The system of $N$ scalar particles with Grassmann-valued color charges plus the color SU(3) Yang-Mills field is reformulated on spacelike hypersurfaces. The Dirac observables are found and the physical invariant mass of the system in the Wigner-covariant rest-frame instant form of dynamics (covariant Coulomb gauge) is given. From the reduced Hamilton equations we extract the second order equations of motion both for the reduced transverse color field and the particles. Then, we study this relativistic scalar quark model, deduced from the classical QCD Lagrangian and with the color field present,
in the $N=2$ (meson) case. A special form of the requirement of having only
color singlets, suited for a field-independent quark model, produces a “pseu-
doclassical asymptotic freedom” and a regularization of the quark self-energy.

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I. INTRODUCTION

The Dirac observables for Yang-Mills theory with Grassmann-valued fermions [1], Abelian and non-Abelian S(2) Higgs models [2,3] and for the SU(3)xSU(2)xU(1) standard model of elementary particles [4] and the corresponding physical Hamiltonians have been found in a noncovariant way equivalent to a generalized Coulomb gauge, following the scheme used by Dirac [5] to do the canonical reduction of the electromagnetic field with charged fermions. See Refs. [6] for reviews of the method and of the program, which aims to get a unified description of the standard model and of tetrad gravity in terms of Dirac’s observables.

Then, the problem of how to covariantize these results was started. In Ref. [7], the system of N scalar particles with Grassmann electric charges plus the electromagnetic field was described by defining it on arbitrary spacelike hypersurfaces, which give a covariant 3+1 decomposition of Minkowski spacetime $M^4$, following Refs. [8,9] (see also Ref. [10], where a theoretical study of this problem is done in curved spacetimes). The new configuration variables are the points $z^\mu(\tau, \vec{\sigma})$ of the spacelike hypersurface $\Sigma_\tau$ [the only ones carrying Lorentz indices] and a set of Lorentz invariant variables containing a 3-vector $\vec{\eta}_i(\tau)$ for each particle [$x^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$] and the electromagnetic gauge potentials $A_\mu(\tau, \vec{\sigma}) = \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^A} A_\mu(z(\tau, \vec{\sigma}))$, which know implicitly the embedding of $\Sigma_\tau$ into $M^4$. One has to choose the sign of the energy of each particle, because there are not mass-shell constraints (like $p_i^2 - m_i^2 \approx 0$) among the constraints of this formulation, due to the fact that one has only 3 degrees of freedom for particle since the intersection of a timelike trajectory and of the spacelike hypersurface $\Sigma_\tau$, with Lorentz scalar ‘time’ parameter $\tau$ (labelling the leaves of the foliation of $M^4$ with the $\Sigma_\tau$ all diffeomorphic to a given $\Sigma$), is determined by 3 numbers: $\vec{\sigma} = \vec{\eta}_i(\tau)$.

Besides a Lorentz scalar form of the electromagnetic first class constraints, one has 4 further first class constraints $H_\mu(\tau, \vec{\sigma}) \approx 0$ implying the independence of the description from the choice of the spacelike hypersufaces foliating $M^4$. Being in special relativity, it is
convenient to restrict ourselves to arbitrary spacelike hyperplanes \( z^{\mu}(\tau, \vec{\sigma}) = x^{\mu}_{s}(\tau) + b^{\mu}_{\tau}(\tau)\vec{\sigma}^{\tau}. \) Since they are described by only 10 variables [an origin \( x^{\mu}_{s}(\tau) \) and 3 orthogonal spacelike unit vectors generating the fixed constant timelike unit normal to the hyperplane], we remain only with 10 first class constraints determining the 10 variables conjugate to the hyperplane [they are a 4-momentum \( p^{\mu}_{s} \) and the 6 independent degrees of freedom hidden in a spin tensor \( S^{\mu\nu}_{s} \)] in terms of the variables of the particles and of the electromagnetic field. We can make the canonical reduction of the electromagnetic field variables to transverse gauge potentials and electric fields at the hypersurface as well as at the hyperplane level.

If we now restrict ourselves to timelike \((p^{2}_{s} > 0)\) 4-momenta [the set of particles plus electromagnetic field configurations with \( p^{\mu}_{s} \) not timelike is of zero measure in the space of all configurations], we can restrict the description to the so-called Wigner hyperplanes orthogonal to \( p^{\mu}_{s} \) itself. To get this result, we must boost at rest all the variables with Lorentz indices by using the standard Wigner boost \( L^{\mu\nu}(p_{s}, \vec{p}_{s}) \) for timelike Poincaré orbits, and then add the gauge-fixings \( b^{\mu}_{\tau}(\tau) - L^{\mu\tau}(p_{s}, \vec{p}_{s}) \approx 0. \) Since these gauge-fixings depend on \( p^{\mu}_{s} \), the final canonical variables, apart \( p^{\mu}_{s} \) itself, are of 3 types: i) there is a non-covariant center-of-mass variable \( \tilde{x}^{\mu}(\tau) \) [the classical basis of the Newton-Wigner position operator]; ii) all the 3-vector variables become Wigner spin 1 3-vectors [boosts in \( M^{4} \) induce Wigner rotations on them]; iii) all the other variables are Lorentz scalars [in the case under consideration they are absent after the canonical reduction to the transverse electromagnetic degrees of freedom]. Only the 4 first class constraints determining \( p^{\mu}_{s} \) are left. One obtains in this way a new kind of instant form of the dynamics (see Ref. [11]), the Euclidean covariant 1-time rest-frame instant form. It is the special relativistic generalization of the nonrelativistic separation of the center of mass from the relative motion \([H = \tilde{p}^{2}_{s}/2M + H_{rel}]\). The role of the center of mass is taken by the Wigner hyperplane, identified by the point \( \tilde{x}^{\mu}(\tau) \) and by its normal \( p^{\mu}_{s} \). The 4 first class constraints can be put in the following form: i) the vanishing of the total (Wigner spin 1) 3-momentum of the particles plus electromagnetic field \( \vec{p}_{[\text{system}]} \approx 0 \), saying that the Wigner hyperplane \( \Sigma_{W}(\tau) \) is the intrinsic rest frame [instead, \( \vec{p}_{s} \) is left arbitrary, since it reflects the orientation of the Wigner hyperplane with
respect to arbitrary reference frames in Minkowski spacetime; ii) \( \pm \sqrt{p_s^2 - M[system]} \approx 0 \), saying that the invariant mass \( M \) of the system replaces the nonrelativistic Hamiltonian \( H_{rel} \) for the relative degrees of freedom, after the addition of the gauge-fixing \( T_s - \tau \approx 0 \) [identifying the time parameter \( \tau \) with the Lorentz scalar time of the center of mass in the rest frame; \( M \) generates the evolution in this time]. When one is able, as in the case of \( N \) free particles, to find the (Wigner spin 1) 3-vector \( \vec{\eta}(\tau) \) conjugate to \( \vec{p}[system](\approx 0) \), the gauge-fixing \( \vec{\eta} \approx 0 \) eliminates the gauge variables describing the 3-dimensional intrinsic center of mass inside the Wigner hyperplane \( [\vec{\eta} \approx 0 \text{ forces it to coincide with } x_s^\mu(\tau) = z^\mu(\tau, \vec{\sigma} = \vec{\eta} = 0) \) and breaks the translation invariance \( \vec{\sigma} \mapsto \vec{\sigma} + \vec{a} \), so that we remain only with Newtonian-like degrees of freedom with rotational covariance: i) a 3-coordinate (not Lorentz covariant) \( \vec{z}_s = \sqrt{p_s^2(x_s - \frac{\vec{p}_s}{p_s^2}\vec{x}^o)} \) and its conjugate momentum \( \vec{k}_s = \vec{p}_s/\sqrt{p_s^2} \) for the absolute center of mass in Minkowski spacetime; ii) a set of relative conjugate pairs of variables with Wigner covariance inside the Wigner hyperplane . As noted in Ref. [7], the noncovariance of the center of mass of extended relativistic systems defines a classical intrinsic unit of length \( \rho = \sqrt{-W^2/cP^2} \) determined by the Poincaré Casimirs] to be used as an ultraviolet cutoff in the spirit of Dirac and Yukawa in future attempts of quantization. Let us remark that this ultraviolet cutoff exists also in asymptotically flat general relativity, taking into account the asymptotic Poincaré charges.

In this paper we will extend these results to \( N \) scalar particles with Grassmann SU(3) color charges plus the SU(3) Yang-Mills field. The resulting covariantization provides the tools to covariantize the bosonic part of the SU(3)xSU(2)xU(1) model. To complete its covariantization, the description of Dirac and chiral fields on spacelike hypersurfaces in Minkowski spacetime is needed (this problem is under investigation [12]).

The final result will be an expression for the physical invariant mass of the system scalar quarks plus the color SU(3) Yang-Mills field in terms of gauge invariant quantities (covariant generalized Coulomb gauge in the rest-frame instant form of dynamics). This pseudoclassical expression is the starting point for defining a relativistic quark model derived
from classical QCD. Till now, only the nonrelativistic quantum quark model is available [13] and there is no satisfactory derivation of it from QCD, notwithstanding its phenomenological relevance. Instead, we have here the full model with relativistic scalar quarks and classical color field and the classical reduced Yang-Mills equations with transverse sources analogue of the equations \( \Box \vec{A}_\perp = \vec{j}_\perp \) of the electromagnetic case in the rest frame [14]: if one could guess a reasonable solution of the equations for the color field and put it into the invariant mass, one would obtain an effective invariant mass for a true relativistic quark model, even still with scalar quarks. Since this is not yet possible, we shall limit ourselves to study some properties of pseudoclassical mesons (N=2). In particular, we will show that the requirement of having only color singlets, realized in a way suited to define a quark model without color field, immediately produces a kind of “pseudoclassical asymptotic freedom” besides regularizing the quark self-energy. Also some comments on the difficult problem of confinement are done.

In Section II we give some definitions and some results on spacelike hypersurfaces. In Section III the Lagrangian of the system on spacelike hypersurfaces is given and the Hamiltonian formalism is developed till the reduction to the Wigner hyperplane. In Section IV the Dirac observables of the system are found and the final form of the physical invariant mass is given. In Section V the reduced Hamilton equations and then the associated Euler-Lagrange equations in the rest frame are given. In Section VI the relativistic quark model in the case N=2 (mesons) is defined and it is shown that there is pseudoclassical asymptotic freedom.

In the final Section there are some conclusions and some comments on confinement and the open problems.
II. DYNAMICS ON SPACELIKE HYPERSURFACES

In this Section we will introduce the background material from Ref. [7] needed in the description of physical systems on spacelike hypersurfaces, integrating it with the definitions of non-Abelian SU(3) Yang-Mills fields [1].

Let \( \{ \Sigma_\tau \} \) be a one-parameter family of spacelike hypersurfaces foliating Minkowski spacetime \( M^4 \) and giving a 3+1 decomposition of it. At fixed \( \tau \), let \( z^\mu(\tau, \vec{\sigma}) \) be the coordinates of the points on \( \Sigma_\tau \) in \( M^4 \), \( \{ \vec{\sigma} \} \) a system of coordinates on \( \Sigma_\tau \). If \( \sigma^\tilde{A} = (\tau; \vec{\sigma} = \{ \sigma^\tilde{r} \}) \) [the notation \( \tilde{A} = (\tau, \tilde{\tau}) \) with \( \tilde{\tau} = 1, 2, 3 \) will be used; note that \( \tilde{A} = \tau \) and \( \tilde{A} = \tilde{\tau} = 1, 2, 3 \) are Lorentz-scalar indices] and \( \partial_{\tilde{A}} = \partial/\partial\sigma^{\tilde{A}} \), one can define the vierbeins

\[
{z}_\mu^A(\tau, \vec{\sigma}) = \partial_{\tilde{A}} z^\mu(\tau, \vec{\sigma}), \quad \partial_B z^\mu_A - \partial_A z^\mu_B = 0, \tag{1}
\]

so that the metric on \( \Sigma_\tau \) is

\[
g^A_B(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma}) \eta_{\mu\nu} z^\nu_B(\tau, \vec{\sigma}), \quad g_{\tau\tau}(\tau, \vec{\sigma}) > 0,
\]

\[
g(\tau, \vec{\sigma}) = -\text{det} \| g^A_B(\tau, \vec{\sigma}) \| = (\text{det} \| z^\mu_A(\tau, \vec{\sigma}) \|)^2,
\]

\[
\gamma(\tau, \vec{\sigma}) = -\text{det} \| g_{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}) \|. \tag{2}
\]

If \( \gamma^{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}) \) is the inverse of the 3-metric \( g_{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}) \) \( [\gamma^{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}) g_{\tilde{a}\tilde{b}}(\tau, \vec{\sigma}) = \delta^\tilde{A}_\tilde{B}] \), the inverse \( g^{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}) \) of \( g_{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}) \) \( [g^{\tilde{A}\tilde{C}}(\tau, \vec{\sigma}) g_{\tilde{c}\tilde{b}}(\tau, \vec{\sigma}) = \delta^\tilde{A}_\tilde{B}] \) is given by

\[
g^{\tau\tau}(\tau, \vec{\sigma}) = \frac{\gamma(\tau, \vec{\sigma})}{g(\tau, \vec{\sigma})},
\]

\[
g^{\tau\tilde{r}}(\tau, \vec{\sigma}) = -\frac{\gamma^{\tilde{A}\tilde{B}}}{g} g_{\tau\tilde{a}} \gamma^{\tilde{A}\tilde{B}}(\tau, \vec{\sigma}),
\]

\[
g^{\tilde{r}\tilde{s}}(\tau, \vec{\sigma}) = \gamma^{\tilde{r}\tilde{s}}(\tau, \vec{\sigma}) + \frac{\gamma^{\tilde{r}\tilde{a}}}{g} g_{\tau\tilde{a}} \gamma^{\tilde{A}\tilde{B}} g_{\tau\tilde{b}}(\tau, \vec{\sigma}), \tag{3}
\]

so that \( 1 = g^{\tau\tau}(\tau, \vec{\sigma}) g_{\tilde{c}\tau}(\tau, \vec{\sigma}) \) is equivalent to

\[
g(\tau, \vec{\sigma}) \gamma(\tau, \vec{\sigma}) = g_{\tau\tau}(\tau, \vec{\sigma}) - \gamma^{\tilde{r}\tilde{s}}(\tau, \vec{\sigma}) g_{\tau\tilde{r}}(\tau, \vec{\sigma}) g_{\tau\tilde{s}}(\tau, \vec{\sigma}). \tag{4}
\]

We have

\[
z^\mu_r(\tau, \vec{\sigma}) = (\sqrt{\frac{g}{\gamma}} l^\mu + g_{\tau\tau} \gamma^{\tilde{r}\tilde{s}} z^\mu_s)(\tau, \vec{\sigma}), \tag{5}
\]
and

\[ \eta^{\mu\nu} = z^{\mu}_A(\tau, \sigma)g^{AB}(\tau, \sigma)z^{\nu}_B(\tau, \sigma) = \]
\[ = (l^{\mu}l^{\nu} + z^{\mu}_r \gamma^{rs} z^{\nu}_s)(\tau, \sigma), \tag{6} \]

where

\[ l^{\mu}(\tau, \sigma) = \left( \frac{1}{\sqrt{\gamma}} \epsilon_{\alpha\beta\gamma} z^{\alpha}_1 z^{\beta}_2 z^{\gamma}_3 \right)(\tau, \sigma), \]
\[ l^2(\tau, \sigma) = 1, \quad l_\mu(\tau, \sigma)z^{\mu}_r(\tau, \sigma) = 0, \tag{7} \]

is the unit (future pointing) normal to \( \Sigma(\tau) \) at \( z^{\mu}(\tau, \sigma) \).

For the volume element in Minkowski spacetime we have

\[ d^4z = z^{\mu}_r(\tau, \sigma)d\tau d^3\Sigma_\mu = d\tau [z^{\mu}_r(\tau, \sigma)l_\mu(\tau, \sigma)]\sqrt{\gamma(\tau, \sigma)}d^3\sigma = \]
\[ = \sqrt{g(\tau, \sigma)}d\tau d^3\sigma. \tag{8} \]

Let us remark that according to the geometrical approach of Ref. [10], one can use Eq.(5) in the form \( z^{\mu}_r(\tau, \sigma) = N(\tau, \sigma)l^{\mu}(\tau, \sigma) + N^r(\tau, \sigma)z^{\mu}_r(\tau, \sigma) \), where \( N = \sqrt{g/\gamma} = \sqrt{g_{\tau\tau} - \gamma^{rs}g_{\tau s}g_{\tau s}} \) and \( N^r = g_{rs}\gamma^{rs} \) are the standard lapse and shift functions, so that \( g_{\tau\tau} = N^2 + g_{rs}N^r N^s, g_{\tau s} = g_{rs}N^s \), \( g_{\tau r} = N^{-2}, g_{\tau r} = -N^r/N^2, g_{rs} = \gamma^{rs} + \frac{N^r N^s}{N^2} \).

\[ \frac{\partial}{\partial z^{\mu}_r} = l_\mu \frac{\partial}{\partial N} + z^{\nu}_s \gamma^{rs} \frac{\partial}{\partial N^r}, \quad d^4z = N\sqrt{\gamma}d\tau d^3\sigma. \]

The rest frame form of a timelike fourvector \( p^{\mu} \) is \( \hat{p}^{\mu} = \eta\sqrt{\hat{p}^2}(1; \hat{0}) = \eta^{\mu\nu}\eta\sqrt{\hat{p}^2}, \hat{p}^2 = p^2 \),

where \( \eta = \text{sign} \ p^0 \). The standard Wigner boost transforming \( \hat{p}^{\mu} \) into \( p^{\mu} \) is

\[ L^{\nu}_\mu(p, \hat{p}) = e^{\mu}_\nu(u(p)) = \]
\[ = \eta^{\mu}_\nu + 2\hat{p}_\mu \hat{p}_\nu \frac{(p^{\mu} + \hat{p}^{\mu})(p_\nu + \hat{p}_\nu)}{p \cdot \hat{p} + p^2} = \]
\[ = \eta^{\mu}_\nu + 2u^{\mu}(p)u_\nu(\hat{p}) - \frac{(u^{\mu}(p) + u^{\mu}(\hat{p}))(u_\nu(p) + u_\nu(\hat{p}))}{1 + u^0(p)}, \]

\[ \nu = 0 \quad e^{\mu}_0(u(p)) = u^{\mu}(p) = p^\mu/\eta \sqrt{p^2}, \]
\[ \nu = r \quad e^{\mu}_r(u(p)) = (-u_r(p); \delta^i_r - \frac{u^i(p)u_r(p)}{1 + u^0(p)}). \tag{9} \]
The inverse of \( L^\mu_\nu(p, \vec{p}) \) is \( L^\mu_\nu(\vec{p}, p) \), the standard boost to the rest frame, defined by

\[
L^\mu_\nu(\vec{p}, p) = L^\mu_\nu(p, \vec{p}) = L^\mu_\nu(p, \vec{p})|_{\vec{p} \to -\vec{p}}. \tag{10}
\]

Therefore, we can define the following vierbeins [the \( \epsilon^\mu_\nu(u(p)) \)'s are also called polarization vectors; the indices \( r, s \) will be used for \( A=1,2,3 \) and \( \bar{o} \) for \( A=0 \)]

\[
\begin{align*}
\epsilon_A^\mu(u(p)) &= L^\mu_A(p, \vec{p}), \\
\epsilon^r_\mu(u(p)) &= \eta_{\mu\nu} \epsilon^\nu_r(u(p)) = u_\mu(p), \\
\epsilon^r_\mu(u(p)) &= -\delta^{rs} \eta_{\mu\nu} \epsilon^\nu_r(u(p)) = (\delta^{rs} u_s(p); \delta^r_j - \delta^{rs} \delta_{jh} u_h(p)) \frac{u^h(p) u_s(p)}{1 + u^o(p)}, \\
\epsilon^A_\mu(u(p)) &= \eta_{AB} \epsilon^\nu_B(u(p)), \\
\epsilon_{\bar{o}}^A(u(p)) &= u_A(p),
\end{align*}
\tag{11}
\]

which satisfy

\[
\begin{align*}
\epsilon_A^\mu(u(p))\epsilon_A^\nu(u(p)) &= \eta^\mu_\nu, \\
\epsilon^A_\mu(u(p))\epsilon^A_\nu(u(p)) &= \eta^A_\nu, \\
\eta^\mu_\nu &= \epsilon_A^\mu(u(p))\eta^{AB} \epsilon^\nu_B(u(p)) = u^\mu(p) u^\nu(p) - \sum_{r=1}^3 \epsilon^\mu_r(u(p)) \epsilon^\nu_r(u(p)), \\
\eta_{AB} &= \epsilon_A^\mu(u(p))\eta_{\mu\nu} \epsilon^\nu_B(u(p)), \\
p_{\alpha} \frac{\partial}{\partial p_{\alpha}} \epsilon_A^\mu(u(p)) &= p_{\alpha} \frac{\partial}{\partial p_{\alpha}} \epsilon_A^\mu(u(p)) = 0. \tag{12}
\end{align*}
\]

The Wigner rotation corresponding to the Lorentz transformation \( \Lambda \) is

\[
R^\mu_\nu(\Lambda, p) = [L(\vec{p}, p) \Lambda^{-1} L(\Lambda p, \vec{p})]_\nu^\mu = \begin{pmatrix} 1 & 0 \\ 0 & R^i_j(\Lambda, p) \end{pmatrix},
\]

\[
R^i_j(\Lambda, p) = (\Lambda^{-1})^i_j - \frac{(\Lambda^{-1})^i_\alpha p_\beta (\Lambda^{-1})^\beta_j}{p^\alpha + \eta \sqrt{p^2}} - \frac{p^i}{p^\alpha + \eta \sqrt{p^2}} [(\Lambda^{-1})^i_\alpha - \frac{((\Lambda^{-1})^o_\alpha - 1) p_\beta (\Lambda^{-1})^\beta_j}{p^\alpha + \eta \sqrt{p^2}}]. \tag{13}
\]

The polarization vectors transform under the Poincaré transformations \((a, \Lambda)\) in the following way
\[\epsilon^\nu_\mu(u(\Lambda p)) = (R^{-1})_\nu^s A^\nu_\mu \epsilon^\nu_s(u(p)). \quad (14)\]

On the hypersurface \(\Sigma_\tau\), we describe the color SU(3) potential and field strength with Lorentz-scalar variables \(A_{a\hat{A}}(\tau, \vec{\sigma})\) and \(F_{a\hat{A}\hat{B}}(\tau, \vec{\sigma})\) respectively: they contain the embedding \(\Sigma(\tau) \to M^4\) and are defined by

\[A_{a\hat{A}}(\tau, \vec{\sigma}) = z^\mu_\hat{A}(\tau, \vec{\sigma}) A_\mu(z(\tau, \vec{\sigma})), \]
\[F_{a\hat{A}\hat{B}}(\tau, \vec{\sigma}) = \partial_\hat{A} A_{a\hat{B}}(\tau, \vec{\sigma}) - \partial_\hat{B} A_{a\hat{A}}(\tau, \vec{\sigma}) + c_{abc} A_{b\hat{A}}(\tau, \vec{\sigma}) A_{c\hat{B}}(\tau, \vec{\sigma}) =
\]
\[= z^\mu_\hat{A}(\tau, \vec{\sigma}) z^\nu_\hat{B}(\tau, \vec{\sigma}) F_{a\mu\nu}(z(\tau, \vec{\sigma})) = z^\mu_\hat{A}(\tau, \vec{\sigma}) z^\nu_\hat{B}(\tau, \vec{\sigma}) |\partial_\mu A_{a\nu}(z(\tau, \vec{\sigma})) - \partial_\nu A_{a\mu}(z(\tau, \vec{\sigma})) + c_{abc} A_{b\mu}(z(\tau, \vec{\sigma})) A_{c\nu}(z(\tau, \vec{\sigma}))]. \quad (15)\]

We could have written \(A_\mu = z^\hat{A}_\mu A_\hat{A} = l_\mu A_l + z^\hat{A}_\mu A_\hat{A} \ [z^\hat{A}_\mu \text{ are the inverse vierbeins}], \) so to get

\[A_\tau(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) A_l(\tau, \vec{\sigma}) + N^\tau(\tau, \vec{\sigma}) A_\hat{A}(\tau, \vec{\sigma}), \quad (16)\]

and we could have used \(A_l(\tau, \vec{\sigma})\) as the genuine field configuration variable independent from the motion of the embedded hypersurface, as suggested in Ref. [10]. However, this more geometric formulation is equivalent to the simpler one of Ref. [7] [see its Appendix C] for spin 1 fields, so that in this paper we shall go on to use \(A_\tau\) rather than \(A_l\).

