), \bar{\sigma}) \times \tilde{B}_{\text{IN}}(\tau, \bar{\sigma})) = 0,
\]

\[
E_{\text{rel}} = \frac{\eta m}{\sqrt{1 - \tilde{\eta}^2(\tau)^2}} + \int d^3\sigma [\frac{\tilde{E}_{\perp \text{IN}}^2 + \tilde{B}_{\text{IN}}^2}{2} + \\
+ Q(\tilde{E}_{\perp \text{IN}} \cdot \tilde{E}_{\perp (+)}(\tau_+, \bar{\sigma}) + \tilde{B}_{\text{IN}} \cdot \tilde{B}_{\text{IN}}(\tau_+, \bar{\sigma}))(\tau, \bar{\sigma}) = \text{const.}
\]

(71)
\[
\frac{d}{d\tau} \sqrt{\eta m} \frac{\eta m}{\sqrt{1 - \eta (\tau)^2}} = Q\hat{\eta}(\tau) \cdot \tilde{E}_{\perp IN}(\tau, \tilde{\eta}(\tau)) + \int_{S_{as}} d\Sigma \tilde{n} \cdot \tilde{E}_{\perp IN} \times \tilde{B}_{IN} + Q(\tilde{E}_{\perp IN} \times \tilde{B}_{(+)\tau} + \tilde{E}_{\perp (+)\tau} \times \tilde{B}_{IN})(\tau, \tilde{\sigma}). \tag{72}
\]

The first of Eqs.(71) replaces the Abraham-Lorentz-Dirac equation [see for instance Ref. [10]] for an electron in an external electromagnetic field \([Q = e\theta^u\theta]\)
\[
\frac{d}{d\tau} mu^\mu = eF_{IN}^\mu(x)u_\nu + \frac{2}{3} \frac{e^2}{4\pi}(\hat{u}^\mu - (u \cdot \hat{u})u^\mu), \quad u^\mu = \frac{\hat{j}^\mu}{\sqrt{x^2}.} \tag{73}
\]

The \(e^2\) term contains: i) the term \((u \cdot \hat{u})u^\mu = -\hat{u}^2 u^\mu\) associated with the Larmor emission of radiation; ii) the Schott term \(\hat{u}^\mu\) producing violations of Einstein causality (either runaway solutions or pre-acceleration). They are inseparable because, given the Larmor term, the requirement of manifest Lorentz covariance forces the appearance of the Schott term. The rest-frame instant form of dynamics has manifest Wigner covariance, avoids the covariance problems in the simultaneous description of particles and fields and has no term of order \(Q^2\) (only terms \(Q_iQ_j\) with \(i \neq j\)) at the pseudoclassical level of description of the electric charge.

Let us now come back to the general case with \(\tilde{\lambda}(\tau) \neq 0\), for which we have the Lagrangian (46). If we put \(\tilde{\lambda}(\tau) = \frac{d}{d\tau}\tilde{g}(\tau)\), then Eqs.(46)-(50) become

\[
L_R(\tau) = -\sum_{i=1}^N \eta_i m_i \sqrt{1 - (\dot{\eta}_i(\tau) + \dot{\tilde{g}}(\tau))^2} + \frac{1}{2} \sum_{i \neq j} \frac{Q_iQ_j}{4\pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)|} +
\]

\[
+ \int d^3\sigma \sum_{i=1}^N \delta^3(\tilde{\sigma} - \tilde{\eta}_i(\tau))Q_i[\dot{\tilde{\eta}}_i(\tau) + \dot{\tilde{g}}(\tau)] \cdot \tilde{\lambda}_i(\tau, \tilde{\sigma}) +
\]

\[
+ \int d^3\sigma \frac{1}{2} \left[\left(\frac{\partial}{\partial \tau} + \frac{d\tilde{g}(\tau)}{d\tau} \cdot \frac{\partial}{\partial \tilde{\sigma}}\right) \tilde{\lambda}_i(\tau, \tilde{\sigma})\right]^2 - \tilde{B}_i^2(\tau, \tilde{\sigma}),
\]

\[
\frac{d}{d\tau} \left[\tilde{\eta}_i(\tau) + \frac{\dot{\tilde{g}}(\tau)}{\sqrt{1 - (\dot{\tilde{g}}(\tau))^2}}\right] + Q_i\tilde{\lambda}_i(\tau, \tilde{\eta}_i(\tau)) \overset{\circ}{=} -\sum_{k \neq i} \frac{Q_iQ_k}{4\pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_k(\tau)|^3} + Q_i \sum_u \left[\tilde{\eta}^u_i(\tau) + \dot{\tilde{g}}^u(\tau)\right] \frac{\partial}{\partial \tilde{\eta}_i} \tilde{\lambda}_i(\tau, \tilde{\eta}_i(\tau)),
\]

\[
[\left(\frac{\partial}{\partial \tau} + \frac{d\tilde{g}(\tau)}{d\tau} \cdot \frac{\partial}{\partial \tilde{\sigma}}\right)^2 + \Delta] \tilde{\lambda}_i^u(\tau, \tilde{\sigma}) \overset{\circ}{=}
\]

28
\[
\sum_{i=1}^{N} \eta_i m_i \left( \frac{\partial \tilde{A}_i}{\partial \tau} + \frac{\partial \tilde{g}(\tau)}{\partial \tau} \right) + \frac{\partial \tilde{\sigma}}{\partial \tilde{\sigma}} \right] + Q_i \tilde{A}_i(\tau, \tilde{\eta}_i(\tau)) + \int d^3 \sigma \left[ \frac{\partial \tilde{A}_i}{\partial \sigma} \cdot \left( \frac{\partial \tilde{g}(\tau)}{\partial \tau} + \frac{\partial \tilde{\sigma}}{\partial \tilde{\sigma}} \right) \tilde{A}_i \right] \right](\tau, \tilde{\sigma}) = 0,
\]

\[
E_{rel} = \sum_{i=1}^{N} \frac{\eta_i m_i}{\sqrt{1 - (\tilde{\eta}_i(\tau) + \tilde{g}(\tau))^2}} + \sum_{i>j} \frac{Q_i Q_j}{4\pi \sqrt{\tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau)}} + \int d^3 \sigma \left[ \frac{\partial \tilde{\sigma}}{\partial \tau} \right] \left( \frac{\partial \tilde{g}(\tau)}{\partial \tau} \left[ \tilde{A}_i \right] \right) + \tilde{B}^2(\tau, \tilde{\sigma}) = \text{const.} \ (74)
\]

If we define the transformation \( \tau' = \tau \), \( \tilde{\sigma}' = \tilde{\sigma} + \tilde{g}(\tau') \), we get \( \frac{\partial}{\partial \tau'} = \frac{\partial}{\partial \tau} + \frac{d \tilde{g}(\tau)}{d \tau} \cdot \frac{\partial}{\partial \tilde{\sigma}} \) and the equation of motion for the transverse vector potential and its solution become

\[
\Box \tilde{A}_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) \equiv \sum_{i=1}^{N} Q_i P^r_{i}(\tilde{\sigma}') \left[ \tilde{g}_i^\prime(\tau') + \tilde{g}(\tau') \right] \delta^3(\tilde{\sigma}' + \tilde{g}(\tau') - \tilde{\eta}_i(\tau')) \]

\[
\tilde{A}_i^r(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) = \tilde{A}_i^r(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) + \sum_{i=1}^{N} \frac{Q_i}{2\pi} \tilde{P}^r_{i}(\tilde{\sigma}') \int d\tau \tilde{d}^3 \sigma \tilde{\theta}(\tau' - \tilde{\tau}) \left[ \tilde{g}_i^\prime(\tau') + \tilde{g}(\tau') \right] \delta^3(\tilde{\sigma}' + \tilde{g}(\tau') - \tilde{\eta}_i(\tau')) = \tilde{A}_i^r(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) + \sum_{i=1}^{N} \frac{Q_i}{2\pi} P^r_{i}(\tilde{\sigma}') \int d\tau \tilde{d}^3 \sigma \tilde{\theta}(\tau' - \tilde{\tau}) \left[ \tilde{g}_i^\prime(\tau') + \tilde{g}(\tau') \right]. \ (75)
\]