The generators \(\hat{T}^a\) of the Lie algebra su(3) of color \([\hat{A}_\mu = A_{a\hat{A}} \hat{T}^a]\) in the 8-dimensional adjoint representation of SU(3) and those \(T^a\) in the 3-dimensional fundamental one are \(c_{abc}\) are the SU(3) totally antisymmetric structure constants]

\[\hat{T}^a = -\hat{T}^a, \quad (\hat{T}^a)_{bc} = c_{abc}, \quad [\hat{T}^a, \hat{T}^b] = c_{abc} \hat{T}^c, \]
\[T^a = -T^a, \quad [T^a, T^b] = c_{abc} T^c, \quad T^a = -\frac{i}{2} \lambda_a, \quad (17)\]

where the \(\lambda_a\)'s era the 3 \(\times\) 3 Gell-Mann matrices.

The covariant derivative associated with \(A_{a\mu}\) is

\[(\hat{D}^A_\mu)_{ac} = \delta_{ac} \partial_\mu + c_{abc} A_{b\mu} = (\partial_\mu - \hat{A}_\mu)_{ac} \quad (18)\]
and the gauge transformations are defined as \( [\hat{F}_{\mu\nu} = F_{\mu\nu}T^a] \)

\[
\hat{A}_\mu(x) \mapsto \hat{A}^U_\mu(x) = U^{-1}(x)\hat{A}_\mu(x)U(x) + U^{-1}(x)\partial_\mu U(x) = \\
= \hat{A}_\mu(x) + U^{-1}(x)(\partial_\mu U(x) + [\hat{A}_\mu(x), U(x)]),
\]

\[
\hat{F}_{\mu\nu}(x) \mapsto \hat{F}^U_{\mu\nu}(x) = U^{-1}(x)\hat{F}_{\mu\nu}(x)U(x) = \hat{F}_{\mu\nu}(x) + U^{-1}(x)[\hat{F}_{\mu\nu}(x), U(x)].
\]

(19)

Here \( U \) is the realization in the adjoint representation of the SU(3) gauge transformations.

The scalar particles with Minkowski coordinates \( x^\mu_i(\tau), i=1,...,N, \) are identified on the spacelike hypersurface \( \Sigma_\tau \) by 3 numbers \( \vec{\eta}_i(\tau), i=1,...,N, \) by the equation \( x^\mu_i(\tau, \vec{\eta}_i(\tau)) = z^\mu(\tau, \vec{\eta}_i(\tau)) \) [so that \( \dot{x}_i^\mu(\tau) = z_i^\mu(\tau, \vec{\eta}_i(\tau)) + z_i^\mu(\tau, \vec{\eta}_i(\tau))\dot{\vec{\eta}}_i(\tau) \) and \( \eta_i = \dot{x}_i^\mu(\tau) \)]. As shown in Ref. [7] this implies that the mass shell constraint \( p_i^2 - m_i^2 = 0 \) has been solved and a choice of the sign of the energy, \( \eta_i = \text{sign} p_i^\mu, \) has been done.

In this paper we consider the case of \( N \) relativistic scalar particles with the color SU(3) charge of each particle described in a pseudoclassical way [15] [see also Refs. [16] for pseudoclassical mechanics] by means of 3 pairs of complex conjugate Grassmann variables \( \theta_{i\alpha}(\tau), \theta_{i\alpha}^*(\tau), \alpha = 1, 2, 3, \) which belong to the fundamental representation of SU(3). They satisfy

\[
\theta_{i\alpha}\theta_{i\beta} + \theta_{i\beta}\theta_{i\alpha} = 0,
\]

\[
\theta_{i\alpha}^*\theta_{j\beta}^* + \theta_{i\beta}^*\theta_{j\alpha}^* = 0,
\]

\[
\theta_{i\alpha}\theta_{j\beta}^* + \theta_{j\beta}^*\theta_{i\alpha} = 0,
\]

(20)

and the Grassmann variables of different particles are assumed to commute

\[
\theta_{i\alpha}\theta_{j\beta} = \theta_{j\beta}\theta_{i\alpha}, \quad i \neq j
\]

\[
\theta_{i\alpha}^*\theta_{j\beta}^* = \theta_{j\beta}^*\theta_{i\alpha}^*;
\]

\[
\theta_{i\alpha}\theta_{j\beta}^* = \theta_{j\beta}^*\theta_{i\alpha}.
\]

(21)

The color charges of the particles are \( Q_{ia}(\tau) = i\sum_{i=1}^{3} \theta_{i\alpha}^*(\tau)(T^a)_{\alpha\beta}\theta_{i\beta}(\tau) \) \( [Q_{ia}^* = Q_{ia} \) since \( T^{a\dagger} = -T^a] \).
At the quantum level [15] (see also Ref. [17]) the Grassmann variables $\theta_{i\alpha}^*, \theta_{i\alpha}$ go into Fermi oscillators $b_{i\alpha}^\dagger, b_{i\alpha}$ satisfying the anticommutation relations $[b_{i\alpha}, b_{j\beta}^\dagger] = \delta_{ij} \delta_{\alpha\beta}$, $[b_{i\alpha}, b_{j\beta}] = [b_{i\alpha}^\dagger, b_{j\beta}^\dagger] = 0$. For each particle there is a 8-dimensional Hilbert space of charge states [the charge operator is $\hat{Q}_{ia} = i \sum_{\alpha\beta} b_{i\alpha}^\dagger (T^a)_{\alpha\beta} b_{i\beta}$] with basis $|0_i>$, $b_{i\alpha}^\dagger |0_i>$, $b_{i\alpha} b_{i\beta}^\dagger |0_i>$, $b_{i\alpha} b_{i\beta} b_{i\gamma}^\dagger |0_i>$: the states with k=0,1,2,3, oscillators transform like a completely antisymmetric representation of dimension $\binom{3}{k}$ of SU(3). Therefore, the space of charge states for each particle transforms like the reducible representation $1 \oplus 3 \oplus 3^* \oplus 1$ of SU(3) of dimension $\sum_{k=0}^{3} \binom{3}{k} = 2^3$.

To select the triplet (quark) or antitriplet (antiquark) representation, we shall add to the pseudoclassical theory the constraint

$$N_i = \sum_{\alpha} \theta_{i\alpha}^* \theta_{i\alpha} \approx 0. \quad (22)$$

As shown in Ref. [18], after quantization $N_i$ is replaced by $\hat{N}_i[A_i] = \sum_{\alpha=0}^{3} b_{i\alpha}^\dagger b_{i\alpha} - A_i = \hat{n}_i - A_i$, where $\hat{n}_i$ is the occupation number selecting the $\binom{3}{k}$-dimensional representation of SU(3) and $A_i$ is an arbitrary c-number present due to ordering problems. Therefore, with the prescription $A_i = 1$ or 2, the constraint $N_i \approx 0$ becomes the quantum constraint $\hat{N}_i[1]|0_i> = 0$ or $\hat{N}_i[2]|0_i> = 0$, selecting the triplet or the antitriplet representation respectively for particle ‘i’.

With the constraint $N_i \approx 0$ for each i one has $Q_{ia}Q_{ib}Q_{ic} = 0$ because it is proportional to $N_i^3$ [$\equiv 0$ in the Dirac strong sense]. Moreover, since one has $\sum_a Q_{ia}^2 = -\theta_{i\alpha}^* \theta_{i\beta}^* \theta_{i\delta}^* \theta_{i\gamma}^* \sum_a (T^a)_{\alpha\beta}(T^a)_{\gamma\delta}$ with $\sum_a (T^a)_{\alpha\beta}(T^a)_{\gamma\delta} = \frac{1}{6} \delta_{\alpha\beta} \delta_{\gamma\delta} - \frac{1}{2} \delta_{\alpha\delta} \delta_{\beta\gamma}$ [valid in the fundamental representation of SU(3)], one gets

$$\sum_a Q_{ia}^2 = -\frac{2}{3} N_i^2 \equiv 0. \quad (23)$$
III. THE LAGRANGIAN FOR COLORED PARTICLES PLUS YANG-MILLS FIELD

The system of $N$ colored scalar particles plus the SU(3) Yang-Mills field is described by the action

$$S = \int d\tau d^3\sigma \mathcal{L}(\tau, \vec{\sigma}) = \int d\tau L(\tau),$$

$$L(\tau) = \int d^3\sigma \mathcal{L}(\tau, \vec{\sigma}),$$

$$\mathcal{L}(\tau, \vec{\sigma}) = \frac{i}{2} \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \sum_{\alpha=1}^{3} [\theta_{i\alpha}^*(\tau) \dot{\theta}_{i\alpha}(\tau) - \dot{\theta}_{i\alpha}(\tau) \theta_{i\alpha}(\tau) + \lambda_i(\tau) \sum_{\alpha=1}^{3} \theta_{i\alpha}^*(\tau) \theta_{i\alpha}(\tau)] -$$

$$- \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) [\eta_i m_i \sqrt{g_{rr}(\tau, \vec{\sigma}) + 2 g_{rf}(\tau, \vec{\sigma}) \tilde{\eta}_i^r(\tau) + g_{fs}(\tau, \vec{\sigma}) \tilde{\eta}_i^s(\tau) \tilde{\eta}_i^s(\tau)] -$$

$$- \sum_{a} Q_{ia}(A_{a\dot{\tau}}(\tau, \vec{\sigma}) + A_{a\tau}(\vec{\sigma}) \tilde{\eta}_i^\tau(\tau)) -$$

$$- \frac{1}{4 g_s^2 \sqrt{g(\tau, \vec{\sigma})}} g^{\mathcal{A}\mathcal{C}}(\tau, \vec{\sigma}) g^{\mathcal{B}\mathcal{D}}(\tau, \vec{\sigma}) \sum_{a} F_{a\mathcal{A}\mathcal{B}}(\tau, \vec{\sigma}) F_{a\mathcal{C}\mathcal{D}}(\tau, \vec{\sigma}),$$

where the configuration variables are $z^\mu(\tau, \vec{\sigma}) A_{a\dot{\tau}}(\tau, \vec{\sigma}), \vec{\eta}_i(\tau), \theta_{i\alpha}(\tau)$ and $\theta_{i\alpha}^*(\tau), i=1, \ldots, N$. The particles have Grassmann-valued charges $Q_{ai}(\tau) = i \sum_{\alpha=1}^{3} \theta_{i\alpha}^*(\tau) (T^a)_{\alpha\beta} \theta_{i\beta}(\tau)$.

We have

$$-\frac{1}{4} \sqrt{g} g^{a\mathcal{A}} g^{b\mathcal{B}} \sum_{a} F_{a\mathcal{A}\mathcal{B}} F_{a\mathcal{C}\mathcal{D}} =$$

$$= -\frac{1}{4} \sqrt{g} \sum_{a} [2 (g^{r\tau} g^{s\dot{s}} - g^{r\dot{s}} g^{s\tau}) F_{a\tau r} F_{a\dot{s} s} + 4 g^{r\dot{s}} g^{s\tau} F_{a\tau r} F_{a\dot{s} s} + g^{r\dot{u}} g^{s\dot{v}} F_{a\tau r} F_{a\dot{v} s} F_{a\dot{u} s}] =$$

$$= -\sqrt{N} \sum_{a} \frac{1}{2} \sqrt{g} F_{a\tau r} \gamma^{rs} F_{a\dot{s} s} - \frac{\sqrt{N}}{2} g_{rr} \gamma^{rs} F_{a\tau r} \gamma^{s\dot{s}} F_{a\tau s} - \frac{1}{4} \sqrt{g} \gamma^{rs} F_{a\tau r} \gamma^{s\dot{s}} F_{a\dot{s} s} +$$

$$+ 2 \frac{\sqrt{N}}{4} g_{rr} \gamma^{s\dot{u}} g_{rs} \gamma^{s\dot{v}} F_{a\tau r} \gamma^{s\dot{u}} F_{a\dot{s} s}] = -\frac{\sqrt{N}}{2} (F_{a\tau r} - N^\tau F_{a\dot{s} s}) \gamma^{rs} (F_{a\tau s} - N^s F_{a\dot{u} u}) - \frac{\sqrt{N}}{4} \gamma^{rs} \gamma^{s\dot{u}} F_{a\tau r} F_{a\dot{u} s}. $$

The action is invariant under separate $\tau$- and $\vec{\sigma}$-reparametrizations, since $A_{a\tau}(\tau, \vec{\sigma})$ transforms as a $\tau$-derivative; moreover, it is invariant under the odd phase transformations $\delta \theta_{i\alpha} \to \alpha(T^a)_{\alpha\beta} \theta_{i\beta}$.

The canonical momenta are $[E_{a\tau} = F_{a\tau r}$ and $B_{a\tau} = \frac{1}{2} \epsilon_{\tau\dot{s}i} F_{a\dot{s} i} (\epsilon_{\tau\dot{s}i} = \epsilon^{\dot{s}i})$ are the electric and magnetic fields respectively; for $g_{\mathcal{A}\mathcal{B}} \to \eta_{\mathcal{A}\mathcal{B}}$ one gets $\pi^\tau_a = -E_{a\tau} = E^\tau_a]$

$$\rho_{\mu}(\tau, \vec{\sigma}) = -\frac{\partial \mathcal{L}(\tau, \vec{\sigma})}{\partial z_{\mu}(\tau, \vec{\sigma})} = \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \eta_i m_i$$

$$\frac{z_{\mu\tau}(\tau, \vec{\sigma}) + z_{\mu\tau}(\vec{\sigma}) \tilde{\eta}_i^\tau(\tau)}{\sqrt{g_{rr}(\tau, \vec{\sigma}) + 2 g_{r\tau}(\tau, \vec{\sigma}) \tilde{\eta}_i^r(\tau) + g_{s\tau}(\tau, \vec{\sigma}) \tilde{\eta}_i^s(\tau) \tilde{\eta}_i^s(\tau)}} +$$

13
\[
+ \sqrt{\frac{g(\tau, \bar{\sigma})}{4}} \left[ (g^{r \tau} z_{\tau \mu} + g^{r \bar{\tau}} z_{\bar{\tau} \mu}) (\tau, \bar{\sigma}) g^{A C}(\tau, \bar{\sigma}) g^{B D}(\tau, \bar{\sigma}) \sum_a F_{a A B}(\tau, \bar{\sigma}) F_{a C D}(\tau, \bar{\sigma}) - 2 |z_{\tau \mu}(\tau, \bar{\sigma}) (g^{A r} g^{r C} g^{B D} + g^{A \bar{r}} g^{\bar{r} C} g^{B D})(\tau, \bar{\sigma}) + z_{\bar{r} \mu}(\tau, \bar{\sigma}) (g^{A r} g^{r C} + g^{A \bar{r}} g^{\bar{r} C})(\tau, \bar{\sigma}) g^{B D}(\tau, \bar{\sigma}) \right] 
\]

\[
\pi^*_a(\tau, \bar{\sigma}) = \frac{\partial L}{\partial \partial_{\tau} A_{a r}(\tau, \bar{\sigma})} = 0,
\]

\[
\pi^a(\tau, \bar{\sigma}) = \frac{\partial L}{\partial \partial_{\tau} A_{a \bar{r}}(\tau, \bar{\sigma})} = -g_s^{-2} \frac{\gamma(\tau, \bar{\sigma})}{\sqrt{g(\tau, \bar{\sigma})}} \gamma^{\hat{s}}(\tau, \bar{\sigma}) (F_{a r \hat{s}} + g_{\tau \bar{\sigma}} \gamma^{\hat{s}} A_{a \bar{s} \hat{s}})(\tau, \bar{\sigma}) = \frac{g_s^{-2} \gamma(\tau, \bar{\sigma})}{\sqrt{g(\tau, \bar{\sigma})}} \gamma^{\hat{s}}(\tau, \bar{\sigma}) (E_{a \bar{s}}(\tau, \bar{\sigma}) + g_{\tau \bar{\sigma}}(\tau, \bar{\sigma}) \gamma^{\hat{s} \hat{a}}(\tau, \bar{\sigma}) \epsilon_{\bar{a} \hat{s} \hat{\bar{s}}} B_{a \hat{s}}(\tau, \bar{\sigma})),
\]

\[
\kappa_{i \bar{r}}(\tau) = -\frac{\partial L(\tau)}{\partial \eta^i_{\bar{r}}(\tau)} = \eta_i m_i \frac{g_{\tau \bar{r}}(\tau, \bar{\eta}_i(\tau)) + g_{\tau \bar{s}}(\tau, \bar{\eta}(\tau)) \eta^i \eta^i(\tau) + g_{\tau \bar{s}}(\tau, \bar{\eta}(\tau)) \eta^i(\tau) \eta^i(\tau)}{\sqrt{g_{\tau \tau}(\tau, \bar{\eta}_i(\tau)) + g_{\tau \bar{r}}(\tau, \bar{\eta}(\tau)) \eta^i(\tau) + g_{\tau \bar{s}}(\tau, \bar{\eta}(\tau)) \eta^i(\tau) \eta^i(\tau)}} - \sum_a Q_{i a}(\tau) A_{a r}(\tau, \bar{\eta}_i(\tau)),
\]

\[
\frac{\partial L(\tau)}{\partial \theta^*_{i \alpha}(\tau)} = -\frac{i}{2} \theta^*_{i \alpha}(\tau),
\]

\[
\frac{\partial L(\tau)}{\partial \theta_{i \alpha}(\tau)} = -\frac{i}{2} \theta_{i \alpha}(\tau),
\]

(25)

The Grassmann momenta give rise to the second class constraints \[ \pi_{\theta i \alpha} + \frac{i}{2} \theta^*_{i \alpha} \approx 0, \]
\[ \pi_{\theta^* i\alpha} + \frac{i}{2} \theta_{i\alpha} \approx 0 \{ \pi_{\theta i\alpha} + \frac{i}{2} \theta_{i\alpha}^*, \pi_{\theta^* j\beta} + \frac{i}{2} \theta_{j\beta}^* \} = -i \delta_{ij} \delta_{\alpha\beta}; \] 
\[ \pi_{\theta i\alpha} \text{ and } \pi_{\theta^* i\alpha} \text{ are then eliminated with the help of Dirac brackets} \]

\[ \{ A, B \}^* = \{ A, B \} - i \sum_{\alpha=1}^{3} \left[ \{ A, \pi_{\theta i\alpha} + \frac{i}{2} \theta_{i\alpha}^* \} \{ \pi_{\theta^* i\alpha} + \frac{i}{2} \theta_{i\alpha}^*, B \} \right. \]
\[ + \{ A, \pi_{\theta^* i\alpha} + \frac{i}{2} \theta_{i\alpha}^* \} \{ \pi_{\theta i\alpha} + \frac{i}{2} \theta_{i\alpha}^*, B \} \right] \]

(27)

so that the remaining Grassmann variables have the fundamental Dirac brackets [which we will still denote \{\ldots, \ldots\} for the sake of simplicity]

\[ \{ \theta_{i\alpha}(\tau), \theta_{j\beta}(\tau) \} = \{ \theta_{i\alpha}^*(\tau), \theta_{j\beta}^*(\tau) \} = 0, \]
\[ \{ \theta_{i\alpha}(\tau), \theta_{j\beta}^*(\tau) \} = -i \delta_{ij} \delta_{\alpha\beta}. \]

(28)

These equations imply \( \{ Q_{ia}, Q_{jb} \} = \delta_{ij} c_{abc} Q_{ic} \) and that the gauge transformations \( \delta \theta_{i\alpha} = (\alpha_a T^a)_{\alpha\beta} \theta_{i\beta} \), under which the action is invariant, are generated by the \( Q_{ia} \)'s.

By varying the Lagrange multipliers \( \lambda_i(\tau) \), we get the Grassmann constraints mentioned in Section II

\[ N_i(\tau) = \sum_{\alpha=1}^{3} \theta_{i\alpha}^*(\tau) \theta_{i\alpha}(\tau) \approx 0. \]

(29)

We could also treat the multipliers \( \lambda_i(\tau) \) as configuration variables: we would get the first class constraints \( \pi_{\lambda_i}(\tau) \approx 0 \) and we could get free of the \( \lambda_i \)'s by adding the gauge-fixings \( \lambda_i(\tau) \approx 0 \) and going to the Dirac brackets for the resulting 2N second class constraints.