If \( \tau'_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) \) is the retarded solution of the delta-function and we define the functions \( \tilde{r}_i^r(\tau'_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau'))), \tilde{\sigma} + \tilde{g}(\tau') = \tilde{\sigma}' + \tilde{g}(\tau') - \tilde{\eta}_i(\tau'_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau'))) \) and the associated \( \rho'_i \), we get

\[
\tilde{A}_i^r(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) = \tilde{A}_i^r(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) + \sum_{i=1}^{N} \frac{Q_i}{4\pi} P^r_{i}(\tilde{\sigma}') \frac{\tilde{g}_i^\prime(\tau'_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau')) + \tilde{g}(\tau'))}{\rho'_i(\tau'_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau')))} \tilde{A}_i(\tau', \tilde{\sigma}' + \tilde{g}(\tau')). \ (76)
\]

Therefore, one could recover all the previous results with \( \tilde{\lambda}(\tau) = \tilde{g}(\tau) \neq 0 \).
If we put Eq.(76) with $\tilde{A}_{\perp}^{IN} = 0$ in the Lagrangian of Eq.(74), we get the effective Fokker action in the rest-frame instant form [one should make a careful analysis of the boundary terms in the variation of $L_R(\tau)$ following Ref. [11], before claiming the equivalence of this effective Fokker action to a subspace of solutions of the original theory].

Following Ref. [12] and in particular Refs. [13–15], one can substitute retarded particle coordinates and velocities with instantaneous [in $\tau$] coordinates and accelerations of all orders. There are two methods for doing this in Refs. [13–15]. It would be interesting to check whether (with one of these methods) one can confirm in a Wigner-covariant way the non-covariant result of Gordeyev that starting with an instantaneous expansion in accelerations of retarded equations one gets instantaneous actions for the accelerations of the type $\frac{1}{2}(\text{retarded} + \text{advanced})$, but with the advanced part being a total $\tau$-derivative (so that it does not contribute to the equations of motion). See also Ref. [16].

Let us note Ref. [17] is the only attempt to study the Dirac constraints originating from actions depending from accelerations of all orders [and, assuming they are equivalent to Fokker actions, also originating from Fokker actions].

A connected open problem is the comparison for $N=2$ of the invariant mass $\epsilon_s = H_{rel}$ of Eq.(42), when one puts into it the retarded [or the $\frac{1}{2}(\text{retarded} + \text{advanced})$] Lienard-Wiechert solutions (61), (64), (65), with $\tilde{A}_{\perp}^{IN}(\tau, \vec{\sigma}) = 0$, with the sum of the pair of first class constraints with instantaneous potentials [18–22], whose quantization gives coupled Klein-Gordon equations for two spin zero particles [see Refs. [21–24] for similar equations for Dirac particles deriving from pairs of first class constraints for spinning particles]. These models were generated as phenomenological approximations to the Bethe-Salpeter equation, by reducing it in instantaneous approximations to a 3-dimensional equation (with the elimination of the spurious abnormal sectors of relative energy excitations) of the Lippmann-Schwinger type and then to the equation of the quasipotential approach [see the bibliography of the quoted references], which Todorov [19] reformulated as a pair of first class constraints at the classical level. In Ref. [22] it is directly shown how the normal sectors of the Bethe-
Salpeter equation are connected with the quantization of pairs of first class constraints with instantaneous (in general nonlocal, but approximable with local) potentials like in Todorov’s examples. Instead, in Refs. [21] it is shown how to derive the Todorov potential for the electromagnetic case from Tetrode-Fokker-Feynman-Wheeler electrodynamics with scalar and vector potentials [this theory is connected with \( \frac{1}{2} \) (retarded + advanced) solutions with no incoming radiation (adjunct Lienard-Wiechert fields) of Maxwell equations with particle currents in the Lorentz gauge]; besides the Coulomb potential, at the order \( 1/c^2 \) one gets the Darwin potential (becoming the Breit one at the quantum level), which is known to be phenomenologically correct. With only retarded Lienard-Wiechert potentials Eq. (42) becomes

\[
\epsilon_s = H_{rel} = \eta_1 \sqrt{m_1^2 + [\tilde{\kappa}_1(\tau) - Q_1 Q_2 \tilde{A}_{\perp(2+)}(\tau(2+), \vec{\eta}_1(\tau), \vec{\eta}_2(\tau))]^2} + \\
+ \eta_2 \sqrt{m_2^2 + [\tilde{\kappa}_2(\tau) - Q_1 Q_2 \tilde{A}_{\perp(1+)}(\tau(1+), \vec{\eta}_1(\tau), \vec{\eta}_2(\tau))]^2} + \frac{Q_1 Q_2}{4\pi |\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|} + \\
+ Q_1 Q_2 \int d^3\sigma [\tilde{E}_{\perp(1+)}(\tau(1+), \vec{\sigma}, \vec{\sigma}) \cdot \tilde{E}_{\perp(2+)}(\tau(2+), \vec{\sigma}, \vec{\sigma}) + \\
+ \tilde{B}_{(1+)}(\tau(1+), \vec{\sigma}, \vec{\sigma}) \cdot \tilde{B}_{(2+)}(\tau(2+), \vec{\sigma}, \vec{\sigma})].
\]

(77)

How, also forgetting the last term, to reexpress it only in terms of particle coordinates and momenta [this problem is connected with the previous one of the Hamiltonian formulation of Fokker actions]? How to check with a \( 1/c^2 \) expansion whether the Darwin potential is already present without using \( \frac{1}{2} \) (retarded + advanced) solutions [this is connected with Gordeyev approach]?

Let us remark that, if in Eqs. (47), (49) and (50) one adds the second class constraints \( \tilde{A}_{\perp}(\tau, \vec{\sigma}) = \tilde{\pi}_{\perp}(\tau, \vec{\sigma}) = 0 \) one selects the sector of phase space describing classical bound states without any kind of radiation [namely one looks for solutions of the charged N-body problem with instantaneous Coulomb interaction but without radiation fields]. One gets the equations

\[
\frac{d}{d\tau} \left( \eta_i m_i \sqrt{1 - \vec{\eta}_i^2(\tau)} \right) = - \sum_{k \neq i} \frac{Q_i Q_k [\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)]}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_k(\tau)|^3}
\]
\[
\sum_{i=1}^{N} \eta_i m_i \frac{\dot{\eta}_i(\tau)}{\sqrt{1 - \dot{\eta}_i^2(\tau)}} = 0
\]

\[
E_{rel} = \sum_i \frac{\eta_i m_i}{\sqrt{1 - \dot{\eta}_i^2(\tau)}} + \sum_{i \neq k} \frac{Q_i Q_k}{4\pi |\tilde{\eta}_i(\tau) - \tilde{\eta}_k(\tau)|} = \text{const.} \quad (78)
\]

For \( N=2 \) and \( m_1 = m_2 = m \) one knows the solutions [25]: there is a rosetta motion without spiral fall on the center.

Finally, in the electromagnetic case, the distribution function on the charge Grassmann variables [8] is

\[
\rho(\theta_i, \theta_i^*) = \prod_i (1 + \theta_i^* \theta_i). \quad (79)
\]

It satisfies the positivity condition only on analytic functions of the \( \theta_i \)'s

\[
f(\theta_i) = f_0 + \sum_i f_i \theta_i. \quad (80)
\]

The classical theory is recovered by making the mean of Grassmann-valued observables with this distribution function. For the electric charges one has \(< Q_i > = < e_i \theta_i^* \theta_i > = e_i\). As noted in Refs. [8], the processes of taking the mean of the equations of motion and then solving the classical equations [with the standard electromagnetic divergences and causality pathologies] or of solving the pseudoclassical field equations and then taking the mean [no divergences and no causality problems] do not commute. Since in the latter case there is a regularization of the electromagnetic self-energy, it would be important to learn how to quantize these solutions.
V. SCALAR ELECTRODYNAMICS ON SPACELIKE HYPERSURFACES

Let us consider the action describing a charged Klein Gordon field interacting with the electromagnetic field on spacelike hypersurfaces following the scheme of Ref. [1]

\[ S = \int d\tau d^3\sigma \sqrt{\gamma(\tau, \vec{\sigma})} \phi^* (\partial_\tau + i e A_\tau) \phi + 
\]
+ \frac{1}{2} g^{\tau\rho} g^{\sigma\mu} \partial_\tau A_\rho \partial_\sigma A_\mu \gamma^{\rho\sigma} \phi^* \phi + \frac{1}{4} \epsilon^{\rho\sigma\mu\nu} g_{\phi\phi} \phi^* \phi + \frac{1}{4} \epsilon^{\rho\sigma\mu\nu} g_{\phi\phi} \phi^* \phi - \frac{1}{4} g^{\rho\sigma} \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\rho\sigma} \]

(81)

where the configuration variables are \[ z^\mu(\tau, \vec{\sigma}) \], \[ \phi(\tau, \vec{\sigma}) = \tilde{\phi}(z(\tau, \vec{\sigma})) \] and \[ A_A(\tau, \vec{\sigma}) = z_A^\mu(\tau, \vec{\sigma}) A_\mu(z(\tau, \vec{\sigma})) \] [\( \tilde{\phi}(z) \) and \( A_\mu(z) \) are the standard Klein-Gordon field and electromagnetic potential, which do not know the embedding of the spacelike hypersurface \( \Sigma \) in Minkowski spacetime like \( \phi \) and \( A_A \)].

Since \[ z_\tau^\mu = N l^\mu + N^\rho z_\tau^\rho \], one has \[ \frac{\partial}{\partial z_\tau^\mu} = l_\mu \frac{\partial}{\partial N} + z_\mu \gamma^\rho \frac{\partial}{\partial N_\tau} \]. Therefore, the canonical momenta are

\[ \pi^\tau(\tau, \vec{\sigma}) = \frac{\partial L}{\partial \partial_\tau A_\tau(\tau, \vec{\sigma})} = 0, \]

\[ \pi^\mu(\tau, \vec{\sigma}) = \frac{\partial L}{\partial \partial_\tau A_\mu(\tau, \vec{\sigma})} = - \frac{\sqrt{\gamma(\tau, \vec{\sigma})}}{N(\tau, \vec{\sigma})} \gamma^{\rho\sigma}(\tau, \vec{\sigma})(F_{\tau^\rho} - N^\rho F_{\tau^\rho})(\tau, \vec{\sigma}), \]

\[ \pi^\phi(\tau, \vec{\sigma}) = \frac{\partial L}{\partial \partial_\tau \phi(\tau, \vec{\sigma})} = \frac{\sqrt{\gamma(\tau, \vec{\sigma})}}{N(\tau, \vec{\sigma})} \left[ \partial_\tau + i e A_\tau - N^\tau \right] \phi^* \phi(\tau, \vec{\sigma}), \]

\[ \pi^\phi(\tau, \vec{\sigma}) = \frac{\partial L}{\partial \partial_\tau \phi^*(\tau, \vec{\sigma})} = \frac{\sqrt{\gamma(\tau, \vec{\sigma})}}{N(\tau, \vec{\sigma})} \left[ \partial_\tau - i e A_\tau - N^\tau \right] \phi^* \phi(\tau, \vec{\sigma}), \]
\[ \rho_\mu(\tau, \vec{\sigma}) = -\frac{\partial L}{\partial \partial_\tau z^\mu(\tau, \vec{\sigma})} = \]
\[ = l_\mu(\tau, \vec{\sigma})\left\{ \frac{\pi_\phi \pi_{\phi^*}}{\sqrt{\gamma}} - \sqrt{\gamma}[\gamma^f g_{fs} \phi^* (\partial_s - ie A_s)\phi - m^2 \phi^* \phi] + \frac{1}{2\sqrt{\gamma}} \pi^f g_{fs} \pi^s - \frac{\sqrt{\gamma}}{4} \gamma^f g_{\tilde{a}\tilde{b}} F_{\tilde{a}\tilde{b}} \right\}(\tau, \vec{\sigma}) + \]
\[ + z_\mu(\tau, \vec{\sigma}) \gamma^f (\tau, \vec{\sigma}) \{ \pi_{\phi^*} (\partial_t + ie A_t)\phi^* + \pi_\phi (\partial_t - ie A_t)\phi - F_{\tilde{a}\tilde{b}} \pi^\tilde{a} \}(\tau, \vec{\sigma}). \quad (82) \]

Therefore, one has the following primary constraints

\[ \pi^\tau(\tau, \vec{\sigma}) \approx 0, \]
\[ \mathcal{H}_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - \]
\[ = -l_\mu(\tau, \vec{\sigma})\left\{ \frac{\pi_\phi \pi_{\phi^*}}{\sqrt{\gamma}} - \sqrt{\gamma}[\gamma^f g_{fs} \phi^* (\partial_s - ie A_s)\phi - m^2 \phi^* \phi] + \frac{1}{2\sqrt{\gamma}} \pi^f g_{fs} \pi^s - \frac{\sqrt{\gamma}}{4} \gamma^f g_{\tilde{a}\tilde{b}} F_{\tilde{a}\tilde{b}} \right\}(\tau, \vec{\sigma}) + \]
\[ + z_\mu(\tau, \vec{\sigma}) \gamma^f (\tau, \vec{\sigma}) \{ \pi_{\phi^*} (\partial_t + ie A_t)\phi^* + \pi_\phi (\partial_t - ie A_t)\phi - F_{\tilde{a}\tilde{b}} \pi^\tilde{a} \}(\tau, \vec{\sigma}) \approx 0, \quad (83) \]

and the following Dirac Hamiltonian \([\lambda(\tau, \vec{\sigma}) \text{ and } \lambda^\mu(\tau, \vec{\sigma}) \text{ are Dirac multiplier}]\)

\[ H_D = \int d^3\sigma [-A_\tau(\tau, \vec{\sigma}) \Gamma(\tau, \vec{\sigma}) + \lambda(\tau, \vec{\sigma}) \pi^\tau(\tau, \vec{\sigma}) + \lambda^\mu(\tau, \vec{\sigma}) \mathcal{H}_\mu(\tau, \vec{\sigma})]. \quad (84) \]

By using the Poisson brackets

\[ \{ z^\mu(\tau, \vec{\sigma}), \rho_\nu(\tau, \vec{\sigma}') \} = \eta^\mu_\nu \delta^3(\vec{\sigma} - \vec{\sigma}'), \]
\[ \{ A_\lambda(\tau, \vec{\sigma}), \pi^\beta(\tau, \vec{\sigma}') \} = \eta^\beta_\lambda \delta^3(\vec{\sigma} - \vec{\sigma}'), \]
\[ \{ \phi(\tau, \vec{\sigma}), \pi_\phi(\tau, \vec{\sigma}') \} = \{ \phi^*(\tau, \vec{\sigma}), \pi_{\phi^*}(\tau, \vec{\sigma}') \} = \delta^3(\vec{\sigma} - \vec{\sigma}'), \quad (85) \]

one finds that the time constancy of the primary constraints implies the existence of only one secondary constraint

\[ \Gamma(\tau, \vec{\sigma}) = \partial_t \pi^\tau(\tau, \vec{\sigma}) + ie(\pi_{\phi^*} \phi^* - \pi_\phi \phi) 0(\tau, \vec{\sigma}) \approx 0. \quad (86) \]

One can verify that these constraints are first class with the algebra given in Eqs.(125) of Ref. [1].