From the expression of the momenta we obtain the four primary constraints

\[ \mathcal{H}_\mu(\tau, \vec{\sigma}) = \rho_\mu(\tau, \vec{\sigma}) - l_\mu(\tau, \vec{\sigma}) [T_{\tau\tau}(\tau, \vec{\sigma}) + \]
\[ + \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \times \]
\[ \eta_i \sqrt{m_i^2 - \gamma^{\epsilon\delta}(\tau, \vec{\sigma}) [\kappa_{\epsilon\delta}(\tau) + \sum_{a} Q_{ia}(\tau) A_{a\epsilon}(\tau, \vec{\sigma})] [\kappa_{\delta\epsilon}(\tau) + \sum_{b} Q_{ib}(\tau) A_{b\delta}(\tau, \vec{\sigma})] - \]
\[ - z_{\epsilon\mu}(\tau, \vec{\sigma}) \gamma^{\epsilon\delta}(\tau, \vec{\sigma}) \{ - T_{\tau\epsilon}(\tau, \vec{\sigma}) + \sum_{i=1}^{N} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) [\kappa_{\epsilon\delta} + \sum_{a} Q_{ia}(\tau) A_{a\epsilon}(\tau, \vec{\sigma})] \} \approx 0, \]

(30)

where
\[ T_{\tau\tau}(\tau, \vec{\sigma}) = -\frac{1}{2} \sum_a (g_a^2 - \pi_a^x g_{\delta x} \pi_a^x - \frac{\sqrt{\gamma}}{2} g^a_\delta \gamma^\delta_\mu \Gamma_{a\mu}(\tau, \vec{\sigma})), \]
\[ T_{\tau\delta}(\tau, \vec{\sigma}) = -\sum_a \Gamma_{a\delta}(\tau, \vec{\sigma}) = -\epsilon_{\delta a} \sum_a \rho_a^\delta(\tau, \vec{\sigma}) B_{a0}(\tau, \vec{\sigma}) = \]
\[ = \sum_a [\pi_a^\delta(\tau, \vec{\sigma}) \times \vec{B}_a(\tau, \vec{\sigma})], \quad (31) \]

are the energy density and the Poynting vector respectively. We use the notation \( \sum_a (\pi_a^x \times \vec{B}_a) = (\vec{E} \times \vec{B})_s \) because it is consistent with \( \epsilon_{\delta a} \sum_a \rho_a^\delta B_{a0} \) in the flat metric limit \( g_{A\bar{B}} \rightarrow \eta_{AB} \); in this limit \( T_{\tau\tau} \rightarrow \frac{1}{2} \sum_a (\vec{E}_a^2 + \vec{B}_a^2) \).

Since the canonical Hamiltonian is (we assume boundary conditions for the electromagnetic potential such that all the surface terms can be neglected; see Ref. [1])

\[ H_c = -\sum_{i=1}^N \kappa_i \pi_i^\delta(\tau) - \int d^3 \sigma [\sum_a \pi_a^\delta(\tau, \vec{\sigma}) \partial_\tau A_{a\delta}(\tau, \vec{\sigma}) - \rho_\tau(\tau, \vec{\sigma}) \sigma^\delta(\tau, \vec{\sigma}) - \mathcal{L}(\tau, \vec{\sigma})] = \]
\[ = \int d^3 \sigma \sum_a [\pi_a^\delta(\tau, \vec{\sigma}) A_{a\tau}(\tau, \vec{\sigma}) - A_{a\tau}(\tau, \vec{\sigma}) \Gamma_a(\tau, \vec{\sigma})] = \]
\[ = -\int d^3 \sigma \sum_a A_{a\tau}(\tau, \vec{\sigma}) \Gamma_a(\tau, \vec{\sigma}), \quad (32) \]

with \[ \partial_\tau = \frac{\partial}{\partial \tau} = -\partial^\tau, \quad \vec{\partial} = \{\partial^\rho, \partial^\mu\}, \quad \Delta = -\vec{\partial}^2, \quad \Gamma_A = \{\delta_{ab} \partial^\rho + c_{abc} A_c^\rho \} \]

\[ \Gamma_a(\tau, \vec{\sigma}) = \partial_\tau \pi_a^\delta(\tau, \vec{\sigma}) + c_{abc} A_b^\delta(\tau, \vec{\sigma}) \pi_c^\delta(\tau, \vec{\sigma}) + \sum_{i=1}^N Q_{ia}(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) = \]
\[ = -\hat{D}_{ab}(\tau, \vec{\sigma}) \cdot \pi_b(\tau, \vec{\sigma}) + \sum_{i=1}^N Q_{ia}(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)), \quad (33) \]

we have the Dirac Hamiltonian \( (\lambda^\mu(\tau, \vec{\sigma}), \lambda_\tau(\tau, \vec{\sigma}) \) and \( \mu_i(\tau) \) are Dirac’s multipliers)

\[ H_D = \int d^3 \sigma [\lambda^\mu(\tau, \vec{\sigma}) \mathcal{H}_\mu(\tau, \vec{\sigma}) + \sum_a \lambda_{a\tau}(\tau, \vec{\sigma}) \pi^a(\tau, \vec{\sigma}) - \sum_a A_{a\tau}(\tau, \vec{\sigma}) \Gamma_a(\tau, \vec{\sigma})] + \]
\[ + \sum_{i=1}^N \mu_i(\tau) N_i(\tau). \quad (34) \]

The Lorentz scalar constraints \( \pi^a(\tau, \vec{\sigma}) \approx 0 \) are generated by the electromagnetic gauge invariance of the action; their time constancy will produce the only secondary constraints (Gauss laws)

\[ \Gamma_a(\tau, \vec{\sigma}) \approx 0. \quad (35) \]
The six constraints \( \mathcal{H}_\mu(\tau, \vec{\sigma}) \approx 0, \pi_\mu(\tau, \vec{\sigma}) \approx 0, \Gamma_a(\tau, \vec{\sigma}) \approx 0 \) are first class with the only non vanishing Poisson brackets

\[
\{ \mathcal{H}_\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma}') \} =
\{ [l_\mu(\tau, \vec{\sigma})z_{\theta^\nu}(\tau, \vec{\sigma}) - l_\nu(\tau, \vec{\sigma})z_{\theta^\mu}(\tau, \vec{\sigma})] \frac{\pi^i(\tau, \vec{\sigma})}{\sqrt{\gamma(\tau, \vec{\sigma})}} -
- z_{\mu^0}(\tau, \vec{\sigma})\gamma^\bar{a}(\tau, \vec{\sigma}) \sum_a F_{a\bar{s}}(\tau, \vec{\sigma})\gamma^\bar{s}(\tau, \vec{\sigma})z_{\theta^\nu}(\tau, \vec{\sigma}) \} \Gamma_a(\tau, \vec{\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}') \approx 0. \tag{36}
\]

Moreover, since \( \{ Q_{ia}, N_j \} = 0 \), also the constraints \( N_i \approx 0 \) are first class and, being constants of the motion for each \( i \), they do not generate secondaries.

Let us remark that the simplicity of Eqs.(36) is due to the use of Cartesian coordinates: had we used the constraints \( \mathcal{H}_l(\tau, \vec{\sigma}) = l_\mu(\tau, \vec{\sigma})\mathcal{H}_\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma})\mathcal{H}_\mu(\tau, \vec{\sigma}) \) (i.e. nonholonomic coordinates), so that their associated Dirac multipliers \( \lambda_l(\tau, \vec{\sigma}), \lambda_\nu(\tau, \vec{\sigma}) \) would have been the lapse and shift functions of general relativity, one would have obtained the universal algebra of Ref. [8].

The ten conserved Poincaré generators are

\[
P^\mu = p^\mu_s = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}),
J^{\mu\nu} = J^{\mu\nu}_s = \int d^3\sigma (z^\mu(\tau, \vec{\sigma})\rho^\nu(\tau, \vec{\sigma}) - z^\nu(\tau, \vec{\sigma})\rho^\mu(\tau, \vec{\sigma})), \tag{37}
\]

so that the total momentum is built starting from the existing energy momentum densities on the hypersurface

\[
\int d^3\sigma \mathcal{H}_\mu(\tau, \vec{\sigma}) = p^\mu_s - \int d^3\sigma l_\mu(\tau, \vec{\sigma})[T_{\tau\tau}(\tau, \vec{\sigma}) +
+ \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)) \times
\eta_i \sqrt{m_i^2 - \gamma^\bar{s}\gamma(\tau, \vec{\sigma})[\kappa_{i\bar{s}}(\tau) + \sum_a Q_{ia}(\tau)A_{a\bar{s}}(\tau, \vec{\sigma})][\kappa_{i\bar{s}}(\tau) + \sum_b Q_{ib}(\tau)A_{b\bar{s}}(\tau, \vec{\sigma})]} -
- \int d^3\sigma z_{\mu^0}(\tau, \vec{\sigma})\gamma^\bar{s}(\tau, \vec{\sigma})[T_{\tau\bar{s}}(\tau, \vec{\sigma}) +
+ \sum_{i=1}^N \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))[\kappa_{i\bar{s}}(\tau) + \sum_a Q_{ia}(\tau)A_{a\bar{s}}(\tau, \vec{\sigma})]] \approx 0. \tag{38}
\]

We add the gauge-fixings to arbitrary hyperplanes as in Ref. [7]
\[ \zeta^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) - x^\mu_s(\tau) - b^\mu_\tau(\tau)\sigma^\tau \approx 0, \]  

(39)

where \( b^\mu_\tau(\tau) \), \( \bar{\tau} = 1, 2, 3 \), are three orthonormal vectors such that the constant (future pointing) normal to the hyperplane is

\[ l^\mu(\tau, \vec{\sigma}) \approx l^\mu = b^\mu_\tau = \epsilon^\mu_{\alpha\beta\gamma} b^\alpha_1(\tau) b^\beta_2(\tau) b^\gamma_3(\tau). \]  

(40)

Therefore, we get

\[ z^\mu_\tau(\tau, \vec{\sigma}) \approx b^\mu_\tau(\tau), \]
\[ z^\mu_s(\tau, \vec{\sigma}) \approx \dot{x}^\mu_s(\tau) + \dot{b}^\mu_\tau(\tau)\sigma^\tau, \]
\[ g_{\bar{s}s}(\tau, \vec{\sigma}) \approx -\delta_{\bar{s}s}, \quad \gamma^{\bar{s}s}(\tau, \vec{\sigma}) \approx -\delta^{\bar{s}s}, \quad \gamma(\tau, \vec{\sigma}) \approx 1. \]  

(41)

By introducing the Dirac brackets for the resulting second class constraints [now we have

\[ \{\eta^\tau_i(\tau), \kappa^\tau_j(\tau)\} = \delta_{ij} \delta^{\tau\bar{s}} \]

\[ \{A, B\}^* = \{A, B\} - \int d^3\sigma[\{A, \zeta^\mu(\tau, \vec{\sigma})\}\{H_\mu(\tau, \vec{\sigma}), B\} - \{A, H_\mu(\tau, \vec{\sigma})\}\{\zeta^\mu(\tau, \vec{\sigma}), B\}], \]  

(42)

we find \( \{x^\mu_s(\tau), p^\nu_s(\tau)\}^* = -\eta^{\mu\nu} \) [with the assumption \( \{b^\mu_\tau(\tau), p^\nu_\tau(\tau)\} = 0 \].

The ten degrees of freedom describing the hyperplane are \( x^\mu_s(\tau) \) with conjugate momentum \( p^\mu_s \) and six variables \( \phi_\lambda(\tau), \lambda = 1, \ldots, 6 \), which parametrize the orthonormal tetrad \( b^\mu_\lambda(\tau) \),

with their conjugate momenta \( T_\lambda(\tau) \).

The preservation of the gauge-fixings \( \zeta^\mu(\tau, \vec{\sigma}) \approx 0 \) in time implies

\[ \frac{d}{d\tau}\zeta^\mu(\tau, \vec{\sigma}) = \{\zeta^\mu(\tau, \vec{\sigma}), H_D\} = -\lambda^\mu(\tau, \vec{\sigma}) - \dot{x}^\mu_s(\tau) - \dot{b}^\mu_\tau(\tau)\sigma^\tau \approx 0, \]  

(43)

so that one has [using \( \dot{b}^\mu_\tau = 0 \) and \( \dot{b}^\mu_\tau(\tau)b^\nu_\tau(\tau) = -b^\mu_\tau(\tau)\dot{b}^\nu_\tau(\tau) \)]

\[ \lambda^\mu(\tau, \vec{\sigma}) \approx \ddot{\lambda}^\mu(\tau) + \ddot{\lambda}^\mu_\nu(\tau)b^\nu_\tau(\tau)\sigma^\tau, \]
\[ \ddot{\lambda}^\mu(\tau) = -\ddot{x}^\mu_s(\tau), \]
\[ \ddot{\lambda}^\mu_\nu(\tau) = -\ddot{\lambda}^\mu_\nu(\tau) = \frac{1}{2}[\dot{b}^\mu_\tau(\tau)b^\nu_\tau(\tau) - b^\mu_\tau(\tau)\dot{b}^\nu_\tau(\tau)]. \]  

(44)

Thus, the Dirac Hamiltonian becomes
and only the following first class constraints are left [now we remain with the variables $x_\mu^s, p_\mu^s, b_\mu^A, S_\mu^s, A_\mu, \pi^A, \bar{\eta}_i, \bar{\kappa}_i, \theta_{i\alpha}, \theta^*_{i\alpha}]$

$$H_D = \lambda^\mu(\tau)\tilde{H}_\mu(\tau) - \frac{1}{2} \lambda^{\mu\nu}(\tau)\tilde{H}_{\mu\nu}(\tau) + \sum_{i=1}^{N} \mu_i(\tau)N_i(\tau), \quad (45)$$

$$\tilde{H}^\mu(\tau) = \int d^3\sigma \tilde{H}^\mu(\tau, \bar{\sigma}) = p_\mu^s - l_\mu \{ \frac{1}{2} \int d^3\sigma \sum_a [g_s^2 \pi_a^2(\tau, \bar{\sigma}) + g_s^{-2} \tilde{B}_a^2(\tau, \bar{\sigma})] + \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + [\bar{\kappa}_i(\tau) + \sum_a Q_{ia}(\tau)\bar{A}_a(\tau, \bar{\eta}_i(\tau))]^2} \} - b_\mu^s(\tau) \{ \int d^3\sigma \sum_a [\pi_a(\tau, \bar{\sigma}) \times \tilde{B}_a(\tau, \bar{\sigma})] + \sum_{i=1}^{N} [\kappa_i(\tau) + \sum_a Q_{ia}(\tau)A_{ia}(\tau, \bar{\eta}_i(\tau))] \} \approx 0,$$

$$\tilde{H}^{\mu\nu}(\tau) = b_\mu^s(\tau) \int d^3\sigma \bar{\sigma}^{\mu} \tilde{H}^\nu(\tau, \bar{\sigma}) - b_\nu^s(\tau) \int d^3\sigma \bar{\sigma}^{\nu} \tilde{H}^\mu(\tau, \bar{\sigma}) =$$

$$= S_{\mu\nu}^s(\tau) - [b_\mu^s(\bar{\sigma})b_\nu^s(\bar{\sigma}) - b_\nu^s(\bar{\sigma})b_\mu^s(\bar{\sigma})] \left\{ \frac{1}{2} \int d^3\sigma \bar{\sigma}^{\mu\nu} \sum_a [g_s^2 \pi_a^2(\tau, \bar{\sigma}) + g_s^{-2} \tilde{B}_a^2(\tau, \bar{\sigma})] + \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 + [\bar{\kappa}_i(\tau) + \sum_a Q_{ia}(\tau)\bar{A}_a(\tau, \bar{\eta}_i(\tau))]^2} \} +$$

$$+ [b_\mu^s(\tau)b_\nu^s(\tau) - b_\nu^s(\tau)b_\mu^s(\tau)] \left\{ \int d^3\sigma \bar{\sigma}^{\mu\nu} \sum_a [\pi_a(\tau, \bar{\sigma}) \times \tilde{B}_a(\tau, \bar{\sigma})] \right\} \approx 0,$$

$$\pi_a^T(\tau, \bar{\sigma}) \approx 0, \quad \Gamma_a(\tau, \bar{\sigma}) \approx 0, \quad N_i \approx 0. \quad (46)$$

Here $S_{\mu\nu}^s$ is the spin part of the Lorentz generators

$$J_{\mu}^s = x_\mu^s p_\mu^s - x_\mu^s p_\mu^s + S_{\mu\nu}^s,$$

$$S_{\mu\nu}^s = b_\mu^s(\tau) \int d^3\sigma \bar{\sigma}^{\mu\nu} \rho^\nu(\tau, \bar{\sigma}) - b_\nu^s(\tau) \int d^3\sigma \bar{\sigma}^{\nu\mu} \rho^\mu(\tau, \bar{\sigma}). \quad (47)$$

As shown in Ref. [7] instead of finding $\phi_\lambda(\tau), T_\lambda(\tau)$, one can use the redundant variables $b_\mu^A(\tau), S_{\mu\nu}^s(\tau)$, with the following Dirac brackets assuring the validity of the orthonormality condition $\eta^{\mu\nu} - b_\mu^A \eta^{\bar{A}b} b_B^{\nu} = 0$ [with the structure constants of the Lorentz group]
The Dirac brackets of the left constraints are

\[
\{ \mathcal{H}^\mu(\tau), \mathcal{H}^\nu(\tau) \}^* = \sum_a \int d^3 \sigma \left\{ [b^\mu_{\tau}(\tau) - b^\mu_{\tau}(\tau)] g_s^2 \pi_{\alpha \rho}(\tau, \sigma) - b^\nu_{\tau}(\tau) F_{\alpha \dot{\beta}}(\tau, \sigma) b^\dot{\beta}_{\tau}(\tau) \right\} \Gamma_a(\tau, \sigma),
\]

\[
\{ \mathcal{H}^\mu(\tau), \mathcal{H}^{\alpha \beta}(\tau) \}^* = - \sum_a \int d^3 \sigma \sigma^i \left\{ \left( [b^\mu_{\tau}(\tau) b^\beta_{\tau}(\tau) - b^\beta_{\tau}(\tau) b^\beta_{\tau}(\tau)] g_s^2 \pi_{\alpha \rho}(\tau, \sigma) - b^\alpha_{\tau}(\tau) F_{\alpha \dot{\beta}}(\tau, \sigma) b^\dot{\beta}_{\tau}(\tau) \right) + b^\nu_{\tau}(\tau) (b^\alpha_{\tau}(\tau) b^\beta_{\tau}(\tau) - b^\beta_{\tau}(\tau) b^\beta_{\tau}(\tau)) \right\} \Gamma_a(\tau, \sigma),
\]

\[
\{ \mathcal{H}^{\alpha \beta}(\tau), \mathcal{H}^{\alpha \beta}(\tau) \}^* = C_{\gamma \delta}^{\alpha \beta} \mathcal{H}^{\gamma \delta}(\tau) + \sum_a \int d^3 \sigma \sigma^i \sigma^j \left\{ [b^\alpha_{\tau}(\tau) b^\beta_{\tau}(\tau) - b^\beta_{\tau}(\tau) b^\beta_{\tau}(\tau)] g_s^2 \pi_{\alpha \rho}(\tau, \sigma) - b^\alpha_{\tau}(\tau) F_{\alpha \dot{\beta}}(\tau, \sigma) b^\dot{\beta}_{\tau}(\tau) \right\} \Gamma_a(\tau, \sigma),
\]

Let us now restrict ourselves to configurations with \( p_s^2 > 0 \) and let us use the Wigner boost \( L^\mu(\tilde{p}_s, p_s) \) to boost to rest the variables \( b^\mu_A, S^\mu_{s} \) of the following non-Darboux basis

\[
x^\mu_s, p^\mu_s, b^\mu_A, S^\mu_s, \eta^\pm_s, \kappa^\pm_s
\]

of the Dirac brackets. The following new non-Darboux basis is obtained \( [\tilde{x}^\mu_s \) is no more a 4-vector]

\[
\tilde{x}^\mu_s = x^\mu_s + \frac{1}{2} \epsilon^A_\nu (u(p_s)) \eta_{AB} \frac{\partial B(u(p_s))}{\partial p_{s\mu}} S^\rho_{s\nu} = \\
= x^\mu_s - \frac{1}{\eta \sqrt{p_s^2} (p_s^\rho + \eta \sqrt{p_s^2})} \left[ p_s^\sigma S^\rho_{s\mu} + \eta \sqrt{p_s^2} (S^\rho_{s\mu} - S^\rho_{s\nu} p^\nu_s P^\mu_s) \right] = \\
= x^\mu_s - \frac{1}{\eta_s \sqrt{p_s^2}} \left[ \eta^A_s (S^\rho_{sA} - \frac{S^A_{s\rho} P^\rho_s}{p_s^\rho + \eta_s \sqrt{p_s^2}}) \right] + \frac{p_s^\mu}{\eta_s \sqrt{p_s^2}} + 2 \eta_s \sqrt{p_s^2} \eta^\rho \cdot \bar{S}_{s\rho} P^\rho_s,
\]

\( \eta_s = \sqrt{1 - \eta^2} \)
\[ p^\mu_s = p^\mu, \]
\[ \eta^\mu_i = \eta^\mu_i, \]
\[ \kappa^\mu_i = \kappa^\mu_i, \]
\[ b^A_r = \epsilon^A_{\mu}(u(p_s)) b^{\mu}_r, \]
\[ S_{\mu\nu} = S_{\mu\nu}^{is} - \frac{1}{2} \epsilon^A_{\rho}(u(p_s)) \eta_{AB} \left( \frac{\partial \epsilon^B_{\sigma}(u(p_s))}{\partial p^\mu_s} p^\nu_s - \frac{\partial \epsilon^B_{\sigma}(u(p_s))}{\partial p^\nu_s} p^\mu_s \right) S_{\sigma}^{is} = \]
\[ = S_{\mu\nu} + \frac{1}{\eta \sqrt{p^2_s} (p^2_s + \eta \sqrt{p^2_s})} \left[ p_{\sigma}^s (S_{\lambda}^{is} p^\lambda_r - S_{\lambda}^{is} p^\lambda_s) + \eta \sqrt{p^2_s} (S_{\lambda}^{is} p^\lambda_s - S_{\lambda}^{is} p^\lambda_s) \right], \]
\[ J_{\mu\nu}^r = \tilde{x}^\mu_s p^\nu_r - \tilde{x}^\nu_s p^\mu_r + \tilde{S}_{\mu\nu}^s = \tilde{L}_{\mu\nu}^r + \tilde{S}_{\mu\nu}^s. \]  