The Poincare’ generators are like in Eq.(28).

34
Following Ref. [1] (see also Sections I, II of this paper), we can restrict ourselves to spacelike hyperplanes $z^\nu(\tau, \vec{\sigma}) = x^\nu_s(\tau) + b^\nu_\tau \sigma^\tau$ where the normal $l^\nu = \epsilon^\nu_{\alpha\beta\gamma} b^\alpha_1(\tau) b^\beta_2(\tau) b^\gamma_3(\tau)$ is $\tau$-independent. Using the results of that paper one finds that $J^\mu_{\nu s} = x^\mu_s p^\nu_s - x^\nu_s p^\mu_s + S^\mu_{\nu s}$ and that the constraints are reduced to the following ones

$$\pi^7(\tau, \vec{\sigma}) \approx 0,$$

$$\Gamma(\tau, \vec{\sigma}) = -\tilde{\partial} \Pi(\tau, \vec{\sigma}) + ie[\pi_{\phi^*}, \phi^* - \pi_\phi](\tau, \vec{\sigma}) \approx 0,$$

$$\tilde{\mathcal{H}}^{\mu}(\tau, \vec{\sigma}) = \int d^3 \sigma \tilde{\mathcal{H}}^{\mu}(\tau, \vec{\sigma}) =$$

$$= p^\mu_s - l^\mu \left\{ \frac{1}{2} \int d^3 \sigma [\tilde{\pi}^2 + \tilde{B}^2]  +$$

$$+ \int d^3 \sigma [\pi_{\phi^*} \phi^* + (\tilde{\partial} + ie \tilde{A}) \phi^* \cdot (\tilde{\partial} - ie \tilde{A}) \phi + m^2 \phi^* \phi](\tau, \vec{\sigma}) \right\} -$$

$$- b^\mu_\tau(\tau) \left\{ \int d^3 \sigma (\tilde{\pi} \times \tilde{B}) \varphi(\tau, \vec{\sigma}) + \int d^3 \sigma [\pi_{\phi^*}(\partial_\varphi + ie A_\varphi) \phi^* +$$

$$+ \pi_\phi(\partial_\varphi - ie A_\varphi) \phi](\tau, \vec{\sigma}) \right\} \approx 0,$$

$$\tilde{\mathcal{H}}^{\mu\nu}(\tau, \vec{\sigma}) =$$

$$= \frac{1}{2} \int d^3 \sigma \tilde{\mathcal{H}}^{\mu}(\tau, \vec{\sigma}) - b^\nu_s(\tau) \int d^3 \sigma \tilde{\mathcal{H}}^{\mu}(\tau, \vec{\sigma}) =$$

$$= S^\mu_{\nu s} - (b^\mu_\tau(\tau) l^\nu - b^\nu_\tau(\tau) l^\mu) \frac{1}{2} \int d^3 \sigma \tilde{\pi}^2 + \tilde{B}^2)(\tau, \vec{\sigma}) +$$

$$+ \int d^3 \sigma \tilde{\pi} \left\{ \pi_{\phi^*} \pi_\phi + (\tilde{\partial} + ie \tilde{A}) \phi^* \cdot (\tilde{\partial} - ie \tilde{A}) \phi + m^2 \phi^* \phi \right\}(\tau, \vec{\sigma}) +$$

$$+ (b^\mu_\tau(\tau) b^\nu_\tau(\tau) - b^\nu_\tau(\tau) b^\mu_\tau(\tau)) \left\{ \int d^3 \sigma \tilde{\pi} \times \tilde{B} \right\}(\tau, \vec{\sigma}) +$$

$$+ \int d^3 \sigma \tilde{\pi} \left\{ \pi_{\phi^*}(\partial_\varphi + ie A_\varphi) \phi^* + \pi_\phi(\partial_\varphi - ie A_\varphi) \phi \right\}(\tau, \vec{\sigma}) \right\} \approx 0. \quad (87)$$

The configuration variables are reduced from $z^\nu(\tau, \vec{\sigma}), A_\lambda(\tau, \vec{\sigma}), \phi(\tau, \vec{\sigma}), \phi^*(\tau, \vec{\sigma})$ to $x^\mu_s(\tau)$, to the six independent degrees of freedom hidden in the orthonormal tetrad $b^\mu_\lambda \ [b^\mu_\tau = l^\mu]$, $A_\lambda(\tau, \vec{\sigma}), \phi(\tau, \vec{\sigma}), \phi^*(\tau, \vec{\sigma})$, with the associated momenta [six degrees of freedom hidden in $S^\mu_{\nu s}$ are the momenta conjugate to those hidden in the tetrad; see Ref. [1] for the associated Dirac brackets].

If one selects all the configurations of the system with timelike total momentum $[p^2_s > 0]$, one can restrict oneself to the special Wigner hyperplanes orthogonal to $p^\mu_s$ itself. The effect of this gauge fixing is a canonical reduction to a phase space spanned only by the variables $\tilde{x}^\mu_s(\tau), p^\mu_s, A_\tau(\tau, \vec{\sigma}), \pi^\tau(\tau, \vec{\sigma}), \vec{A}(\tau, \vec{\sigma}), \vec{\pi}(\tau, \vec{\sigma}), \phi(\tau, \vec{\sigma}), \pi_\phi(\tau, \vec{\sigma}), \phi^*(\tau, \vec{\sigma}), \pi_{\phi^*}(\tau, \vec{\sigma})$, with standard Dirac brackets.
The only surviving constraints are \[ \epsilon_s = \eta_s \sqrt{p_s^2} \]

\[ \pi^\tau(\tau, \bar{\sigma}) \approx 0, \]
\[ \Gamma(\tau, \bar{\sigma}) = -\bar{\partial}\pi(\tau, \bar{\sigma}) + ie[\pi_{\bar{\phi}^*}^* - \pi_{\phi^*}^*](\tau, \bar{\sigma}) \approx 0, \]
\[ \mathcal{H}(\tau) = \epsilon_s - \left\{ \frac{1}{2} \int d^3\sigma (\bar{\pi}^2 + \bar{\mathcal{B}}^2)(\tau, \bar{\sigma}) + \int d^3\sigma [\pi_{\bar{\phi}^*}^* \pi_{\phi^*}^* + (\bar{\partial} + ie\bar{A})\phi^* \cdot (\bar{\partial} - ie\bar{A})\phi + m^2 \phi^* \phi](\tau, \bar{\sigma}) \right\} \approx 0, \]
\[ \mathcal{H}_p(\tau) = \int d^3\sigma (\bar{\pi} \times \bar{\mathcal{B}})(\tau, \bar{\sigma}) + \int d^3\sigma [\pi_{\bar{\phi}^*}^* (\bar{\partial} + ie\bar{A})\phi^* + \pi_{\phi^*}^* (\bar{\partial} - ie\bar{A})\phi](\tau, \bar{\sigma}) \approx 0. \] (88)

Always following Ref. [1], it can be shown that the Lorentz generators take the following form

\[ J_{ij}^s = \bar{x}_i p_j^s - \bar{x}_j p_i^s + \delta^{ir} \delta^{js} \bar{S}_{rs}^s, \]
\[ J_{oi}^s = \bar{x}_i p_o^s - \bar{x}_o p_i^s - \delta^{ir} \bar{S}_{rs}^s p_o^s + \frac{1}{p_s^0 + \epsilon_s}, \]
\[ \bar{S}_{rs}^s = \int d^3\sigma \{ \sigma^r (\bar{\mathcal{B}})^s(\tau, \bar{\sigma}) - \sigma^s (\bar{\mathcal{B}})^r(\tau, \bar{\sigma}) \} + \int d^3\sigma [\sigma^r [\pi_{\bar{\phi}^*}^* (\bar{\partial} + ieA^s)\phi^* + \pi_{\phi^*}^* (\bar{\partial} - ieA^s)\phi](\tau, \bar{\sigma}) - (r \leftrightarrow s)]. \] (89)

To make the reduction to Dirac’s observables with respect to the electromagnetic gauge transformations, let us recall [9,6] that the electromagnetic gauge degrees of freedom are described by the two pairs of conjugate variables \( A_r(\tau, \bar{\sigma}), \pi_r(\tau, \bar{\sigma}) [\approx 0], \eta_{em}(\tau, \bar{\sigma}) = -\frac{1}{\Delta} \frac{\partial}{\partial \bar{\sigma}} \cdot \bar{A}(\tau, \bar{\sigma}), \Gamma(\tau, \bar{\sigma}) [\approx 0] \), so that we have the decompositions

\[ A^r(\tau, \bar{\sigma}) = \frac{\partial}{\partial \sigma^r} \eta_{em}(\tau, \bar{\sigma}) + A^r_{\perp}(\tau, \bar{\sigma}), \]
\[ \pi^r(\tau, \bar{\sigma}) = \pi^r_{\perp}(\tau, \bar{\sigma}) + \frac{1}{\Delta} \frac{\partial}{\partial \sigma^r} [-\Gamma(\tau, \bar{\sigma}) + ie(\pi_{\bar{\phi^*}}^* - \pi_{\phi^*})(\tau, \bar{\sigma})] \approx 0, \]
\[ \{ A^r_{\perp}(\tau, \bar{\sigma}), \pi^r_{\perp}(\tau, \bar{\sigma}) \} = -P^r_{\perp}(\bar{\sigma}) \delta^3(\bar{\sigma} - \bar{\sigma}'), \]

where \( P^r_{\perp}(\bar{\sigma}) = \delta^{rs} + \frac{\partial^r \partial^s}{\Delta}, \Delta = -\bar{\partial}^2 \). Then, we have
\[ \int d^3 \sigma \quad \pi^2(\tau, \vec{\sigma}) = \int d^3 \sigma \pi_1^2(\tau, \vec{\sigma}) - \frac{e^2}{4\pi} \int d^3 \sigma_1 d^3 \sigma_2 \frac{i(\pi_{\phi}\phi^* - \pi_{\phi}\phi)(\tau, \vec{\sigma}_1) i(\pi_{\phi}\phi^* - \pi_{\phi}\phi)(\tau, \vec{\sigma}_2)}{|\vec{\sigma}_1 - \vec{\sigma}_2|}. \] (91)

Since we have

\[ \{\phi(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}')\} = ie\phi(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}'), \]
\[ \{\pi_{\phi}(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}')\} = -ie\pi_{\phi}(\tau, \vec{\sigma})\delta^3(\vec{\sigma} - \vec{\sigma}'), \] (92)

the Dirac observables for the Klein-Gordon field are

\[ \hat{\phi}(\tau, \vec{\sigma}) = [\hat{\phi}^*(\tau, \vec{\sigma})]^* = e^{ie\eta_{\phi}(\tau, \vec{\sigma})}\phi(\tau, \vec{\sigma}), \]
\[ \hat{\pi}_{\phi}(\tau, \vec{\sigma}) = [\hat{\pi}_{\phi}^*(\tau, \vec{\sigma})]^* = e^{-ie\eta_{\phi}(\tau, \vec{\sigma})}\pi_{\phi}(\tau, \vec{\sigma}), \] (93)

The constraints take the following form

\[ \mathcal{H}(\tau) = \epsilon_s - \{ \frac{1}{2} \int d^3 \sigma (\pi_1^2 + B^2)(\tau, \vec{\sigma}) + \int d^3 \sigma [\hat{\pi}_{\phi}^* \hat{\pi}_{\phi} + (\vec{\partial} + ie\vec{A}_1)\hat{\phi}^* \cdot (\vec{\partial} - ie\vec{A}_1)\hat{\phi} + m^2\hat{\phi}^* \hat{\phi}](\tau, \vec{\sigma}) - \frac{e^2}{8\pi} \int d^3 \sigma_1 d^3 \sigma_2 \frac{i(\pi_{\phi^*}\phi^* - \pi_{\phi}\phi)(\tau, \vec{\sigma}_1) i(\pi_{\phi^*}\phi^* - \pi_{\phi}\phi)(\tau, \vec{\sigma}_2)}{|\vec{\sigma}_1 - \vec{\sigma}_2|} \}, \]
\[ \bar{\mathcal{H}}_p(\tau) = \int d^3 \sigma (\pi_1 \times \vec{B})(\tau, \vec{\sigma}) + \int d^3 \sigma (\hat{\pi}_{\phi}^* \vec{\partial} \hat{\phi}^* + \hat{\pi}_{\phi} \vec{\partial} \hat{\phi})(\tau, \vec{\sigma}) \approx 0, \] (94)

where the Coulomb self-interaction appears in the invariant mass and where the 3 constraints defining the rest frame do not depend on the interaction since we are in an instant form of the dynamics. The final form of the rest-frame spin tensor is

\[ \bar{S}_{rs}^s = \int d^3 \sigma \{\sigma^r[(\pi_1 \times \vec{B})^s + \hat{\pi}_{\phi^*} \vec{\partial} \hat{\phi}^* + \hat{\pi}_{\phi} \vec{\partial} \hat{\phi}] - (r \leftrightarrow s)\}(\tau, \vec{\sigma}). \] (95)

If we go to the gauge \( \chi = T_s - \tau \approx 0 \), we can eliminate the variables \( \epsilon_s, T_s \), and the \( \tau \)-evolution (in the Lorentz scalar rest-frame time) is governed by the Hamiltonian
\[ H_R = H_{rel} - \tilde{\lambda}(\tau) \cdot \mathcal{H}_p(\tau), \]

\[
H_{rel} = \frac{1}{2} \int d^3\sigma (\tilde{\pi}_\perp^2 + \tilde{B}^2)(\tau, \tilde{\sigma}) + \\
+ \int d^3\sigma [\tilde{\pi}_\phi \cdot \tilde{\pi}_\phi + (\tilde{\partial} + ie\tilde{A}_\perp) \hat{\phi}^* \cdot (\tilde{\partial} - ie\tilde{A}_\perp) \hat{\phi} + m^2 \hat{\phi}^* \hat{\phi}](\tau, \tilde{\sigma}) - \\
- \frac{e^2}{8\pi} \int d^3\sigma_1 d^3\sigma_2 \frac{i(\tilde{\pi}_\phi \cdot \hat{\phi}^* - \tilde{\pi}_\phi \hat{\phi}^*)(\tau, \tilde{\sigma}_1) i(\tilde{\pi}_\phi \cdot \hat{\phi} - \tilde{\pi}_\phi \hat{\phi})(\tau, \tilde{\sigma}_2)}{|\tilde{\sigma}_1 - \tilde{\sigma}_2|}. \tag{96}
\]