We have
\[
\{ \tilde{x}^\mu_s, p^r_s \}^* = -\eta^{\mu r}, \\
\{ \tilde{S}^{is}_r, b^A_{s} \}^* = \frac{\delta^{is} (p^r_s b^A_{s} - p^r_s b^A_{s})}{p^r_s + \eta_s \sqrt{p^2_s}}, \\
\{ \tilde{S}^{ij}_s, b^A_{s} \}^* = (\delta^{ij} \delta^{is} - \delta^{is} \delta^{ij}) b^A_{s}, \\
\{ \tilde{S}^{\mu\nu}_s, \tilde{S}^{\alpha\beta}_s \}^* = C^{\mu\nu\alpha\beta}_{\gamma\delta} \tilde{S}^{\gamma\delta}_s. 
\]

As shown in Ref. [7], under Poincaré transformations \((a, \Lambda)\) we get
\[
p^{\nu}_s = \Lambda^{\nu}_r p^r, \\
\tilde{x}^{\mu}_s = \Lambda^{\mu}_r \tilde{x}^{r}_s + \frac{1}{2} \tilde{S}^{rs}_{s,r} \Lambda^{\nu}_r \Lambda^{\alpha}_s \left( \frac{\partial}{\partial p^{\nu}_s} R^{s}_{\nu} \left( \Lambda, p_s \right) \right) + a^{\mu} = \\
= \Lambda^{\mu}_r \{ \tilde{x}^{r}_s + \tilde{S}^{rs}_{s,r} \left( \Lambda^{\nu}_r \Lambda^{\alpha}_s - \frac{(\Lambda^{\nu}_o - 1) p_{s,r}}{p^2_s + \eta \sqrt{p^2_s}} \right) - \\
- \frac{\left( p^{\nu}_s + \eta_s \sqrt{p^2_s} p_{s,r} \Lambda^{\alpha}_s \right)}{\eta \sqrt{p^2_s} (p^2_s + \eta \sqrt{p^2_s})} \} + a^{\mu}. \tag{52} \]

Therefore, \( \tilde{x}^{\mu}_s \) is not a 4-vector: its infinitesimal transformation properties under Lorentz transformations generated by \( J^{\mu\nu}_s \) are
\[\{\tilde{x}_\mu, J^\alpha_s\} = \eta^{\mu\alpha}\tilde{x}_\mu^\alpha - \eta^{\mu\beta}\tilde{x}_\mu^\beta + \{\tilde{x}_\mu, \tilde{S}^\alpha_s\},\]

\[\{\tilde{x}_\mu, \tilde{S}^\alpha_s\} = -\frac{1}{p^\mu_s + \eta \sqrt{p^2_s}} [\eta^{\mu j}\tilde{S}^{ij}_s + (p^\mu_s + \eta \eta \sqrt{p^2_s})\tilde{S}^{ik}_s p^k_s],\]

\[\{\tilde{x}_\mu, \tilde{S}^{ij}_s\} = 0. \quad (53)\]

We can define [the new variables are \(\tilde{x}^\mu, p^\mu_s, b^A, \tilde{S}^{\mu\nu}_s, A_{a\bar{A}}, \pi^A, \eta_i, \bar{\kappa}_i, \theta^\imath, \theta^\imath\)]

\[\tilde{S}^{AB}_s = \epsilon_\mu^A(u(p_s))\epsilon_\nu^B(u(p_s))S^{\mu\nu}_s \approx\]

\[\approx [b^A_{p}^{(A)}(\tau)b^B_{p}^{(A)}(\tau) - b^{(B)}_{p}^{(p)}(\tau)b^{(A)}_{p}^{(p)}(\tau)] \left[\frac{1}{2} \int d^3\sigma \sigma^\imath \sum_a [g_s^2 \pi^2_a(\tau, \sigma) + g_s^{-2} \bar{B}^2_a(\tau, \sigma)] + \right.\]

\[+ \sum_{i=1}^N \eta^\imath_i(\tau) \eta_i^\imath \sqrt{\bar{m}^2_i + [\bar{\kappa}^\imath_i(\tau) + \sum_a Q_{ia}(\tau) A_a(\tau, \bar{\eta}_i(\tau))]^2} - \]

\[- [b^A_{p}^{(A)}(\tau)b^B_{p}^{(A)}(\tau) - b^{(B)}_{p}^{(p)}(\tau)b^{(A)}_{p}^{(p)}(\tau)] \left[\int d^3\sigma \sigma^\imath \sum_a [\pi^\imath_a(\tau, \sigma) \times \bar{B}_a(\tau, \sigma)]_s + \right.\]

\[+ \sum_{i=1}^N \eta^\imath_i(\tau)[\kappa^\imath_i(\tau) + \sum_a Q_{ia}(\tau) A^\imath_a(\tau, \bar{\eta}_i(\tau))] \]. \quad (54)

Let us now add six more gauge-fixings by selecting the special family of spacelike hyperplanes orthogonal to \(p^\mu_s\) (this is possible for \(p^2_s < 0\), which can be called the ‘Wigner foliation’ of Minkowski spacetime. This can be done by requiring (only six conditions are independent)

\[T^\mu_{A}(\tau) = b^\mu_{A}(\tau) - \epsilon^\mu_{A=A}(u(p_s)) \approx 0\]

\[\Rightarrow b^\mu_{A}(\tau) = \epsilon_{A}(u(p_s))b^\mu_{A}(\tau) \approx \eta^\imath_A. \quad (55)\]

Now the tetrad \(b^\mu_{A}\) has become \(\epsilon_{A}(u(p_s))\) and the indices ‘\(\imath\)’ are forced to coincide with the Wigner spin-1 indices ‘\(r\)’, while \(\bar{\sigma} = \tau\) is a Lorentz-scalar index. The final Wigner-covariant variables are \(\tilde{x}^\mu, p^\mu_s, A_{a\bar{A}}, \pi^\imath, A_{a\bar{A}}, \pi^\imath, \kappa^\imath, \theta^\imath, \theta^\imath\). One has

\[\tilde{S}^{AB}_s \approx (\eta^\imath_A \eta^\imath_B - \eta^\imath_A \eta^\imath_B) \left[\frac{1}{2} \int d^3\sigma \sigma^\imath \sum_a [g_s^2 \pi^2_a(\tau, \sigma) + g_s^{-2} \bar{B}^2_a(\tau, \sigma)] + \right.\]

\[+ \sum_{i=1}^N \eta^\imath_i(\tau) \eta_i^\imath \sqrt{\bar{m}^2_i + [\bar{\kappa}^\imath_i(\tau) + \sum_a Q_{ia}(\tau) A_a(\tau, \bar{\eta}_i(\tau))]^2} - \]

\[- (\eta^\imath_A \eta^\imath_B - \eta^\imath_A \eta^\imath_B) \left[\int d^3\sigma \sigma^\imath \sum_a [\pi^\imath_a(\tau, \sigma) \times \bar{B}_a(\tau, \sigma)]_s + \right.\]

\[+ \sum_{i=1}^N \eta^\imath_i(\tau)[\kappa^\imath_i(\tau) + \sum_a Q_{ia}(\tau) A^\imath_a(\tau, \bar{\eta}_i(\tau))] \].
\[\bar{S}_s^{rs} \approx \sum_{i=1}^{N} (\eta_i^s(\tau) [\kappa_i^s(\tau)] + \sum_a Q_{ia}(\tau) A_a^s(\tau, \tilde{\eta}_i(\tau))] -
- \sum_{i=1}^{N} \eta_i^s(\tau) [\kappa_i^s(\tau)] + \sum_a Q_{ia}(\tau) A_a^s(\tau, \tilde{\eta}_i(\tau))] +
+ \int d^3\sigma \sum_a (\sigma^r [\tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma})]^s - \sigma^s [\tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma})]^r),\]

\[\bar{S}_{\mu}^{r\sigma} \approx -\tilde{S}_s^{r\sigma} = -\sum_{i=1}^{N} \eta_i^r(\tau) \eta_i \sqrt{m_i^2 + [\tilde{\kappa}_i(\tau) + \sum_a Q_{ia}(\tau) A_a(\tau, \tilde{\eta}_i(\tau))]^2} -
- \frac{1}{2} \int d^3 \sigma \sigma^r \sum_a [g_s^2 \pi_a^s(\tau, \tilde{\sigma}) + g_s^{-2} \tilde{B}_a^2(\tau, \tilde{\sigma})],\]

\[J_{ij}^{ij} \approx \bar{x}_s^i p_s^j - \bar{x}_s^j p_s^i + \delta^{ij} \delta^{rs} \bar{S}_s^{rs};
J_{si}^{oi} \approx \bar{x}_s^i p_s^o - \delta^{ij} \delta^{rs} \bar{S}_s^{rs} \frac{p_s^o}{p_s^o + \eta_s \sqrt{b_s^2}}.\]  

(56)

The time constancy of \(T_\lambda^\mu \approx 0\) with respect to the Dirac Hamiltonian gives

\[\frac{d}{d\tau} [b_\mu^\tau(\tau) - \epsilon_\nu^\tau(u(p_s))] = \left\{ b_\mu^\nu(\tau) - \epsilon_\nu^\mu(u(p_s)), H_D \right\}^* =
= \frac{1}{2} \tilde{\lambda}^\alpha_\beta(\tau) \{ b_\mu^\nu(\tau), S_{\alpha_\beta}(\tau) \}^* = \tilde{\lambda}^{\alpha_\beta}(\tau)b_{\nu_\alpha}(\tau) \approx 0
\Rightarrow \tilde{\lambda}^{\mu_\nu}(\tau) \approx 0,\]  

(57)

so that the independent gauge-fixings contained in Eqs.(55) and the constraints \(\tilde{H}_\mu^{\mu_\nu}(\tau) \approx 0\) form six pairs of second class constraints.

Now we have [remember that \(\dot{x}_s^\mu(\tau) = -\tilde{\lambda}^\mu(\tau)\)]

\[l_\mu = b_\mu^\tau(\tau),\]
\[z_\mu^\tau(\tau) = \dot{\tilde{x}}_s^\mu(\tau) = \sqrt{g(\tau) u^{\mu}(p_s)} - \dot{x}_s^\mu(\tau) \epsilon_\nu^\mu(u(p_s)) \epsilon_\nu^\tau(u(p_s)),\]
\[g(\tau) = [\dot{x}_s^\mu(\tau) u^{\mu}(p_s)]^2,\]
\[g_{\tau\tau} = \dot{x}_s^2, \quad g^{\tau\tau} = \frac{1}{g}, \quad g^{\tau\tau} = \frac{1}{g} \dot{x}_s \delta_{\tau_\mu} \epsilon_\nu^\tau(u(p_s)),\]
\[g_{\tau\tau} = \dot{x}_s \delta_{\tau_\mu} \epsilon_\nu^\tau(u(p_s)), \quad g^{\tau\tau} = -\delta_{\tau_\mu} + \frac{\delta_{\tau_\mu} \delta^{\nu_\tau}}{g(\tau)} \dot{x}_s \epsilon_\nu^\tau(u(p_s)) \dot{x}_s \epsilon_\nu^\tau(u(p_s)).\]  

(58)

On the hyperplane \(\Sigma_{W_\tau}\) all the degrees of freedom \(z_\mu(\tau, \tilde{\sigma})\) are reduced to the four degrees of freedom \(\tilde{x}_s^\mu(\tau),\) which replace \(x_s^\mu.\) The Dirac Hamiltonian is now
\[H_D = \tilde{\lambda}^\mu(\tau) \tilde{H}_\mu(\tau) +
+ \sum_{i=1}^{N} \mu_i(\tau) N_i(\tau) + \int d^3\sigma \sum_a [\lambda_{ar}(\tau, \tilde{\sigma}) \pi_a^\tau(\tau, \tilde{\sigma}) + \lambda_a(\tau, \tilde{\sigma}) \Gamma_a(\tau, \tilde{\sigma})]\]  

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To find the new Dirac brackets, one needs to evaluate the matrix of the old Dirac brackets of the second class constraints (without extracting the independent ones)

\[
C = \begin{pmatrix}
\{\hat{H}^{\alpha\beta}, \hat{H}^{\gamma\delta}\}^* & \{\hat{H}^{\alpha\beta}, T^\alpha_B\}^* \\
= \delta_{BB}[\eta^{r\beta}e^\alpha_B(u(p_s)) - \eta^{s\alpha}e^\beta_B(u(p_s))] \\
\{T^\alpha_A, \hat{H}^{\gamma\delta}\}^* & \{T^\alpha_A, T^\beta_B\}^* = 0 \\
= \delta_{AA}[\eta^{r\gamma}e^\delta_A(u(p_s)) - \eta^{s\delta}e^\gamma_A(u(p_s))] \\
\end{pmatrix}
\]

(59)

Since the constraints are redundant, this matrix has the following left and right null eigenvectors: \( \begin{pmatrix} a_{\alpha\beta} = a_{\beta\alpha} \\ 0 \end{pmatrix} \) [\( a_{\alpha\beta} \) arbitrary], \( \begin{pmatrix} 0 \\ \epsilon^B_D(u(p_s)) \end{pmatrix} \). Therefore, one has to find a left and right quasi-inverse \( \tilde{C} \), \( \tilde{C}C = CC = D \), such that \( \tilde{C} \) and \( D \) have the same left and right null eigenvectors. One finds

\[
\tilde{C} = \begin{pmatrix}
0_{\gamma\delta\mu\nu} & \frac{1}{4}[\eta_{\gamma\tau}e^D_\delta(u(p_s)) - \eta_{\delta\tau}e^D_\gamma(u(p_s))] \\
\frac{1}{2}[\eta_{\alpha\nu}e^B_\mu(u(p_s)) - \eta_{\alpha\mu}e^B_\nu(u(p_s))] & 0^{BD}_{\sigma\tau}
\end{pmatrix}
\]

(60)

and the new Dirac brackets are

\[
\{A, B\}^{**} = \{A, B\}^* - \frac{1}{4}[\{A, \hat{H}^{\gamma\delta}\}^*[\eta_{\gamma\tau}e^D_\delta(u(p_s)) - \eta_{\delta\tau}e^D_\gamma(u(p_s))]\{T^\tau_D, B\}^* + \{A, T^\sigma_B\}^*[\eta_{\sigma\nu}e^B_\mu(u(p_s)) - \eta_{\sigma\mu}e^B_\nu(u(p_s))]\{\hat{H}^{\mu\nu}, B\}^*].
\]

(61)

While the check of \( \{\hat{H}^{\alpha\beta}, B\}^{**} = 0 \) is immediate, we must use the relation \( b_A T^\mu_D e^{D\rho} = -T^\rho_A \) [at this level we have \( T^\mu_A = T^\mu_A \)] to check \( \{T^\alpha_A, B\}^{**} = 0 \).

Then, we find the following brackets for the remaining variables \( \tilde{x}^\mu_s, p^\mu_s, \eta^\gamma_i, \kappa^\gamma_i \) [the metric \( \gamma^{rs} = -\delta^{rs} \) will be used, so that \( \{\eta^\gamma_i, \cdot\} = \partial/\partial \kappa^\gamma_i = -\partial/\partial \kappa_{ir} \)]

\[
\{\tilde{x}^\mu_s, p^\mu_s\}^{**} = -\eta^{\mu\nu},
\]

\[
\{\eta^\gamma_i, \kappa^\gamma_j\}^{**} = \delta_{ij}\delta^{rs} = -\delta_{ij}\gamma^{rs},
\]

(62)

and the following form of the Poincaré generators \([\{\hat{I}^{\mu\nu}_s, \tilde{S}^{\mu\nu}_s\}]^{**} \neq 0 \)
\begin{equation}
\frac{P^\mu_s}{J^\mu_s = \tilde{x}^\mu_s p_s^\mu - \tilde{x}^\nu_s p_s^\nu + \tilde{S}_{\mu\nu} = \tilde{L}_{\mu\nu} + \tilde{S}_{\mu\nu}},
\end{equation}
\begin{equation}
\tilde{S}_{oi} = -\frac{\delta_{ir} \bar{S}_{rs} p_s^o}{p_s^o + \eta_s \sqrt{p_s^2}},
\end{equation}
\begin{equation}
\tilde{S}_{ij} = \delta_{ir} \delta_{js} \bar{S}_{st}.
\end{equation}

Therefore, \( \tilde{x}^\mu_s \) is not a fourvector and \( \tilde{\eta}_i, \tilde{\kappa}_i \) transform as Wigner spin-1 3-vectors. Indeed, as shown in Ref. [7], under global Poincaré and infinitesimal Lorentz transformations one has
\begin{equation}
\eta_{ir}' = \eta_{is} R_{sr}^i(\Lambda, p),
\end{equation}
\begin{equation}
\kappa_{ir}' = \kappa_{is} R_{sr}^i(\Lambda, p),
\end{equation}
\begin{equation}
\{ \eta_{r}^r, J_{oi}^s \} = -\frac{\delta_{is} (p_s^o \eta_i^s - p_s^2 \eta_i^r)}{p_s^o + \eta \sqrt{p_s^2}},
\end{equation}
\begin{equation}
\{ \eta_{r}^r, J_{ij}^s \} = \delta_{ir} \delta_{js} (\delta_{is} \delta_{jr} - \delta_{ir} \delta_{js}) \eta_i^s,
\end{equation}
\begin{equation}
\{ \kappa_{r}^r, J_{oi}^s \} = -\frac{\delta_{is} (p_s^o \kappa_i^s - p_s^2 \kappa_i^r)}{p_s^o + \eta \sqrt{p_s^2}},
\end{equation}
\begin{equation}
\{ \kappa_{r}^r, J_{ij}^s \} = (\delta_{is} \delta_{jr} - \delta_{ir} \delta_{js}) \kappa_i^s.
\end{equation}

The only left first class constraints are
\begin{equation}
\tilde{\mathcal{H}}^\mu(\tau) = p_s^\mu - u^\mu(u(p_s)) \left[ \frac{1}{2} \int d^3 \sigma \sum_a [g_s^2 \bar{\pi}_a(\tau, \bar{\sigma}) + g_s^{-2} \bar{B}_a(\tau, \bar{\sigma})] + \right.
\end{equation}
\begin{equation}
+ \sum_{i=1}^N \eta_i \sqrt{m_i^2 + |\bar{\kappa}_i(\tau)|^2} + \sum_a Q_{ia}(\tau) \bar{A}_a(\tau, \bar{\eta}_i(\tau)) \right] -
\end{equation}
\begin{equation}
- \epsilon_{r}^s(u(p_s)) \left[ \int d^3 \sigma \sum_a [\bar{\pi}_a(\tau, \bar{\sigma}) \times \bar{B}_a(\tau, \bar{\sigma})] ^r + \right.
\end{equation}
\begin{equation}
+ \sum_{i=1}^N [\kappa_i^r(\tau) + \sum_a Q_{ia}(\tau) \bar{A}_a(\tau, \bar{\eta}_i(\tau))] \right] \approx 0,
\end{equation}
\begin{equation}
\pi_a(\tau, \bar{\sigma}) \approx 0,
\end{equation}
\begin{equation}
\Gamma_a(\tau, \bar{\sigma}) \approx 0,
\end{equation}
\begin{equation}
N_i(\tau) \approx 0,
\end{equation}
\begin{equation}
\{ \tilde{\mathcal{H}}^\mu, \tilde{\mathcal{H}}^\nu \} = \sum_a \int d^3 \sigma \{ [\epsilon_{r}^s(u(p_s)) \epsilon_{r}^\nu(u(p_s)) - \epsilon_{r}^s(u(p_s)) \epsilon_{r}^\nu(u(p_s))] \pi_{ar}(\tau, \bar{\sigma}) + \right.
\end{equation}
\begin{equation}
\approx 0.
\end{equation}
\[+\epsilon^\mu_\tau(p_s)F_{ars}(\tau, \vec{\sigma})\epsilon^\nu_s(p_s)\} \Gamma_a(\tau, \vec{\sigma}), \quad (65)\]

or

\[
\mathcal{H}(\tau) = \eta_s \sqrt{p_s^2} - \left[ \sum_{i=1}^N \eta_i \sqrt{m_i^2 + [\vec{k}_i(\tau) + \sum_a Q_{ia}(\tau) \vec{A}_a(\tau, \vec{n}_i(\tau))]^2} + \frac{1}{2} \int d^3\sigma \sum_a [g_s^2 \vec{\pi}_a^2(\tau, \vec{\sigma}) + g_s^{-2} \vec{B}_a^2(\tau, \vec{\sigma})] \right] \approx 0,
\]

\[
\vec{H}_p(\tau) = \sum_{i=1}^N [\vec{k}_i(\tau) + \sum_a Q_{ia}(\tau) \vec{A}_a(\tau, \vec{n}_i(\tau))] + \int d^3\sigma \sum_a \vec{\pi}_a(\tau, \vec{\sigma}) \times \vec{B}_a(\tau, \vec{\sigma}) \approx 0, \quad (66)
\]

\[
\pi^\tau_a(\tau, \vec{\sigma}) \approx 0,
\]

\[
\Gamma_a(\tau, \vec{\sigma}) \approx 0,
\]

\[
N_i(\tau) \approx 0.
\]

The first one gives the mass spectrum of the isolated system, while the other three say that the total (Wigner spin 1) 3-momentum of the N particles on the hyperplane \(\Sigma_{W\tau}\) vanishes. The Dirac Hamiltonian is now

\[
H_D = \lambda(\tau) \mathcal{H}(\tau) - \vec{\lambda}(\tau) \cdot \vec{H}_p(\tau) + \sum_{i=1}^N \mu_i(\tau) N_i(\tau) + \int d^3\sigma \sum_a [\lambda_{ar}(\tau, \vec{\sigma}) \pi^\tau_a(\tau, \vec{\sigma}) + \lambda_a(\tau, \vec{\sigma}) \Gamma_a(\tau, \vec{\sigma})]
\]

and we have \(\dot{\vec{x}}^\mu_s = \{\vec{x}^\mu_s, H_D\}^{**} = -\lambda(\tau) u^\mu(p_s)\). Therefore, while the old \(x^\mu_s\) had a velocity \(\dot{x}^\mu_s\) not parallel to the normal \(l^\mu = u^\mu(p_s)\) to the hyperplane as shown by Eqs.(58), the new \(\vec{x}_s\) has \(\vec{x}_s || l^\mu\) and no classical zitterbewegung. Moreover, we have that \(T_s = l \cdot \vec{x}_s = l \cdot x_s\) is the Lorentz-invariant rest frame time.