In the gauge \( \tilde{\lambda}(\tau) = 0 \), the Hamilton equations are

\[
\partial_\tau \dot{\hat{\phi}}(\tau, \tilde{\sigma}) \overset{\circ}{=} \tilde{\pi}_\phi(\tau, \tilde{\sigma}) + \\
+ \frac{ie^2}{4\pi} \hat{\phi}(\tau, \tilde{\sigma}) \int d^3\sigma \frac{i(\tilde{\pi}_\phi \cdot \hat{\phi}^* - \tilde{\pi}_\phi \hat{\phi})(\tau, \tilde{\sigma})}{|\tilde{\sigma} - \tilde{\sigma}|},
\]

\[
\partial_\tau \pi_{\phi^*}(\tau, \tilde{\sigma}) \overset{\circ}{=} \frac{i}{2} \left[ (\tilde{\partial} - ie\tilde{A}_\perp(\tau, \tilde{\sigma}))^2 - m^2 \right] \hat{\phi}(\tau, \tilde{\sigma}) + \\
+ \frac{ie^2}{4\pi} \pi_{\phi^*}(\tau, \tilde{\sigma}) \int d^3\sigma \frac{i(\tilde{\pi}_\phi \cdot \hat{\phi}^* - \tilde{\pi}_\phi \hat{\phi})(\tau, \tilde{\sigma})}{|\tilde{\sigma} - \tilde{\sigma}|},
\]

\[
\partial_\tau A^*_{\perp}(\tau, \tilde{\sigma}) \overset{\circ}{=} -\pi^*_{\perp}(\tau, \tilde{\sigma}),
\]

\[
\partial_\tau \pi^*_{\perp}(\tau, \tilde{\sigma}) \overset{\circ}{=} \Delta A^*_{\perp}(\tau, \tilde{\sigma}) + \\
+ ieP_{\tau \perp}(\tilde{\sigma}) \left[ \hat{\phi}^* (\partial^* - ieA^*_{\perp}) \hat{\phi} - \hat{\phi} (\partial^* + ieA^*_{\perp}) \hat{\phi}^* \right](\tau, \tilde{\sigma}). \tag{97}
\]

The equations for \( \hat{\phi}^* \) and \( \pi_{\phi} \) are the complex conjugate of those for \( \hat{\phi} \) and for \( \pi_{\phi^*} \).

By using the results of Ref. [6], we have the following inversion formula

\[
\tilde{\pi}_{\phi^*} \overset{\circ}{=} \partial_\tau \hat{\phi} + ie^2 \hat{\phi} \frac{1}{\Delta} i(\tilde{\pi}_{\phi^*} \cdot \hat{\phi} - \tilde{\pi}_{\phi} \hat{\phi}) = \\
= \partial_\tau \hat{\phi} + ie^2 \hat{\phi} \frac{1}{\Delta + 2e^2 \hat{\phi}^* \hat{\phi}} i(\hat{\phi}^* \partial_\tau \hat{\phi} - \hat{\phi} \partial_\tau \hat{\phi}^*), \tag{98}
\]

since we have \( i(\hat{\phi}^* \partial_\tau \hat{\phi} - \hat{\phi} \partial_\tau \hat{\phi}^*) = [1 + 2e^2 \hat{\phi}^* \hat{\phi}] i(\tilde{\pi}_{\phi^*} \cdot \hat{\phi} - \tilde{\pi}_{\phi} \hat{\phi}) \) and where use has been done of the operator identity \( \frac{1}{\Delta + 2e^2 \hat{\phi}^* \hat{\phi}} = \frac{1}{\Delta} [1 - B \frac{1}{A} + B \frac{1}{A} B \frac{1}{A} - ...] = \frac{1}{\Delta + B} \) (valid for \( B \) a small perturbation of \( A \)) for \( A = \Delta \) and \( B = 2e^2 \hat{\phi}^* \hat{\phi} \).

Using this formula, we get the following second order equations of motion

\[
\{ [\partial_\tau + ie^2 \frac{1}{\Delta + 2e^2 \hat{\phi}^* \hat{\phi}} i(\hat{\phi}^* \partial_\tau \hat{\phi} - \hat{\phi} \partial_\tau \hat{\phi}^*)]^2 \} \hat{\phi} = 0,
\]

38
\[ [\partial^2 + \Delta] A^*_\perp = ie P^r_s(\vec{\sigma}) [\hat{\phi}^s (\partial^s - ieA^*_\perp) \hat{\phi} - \hat{\phi} (\partial^s + ieA^*_\perp) \hat{\phi}^s]. \] (99)

We see that the non-local velocity-dependent self-energy is formally playing the role of a scalar potential.

The previous results can be reformulated in the two-component Feshbach-Villars formalism for the Klein-Gordon field [5] [see also Ref. [26,27]]. If we put \((\tau_i \text{ are the Pauli matrices}) \)

\[
\hat{\phi} = \frac{1}{\sqrt{2}} [\varphi + \chi], \\
i \hat{\pi}_{\phi^*} = \frac{1}{\sqrt{2}} [\varphi - \chi], \\
\varphi = \frac{1}{\sqrt{2}} [\hat{\phi} + \frac{i}{m} \hat{\pi}_{\phi^*}], \\
\chi = \frac{1}{\sqrt{2}} [\hat{\phi} - \frac{i}{m} \hat{\pi}_{\phi^*}], \\
\Phi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{100}
\]

the Hamilton equations for the Klein-Gordon field become

\[
i \partial_{\tau} \varphi = \frac{1}{2m} (-i \vec{\sigma} - e\vec{A}_{\perp})^2 (\varphi + \chi) + (m + K) \varphi, \\
i \partial_{\tau} \chi = -\frac{1}{2m} (-i \vec{\sigma} - e\vec{A}_{\perp})^2 (\varphi + \chi) + (-m + K) \chi,
\]

\[
K(\tau, \vec{\sigma}) = -\frac{me^2}{4\pi} \int d^3\sigma_1 \frac{(\varphi^s \varphi - \chi^s \chi)(\tau, \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|} = -\frac{me^2}{4\pi} \int d^3\sigma_1 \frac{(\Phi^s \tau_3 \Phi)(\tau, \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|}. \tag{101}
\]

In the \(2 \times 2\) matrix formalism we have

\[
i \partial_{\tau} \Phi = \left[ \frac{1}{2m} (-i \vec{\sigma} - e\vec{A}_{\perp})^2 (\tau_3 + i \tau_2) + m \tau_3 + K \right] \Phi = H \Phi. \tag{102}
\]
Since $\rho = \frac{i}{m} \Phi^* \tau_3 \Phi = \frac{i}{m}(\varphi^* \varphi - \chi^* \chi) = \frac{i}{m}(\hat{\pi}_a \hat{\phi}^* - \hat{\pi}_a \hat{\phi})$ is the density of the conserved charge $e/m$ (see the Gauss law), the normalization of $\Phi$ can be taken as $\int d^3 \sigma (\Phi^* \tau_3 \Phi)(\tau, \vec{\sigma}) = \frac{e}{m}$.