For \(N \geq 2\), let us perform the following canonical transformation \([a = 1, \ldots, N - 1]\)

\[
\begin{array}{cccc}
\vec{x}_s^\mu & p_s^\mu & \vec{x}_s^\nu & p_s^\nu \\
\vec{n}_i & \vec{k}_i & \eta_i & \vec{n}_i
\end{array} \quad \rightarrow \quad
\begin{array}{cccc}
\vec{z}_s & \vec{k}_s & \epsilon_s \\
\vec{\rho}_\alpha & \vec{\pi}_\alpha & \\
\vec{n}_+ & \vec{k}_+ & \\
T_s = p_s \cdot \vec{x}_s & \frac{p_s \cdot x_s}{\eta_s \sqrt{p_s^2}}
\end{array}
\]

\[
T_s = \frac{p_s \cdot \vec{x}_s}{\eta_s \sqrt{p_s^2}} = \frac{p_s \cdot x_s}{\eta_s \sqrt{p_s^2}}
\]
\[ \epsilon_s = \eta_s \sqrt{p_s^2}, \]
\[ \hat{z}_s = \eta_s \sqrt{p_s^2} (\hat{x}_s - \frac{\bar{\eta}_s \bar{\eta}_s^{\alpha}}{p_s^{\alpha} \bar{p}_s^{\alpha}}), \]
\[ \bar{k}_s = \frac{\bar{p}_s}{\eta_s \sqrt{p_s^2}}, \]
\[ \bar{\eta}_+ = \frac{1}{N} \sum_{i=1}^{N} \bar{\eta}_i, \]
\[ \bar{\kappa}_+ = \sum_{i=1}^{N} \bar{\kappa}_i, \]
\[ \hat{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \hat{\gamma}_{ai} \bar{\eta}_i, \]
\[ \hat{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{\gamma}_{ai} \bar{\kappa}_i, \]
\[ \sum_{i=1}^{N} \hat{\gamma}_{ai} = 0, \quad \sum_{i=1}^{N} \hat{\gamma}_{ai} \hat{\gamma}_{bi} = \delta_{ab}, \]
\[ \sum_{a=1}^{N-1} \hat{\gamma}_{ai} \hat{\gamma}_{aj} = \delta_{ij} - \frac{1}{N}, \quad \text{(68)} \]

whose inverse is
\[ \bar{\rho}_a = \sqrt{N} \sum_{i=1}^{N} \hat{\gamma}_{ai} \bar{\kappa}_i, \]
\[ \bar{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \hat{\gamma}_{ai} \bar{\eta}_i, \]
\[ \bar{\kappa}_i = \frac{1}{N} \bar{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \hat{\gamma}_{ai} \bar{\pi}_a, \quad \text{(69)} \]

The new form of the constraints is
\[ \mathcal{H}(\tau) = \epsilon_s - \left\{ \sum_{i=1}^{N} \eta_i \times \right\} \]
\[ \sqrt{m_t^2 + \left[ \frac{1}{N} \bar{\kappa}_+ (\tau) + \sqrt{N} \sum_{a=1}^{N-1} \hat{\gamma}_{ai} \bar{\pi}_a (\tau) + \sum_a Q_{ia} (\tau) \bar{A}_a (\tau; \bar{\eta}_+ (\tau) + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \hat{\gamma}_{ai} \bar{\pi}_a (\tau) \right]^2 +} \]
\[ \frac{1}{2} \int d^3 \sigma \sum_a [g_a^2 \pi_a^2(\tau, \vec{\sigma}) + g_a^{-2} \vec{B}_a^2(\tau, \vec{\sigma})] = \epsilon_s - E_{(P+I)s} - E_{(F)s} \approx 0, \]

\[ \bar{\mathcal{H}}_p(\tau) = \bar{\kappa}_+(\tau) + \sum_{i=1}^N \sum_a Q_{ia}(\tau) \bar{A}_a(\tau, \bar{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \bar{\gamma}_{ai} \bar{\rho}_a(\tau)) + \]

\[ + \int d^3 \sigma \sum_a \bar{\pi}_a(\tau) \times \vec{B}_a(\tau, \vec{\sigma}) = \vec{P}_{(P+I)s} + \vec{P}_{(F)s} \approx 0, \]

\[ \vec{\pi}_a(\tau, \vec{\sigma}) \approx 0, \]

\[ \Gamma_a(\tau, \vec{\sigma}) \approx 0, \]

\[ N_i(\tau) \approx 0, \quad (70) \]

where \( E_{(F)s} = \frac{1}{2} \int d^3 \sigma \sum_a [g_a^2 \pi_a^2(\tau, \vec{\sigma}) + g_a^{-2} \vec{B}_a^2(\tau, \vec{\sigma})] \) and \( \vec{P}_{(F)s} = \int d^3 \sigma \sum_a \bar{\pi}_a(\tau, \vec{\sigma}) \times \vec{B}_a(\tau, \vec{\sigma}) \)

are the rest-frame field energy and three-momentum respectively [now we have \( \bar{\pi}_a(\tau, \vec{\sigma}) = g_a^{-2} \vec{E}_a(\tau, \vec{\sigma}) \)], while \( E_{(P+I)s} \) and \( \vec{P}_{(P+I)s} \) denote the particle+interaction total rest-frame energy and three-momentum, before the decoupling from the electromagnetic gauge degrees of freedom.

The final form of the rest-frame spin tensor is

\[ \bar{S}_{rs} = \sum_{i=1}^N \{ (\eta_+^r(\tau) + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \bar{\gamma}_{ai} \bar{\rho}_a^r(\tau)) (\kappa_{+}^s(\tau) + \frac{1}{\sqrt{N}} \sum_{b=1}^{N-1} \bar{\gamma}_{bi} \bar{\pi}_b^s(\tau)) + \]

\[ + \sum_a Q_{ia}(\tau) A_a^r(\tau, \bar{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{c=1}^{N-1} \bar{\gamma}_{ci} \bar{\rho}_c(\tau)) \} - \]

\[ - (\eta_+^r(\tau) + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \bar{\gamma}_{ai} \bar{\rho}_a^r(\tau)) (\kappa_{+}^s(\tau) + \sqrt{N} \sum_{b=1}^{N-1} \bar{\gamma}_{bi} \bar{\pi}_b^s(\tau)) + \]

\[ + \sum_a Q_{ia}(\tau) A_a^r(\tau, \bar{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{c=1}^{N-1} \bar{\gamma}_{ci} \bar{\rho}_c(\tau)) \} + \]

\[ + \int d^3 \sigma \sum_a (\sigma^r [\bar{\pi}_a(\tau, \vec{\sigma}) \times \vec{B}_a(\tau, \vec{\sigma})]^s - \sigma^s [\bar{\pi}_a(\tau, \vec{\sigma}) \times \vec{B}_a(\tau, \vec{\sigma})]^r) = \bar{S}^{rs}_{(P+I)s} + \bar{S}^{rs}_{(F)s}, \]

\[ \bar{S}_{s} = \left[ m_i^2 + \left( \frac{1}{N} \bar{\kappa}_+(\tau) + \sqrt{N} \sum_{a=1}^{N-1} \bar{\gamma}_{ai} \bar{\pi}_a(\tau) + \sum_a Q_{ia}(\tau) \bar{A}_a(\tau, \bar{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \bar{\gamma}_{ai} \bar{\rho}_a(\tau)) \right)^2 \right]^{1/2} \]

\[ - \frac{1}{2} \int d^3 \sigma \sigma^r \sum_a [g_a^2 \bar{\pi}_a^2(\tau, \vec{\sigma}) + g_a^{-2} \vec{B}_a^2(\tau, \vec{\sigma})], \quad (71) \]
while the Dirac Hamiltonian is

\[
H_D = \lambda(\tau) \mathcal{H} - \bar{\lambda}(\tau) \mathcal{H}_p + \int d^3\sigma \sum_a [\lambda_a(\tau, \bar{\sigma}) \pi_a^\tau(\tau, \bar{\sigma}) - A_{a\tau}(\tau, \bar{\sigma}) \Gamma_a(\tau, \bar{\sigma})] + \\
+ \sum_{i=1}^N \mu_i(\tau) N_i(\tau).
\]

(72)

On an arbitrary spacelike hypersurface or on the Wigner hyperplane one has in the free case [but also in the interacting one looking at the four constraints \( \bar{H}^\mu(\tau) \approx 0 \) of Eq.(65) and of Ref. [7]]

\[
p_i^\mu|_{\Sigma, \tau} = \eta_i \sqrt{m_i^2 - \gamma^{i\bar{s}}(\tau, \bar{\sigma}) \kappa_{i\bar{s}}(\tau) \kappa_{i\bar{s}}(\tau)} \hat{l}_i^\mu(\tau, \bar{\sigma}) - z_i^\mu(\tau, \bar{\sigma}) \gamma^{i\bar{s}}(\tau, \bar{\sigma}) \kappa_{i\bar{s}}(\tau)
\]

\[
p_i^\mu|_{\text{Wigner hyperplane}}(\tau) = \eta_i \sqrt{m_i^2 + \bar{\kappa}_i^2(\tau) u^\mu(p_a) + \epsilon_i^\mu(u(p_a)) \kappa_i^\tau(\tau)}
\]

which are solutions of \( p_i^2 - m_i^2 = 0 \).
IV. THE DIRAC OBSERVABLES

In this Section we will make the canonical reduction with respect to the SU(3) Yang-Mills gauge transformations. From now on we shall use the notation \( \{ . , . \} \) for the Dirac brackets \( \{ . , . \}^{**} \) on the Wigner hyperplane \( \Sigma_{W^\ast} \).

The decompositions in the non-Abelian SU(3) gauge potential can be obtained from the second paper in Ref. [1]. For the vector potential we use Eqs.(4-13), (4-16), (4-26), (4-29), (4-30), (4-31), (4-33), (5-21), (5-24), of that paper to get

\[
\begin{align*}
\vec{A}_a(\tau, \vec{\sigma}) &= A_{ab}(\eta^A(\tau, \vec{\sigma})) \vec{\partial} \eta^A_b(\tau, \vec{\sigma}) + (P e^{\Omega^{(\gamma)}_s(\eta^A(\tau, \vec{\sigma}))})_{ab} \vec{A}_{b\perp}(\tau, \vec{\sigma}) = \\
&= \vec{\Theta}_a(\eta^A(\tau, \vec{\sigma})) \vec{\partial} \eta^A(\tau, \vec{\sigma}) + (P e^{\Omega^{(\gamma)}_s(\eta^A(\tau, \vec{\sigma}))})_{ab} \vec{A}_{b\perp}(\tau, \vec{\sigma}), \\
\vec{\partial} \cdot \vec{A}_a(\tau, \vec{\sigma}) &= 0
\end{align*}
\]

\[
\begin{align*}
\vec{T}^a A_{ab}(\eta^A(\tau, \vec{\sigma})) \vec{\partial} \eta^A_b(\tau, \vec{\sigma}) \cdot d\vec{\sigma} &= H_b(\eta^A(\tau, \vec{\sigma})) \vec{\partial} \eta^A_b(\tau, \vec{\sigma}) \cdot d\vec{\sigma} = \\
&= \Theta_a(\eta^A(\tau, \vec{\sigma})) \vec{\partial} \eta^A(\tau, \vec{\sigma})) \vec{T}^a = d(\gamma) \Omega^{(\gamma)}(\eta^A(\tau, \vec{\sigma})), \quad \Theta_a = \vec{\Theta}_a \cdot d\vec{\sigma},
\end{align*}
\]

\[
\begin{align*}
\Omega^{(\gamma)}(\eta^A(\tau, \vec{\sigma}))) &= \Omega^{(\gamma)}(\eta^A(\tau, \vec{\sigma})) \vec{T}^a = \\
&= (\gamma) \int_{0}^{\eta^A(\tau, \vec{\sigma}, s)} H_b(\eta^A(\tau, \vec{\sigma}; s)) \vec{\partial} \eta^A_b(\tau, \vec{\sigma}; s).
\end{align*}
\]

If \( \eta_a \) are coordinates in a chart of the group manifold of SU(3), the matrices \( A_{ab}(\eta) \) satisfy the Maurer-Cartan equations, which can be written in the zero curvature form

\[
\frac{\partial H_a(\eta)}{\partial \eta_b} - \frac{\partial H_b(\eta)}{\partial \eta_a} + [H_a(\eta), H_b(\eta)] = 0.
\]

We shall use only canonical coordinates of the first kind, defined by \( A_{ab}(\eta) \eta_b = \eta_a \) so that \( A(\eta) = e^{T\eta / T_\eta} \) with \( (T \eta)_{ab} = (T^c)_{ab} \eta_c = c_{abc} \eta_c \).

If \( \theta_a = A_{ab}(\eta) \eta_b \) are the left-invariant (or Maurer-Cartan) one-forms on SU(3), the abstract Maurer-Cartan equations are \( d\theta_a = -\frac{1}{2} c_{abc} \theta_b \land \theta_c \); then, by using the preferred line \( \gamma(a) \) (s is the parameter along it) defining the canonical coordinates of the first kind in a neighbourhood of the identity I of SU(3), one can define \( d(\gamma) \omega(\gamma)(\eta(s)) = \theta_a(\eta(s)) \) \( [d(\gamma)] \) is the exterior derivative along \( \gamma \) with \( \omega(\gamma)(\eta(s)) = \omega_a(\gamma)(\eta(s)) \vec{T}^a = (\gamma) \int_{0}^{\eta(s)} \vec{T}^a A_{ab}(\eta) \eta_b = \frac{30}{30} \).
In our case of a trivial principal SU(3)-bundle $P(\Sigma(\tau), SU(3))$ over the spacelike hypersurface $\Sigma_\tau$ [diffeomorphic to $R^3$] of Minkowski spacetime, it is shown in Ref. [1] that $\Theta_a(\eta^{(A)}(\tau, \vec{\sigma}), \partial_\eta(\eta^{(A)}(\tau, \vec{\sigma})))$ and $\Omega_a^{(\gamma)}(\eta^{(A)}(\tau, \vec{\sigma}))$ are just the extension of these SU(3) objects: in the second paper of Ref. [1], a connection-dependent coordinatization $(\tau, \vec{\sigma}; h(\eta^{(A)}(\tau, \vec{\sigma})))$ of the principal bundle is given with the SU(3) fibers parametrized with parallely transported (with respect to the given connection) canonical coordinates of the first kind from a reference fiber over an arbitrarily chosen origin in $R^3$. The functions $\eta_a(\tau, \vec{\sigma})$ and their gradients $\partial_\eta(\eta^{(A)}(\tau, \vec{\sigma}))$ vanish on the identity cross section $\sigma_I$ of the trivial principal bundle. The path $\hat{\gamma}$ is a surface (in the total bundle space) of preferred paths, associated with these generalized canonical coordinates of the first kind, starting from the identity cross section $\sigma_I$ till a cross section parametrized by the parameter $s$, in a tubolar neighbourhood of $\sigma_I$. The operator $d(\hat{\gamma})$ is the exterior derivative on the principal SU(3)-bundle total space restricted to $\hat{\gamma}$; it can be identified with the vertical derivative on the principal bundle and with the Hamiltonian BRST operator. With these conventions, one has $\{\eta_a(\tau, \vec{\sigma})\} = \{\partial_\eta(\eta^{(A)}(\tau, \vec{\sigma}))\} = -B_{ab}(\eta^{(A)}(\tau, \vec{\sigma}))\frac{\delta}{\delta\eta_a(\eta^{(A)}(\tau, \vec{\sigma}))} [B(\eta) = A^{-1}(\eta)]$ with the functional derivative to be interpreted as a directional derivative along the surface of paths $\hat{\gamma}$. The longitudinal gauge variables have a complicated formal implicit expression given in Eq.(4-49) of the second paper of Ref. [1] and satisfy $\{\eta_a(\tau, \vec{\sigma}), \Gamma_{ab}(\tau, \vec{\sigma})\} = -\delta_{ab}\delta^3(\vec{\sigma} - \vec{\sigma'})$, where $\tilde{\Gamma}_a(\tau, \vec{\sigma}) = \Gamma_{b}(\tau, \vec{\sigma})A_{ba}(\eta^{(A)}(\tau, \vec{\sigma}))$ are the Abelianized Gauss laws $\{\tilde{\Gamma}_a(\tau, \vec{\sigma}), \tilde{\Gamma}_{b}(\tau, \vec{\sigma})\} = 0$. In Eq.(73), $A_{ab}(\eta^{(A)}(\tau, \vec{\sigma}))\partial_\eta(\eta^{(A)}(\tau, \vec{\sigma}))$ is the pure gauge part (saturated with $d\vec{\sigma}$ it is the BRST ghost) of the vector potential $\vec{A}_a(\tau, \vec{\sigma})$: the magnetic field $\vec{B}_a(\tau, \vec{\sigma})$ is generated only by the second term of Eq.(73). In this sense, $\eta_a(\tau, \vec{\sigma}) = 0$ is the true generalized non-Abelian Coulomb gauge with all the same properties of the Abelian Coulomb gauge. In suitable weighted Sobolev spaces, as discussed in Ref. [1], this gauge-fixing is well defined, since all the connections over the principal SU(3)-bundle are completely irreducible [their holonomy bundles (i.e. the set of points of $P(R^3, SU(3))$ which can be joined by horizontal
curves) coincide with the principal bundle itself] and there is no form of Gribov ambiguity (i.e. of stability subgroups of the group of gauge transformations for special connections and/or field strengths). In these spaces, the covariant divergence is an elliptic operator without zero modes \([19]\) and its Green function 
\[\vec{\zeta}^{(A)}_{ab}(\vec{\sigma},\vec{\sigma}';\tau)\]
is globally defined

\[
\vec{D}_{ab}(\tau,\vec{\sigma}) \cdot \vec{\zeta}^{(A)}_{bc}(\vec{\sigma},\vec{\sigma}';\tau) = -\delta_{ac}\delta^3(\vec{\sigma} - \vec{\sigma}')
\]

\[
\vec{c}(\vec{x}) = \vec{\partial} c(\vec{x}) = \frac{\vec{x}}{4\pi|\vec{x}|^3}, \quad \triangle = -\vec{\partial}^2, \quad c(\vec{x}) = \frac{1}{\triangle} \delta^3(\vec{x}) = \frac{-1}{4\pi|\vec{x}|}.
\]  

(74)

The path ordering is along the straighline (flat geodesic) joining \(\vec{y}\) and \(\vec{x}\) when \(\Sigma_{\tau}\) is a hyperplane like \(\Sigma_{W_{\tau}}\).

Therefore, we have

\[
\vec{\pi}_a(\tau,\vec{\sigma}) = -\frac{\vec{\partial}}{\triangle} \vec{\partial} \cdot \vec{\pi}_a(\tau,\vec{\sigma}) + \vec{\pi}_{a,\perp}(\tau,\vec{\sigma}) =
\]

\[
= \vec{\pi}_{a,D\perp}(\tau,\vec{\sigma}) + \int d^3\sigma' \vec{\zeta}^{(A)}_{ab}(\vec{\sigma},\vec{\sigma}';\tau)[\Gamma_b(\tau,\vec{\sigma}') - \sum_{i=1}^{N} Q_{ib}(\tau)\delta^3(\vec{\sigma}' - \vec{\eta}_i(\tau))] - \sum_{i=1}^{N} Q_{ib}(\tau)\delta^3(\vec{\sigma}' - \vec{\eta}_i(\tau))]
\]

\[
\vec{\partial} \cdot \vec{\pi}_{a,\perp}(\tau,\vec{\sigma}) = \vec{D}_{ab}(\tau,\vec{\sigma}) \cdot \vec{\pi}_{b,D\perp}(\tau,\vec{\sigma}) = 0.
\]  

(75)

It is shown in Eqs. (5-7), (5-8), (5-10), of the second paper in Ref. [1] that we have

\[
\vec{\partial} \cdot \vec{\pi}_a(\tau,\vec{\sigma}) = \int d^3\sigma' \vec{\zeta}^{(A)}_{ab}(\vec{\sigma},\vec{\sigma}';\tau)
\]

\[
[\delta_{ef}\vec{A}_e(\tau,\vec{\sigma}') \cdot \vec{\pi}_{f,\perp}(\tau,\vec{\sigma}') + \Gamma_b(\tau,\vec{\sigma}') - \sum_{i=1}^{N} Q_{ib}(\tau)\delta^3(\vec{\sigma}' - \vec{\eta}_i(\tau))] - \sum_{i=1}^{N} Q_{ib}(\tau)\delta^3(\vec{\sigma}' - \vec{\eta}_i(\tau)),
\]

\[\Gamma_b(\tau,\vec{\sigma}') - \sum_{i=1}^{N} Q_{ib}(\tau)\delta^3(\vec{\sigma}' - \vec{\eta}_i(\tau))],
\]

\[
\vec{\pi}_{a,D\perp}(\tau,\vec{\sigma}) = \int d^3\sigma' [\delta_{ij}\delta_{ab}\delta^3(\vec{\sigma} - \vec{\sigma}')] -
\]

\[
- \frac{\vec{\partial}^i}{\Delta_{\sigma}} \frac{\vec{\partial}}{\Delta_{\sigma}} \cdot \vec{\zeta}^{(A)}_{ac}(\vec{\sigma},\vec{\sigma}';\tau)\delta_{ij}\delta_{ab}\delta^3(\vec{\sigma}')\vec{\pi}_{b,\perp}(\tau,\vec{\sigma}')
\]

\[
\Rightarrow \vec{\pi}_{a,\perp}(\tau,\vec{\sigma}) = P^{ij}_{\perp}(\vec{\sigma})\vec{\pi}_{a,D\perp}(\tau,\vec{\sigma}),
\]  

(76)

where \(P^{ij}_{\perp}(\vec{\sigma}) = \delta^{ij} - \vec{\partial}^i\vec{\partial}^j/\triangle\). Moreover, Eqs.(5-21) and (5-25) of that paper give
\[ \pi_{a,D \perp} (\tau, \vec{\sigma}) = (P e^{\Omega^{(\gamma)} (\eta (A) (\tau, \vec{\sigma}))})_{ab} \pi_{b,D \perp} (\tau, \vec{\sigma}) =
( P e^{\Omega^{(\gamma)} (\eta (A) (\tau, \vec{\sigma}))})_{ab} \pi_{b,\perp} (\tau, \vec{\sigma}) - \frac{\vec{\partial}}{\Delta} \cdot \vec{\pi}_{b,D \perp} (\tau, \vec{\sigma}), \]

\[ \pi^i_{a,\perp} (\tau, \vec{\sigma}) = P^i_{\perp} (\vec{\sigma}) \pi^i_{a,D \perp} (\tau, \vec{\sigma}) \]

\[ \{ \pi_{a,D \perp} (\tau, \vec{\sigma}), \Gamma_b (\tau, \vec{\sigma}) \} = 0. \quad (77) \]