As shown in Ref. [5], when we put $\vec{A}_\perp = K = 0$, the free Klein-Gordon field has the Hamiltonian $H_o = \frac{p^2}{2m}(\tau_3 + i\tau_2) + m\tau_3$ in the momentum representation and this Hamiltonian can be diagonalized ($p^\tau = +\sqrt{m^2 + \vec{p}^2}$)

$$H_{o,U} = U^{-1}(\vec{p}) H_o U(\vec{p}) = p^\tau \tau_3 = \begin{pmatrix} \sqrt{m^2 + \vec{p}^2} & 0 \\ 0 & -\sqrt{m^2 + \vec{p}^2} \end{pmatrix},$$

$$\Phi_{U}(\tau, \vec{p}) = U^{-1}(\vec{p}) \Phi(\tau, \vec{p}),$$

$$i \partial_\tau \Phi_U = H_{o,U} \Phi_U,$$

$$U(\vec{p}) = \frac{1}{2\sqrt{mp^2}} [(m + p^\sigma)1 - (m - p^\sigma)\tau_1],$$

$$U^{-1}(\vec{p}) = \frac{1}{2\sqrt{mp^2}} [(m + p^\sigma)1 + (m - p^\sigma)\tau_1].$$

(103)

Like in the case of the Foldy-Wouthuysen transformation for particles of spin $1/2$, also in the spin 0 case the exact diagonalization of the Hamiltonian cannot be achieved in presence of an arbitrary external electromagnetic field [5].

Now, Eq.(102) has the following form after Fourier transform

$$i \partial_\tau \Phi(\tau, \vec{p}) = \tilde{H} \Phi(\tau, \vec{p}),$$

$$\tilde{H} = \frac{1}{2m} [\vec{p}^2 - e \int d^3 k \tilde{A}_\perp(\tau, \vec{k}) e^{-\vec{k} \cdot \vec{\phi}}]2(\tau_3 + i\tau_2) + m\tau_3 +$$

$$+ \int d^3 k K(\tau, \vec{k}) e^{-\vec{k} \cdot \vec{\phi}} \mathbb{1}.$$ (104)

If we put $\Phi(\tau, \vec{p}) = U(\vec{p}) \Phi_U(\tau, \vec{p})$ with the same $U(\vec{p})$ of the free case, we get [see Ref. [5]]

$$i \partial_\tau \Phi_U(\tau, \vec{p}) = \sqrt{m^2 + \vec{p}^2} \tau_3 \Phi_U(\tau, \vec{p}) +$$

$$+ \int d^3 k K(\tau, \vec{p} - \vec{k}) \left( \sqrt{m^2 + \vec{p}^2} + \sqrt{m^2 + \vec{k}^2} \right) \mathbb{1} + \left( \sqrt{m^2 + \vec{p}^2} - \sqrt{m^2 + \vec{k}^2} \right) \tau_1$$

$$\mathbb{1} \Phi_U(\tau, \vec{k}) +$$
\[ + \int d^3k \frac{m}{2\sqrt{m^2 + \vec{p}^2\sqrt{m^2 + \vec{k}^2}}} \left[ -\frac{e}{m} \vec{k} \cdot \vec{A}_\perp(\tau, \vec{p} - \vec{k}) + \frac{e^2}{2m} (\vec{A}_\perp^2)(\tau, \vec{p} - \vec{k}) \right] \\
(\mathbb{1} + \tau_1)\Phi_U(\tau, \vec{k}), \]

(105)

where \((\vec{A}_\perp^2)(\tau, \vec{p})\) means the Fourier transform of \(\vec{A}_\perp^2(\tau, \vec{\sigma})\).

In ref. [5], it is shown that this Hamiltonian cannot be diagonalized, because the separation of positive and negative energies is inhibited by effects which (in a second quantized formalism) can be ascribed to the vacuum polarization, namely to the pair production. This (i.e. the nonseparability of positive and negative energies) is also the source of the zitterbewegung effects for localized Klein-Gordon wave packets as discussed in Ref. [5].

Let us come back to the constraint (94) giving the invariant mass of the full system. With an integration by parts it can be rewritten as

\[
\epsilon_s - \frac{1}{2} \int d^3\sigma (\vec{\pi}_\perp^2 + \vec{B}^2(\tau, \vec{\sigma})) - \\
- \int d^3\sigma \Phi^*(\tau, \vec{\sigma}) \tau_3 \left[ \frac{1}{2} (-i\vec{\partial} - e\vec{A}_\perp(\tau, \vec{\sigma}))^2(\tau_3 + i\tau_2) + m^2 \tau_3 \right] \Phi(\tau, \vec{\sigma}) + \\
+ \int d^3\sigma \Phi^*(\tau, \vec{\sigma}) \tau_3 \left[ \frac{e^2 m^2}{8\pi} \int d^3\sigma_1 \frac{(\Phi^\ast \tau_3 \Phi)(\tau, \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|} \mathbb{1} \right] \Phi(\tau, \vec{\sigma}) \approx 0. \tag{106}
\]

If we suppose that \(\Phi(\tau, \vec{\sigma})\) is normalized to \(\int d^3\sigma \Phi^*(\tau, \vec{\sigma}) \tau_3 \Phi(\tau, \vec{\sigma}) = 1/m\) [this is a charge normalization compatible with the nonlinear equations of motion, because the electric charge is conserved], we can rewrite the previous formula as

\[
\int d^3\sigma \Phi^*(\tau, \vec{\sigma}) \tau_3 \left\{ \epsilon_s - \frac{1}{2} \int d^3\sigma (\vec{\pi}_\perp^2 + \vec{B}^2(\tau, \vec{\sigma})) + \\
e^2 m \int d^3\sigma_1 \frac{(\Phi^\ast \tau_3 \Phi)(\tau, \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|} \mathbb{1} - \\
- \frac{1}{2m} (-i\vec{\partial} - e\vec{A}_\perp(\tau, \vec{\sigma}))^2(\tau_3 + i\tau_2) + m\tau_3 \right\} \Phi(\tau, \vec{\sigma}) \approx 0. \tag{107}
\]

If we assume that the nonlinear equations for the reduced Klein-Gordon field have solutions of the form \(\Phi(\tau, \vec{\sigma}) = \Phi(\tau, \vec{p}) e^{i\vec{p} \cdot \vec{\sigma}} + \Phi_1(\tau, \vec{\sigma})\) with \(\Phi_1\) negligible, namely that the global form of the nonlinear wave admits a sensible eikonal approximation, then, neglecting \(\Phi_1\), we get approximately

41
\[
\int d^3\sigma \Phi^*(\tau, \vec{\sigma}) \tau_3 \left\{ \left[ \epsilon_s - \frac{1}{2} \right] \int d^3\sigma (\vec{\pi}_\perp^2 + \vec{B}^2) + e^2m \frac{1}{8\pi} \int d^3\sigma_1 \frac{\Phi^* \tau_3 \Phi(\tau, \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|} \right\} - \frac{1}{2m} (\vec{p} - e\vec{A}_\perp(\tau, \vec{\sigma}))^2 (\tau_3 = i\tau_2 + m\tau_3) \right\} \Phi(\tau, \vec{\sigma}) \approx 0. \quad (108)
\]

If we now redefine \(\Phi_U(\tau, \vec{\sigma}) = U^{-1}(\vec{p} - e\vec{A}_\perp(\tau, \vec{\sigma})) \Phi(\tau, \vec{\sigma})\) with the same \(U\) of Eq.(103), we get

\[
\int d^3\sigma \Phi_U^*(\tau, \vec{\sigma}) \begin{pmatrix} \mathcal{H}_+(\tau, \vec{\sigma}) & 0 \\ 0 & \mathcal{H}_-(\tau, \vec{\sigma}) \end{pmatrix} \Phi_U(\tau, \vec{\sigma}) \approx 0,
\]

\[
\mathcal{H}_\pm(\tau, \vec{\sigma}) = \epsilon_s \mp \sqrt{m^2 + (\vec{p} - e\vec{A}_\perp(\tau, \vec{\sigma}))^2} + e^2m \frac{1}{8\pi} \int d^3\sigma_1 \frac{\Phi^* \tau_3 \Phi(\tau, \vec{\sigma}_1)}{|\vec{\sigma} - \vec{\sigma}_1|} - \frac{1}{2} \int d^3\sigma_1 (\vec{\pi}_\perp^2 + \vec{B}^2)(\tau, \vec{\sigma}_1). \quad (109)
\]

where \(\mathcal{H}_\pm(\tau, \vec{\sigma}) \approx 0\) are the constraints (30) for \(N=1\) for the invariant mass of charged scalar particles plus the electromagnetic field given in Ref. [1] for the two possible signs of the energy \(\eta = \pm\). The Klein-Gordon self-energy should go in the particle limit (eikonal approximation of filed theory) in the Coulomb self-energy of the classical particle, which is assent in Ref. [1] because it is regularized by assuming that the particle electric charge \(Q\) is pseudoclassically described by Grassmann variables so that \(Q^2 = 0\). Therefore, the particle description of Ref. [1] is valid only when one disregards the effects induced by vacuum polarization and pair production and uses a strong eikonal approximation neglecting diffractive effects. It would be interesting to investigate whether there are special electromagnetic fields for which Eq.(104) can be diagonalized with a field-dependent matrix more general of \(U(\vec{p})\) and whether Eq.(109) can be generalized with non-minimal coupling terms in these cases.
VI. CONCLUSIONS.

In this paper we ended the rest-frame description of scalar charged, either particle or field, systems interacting with the electromagnetic field. For particles we deduced the final reduced equations of motion and gave indications of how to attack the problem of getting an equivalent Fokker-type description. Due to the pseudoclassical nature of the Grassmann-valued electric charges, the causality problems of the Abraham-Lorentz-Dirac equation get a solution without damaging the macroscopic results of radiation theory based on the Larmor equation.

We left the signs $\eta_i$ of the energies of the particles arbitrary. Actually, we can put all signs $\eta_i = +1$, with the classical antiparticles moving forward in $\tau$ and having opposite electric charge (opposite ratio charge/mass) with respect to the classical particles [28–30].

We showed which should be the starting point for the connection with the two-body equations of Refs. [21–24]: is the Darwin potential at order $1/c^2$ already contained in our reduced theory? Let us add that one can introduce further instantaneous interactions (besides the Coulomb one) in Eqs.(30), at least for $N=2$, in such a way that the constraints remain first class. As shown in Eq.(111) of Ref. [1] one can introduce any additive potential depending on $|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|$ in the first os Eqs.(30) and the constraints remain first class, because the other constraints in Eq.(30) are interaction independent and contain only $\tilde{\kappa}_+(\tau) = \sum_{i=1}^{N} \tilde{\kappa}_i(\tau)$; instead an interaction depending also on the transverse electromagnetic field should commute with $\int d^3\sigma (\vec{\pi}_\perp \times \tilde{\vec{B}})(\tau, \vec{\sigma})$. Once one has the first class constraints on the Wigner hyperplane, one can try to go backward and deduce the constraints on general spacelike hypersurfaces. This could simulate more general interactions maybe coming from a partial diagonalization of Feshbach-Villars’ Eqs.(105) with a consistent truncation of the off-diagonal terms connected with pair production. Moreover, one should study the separation of positive and negative particle energies in the pairs of first class constraints of Refs. [21,22] to see whether it is possible to get a $4 \times 4$ formalism generalizing Eq.(109) in this two-particle sector. One should obtain 4 first class constraints for the 4 branches
of the mass spectrum for $m_1 \neq m_2$ [for $m_1 = m_2$ the situation is more complex [1]] to be compared with the results of Ref. [1], remembering that in general the theory of interacting particles on spacelike hypersurfaces is not completely equivalent to the one with mass-shell constraints [non-timelike branches of the mass spectrum are eliminated by construction on the hypersurface].

We also gave the rest-frame formulation of scalar electrodynamics and we showed that the approximation with scalar charged particles emerges in an eikonal approximation after a Feshbach-Villars reformulation.

Let us remark that in the electromagnetic case all the dressing with Coulomb clouds [of the scalar particles and of charged Klein-Gordon fields in this paper and of Grassmann-valued Dirac fields in Ref. [9]] are done with the Dirac phase $\eta_{em} = -\frac{1}{\Delta} \vec{\partial} \cdot \vec{A}$ [31]. The same phase is used in Ref. [32] to dress fermions in QED. Also in Ref. [33] the solution of the quantum Gauss law constraint on Schroedinger functional $\Psi[A]$ in the case of two static particles of opposite charges, is able to reproduce the Coulomb potential and the Coulomb self-energy with the same mechanism as in Eq.(91) only if $\Psi[A] = e^{i\eta_{em}} \Phi[A]$ with $\Phi[A]$ gauge invariant and not with $\Psi'[A] = e^{i \int_{x_0}^{x_1} d\vec{x} \cdot \vec{A}(x^o, \vec{x})} \Phi'[A]$ with a phase factor resembling the Wilson loop operator [one has $\Phi[A] = e^{i \int_{x_0}^{x_1} d\vec{x} \cdot \vec{A}_{\perp}(x^o, \vec{x})} \Phi'[A]$, namely the Wilson line operator has been broken in the gauge part plus the gauge invariant part using Eq.(90)].

The next step would be the elimination of the 3 constraints $\vec{H}_p(\tau) \approx 0$ defining the intrinsic rest frame. This requires the introduction of 3 gauge-fixings identifying the Wigner 3-vector describing the intrinsic 3-center of mass on the Wigner hyperplane. However, till now these gauge-fixings are known only in the case of an isolated system containing only particles. When the center of mass canonical decomposition of linear classical field theories will be available (see Ref. [34] for the Klein-Gordon field), its reformulation on spacelike hypersurfaces will allow the determination of these gauge-fixings also when fields are present and a Hamiltonian description with only Wigner-covariant relative variables with an explicit control on the action-reaction balance between fields and particles or between two types of fields.
Finally, one has to start the quantization program of relativistic scalar charged particles plus the electromagnetic field in the rest-frame Coulomb gauge based on the Hamiltonian $H_{rel}$ of Eq.(42). On the particle side, the complication is the quantization of the square roots associated with the relativistic kinetic energy terms. On the field side, the obstacle is the absence (notwithstanding the absence of no-go theorems) of a complete regularization and renormalization procedure of electrodynamics in the Coulomb gauge: see Refs. [35,32] for the existing results for QED. However, as shown in Refs. [1,9], the rest-frame instant form of dynamics automatically gives a physical ultraviolet cutoff: it is the Møller radius $\rho = \sqrt{-W^2c/P^2} = |\vec{S}|c/\sqrt{P^2}$ ($W^2 = -P^2\vec{S}^2$ is the Pauli-Lubanski Casimir), namely the classical intrinsic radius of the worldtube, around the covariant noncanonical Fokker-Price center of inertia, inside which the noncovariance of the canonical center of mass $\tilde{x}^\mu$ is concentrated. At the quantum level $\rho$ becomes the Compton wavelength of the isolated system multiplied its spin eigenvalue $\sqrt{s(s+1)}$, $\rho \mapsto \hat{\rho} = \sqrt{s(s+1)}\hbar/M$ with $M = \sqrt{P^2}$. 
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