Therefore, the color SU(3) canonical pairs of Dirac’s observables turn out to be \( \tilde{A}_{a,\perp} (\tau, \vec{\sigma}) \), \( \tilde{\pi}_{a,\perp} (\tau, \vec{\sigma}) \). They satisfy the Poisson brackets

\[ \{ \tilde{A}^i_{a,\perp} (\tau, \vec{\sigma}), \tilde{\pi}^i_{b,\perp} (\tau, \vec{\sigma}) \} = -\delta_{ab} P^i_{\perp} (\vec{\sigma}) \delta^3 (\vec{\sigma} - \vec{\sigma}'). \quad (78) \]

Instead, the gauge sector is given by the pairs \( A_{a,\tau} (\tau, \vec{\sigma}), \pi^i_{a,\tau} (\tau, \vec{\sigma}) \approx 0, \eta^a (\tau, \vec{\sigma}), \tilde{\Gamma}_a (\tau, \vec{\sigma}) \approx 0. \]

Let us now look for the Dirac observables of the particles. First of all, the Grassmann variables are not gauge invariant because one has

\[ \{ \theta_{\alpha a} (\tau), \Gamma_a (\tau \vec{\sigma}) \} = (T^a)_{\alpha \beta} \theta_{\beta} \delta^3 (\vec{\sigma} - \vec{\eta}_a (\tau)), \]

\[ \{ \theta^*_{\alpha a} (\tau), \Gamma_a (\tau \vec{\sigma}) \} = -\theta^*_{\beta} (T^a)_{\beta \alpha} \delta^3 (\vec{\sigma} - \vec{\eta}_a (\tau)). \quad (79) \]

The Grassmann Dirac observables are

\[ \tilde{\theta}_{\alpha a} (\tau) = [P e^{\Omega^{(\gamma)} (\eta (A) (\tau, -\vec{\eta}_a (\tau))] T^a]_{\alpha \beta} \theta_{\beta} (\tau), \]

\[ \Rightarrow \{ \tilde{\theta}_{\alpha a} (\tau), \Gamma_a (\tau, \vec{\sigma}) \} = 0, \]

\[ \tilde{\theta}^*_{\alpha a} (\tau) = \theta^*_{\beta} (\tau) [P e^{\Omega^{(\gamma)} (\eta (A) (\tau, -\vec{\eta}_a (\tau))] T^a]_{\beta \alpha}, \]

\[ \Rightarrow \{ \tilde{\theta}^*_{\alpha a} (\tau), \Gamma_a (\tau, \vec{\sigma}) \} = 0, \]

\[ \{ \tilde{\theta}_{\alpha a} (\tau), \tilde{\theta}^*_i_{\beta} (\tau) \} = -i \delta_{ij} \delta_{\alpha \beta}, \quad (80) \]

with the path ordering evaluated in the fundamental representation of SU(3). The Dirac observables for the non-Abelian charges of the particles are
\[\hat{Q}_{ia} = i\hat{\theta}_{ia}^* (T^a)_{\alpha\beta} \hat{\theta}_{i\beta} = Q_{ib} [P e^{\Omega_{ia}^{(5)} (\eta^{(a)}(\tau, \tilde{\eta}(\tau))) T^a}]_{ba},\]

\[\{\hat{Q}_{ia}, \Gamma_b(\tau, \vec{\sigma})\} = 0,\]

\[\{\hat{Q}_{ia}, \hat{Q}_{ja}\} = \delta_{ij} c_{abc} \hat{Q}_{ic},\]  

(81)

where in the second line the path ordering is evaluated in the adjoint representation of SU(3), since one has used the identity

\[e^{u_k T^k} e^{u_c T^c} = T^c (e^{-u_k T^k})_{ca}.\]  

(82)

By using Eqs.(75) and (76), for \(\eta_a^{(A)}(\tau, \vec{\sigma}) = \Gamma_a(\tau, \vec{\sigma}) = 0\) [so that also \(\tilde{\partial} \eta_a^{(A)}(\tau, \vec{\sigma}) = 0\), \(\Omega_s^{(5)}(\eta^{(A)}(\tau, \vec{\sigma})) = 0\) as shown in the second paper of Ref. [1]], namely in the generalized Coulomb gauge [we still go on to use the notation \{,\} for the new Dirac brackets with respect to the second class constraints \(\eta_a^{(A)}(\tau, \vec{\sigma}) \approx 0, \tilde{\Gamma}_a(\tau, \vec{\sigma}) \approx 0\); instead the temporal variables \(A_{a\tau}\) and \(\pi_{a\tau}^*\) simply decouple], we get

\[\pi_a(\tau, \vec{\sigma}) \rightarrow \tilde{\pi}_a(\tau, \vec{\sigma}) = \pi_a(\vec{\sigma}) - \frac{\tilde{\partial}}{\Delta} \int d^3\sigma' \tilde{\partial}_{\sigma} \cdot \tilde{\zeta}_{ab}^{(A)\perp}(\vec{\sigma}, \vec{\sigma}'; \tau) [c_{bce} \tilde{A}_c^{(h)}(\tau, \vec{\sigma}') \tilde{\pi}_e^{(h)}(\tau, \vec{\sigma}') - \sum_{i=1}^N \hat{Q}_{ib} \delta^3(\vec{\sigma}' - \tilde{\eta}_i(\tau))],\]

\[\tilde{\zeta}_{ab}^{(A)\perp}(\vec{\sigma}, \vec{\sigma}'; \tau) = \tilde{c}(\vec{\sigma} - \vec{\sigma}') (P e^{\int_{\sigma'}^\tau d\vec{x}' \cdot \tilde{A}_{\perp}^{(h)}(\tau, \vec{\sigma}')} T_{a b})_{\tau}.\]  

(83)

While in the electromagnetic case it is possible to get the physical Hamiltonian without imposing the Coulomb gauge-fixing \(\tilde{\eta}_{em}(x) \approx 0\) as in Ref. [7] (namely it is obtained by a canonical decoupling of the gauge degrees of freedom), this is too difficult in the non-Abelian case. Therefore, we shall evaluate the physical quantities by imposing the generalized Coulomb gauge-fixings \(\eta_a^{(A)}(\tau, \vec{\sigma}) \approx 0\). Conceptually, the canonical decoupling of the gauge degrees of freedom gives the same results for the physical quantities.

As shown in Refs. [1,2], the Noether identities implied by the second Noether theorem, applied to the color SU(3) gauge group, give the following result for the weak improper

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conserved non-Abelian Noether charges $Q_a$ and for the strong improper conserved ones $Q_{(s)a}$

$$Q_a = g_s^{-2} c_{abc} \int d^3 \sigma F^a_{\mu k} (\tau, \vec{\sigma}) A^k (\tau, \vec{\sigma}) - \sum_{i=1}^N Q_{ia} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)),$$

$$\overset{\circ}{Q_{(s)a}} = \int d^2 \Sigma \cdot E_\sigma (\tau, \vec{\sigma}), \quad (84)$$

Then, we get [see Eqs.(6-33)-(6-35) in the second paper of Ref. [1]] the following Dirac’s observables

$$Q_a \rightarrow_{(A)} \hat{Q}_a = \hat{Q}^{(YM)}(\tau) - \sum_{i=1}^N \hat{Q}_{ia}(\tau) =$$

$$\int d^3 \sigma \hat{p}_a (\tau, \vec{\sigma}) = \int d^3 \sigma [\hat{p}^{(YM)}_a (\tau, \vec{\sigma}) + \sum_{i=1}^N \hat{p}_{ia}(\tau, \vec{\sigma})],$$

$$\hat{p}^{(YM)}_a (\tau, \vec{\sigma}) = -c_{abc} \hat{A}_{b\perp} (\tau, \vec{\sigma}) \cdot \hat{\pi}_{c\perp} (\tau, \vec{\sigma}),$$

$$\hat{p}_{ia}(\tau, \vec{\sigma}) = -\hat{Q}_{ia}(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)).$$

\{\hat{Q}_a, \hat{Q}_b\} = c_{abc} \hat{Q}_c, \quad \{\hat{Q}^{(YM)}_a(\tau), \hat{Q}^{(YM)}_b(\tau)\} = c_{abc} \hat{Q}^{(YM)}_c(\tau),

\{\hat{A}_{a\perp}(\tau, \vec{\sigma}), \hat{Q}_b(\tau)\} = c_{abc} \hat{A}_{c\perp}(\tau, \vec{\sigma})

\{\hat{\pi}_{a\perp}(\tau, \vec{\sigma}), \hat{Q}_b(\tau)\} = c_{abc} \hat{\pi}_{c\perp}(\tau, \vec{\sigma}). \quad (85)$$

While particle’s 3-positions $\vec{\eta}_i(\tau)$ are gauge invariant, for particle’s momenta we have

$$\{\kappa_i(\tau), \Gamma_a(\tau, \vec{\sigma})\} = -Q_{ia}(\tau) \frac{\partial}{\partial \eta_i} \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau)). \quad (86)$$

To find the Dirac observables $\tilde{\kappa}_i(\tau)$ [namely to dress the particles with a color cloud], we note that, anagously to the electromagnetic case [7], we expect that the form of the minimal coupling is mantained, namely

$$\tilde{\kappa}_i(\tau) + Q_{ia}(\tau) \tilde{A}_a(\tau, \vec{\eta}_i(\tau)) = \tilde{\kappa}_i(\tau) + \hat{Q}_{ia}(\tau) \tilde{A}_{a\perp}(\tau, \vec{\eta}_i(\tau)). \quad (87)$$

Since we have
\[ \hat{Q}_{ab}(\tau) \tilde{A}_{a\perp}(\tau, \tilde{h}_i(\tau)) = Q_{ia}(\tau) [\iota \hat{P} e^{\iota_{ia}(\eta^{(A)}(\tau, -\tilde{h}_i(\tau)))} \hat{r}_{ua}]_{ab} \tilde{A}_{b\perp}(\tau, \tilde{h}_i(\tau)), \]  

Eqs.(73) suggest the following expression for the gauge invariant momenta

\[ \tilde{k}_i(\tau) = \tilde{\kappa}_i(\tau) + \sum_a Q_{ia}(\tau) \tilde{\Theta}_a(\eta^{(A)}(\tau, \tilde{h}_i(\tau)), \tilde{\partial} \eta^{(A)}(\tau, \tilde{h}_i(\tau))). \]

Since we have

\[
\{ \Theta^r_a(\eta^{(A)}(\tau, \sigma), \tilde{\partial} \eta^{(A)}(\tau, \sigma)), \Gamma_b(\tau, \sigma') \} = -B_{ab}(\eta^{(A)}(\tau, \sigma')) \frac{\delta[A_{av}(\eta^{(A)}) \partial' \eta^{(A)}(\tau, \sigma')]}{\delta \eta^{(A)}(\tau, \sigma')} = c_{abc} \Theta^r_c(\eta^{(A)}(\tau, \tilde{h}_i(\tau)), \tilde{\partial} \eta^{(A)}(\tau, \tilde{h}_i(\tau))) \delta^3(\tilde{\sigma} - \tilde{\sigma}') + \\
+ \delta_{ab} \partial \delta^3(\tilde{\sigma} - \tilde{\sigma}'),
\]

we get the gauge invariance of \( \tilde{k}_i(\tau) \)

\[
\{ \tilde{k}_i(\tau), \Gamma_b(\tau, \sigma) \} = \{ \kappa_i^r(\tau), \Gamma_b(\tau, \sigma) \} + \\
+ \{ Q_{ia}(\tau) \Theta^r_a(\eta^{(A)}(\tau, \tilde{h}_i(\tau)), \tilde{\partial} \eta^{(A)}(\tau, \tilde{h}_i(\tau))), \Gamma_b(\tau, \sigma) \} = \\
- Q_{ia}(\tau) \frac{\partial}{\partial \tilde{h}_i^r} \delta^3(\tilde{\sigma} - \tilde{h}_i(\tau)) + \\
+ c_{abc} \Theta^r_c(\eta^{(A)}(\tau, \tilde{h}_i(\tau)), \tilde{\partial} \eta^{(A)}(\tau, \tilde{h}_i(\tau))) Q_{ic}(\tau) \delta^3(\tilde{\sigma} - \tilde{h}_i(\tau)) + \\
+ Q_{ia}(\tau) c_{abc} \Theta^r_c(\eta^{(A)}(\tau, \tilde{h}_i(\tau)), \tilde{\partial} \eta^{(A)}(\tau, \tilde{h}_i(\tau))) \delta^3(\tilde{h}_i(\tau) - \sigma) + \\
+ Q_{ib}(\tau) \frac{\partial}{\partial \tilde{h}_i^r} \delta(\tilde{h}_i(\tau) - \sigma) = 0. \tag{91}
\]

We can now rewrite the constraints \( \mathcal{H}(\tau) \approx 0, \tilde{\mathcal{H}}_p(\tau) \approx 0, \) \( N_i \approx 0 \) in terms of the Dirac observables with respect to the SU(3) gauge transformations. The Grassmann constraints become

\[ \tilde{N}_i = \sum_{\alpha=1}^{3} \tilde{\theta}_{i\alpha}(\tau) \tilde{\theta}_{i\alpha}(\tau) \approx 0. \tag{92} \]

As shown in Ref. [1], we have \( \sum_a \tilde{B}_a^2(\tau, \tilde{\sigma}) = \sum_a \tilde{B}_a^2(\tau, \tilde{\sigma}) \) with the chromomagnetic field \( \tilde{B}_a(\tau, \tilde{\sigma}) \) built in terms of \( \tilde{A}_{a\perp}(\tau, \tilde{\sigma}) \). For the chromoelectric field we have from Eqs.(83) and using Eqs.(85) and (74)
\[ \sum_a \pi_a^2(\tau, \vec{\sigma}) \big|_{\eta'h_b = 0} = \]
\[ = \sum_a \left[ \tilde{\pi}_a(\tau, \vec{\sigma}) + \frac{\partial}{\partial \Delta} \int d^3\sigma \partial_\sigma \cdot \zeta^{(A)}_{ab}(\vec{\sigma}, \vec{\sigma}_1; \tau) \bar{\rho}_b(\tau, \vec{\sigma}_1) \right]^2 = \]
\[ = \sum_a \left[ \tilde{\pi}_a(\tau, \vec{\sigma}) - \frac{\partial}{\partial \Delta} \int d^3\sigma_1 (\partial_{ab} \delta^3(\vec{\sigma} - \vec{\sigma}_1) + c_{a,mn} \tilde{A}_{m,\perp}(\tau, \vec{\sigma}) \cdot \zeta^{(A)}_{ab}(\vec{\sigma}, \vec{\sigma}_1; \tau)) \bar{\rho}_b(\tau, \vec{\sigma}_1) \right]^2, \]

\[ \Rightarrow \int d^3\sigma \sum_a \pi_a^2(\tau, \vec{\sigma}) \big|_{\eta'h_b = 0} = \int d^3\sigma \sum_a \tilde{\pi}_a^2(\tau, \vec{\sigma}) + \]
\[ + \int d^3\sigma \sum_a \left[ \frac{\partial}{\partial \Delta} \int d^3\sigma_1 (\partial_{ab} \delta^3(\vec{\sigma} - \vec{\sigma}_1) + c_{a,mn} \tilde{A}_{m,\perp}(\tau, \vec{\sigma}) \cdot \zeta^{(A)}_{ab}(\vec{\sigma}, \vec{\sigma}_1; \tau)) \bar{\rho}_b(\tau, \vec{\sigma}_1) \right]^2. \quad (93) \]

To obtain the last line, we have done an integration by parts and used the transversality of \( \tilde{\pi}_a(\tau, \vec{\sigma}) \). The final result is

\[ g_s^2 \int d^3\sigma \sum_a \pi_a^2(\tau, \vec{\sigma}) \big|_{\eta'h_b = 0} = \]
\[ = \int d^3\sigma \sum_a \tilde{\pi}_a^2(\tau, \vec{\sigma}) + V[\eta_h, \tilde{A}_{a,\perp}, \tilde{\pi}_{a,\perp}](\tau), \quad (94) \]

with the potential given by

\[ V[\eta_h, \tilde{A}_{a,\perp}, \tilde{\pi}_{a,\perp}](\tau) = g_s^2 \int d^3\sigma d^3\sigma_1 d^3\sigma_2 \sum_{a,b,c} \left[ \partial_\sigma \zeta^{(A)}_{cd}(\vec{\sigma}, \vec{\sigma}_1; \tau) \bar{\rho}_a(\tau, \vec{\sigma}_1) \right] \]
\[ \cdot \frac{1}{\Delta_\sigma} \left[ \partial_\sigma \zeta^{(A)}_{eb}(\vec{\sigma}, \vec{\sigma}_2; \tau) \bar{\rho}_b(\tau, \vec{\sigma}_2) \right] = \]
\[ = g_s^2 \int d^3\sigma_1 d^3\sigma_2 \sum_{a,b} \bar{\rho}_a(\tau, \vec{\sigma}_1) K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) \bar{\rho}_b(\tau, \vec{\sigma}_2) = \]
\[ = g_s^2 \int d^3\sigma_1 d^3\sigma_2 \sum_{a,b} \left[ \bar{\rho}_a(Y^M)(\tau, \vec{\sigma}_1) - \sum_{i=1}^N \bar{Q}_{ia}(\tau) \delta^3(\vec{\sigma}_1 - \eta_i(\tau)) \right] \]
\[ K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) \bar{\rho}_b(Y^M)(\tau, \vec{\sigma}_2) - \sum_{j=1}^N \bar{Q}_{jb}(\tau) \delta^3(\vec{\sigma}_2 - \eta_j(\tau)) \],

\[ K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = K_{ba}(\vec{\sigma}_2, \vec{\sigma}_1; \tau) = \]
\[ = \int d^3\sigma_3 d^3\sigma_4 \left\{ \frac{\delta_{ab} \delta^3(\vec{\sigma}_3 - \vec{\sigma}_1) \delta^3(\vec{\sigma}_4 - \vec{\sigma}_2)}{4\pi | \vec{\sigma}_3 - \vec{\sigma}_4 |} + \right. \]
\[ + \frac{\delta^3(\vec{\sigma}_4 - \vec{\sigma}_2) \tilde{A}_{+}(\tau, \vec{\sigma}_3) \cdot \zeta^{(A)}_{ab}(\vec{\sigma}_3, \vec{\sigma}_1; \tau)}{4\pi | \vec{\sigma}_3 - \vec{\sigma}_4 |} + (\sigma_1 \leftrightarrow \sigma_2) + \]
\[ + \left. \frac{\tilde{A}_{+}(\tau, \vec{\sigma}_3) \cdot \zeta^{(A)}_{ab}(\vec{\sigma}_3, \vec{\sigma}_1; \tau) \delta^3(\vec{\sigma}_4 - \vec{\sigma}_2)}{4\pi | \vec{\sigma}_3 - \vec{\sigma}_4 |} \right\}. \quad (95) \]
The first line of this equation agrees with Eq. (6-25) of the second paper in Refs. [1], while, from Eq. (6-27) of that paper, we get that $K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = -G_{\Delta, ab}^{(A)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)$, where $G_{\Delta, ab}^{(A)}$ is the Green function of $\hat{D}_{ac}(\tau, \vec{\sigma}) \cdot \hat{D}_{cb}(\tau, \vec{\sigma})$ [see Eqs. (3-16), (3-20), (3-21), (3-25) of that paper].

The constraint giving the invariant mass of the system takes the form

$$\hat{\mathcal{H}}(\tau) = \epsilon_s - \sum_i \eta_i \sqrt{m_i^2 + (\vec{\kappa}_i(\tau) + \sum_a \hat{Q}_{ai}(\tau) \vec{A}_{a,\perp}(\tau, \vec{\eta}_i(\tau)))^2 -}$$

$$- \sum_a \int d^3\sigma \left( \frac{\tilde{g}_a^2 \tilde{\pi}_{a,\perp}(\tau, \vec{\sigma})}{2} + \frac{\tilde{B}_{a,\perp}(\tau, \vec{\sigma})}{2g_s^2} \right) - \frac{1}{2} V[\vec{\phi}_i, \vec{A}_{a,\perp}, \vec{\pi}_{a,\perp}]^{(A)}(\tau) =$$

$$= \epsilon_s - H_{rel} \approx 0. \quad (96)$$

We see that, since $\tilde{\rho}_a(\tau) = \tilde{\rho}_a^{(YM)}(\tau) + \sum_{i=1}^N \tilde{\rho}_{ia}$, there is a universal interaction kernel $K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)$ which creates the particle-particle, the particle-field and the field-field interaction between the corresponding color charge densities. This interaction kernel contains 3 kinds of instantaneous (i.e. at equal $\tau$ on the Wigner hyperplane) interactions:

i) a Coulomb interaction;

ii) an interaction mediated by an arbitrary center (over whose spatial location is integrated): one color density has a Coulomb interaction with the transverse potential at the center, which simultaneously interacts with the other color density through an instantaneous “Wilson line” along the geodesic (the straightline) on the Wigner hyperplane, i.e. $[\vec{A}_{a,\perp} = \hat{A}_{a,\perp} \hat{T}^a$ with $(\hat{T})_{ab} = c_{cab}$]:

$$[\hat{A}_{a,\perp}(\tau, \vec{\sigma}) \cdot \tilde{\zeta}^{(A)}(\vec{\sigma}, \vec{\sigma}; \tau)]_{ab} = \vec{A}_{c,\perp}(\tau, \vec{\sigma}) c_{cad} \cdot \tilde{c}(\vec{\sigma} - \vec{\sigma}) (P e^{\int_{\vec{\sigma}}^{\vec{\sigma}'} d\vec{\sigma} \cdot \hat{A}_{c,\perp}(\tau, \vec{\sigma}') \hat{T}^c)_{db},$$

where $\vec{\sigma}$ is the position of the center and $\vec{\sigma}$ that of the color density;

iii) an interaction mediated by two arbitrary centers (over whose spatial location is integrated): each color density interacts, through a Wilson line along the geodesic, with the transverse potential at one center and the two centers have a mutual Coulomb interaction.

If we rescale the transverse potentials and electric fields $[\vec{A}_{a,\perp} = g_s \vec{\hat{A}}_{a,\perp}, \vec{\pi}_{a,\perp} = g_s^{-1} \vec{\hat{\pi}}_{a,\perp}]$, we get

$$\hat{\mathcal{H}}(\tau) = \epsilon_s - \sum_i \eta_i \sqrt{m_i^2 + (\vec{\kappa}_i(\tau) + g_s \sum_a \hat{Q}_{ai}(\tau) \vec{A}_{a,\perp}(\tau, \vec{\eta}_i(\tau)))^2 -}$$

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\[ -\sum_a \int d^3\sigma \frac{1}{2} \sum_a [\tilde{\pi}_{a\perp}^2(\sigma, \tilde{\sigma}) + \tilde{B}_a(\sigma, \tilde{\sigma})] - \frac{1}{2} V[\tilde{\eta}_i, g_s \tilde{A}_{a\perp}, g_s^{-1} \tilde{\pi}_{a\perp}](\tau) \approx 0. \]  

(97)

To get the constraint defining the intrinsic rest frame, i.e. \( \tilde{H}_p(\tau) \approx 0 \) of Eq.(66), we must consider the term \( \sum_a \tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma}) = \sum_a \tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma}) \). From Eq.(83) we get

\[ \int d^3\sigma \sum_a \tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma})(\eta_b^{(a)} = \Gamma_b = 0) = \]

\[ = \int d^3\sigma \sum_a [\tilde{\pi}_{a\perp}(\tau, \tilde{\sigma}) + \frac{\tilde{\sigma}}{\Delta} \int d^3\sigma_1 \tilde{\sigma}_{a\perp} \cdot \tilde{\zeta}_{ad}^{(a\perp)}(\sigma, \tilde{\sigma}_1; \tau) \tilde{\rho}_d(\sigma, \tilde{\sigma}_1)] \times \tilde{B}_a(\tau, \tilde{\sigma}). \]  

(98)

Remembering that \( (\tilde{\pi}_a \times \tilde{B}_a)_r = \epsilon_{rmn} \tilde{\pi}_a^m B_a^n = F_{ars} \tilde{\pi}_a^s \), we get

\[ \int d^3\sigma \sum_a [\tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma})]_r \eta_b^{(a)} = \Gamma_b = 0 = \]

\[ = \int d^3\sigma \sum_a F_{ars}(\tau, \tilde{\sigma}) \tilde{\pi}_a^s(\tau, \tilde{\sigma}) + \]

\[ + \int d^3\sigma \sum_a [\partial_r \tilde{A}_{a\perp} - \partial_\tau \tilde{A}_{a\perp} + c_{abc} \tilde{A}_{b\perp} \tilde{A}_{c\perp}](\tau, \tilde{\sigma}) \cdot \]

\[ \cdot \frac{\partial^s}{\Delta} \int d^3\sigma_1 \tilde{\sigma}_{a\perp} \cdot \tilde{\zeta}_{ad}^{(a\perp)}(\sigma, \tilde{\sigma}_1; \tau) \tilde{\rho}_d(\sigma, \tilde{\sigma}_1) = \]

\[ = \int d^3\sigma \sum_a F_{ars}(\tau, \tilde{\sigma}) \tilde{\pi}_a^s(\tau, \tilde{\sigma}) + \]

\[ + \int d^3\sigma \sum_a \tilde{A}_{b\perp}(\tau, \tilde{\sigma}) [\delta_{ba} \Delta - c_{bca} \tilde{A}_{c\perp} \partial_\sigma](\tau, \tilde{\sigma}) \partial_s \cdot \]

\[ \cdot \frac{1}{\Delta} \int d^3\sigma_1 \tilde{\sigma}_{a\perp} \cdot \tilde{\zeta}_{ad}^{(a\perp)}(\sigma, \tilde{\sigma}_1; \tau) \tilde{\rho}_d(\sigma, \tilde{\sigma}_1). \]  

(99)

We have done a first integration by parts to use the transversality of \( \tilde{A}_{a\perp} \) and then a second one on \( \partial_\tau \tilde{A}_{a\perp} \). In the last expression we recognize the Faddeev-Popov operator \( \tilde{K}_{b = \perp}(\tau, \tilde{\sigma}) = -\tilde{\sigma} \cdot \tilde{D}_{b = \perp}(\tau, \tilde{\sigma}) = \delta_{ba} \Delta + c_{bca} \tilde{A}_{c\perp} \cdot \tilde{\sigma} = \delta_{ba} \Delta - c_{bca} \tilde{A}_{c\perp} \cdot \tilde{\sigma}. \) As shown in Eqs.(3-11) and (3-17) of the second paper in Refs. [1], we have \( \tilde{K}_{b = \perp}(\tau, \tilde{\sigma}) \frac{1}{\Delta} \tilde{\sigma} \cdot \tilde{\zeta}_{bc}^{(a\perp)}(\tilde{\sigma}, \tilde{\sigma}_1; \tau) = -\delta_{ac} \Delta^3(\tilde{\sigma} - \tilde{\sigma}_1). \) Therefore, we get

\[ \int d^3\sigma \sum_a [\tilde{\pi}_a(\tau, \tilde{\sigma}) \times \tilde{B}_a(\tau, \tilde{\sigma})]_r \eta_b^{(a)} = \Gamma_b = 0 = \]

\[ = \int d^3\sigma \sum_a [\tilde{F}_{ars} \tilde{\pi}_a^s - \tilde{A}_{a\perp} \tilde{\rho}_a](\tau, \tilde{\sigma}) = \]

\[ = \int \sum_a [(\partial_\tau \tilde{A}_{a\perp} + c_{abc} \tilde{A}_{b\perp} \tilde{A}_{c\perp}) \tilde{\pi}_a^s - \]

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the Faddeev-Popov operator, we obtain

\[ -\tilde{A}_{a \perp} (\vec{c}_{ab} \cdot \vec{\pi}_{c \perp} + \sum_{i=1}^{N} \vec{\rho}_{ia}) (\tau, \vec{\sigma}) = \]

\[ = \sum_{i=1}^{N} \sum_{a} \tilde{Q}_{ia} (\tau) \tilde{A}_{a \perp r} (\tau, \vec{\pi}_{i} (\tau)) + \int d^{3} \sigma \sum_{a} [ (\partial_{\tau} \tilde{A}_{a \perp s} - \partial_{s} \tilde{A}_{a \perp r}) \tilde{\pi}_{a \perp}^{s} (\tau, \vec{\sigma})]. \]  

(100)

Then, the constraints \( \tilde{\mathcal{H}}_{p} (\tau) \approx 0 \) have the form

\[ \tilde{\mathcal{H}}_{p r} (\tau) = \sum_{i=1}^{N} \tilde{k}_{i r} (\tau) + \int d^{3} \sigma \sum_{a} [ (\partial_{\tau} \tilde{A}_{a \perp s} - \partial_{s} \tilde{A}_{a \perp r}) \tilde{\pi}_{a \perp}^{s} ] (\tau, \vec{\sigma}) = \]

\[ = \tilde{k}_{+ r} (\tau) + \int d^{3} \sigma \sum_{a} [ (\partial_{\tau} \tilde{A}_{a \perp s} - \partial_{s} \tilde{A}_{a \perp r}) \tilde{\pi}_{a \perp}^{s} ] (\tau, \vec{\sigma}) \approx 0. \]  

(101)

As in the electromagnetic case of Ref. [7], there is no interaction term in this constraint (neither minimal coupling to particles nor self-coupling of the color field). As shown there, this is the requirement for being in an instant form of the dynamics. The other requirement is that the rest-frame spin tensor \( \tilde{S}_{r s}^{rs} \) must also not depend on the interaction. From Eq.(71), rewritten in terms of \( \tilde{\eta}_{i}, \tilde{k}_{i} \), one has

\[ \tilde{S}_{r s}^{rs} (\tau) = \sum_{i} \eta^{i}_{r} \kappa_{i}^{s} + \sum_{i} \eta^{i}_{s} \sum_{a} Q_{ia} \tilde{A}_{a}^{s} (\vec{\pi}_{i}) - (r \leftrightarrow s) + \]

\[ + \int d^{3} \sigma \sigma^{r} F_{a u}^{r} (\vec{\sigma}) \tilde{\pi}_{a}^{u} (\vec{\sigma}) - (r \leftrightarrow s) = \]

\[ = \sum_{i} \eta^{i}_{r} \kappa_{i}^{s} + \sum_{i} \eta^{i}_{s} \sum_{a} \tilde{Q}_{ia} \tilde{A}_{a \perp}^{s} (\vec{\pi}_{i}) - (r \leftrightarrow s) + \]

\[ + \int d^{3} \sigma \sigma^{r} \tilde{F}_{a u}^{s} (\vec{\sigma}) \tilde{\pi}_{a \perp}^{u} (\vec{\sigma}) - (r \leftrightarrow s). \]  

(102)

By doing various integrations by part, discarding terms symmetric in \( (r \leftrightarrow s) \) and using the Faddeev-Popov operator, we obtain

\[ \int d^{3} \sigma \sigma_{r} \tilde{F}_{a u} (\vec{\sigma}) \tilde{\pi}_{a \perp}^{u} (\vec{\sigma}) - (r \leftrightarrow s) = \]

\[ = \int d^{3} \sigma \sigma_{r} \tilde{F}_{a u} (\vec{\sigma}) \tilde{\pi}_{a \perp}^{u} (\vec{\sigma}) + \]

\[ + \int d^{3} \sigma \sigma_{r} \tilde{F}_{a u} (\vec{\sigma}) \tilde{\pi}_{a \perp}^{u} (\vec{\sigma}) \frac{\tilde{F}_{a u}}{D} \int d^{3} \sigma' \tilde{\sigma} \cdot \tilde{\zeta}_{ab} (\vec{\sigma}, \vec{\sigma}') \tilde{\rho}_{b} (\vec{\sigma}') - (r \leftrightarrow s) = \]

\[ = \int d^{3} \sigma \sigma_{r} \tilde{F}_{a u} (\vec{\sigma}) \tilde{\pi}_{a \perp}^{u} (\vec{\sigma}) + \]

\[ - \int d^{3} \sigma \sigma_{u} (\tilde{F}_{a u}^{r}) \int d^{3} \sigma' \frac{\tilde{\sigma} \cdot \tilde{\zeta}_{ab} (\vec{\sigma}, \vec{\sigma}')}{D} \tilde{\rho}_{b} (\vec{\sigma}') - (r \leftrightarrow s) = \]

\[ = \int d^{3} \sigma \sigma_{r} \tilde{F}_{a u} (\vec{\sigma}) \tilde{\pi}_{a \perp}^{u} (\vec{\sigma}) + \]

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\[
- \int d^3 \sigma \left[ -\partial_s A_{\perp}(\sigma) - \partial_r A_{\perp}(\sigma) - \partial^u \partial_u (A_{\perp}(\sigma)\sigma^r) + c_{abk} A_{\perp}(\sigma) \partial^u (A_{\perp}(\sigma)\sigma^r) \right] \\
\cdot \int d^3 \sigma' \frac{\vec{\partial} \cdot \vec{\zeta}_{ab}(\sigma, \sigma')}{\Delta} \hat{\rho}_b(\sigma') - (r \leftrightarrow s) = \\
= \int d^3 \sigma \sigma_r F_{\mu\nu}(\sigma) \hat{\pi}_{\mu\nu}(\sigma) + \\
+ \int d^3 \sigma \int d^3 \sigma' \left[ \partial_u \partial^u (A_{\perp}(\sigma)\sigma_r) - c_{bka} A_{\perp}(\sigma) \partial^u (A_{\perp}(\sigma)\sigma_r) \right] \frac{\vec{\partial} \cdot \vec{\zeta}_{ab}(\sigma, \sigma')}{\Delta} \hat{\rho}_b(\sigma') + \\
- (r \leftrightarrow s) = \\
= \int d^3 \sigma \sigma_r F_{\perp}(\sigma) \hat{\pi}_{\perp}(\sigma) + \\
+ \int d^3 \sigma \int d^3 \sigma' \hat{A}_{\perp}(\sigma) \sigma_r \left[ \partial_u \delta_{ha} + c_{bka} A_{\perp}(\sigma) \partial^u \right] \frac{\vec{\partial} \cdot \vec{\zeta}_{ab}(\sigma, \sigma')}{\Delta} \hat{\rho}_b(\sigma') + \\
- (r \leftrightarrow s) = \\
= \int d^3 \sigma \sigma_r \hat{F}_{\perp}(\sigma) \hat{\pi}_{\perp}(\sigma) - \int d^3 \sigma \hat{A}_{\perp}(\sigma) \sigma_r \hat{\rho}_b(\sigma) - (r \leftrightarrow s). 
\tag{103}
\]

In conclusion, we get [in the last line we use \( \vec{\mathcal{H}}_p(\tau) \approx 0 \) and go to relative variables]

\[
\bar{S}^{rs}_s = \sum_{i=1}^N \eta^r_i(\tau) \bar{S}^i_s(\tau) + \int d^3 \sigma \sigma_r \sum_a \left[ \partial^s \hat{A}_{\perp}(\tau, \sigma) - \partial_u \hat{A}_{\perp}(\tau, \sigma) \right] \hat{\pi}_{\perp}^u(\tau, \sigma) - \\
- (r \leftrightarrow s) \approx \\
\approx \sum_{a=1}^{N-1} \rho^r_a(\tau) \bar{S}^a_s(\tau) + \int d^3 \sigma (\sigma^r - \eta_{r+}(\tau)) \sum_a \left[ \partial^s \hat{A}_{\perp}(\tau, \sigma) - \partial_u \hat{A}_{\perp}(\tau, \sigma) \right] \hat{\pi}_{\perp}^u(\tau, \sigma) - \\
- (r \leftrightarrow s), \tag{104}
\]

as expected in an instant form also in the non-Abelian case.
V. THE REDUCED HAMILTON-DIRAC EQUATIONS

To write the reduced Hamilton equations, it is convenient to introduce the gauge-fixing

\[ \chi = T_s - \tau \approx 0, \]

whose conservation in time requires \( \lambda(\tau) = -1 \) in Eq.(72). Then, we can eliminate the pair of variables \((T_s, \epsilon_s)\) and describe the evolution in terms of the rest-frame time. The Hamiltonian for this evolution will be

\[ \hat{H}_D = \hat{H}_{rel} - \vec{\lambda}(\tau) \cdot \vec{\mathcal{H}}_p(\tau) + \sum_{i=1}^{N} \mu_i(\tau) \bar{N}_i, \]

with

\[ \hat{H}_{rel} = \sum_i \eta_i \sqrt{m_i^2 + \left( \vec{\kappa}_i(\tau) + \sum_a \vec{Q}_ia(\tau) \vec{A}_{a\perp}(\tau, \vec{n}_i(\tau)) \right)^2} + \]

\[ + \sum_a \int d^3\sigma \left[ \frac{g_s^2 \tau_{a\perp}(\tau, \sigma)}{2} + \frac{\check{B}_{a\perp}(\tau, \sigma)}{2g_s^2} \right] + \frac{1}{2} V[\vec{n}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}](\tau). \]

If we would know the correct form of the gauge-fixing to eliminate the intrinsic center-of-mass degrees of freedom associated with the constraints \( \vec{\mathcal{H}}_p(\tau) \approx 0 \), we could put \( \vec{\lambda}(\tau) = 0 \) and rewrite the invariant mass \( \hat{H}_{rel} \) only in terms of relative variables. See the analogous discussion for the electromagnetic case in Ref. [14]

At this stage of reduction, we get the following Hamilton equations of motion \([\tau \equiv T_s; P^r_{\perp}(\vec{\sigma}) = \delta^r_{\perp} + \partial^\sigma_{\perp}\partial^\sigma_{\perp}/\Delta; \vec{n}_i = (\eta_i^r); \partial/\partial \vec{n}_i = (\partial/\partial \eta_i^r); \) the symbol \( \overset{\circ}{\text{}} \) means evaluated on the solutions of the equations of motion]

\[ \frac{d}{d\tau} \vec{n}_i(\tau) \overset{\circ}{=} \eta_i \frac{\vec{\kappa}_i(\tau) + \sum_a \vec{Q}_ia(\tau) \vec{A}_{a\perp}(\tau, \vec{n}_i(\tau))}{\sqrt{m_i^2 + \left( \vec{\kappa}_i(\tau) + \sum_a \vec{Q}_ia(\tau) \vec{A}_{a\perp}(\tau, \vec{n}_i(\tau)) \right)^2}} - \vec{\lambda} (\tau) \]

\[ \frac{d}{d\tau} \vec{\kappa}_i(\tau) \overset{\circ}{=} - \sum_a \left( \frac{d\eta_i^u(\tau)}{d\tau} + \lambda^u(\tau) \sum_a \vec{Q}_ia(\tau) \frac{\partial \vec{A}_{a\perp}^u(\tau, \vec{n}_i(\tau))}{\partial \vec{n}_i} \right) - \]

\[ - g_s^2 \sum_{a,b} \vec{Q}_ia(\tau) \int d^3\sigma' \frac{\partial K_{ab}(\vec{n}_i(\tau), \vec{\sigma}'; \tau)}{\partial \vec{n}_i} \check{\rho}_b(\tau, \vec{\sigma}'), \]

\[ \frac{d}{d\tau} \vec{\theta}_{\alpha\beta}(\tau) \overset{\circ}{=} \sum_{a} (T^\alpha)_{\alpha\beta} \check{\theta}_{\alpha\beta}(\tau) \cdot \vec{\mathcal{A}}_{b\perp}(\tau, \vec{n}_i(\tau)) - \]

\[ \frac{d}{d\tau} \vec{\pi}_{\alpha\beta}(\tau) \overset{\circ}{=} \sum_{a} (T^\alpha)_{\alpha\beta} \check{\pi}_{\alpha\beta}(\tau) \cdot \vec{\mathcal{A}}_{b\perp}(\tau, \vec{n}_i(\tau)) - \]

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\[ -g_s^2 \int d^3 \sigma K_{ab}(\tilde{\eta}_i(\tau), \sigma; \tau) \tilde{p}_b(\tau, \sigma) - i \mu_1(\tau) \tilde{\theta}_{ia}(\tau), \]
\[ \frac{d}{d \tau} \tilde{\theta}_{ia}(\tau) \overset{\circ}{=} - \sum_a \tilde{\theta}_{ia}^*(\tau) (T^a)_{\beta a} [\tilde{\eta}_i(\tau) \cdot \tilde{A}_{b \perp}(\tau, \tilde{\eta}_i(\tau)) - \frac{d}{d \tau} \tilde{\theta}_{ia}(\tau) ] \]
\[ - g_s^2 \int d^3 \sigma K_{ab}(\tilde{\eta}_i(\tau), \sigma; \tau) \tilde{p}_b(\tau, \sigma) + i \mu_1(\tau) \tilde{\theta}_{ia}(\tau), \]
\[ \frac{d}{d \tau} \tilde{Q}_{ia}(\tau) \overset{\circ}{=} c_{acd} \tilde{Q}_{id}(\tau) [\tilde{\eta}_i(\tau) \cdot \tilde{A}_{c \perp}(\tau, \tilde{\eta}_i(\tau)) - \frac{d}{d \tau} \tilde{\theta}_{ia}(\tau) ] \]
\[ - g_s^2 \int d^3 \sigma K_{cb}(\tilde{\eta}_i(\tau), \sigma; \tau) \tilde{p}_b(\tau, \sigma), \]
\[ , \quad (108) \]
\[ \frac{\partial}{\partial \tau} \tilde{A}_{a \perp r}(\tau, \sigma) \overset{\circ}{=} - g_s^2 \tilde{\pi}_{a \perp r}(\tau, \sigma) - [\tilde{X}(\tau) \cdot \tilde{\partial}] \tilde{A}_{a \perp r}(\tau, \sigma) + \]
\[ + g_s^2 P_{r a}^{rs}(\sigma) \int d^3 \sigma' \ c_{abu} [\tilde{A}_{a \perp}(\tau, \sigma) K_{bc}(\sigma, \sigma'; \tau)) \tilde{p}_c(\tau, \sigma') \]
\[ \frac{\partial}{\partial \tau} \tilde{\pi}_{a \perp}^r(\tau, \sigma) \overset{\circ}{=} g_s^2 P\tilde{A}_{a \perp}(\tau, \sigma) \tilde{A}_{b \perp}(\tau, \sigma) - [\tilde{X}(\tau) \cdot \tilde{\partial}] \tilde{\pi}_{a \perp}^r(\tau, \sigma) - \]
\[ - \sum_{i=1}^N \tilde{Q}_{ia}(\tau) P_{r a}^{rs}(\sigma)(\frac{d \eta_i^r(\tau)}{d \tau} + \lambda^s(\sigma - \eta_i(\tau)) + \]
\[ + \frac{1}{2} P_{r a}^{rs}(\sigma) \frac{\delta V[\tilde{\eta}_i, \tilde{A}_{a \perp}, \tilde{\pi}_{a \perp}]}{\delta \tilde{A}_{a \perp s}(\tau, \sigma)}, \]
\[ \frac{d}{d \tau} \tilde{Q}_{a}^{YM}(\tau) \overset{\circ}{=} - \frac{d}{d \tau} \sum_{i=1}^N \tilde{Q}_{ia}(\tau) = - c_{acd} \sum_{i=1}^N \tilde{Q}_{id}(\tau) [\tilde{\eta}_i(\tau) \cdot \tilde{A}_{c \perp}(\tau, \tilde{\eta}_i(\tau)) - \]
\[ - g_s^2 \int d^3 \sigma K_{cb}(\tilde{\eta}_i(\tau), \sigma; \tau) \tilde{p}_b(\tau, \sigma), \]
\[ , \quad (109) \]
\[ \text{since} \quad \frac{d}{d \tau} \tilde{Q}_{a} \overset{\circ}{=} 0 \quad [\text{it is not known (see also Ref. [1]) how to check this formula by a direct calculation}]. \]

In Eqs.(109) we have
\[ \frac{1}{2} \frac{\delta V[\tilde{\eta}_i, \tilde{A}_{a \perp}, \tilde{\pi}_{a \perp}]}{\delta \tilde{A}_{a \perp s}(\tau, \sigma)} = - g_s^2 P_{r a}^{rs}(\sigma) c_{abu} \tilde{\pi}_{a \perp}^s(\tau, \sigma) \int d^3 \sigma' K_{bc}(\sigma, \sigma'; \tau) \tilde{p}_c(\tau, \sigma') + \]
\[ + g_s^2 P_{r a}^{rs}(\sigma) \int d^3 \sigma_1 d^3 \sigma_2 \tilde{p}_b(\tau, \sigma_1) \]
\[ \{ \int \frac{d^3 \sigma'}{4 \pi |\sigma' - \sigma_2|} [\delta^3(\sigma - \sigma') c_{be f} (\tilde{A}_{f \perp}(\tau, \sigma')) c_{man} s_{nc}(\sigma, \sigma_1; \tau) + \]
\[ + c_{be f} \tilde{A}_{f \perp}(\tau, \sigma') \cdot \tilde{c}_{em}(\tilde{A}_{e \perp}) (\sigma', \sigma; \tau) c_{man} (\tilde{A}_{e \perp})^s(\sigma, \sigma_1; \tau)] \}
\[ + \frac{d^3 \sigma_3 d^3 \sigma_4}{4 \pi |\sigma_3 - \sigma_4|} \{ [\delta^3(\sigma - \sigma_3) c_{be f} (\tilde{A}_{f \perp}(\tau, \sigma_3)) c_{man} s_{nc}(\sigma, \sigma_1; \tau) + \]
\[ + \]
These equations of motion have to be supplemented with the constraints

\[ \{ \frac{\partial}{\partial \tau} \bar{A}_b^\perp (\tau, \bar{\sigma}); \bar{\sigma}_1 \} \} \bar{p}_c (\tau, \bar{\sigma}_2). \]

(110)

because from Eq. (74) we get

\[
\frac{\delta \zeta_{bc}^\perp (\bar{\sigma}_1, \bar{\sigma}_2; \tau)}{\delta \bar{A}_{u_{ls}} (\tau, \bar{\sigma})} = -P^s_{1} (\bar{\sigma}) \delta^3 (\bar{\sigma} - \bar{\sigma}_1) c_{abc}^s \zeta_{bc}^\perp (\bar{\sigma}_1, \bar{\sigma}_2; \tau),
\]

\[
\Rightarrow \frac{\delta \zeta_{bc}^\perp (\bar{\sigma}_1, \bar{\sigma}_2; \tau)}{\delta \bar{A}_{u_{ls}} (\tau, \bar{\sigma})} = P^s_{1} (\bar{\sigma}) \zeta_{bc}^\perp (\bar{\sigma}_1, \bar{\sigma}_2; \tau) c_{abc}^s \zeta_{bc}^\perp (\bar{\sigma}_1, \bar{\sigma}_2; \tau). \tag{111}
\]

These equations of motion have to be supplemented with the constraints

\[
\dot{\mathcal{H}}_{pr} (\tau) = \sum_{i=1}^{N} \dot{\mathcal{H}}_{ip} (\tau) + \int d^3 \sigma \sum_{a} [(\partial_r \bar{A}_{a_{\perp}} - \partial_s \bar{A}_{a_{\perp}r}) \bar{\pi}_{a_{\perp}}^s (\tau, \bar{\sigma})] (\tau, \bar{\sigma}) \approx 0,
\]

\[
\bar{N}_i = \sum_{\alpha=1}^{3} \theta^*_{\alpha} (\tau) \theta_{\alpha} (\tau) \approx 0. \tag{112}
\]

The first line of Eq. (108) can be inverted to get

\[
\bar{\kappa}_i (\tau) = \eta_i m_i \frac{\bar{\eta}_i (\tau) + \bar{\lambda} (\tau)}{\sqrt{1 - (\bar{\eta}_i (\tau) + \bar{\lambda} (\tau))^2}} - \sum_a \bar{Q}_{ia} (\tau) \bar{A}_{a_{\perp}} (\tau, \bar{\eta}_i (\tau)). \tag{113}
\]

The first line of Eqs. (109) coincides with Eq. (6-24) of the second paper in Refs. [1] if in this equation \( \bar{\kappa}_{a_{\perp}} \rightarrow -\bar{\kappa}_{a_{\perp}} , i \bar{\psi}^T T^c \bar{\psi} \rightarrow \sum_{i=1}^{N} \bar{Q}_{ic} \) and if we note (see Ref. [1]) that

\[
\bar{D}_{ib}^{(A_{\perp})k} (\tau, \bar{\sigma}) P^k_{ab} (\bar{\sigma}) \zeta_{bc}^\perp (\bar{\sigma}, \bar{\sigma}_1; \tau) = 0, \quad K_{ab} (\bar{\sigma}_1, \bar{\sigma}_2; \tau) = -G_{ab} (\bar{\sigma}_1, \bar{\sigma}_2; \tau) \text{ and } (\text{see Eq. (3-17) of that paper}) \zeta_{ac}^\perp (\bar{\sigma}, \bar{\sigma}_1; \tau) = -\bar{D}_{ab} (\tau, \bar{\sigma}) G_{\Delta, bc}^{(A_{\perp})} (\bar{\sigma}, \bar{\sigma}_1; \tau) = \bar{D}_{ab} (\tau, \bar{\sigma}) K_{bc} (\bar{\sigma}, \bar{\sigma}_1; \tau). \]

Therefore, Eq. (6-21) of that paper gives its inversion in the form

\[
\bar{\pi}_{a_{\perp}}^r (\tau, \bar{\sigma}) = g_s^{-2} \bar{E}_{a_{\perp}}^r (\tau, \bar{\sigma}) \frac{\partial}{\partial \tau} + \bar{\lambda} (\tau) \cdot \frac{\partial}{\partial \bar{\sigma}_1} \bar{A}_{b_{\perp}}^r (\tau, \bar{\sigma}_1) -
\]

\[
\int d^3 \sigma_1 \mathcal{P}_{ab}^{(A_{\perp})st} \bar{A}_{bc}^s (\tau, \bar{\sigma}_1; \tau) \]
\[- P_\perp^{rs}(\bar{\sigma}) \int d^3 \sigma_1 \zeta_{ab}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) \sum_{i=1}^{N} \dot{\rho}_{ib}(\tau, \bar{\sigma}_1) =
\]
\[= - g_s^{-2} P_\perp^{rs}(\bar{\sigma}) \int d^3 \sigma_1 [\delta^{st} \delta_{ab} \delta^3(\bar{\sigma} - \bar{\sigma}_1) +
\]
\[+ \dot{D}_{ad}^{(A_\perp)_s}(\tau, \bar{\sigma}) K_{de}(\bar{\sigma}, \bar{\sigma}_1; \tau) \dot{D}_{eb}^{(A_\perp)_s}(\tau, \bar{\sigma}_1)] \left[ \frac{\partial}{\partial \tau} + \bar{\lambda}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}_1} \right] \dot{A}_{b\perp}(\tau, \bar{\sigma}_1) +
\]
\[+ P_\perp^{rs}(\bar{\sigma}) c_{ad} \dot{A}_{d\perp}(\tau, \bar{\sigma}) \sum_{i=1}^{N} K_{cb}(\bar{\sigma}, \bar{\eta}_i(\tau); \tau) \dot{Q}_{ib}(\tau), \quad (114)\]

where [see Eq.(3-30) of Ref. [1]] \( P_{ab}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) = \delta^{rs} \delta_{ab} \delta^3(\bar{\sigma} - \bar{\sigma}_1) - \dot{D}_{ad}^{(A_\perp)_s}(\tau, \bar{\sigma}) G_{\triangle de}(\bar{\sigma}, \bar{\sigma}_1; \tau) \dot{D}_{eb}^{(A_\perp)_s}(\tau, \bar{\sigma}_1) =
\]
\[= \delta^{rs} \delta_{ab} \delta^3(\bar{\sigma} - \bar{\sigma}_1) + \zeta_{ac}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) \dot{D}_{cb}^{(A_\perp)_s}(\tau, \bar{\sigma}_1) \text{ is a projector giving an explicit realization of the Mitter-Viallet abstract metric [it satisfies } \int d^3 \sigma_1 P_{ab}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) P_{bc}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) \dot{D}_{cb}^{(A_\perp)_s}(\tau, \bar{\sigma}_1) = \delta^{rs} \delta_{ab} \delta^3(\bar{\sigma} - \bar{\sigma}_1) \text{].}
\]

By using also Eq.(6-23) of Ref. [1], namely \( \dot{D}_{ad}^{(A_\perp)_s}(\tau, \bar{\sigma}) P_\perp^{rs}(\bar{\sigma}) \dot{D}_{eb}^{(A_\perp)_s}(\tau, \bar{\sigma}_1) = 0 \) (when acting on functions of \( \bar{\sigma} \)), we get
\[\dot{\rho}_a(\tau, \bar{\sigma}) = g_s^{-2} c_{abc} \dot{A}_{b\perp}(\tau, \bar{\sigma}) P_\perp^{rs}(\bar{\sigma}) \int d^3 \sigma_1 P_{cd}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) \left[ \frac{\partial}{\partial \tau} + \bar{\lambda}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}_1} \right] \dot{A}_{b\perp}(\tau, \bar{\sigma}_1) +
\]
\[+ \sum_{i=1}^{N} \int d^3 \sigma_1 [\delta^{ad} \delta^3(\bar{\sigma} - \bar{\sigma}_1) + c_{abc} \dot{A}_{b\perp}(\tau, \bar{\sigma}) P_\perp^{rs}(\bar{\sigma}) c_{cd}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau)] \dot{\rho}_{ia}(\tau, \bar{\sigma}_1) =
\]
\[= g_s^{-2} c_{abc} \dot{\tilde{A}}_{b\perp}(\tau, \bar{\sigma}) \cdot \left[ \frac{\partial}{\partial \tau} + \bar{\lambda}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \right] \dot{\tilde{A}}_{c\perp}(\tau, \bar{\sigma}) + \sum_{i=1}^{N} \dot{\rho}_a(\tau, \bar{\sigma}) =
\]
\[= \dot{\rho}_a^{(Y_M)}(\tau, \bar{\sigma}) + \sum_{i=1}^{N} \dot{\rho}_a(\tau, \bar{\sigma}),
\]

\[\dot{\rho}_a^{(Y_M)}(\tau, \bar{\sigma}) = - g_s^{-2} c_{abc} \dot{\tilde{A}}_{b\perp}(\tau, \bar{\sigma}) \cdot \dot{\tilde{E}}_{c\perp}(\tau, \bar{\sigma}). \quad (115)\]

Therefore by putting \( \bar{\lambda}(\tau) = \dot{g}(\tau) \) like in Ref. [14], the equations of motion for \( \dot{\tilde{A}}_{a\perp}(\tau, \bar{\sigma}) \) become
\[\left[ \frac{\partial}{\partial \tau} + \bar{\lambda}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \right] \{ - g_s^{-2} P_\perp^{rs}(\bar{\sigma}) \int d^3 \sigma_1 P_{ab}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) \cdot
\]
\[\left[ \frac{\partial}{\partial \tau} + \dot{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}_1} \right] \dot{A}_{b\perp}(\tau, \bar{\sigma}_1) -
\]
\[- P_\perp^{rs}(\bar{\sigma}) \int d^3 \sigma_1 \zeta_{ab}^{(A_\perp)_s}(\bar{\sigma}, \bar{\sigma}_1; \tau) \sum_{i=1}^{N} \dot{\rho}_{ib}(\tau, \bar{\sigma}_1) \} -
\]
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\[-g^{-2}_s P_{\perp}^{rs}(\bar{\sigma})[\delta_{ab}\triangle + c_{abc}\tilde{A}_{c,\perp}(\tau, \bar{\sigma}) \cdot \vec{\partial}]\tilde{A}_{b,\perp}^{s}(\tau, \bar{\sigma}) \triangleq \]

\[\equiv \sum_{i=1}^{N} \tilde{Q}_{ia}(\tau) P_{\perp}^{rs}(\bar{\sigma})[\tilde{\eta}^{s}_{i}(\tau) + \lambda^{s}(\tau)]\delta^{3}(\bar{\sigma} - \tilde{\eta}_{i}(\tau)) + \]

\[-g^{2}_s P_{\perp}^{rs}(\bar{\sigma})c_{abc}\tilde{\pi}_{e,\perp}^{s}(\tau, \bar{\sigma}) \int d^{3}\sigma' K_{bc}(\bar{\sigma}, \bar{\sigma}'; \tau)\tilde{\rho}_{c}(\tau, \bar{\sigma}') + \]

\[+ g^{2}_s P_{\perp}^{rs}(\bar{\sigma}) \int d^{3}\sigma_{1} d^{3}\sigma_{2} \tilde{\rho}_{b}(\tau, \bar{\sigma}_{1}) \]

\[\{ \int \frac{d^{3}\sigma'}{4\pi|\bar{\sigma} - \sigma'|} [\delta^{3}(\bar{\sigma} - \sigma')c_{abc}\zeta_{ec}^{s}(\vec{\sigma}, \bar{\sigma}_{1}; \tau) + \]

\[+ c_{bcf}\tilde{A}_{f,\perp}(\tau, \vec{\sigma}'), \zeta_{em}^{s}(\vec{\sigma}, \bar{\sigma}; \tau)c_{man}\zeta_{nc}^{s}(\vec{\sigma}, \bar{\sigma}_{1}; \tau)] + (\bar{\sigma}_{1} \leftrightarrow \bar{\sigma}_{2}) + \]

\[+ \int \frac{d^{3}\sigma_{3} d^{3}\sigma_{4}}{4\pi|\sigma_{3} - \sigma_{4}|} \{ [\delta^{3}(\bar{\sigma} - \sigma_{3})c_{abc}\zeta_{eu}^{s}(\vec{\sigma}_{3}, \bar{\sigma}_{1}; \tau) + \]

\[+ c_{bcf}\tilde{A}_{f,\perp}(\tau, \sigma_{3}), \zeta_{em}^{s}(\sigma_{3}, \bar{\sigma}; \tau)c_{man}\zeta_{nu}^{s}(\sigma_{3}, \bar{\sigma}_{1}; \tau)] \cdot \]

\[c_{ers}\tilde{A}_{s,\perp}(\tau, \bar{\sigma}_{1}), \zeta_{ru}^{s}(\sigma_{4}, \bar{\sigma}_{2}; \tau) + \]

\[c_{bcf}\tilde{A}_{f,\perp}(\tau, \sigma_{3}), \zeta_{eu}^{s}(\sigma_{3}, \bar{\sigma}_{1}; \tau) \]

\[\{ \delta^{3}(\bar{\sigma} - \sigma_{4})c_{acn}\zeta_{smu}^{s}(\sigma_{4}, \sigma_{2}; \tau) + \]

\[+ c_{cmn}\tilde{A}_{n,\perp}(\tau, \sigma_{4}), \zeta_{smr}^{s}(\sigma_{4}, \bar{\sigma}; \tau)c_{cras}\zeta_{snu}^{s}(\sigma_{4}, \sigma_{2}; \tau) \}] \} \tilde{\rho}_{c}(\tau, \bar{\sigma}_{2}), \]

(116)

After some manipulations its final form is

\[P_{\perp}^{rs}(\bar{\sigma}) \equiv \{ \int \frac{\partial}{\partial \tau} + \dot{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \int d^{3}\sigma P_{ad}^{(A_{\perp})st}(\bar{\sigma}, \bar{\sigma}; \tau)[\frac{\partial}{\partial \tau} + \dot{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}}] + \]

\[+ \delta^{st} \int d^{3}\sigma \delta^{3}(\bar{\sigma} - \bar{\sigma})\tilde{K}_{ad}^{(A_{\perp})}(\tau, \bar{\sigma}) \} \tilde{A}_{d,\perp}^{t}(\tau, \bar{\sigma}) \triangleq \]

\[\equiv - g^{2}_s P_{\perp}^{rs}(\bar{\sigma})[\sum_{i=1}^{N} \tilde{Q}_{ia}(\tau)[\tilde{\eta}^{s}_{i}(\tau) + \dot{g}^{s}(\tau)]\delta^{3}(\bar{\sigma} - \tilde{\eta}_{i}(\tau)) + \]

\[+ c_{abd}\tilde{A}_{b,\perp}(\tau, \bar{\sigma}) \int d^{3}\sigma' K_{bc}(\bar{\sigma}, \bar{\sigma}' ; \tau)\tilde{\rho}_{c}(\tau, \bar{\sigma}') \} - \]

\[- g^{4}_s P_{\perp}^{rs}(\bar{\sigma})\tilde{A}_{d,\perp}^{s}(\tau, \bar{\sigma}) \int d^{3}\sigma_{1} d^{3}\sigma_{2} \tilde{\rho}_{b}(\tau, \sigma_{1})F_{abcd}(\bar{\sigma}, \bar{\sigma}_{1}, \bar{\sigma}_{2}; \tau) + \]

\[+ c_{auv}c_{vde}K_{eb}(\bar{\sigma}, \bar{\sigma}_{1}; \tau)K_{uc}(\bar{\sigma}, \bar{\sigma}_{2}; \tau)\tilde{\rho}_{c}(\bar{\sigma}_{2}, \bar{\sigma}_{2}), \]

or

\[P_{\perp}^{rs}(\bar{\sigma}) \equiv \{ \int \frac{\partial}{\partial \tau} + \dot{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \int d^{3}\sigma [\delta^{st}\delta_{ad}\delta^{3}(\bar{\sigma} - \bar{\sigma}) + \]

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\[ + \hat{D}_{au}^{(A_{\perp})s}(\tau, \bar{\sigma}) K_{au}(\bar{\sigma}, \bar{\sigma}; \tau) \hat{D}_{vd}^{(A_{\perp})t}(\tau, \bar{\sigma}) \left[ \frac{\partial}{\partial \tau} + \hat{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \right] + \delta^{st} \int d^{3}\bar{\sigma} \delta^{3}(\bar{\sigma} - \bar{\sigma}) \left[ \delta_{ad} \bar{\Delta} + c_{adc} \bar{A}_{c \perp}(\tau, \bar{\sigma}) \cdot \frac{\partial}{\partial \bar{\sigma}} \right] \hat{A}_{d \perp}^{t}(\tau, \bar{\sigma}) = \]

\[ = - g_{s}^{2} P_{d \perp}^{rs}(\bar{\sigma}) \left\{ \sum_{i=1}^{N} \hat{Q}_{ia}(\tau) [\hat{\eta}_{i}^{s}(\tau) + \hat{g}^{s}(\tau)] \delta^{3}(\bar{\sigma} - \bar{\eta}_{i}(\tau)) + c_{abd} \bar{A}_{d \perp}(\tau, \bar{\sigma}) \int d^{3}\sigma' K_{bc}(\bar{\sigma}, \bar{\sigma}'; \tau) \hat{\rho}_{c}(\tau, \bar{\sigma}') \right\} - \]

\[ - g_{s}^{4} P_{d \perp}^{rs}(\bar{\sigma}) \hat{A}_{d \perp}(\tau, \bar{\sigma}) \int d^{3}\sigma_1 d^{3}\sigma_2 \hat{\rho}_{b}(\tau, \bar{\sigma}_1) + \sum_{i=1}^{N} \hat{\rho}_{ib}(\tau, \bar{\sigma}_1)\]

\[ [F_{abcd}(\bar{\sigma}, \bar{\sigma}_1; \tau) + c_{acd} c_{bde} K_{eb}(\bar{\sigma}, \bar{\sigma}_1; \tau) K_{ec}(\bar{\sigma}, \bar{\sigma}_2; \tau)] \hat{\rho}_{c}(\tau, \bar{\sigma}_2)]\]

or

\[ P_{d \perp}^{rs}(\bar{\sigma}) \left\{ \delta^{st} [\delta_{ad} \left( \frac{\partial}{\partial \tau} + \hat{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \right)^{2} + \Delta] + c_{adc} \bar{A}_{c \perp}(\tau, \bar{\sigma}) \cdot \frac{\partial}{\partial \bar{\sigma}} \right\].

\[ \int d^{3}\bar{\sigma} \delta^{3}(\bar{\sigma} - \bar{\sigma}) + \left[ \frac{\partial}{\partial \tau} + \hat{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \right] \hat{D}_{au}^{(A_{\perp})s}(\tau, \bar{\sigma}) \int d^{3}\bar{\sigma} K_{uv}(\bar{\sigma}, \bar{\sigma}; \tau) \hat{D}_{vd}^{(A_{\perp})t}(\tau, \bar{\sigma}) \left[ \frac{\partial}{\partial \tau} + \hat{g}(\tau) \cdot \frac{\partial}{\partial \bar{\sigma}} \right] \hat{A}_{d \perp}^{t}(\tau, \bar{\sigma}) = \]

\[ = - g_{s}^{2} P_{d \perp}^{rs}(\bar{\sigma}) \left\{ \sum_{i=1}^{N} \hat{Q}_{ia}(\tau) [\hat{\eta}_{i}^{s}(\tau) + \hat{g}^{s}(\tau)] \delta^{3}(\bar{\sigma} - \bar{\eta}_{i}(\tau)) + \right. \]

\[ + c_{abd} \bar{A}_{d \perp}(\tau, \bar{\sigma}) \int d^{3}\sigma' K_{bc}(\bar{\sigma}, \bar{\sigma}'; \tau) \hat{\rho}_{c}(\tau, \bar{\sigma}') \}

\[ \left[ \hat{\rho}_{c}(\tau, \bar{\sigma}_1) + \sum_{i=1}^{N} \hat{\rho}_{ic}(\tau, \bar{\sigma}_1) \right] - \]

\[ - g_{s}^{4} P_{d \perp}^{rs}(\bar{\sigma}) \hat{A}_{d \perp}(\tau, \bar{\sigma}) \int d^{3}\sigma_1 d^{3}\sigma_2 \left[ \hat{\rho}_{b}^{(Y M)}(\tau, \bar{\sigma}_1) + \sum_{i=1}^{N} \hat{\rho}_{ib}(\tau, \bar{\sigma}_1)\right] \]

\[ [F_{abcd}(\bar{\sigma}, \bar{\sigma}_1; \tau) + c_{acd} c_{bde} K_{eb}(\bar{\sigma}, \bar{\sigma}_1; \tau) K_{ec}(\bar{\sigma}, \bar{\sigma}_2; \tau)] \hat{\rho}_{c}(\tau, \bar{\sigma}_2)]\]

\[ F_{abcd}(\bar{\sigma}, \bar{\sigma}_1; \tau, \bar{\sigma}_2; \tau) = - c_{abe} c_{cdef} \left( \frac{K_{ef}(\bar{\sigma}, \bar{\sigma}_1; \tau)}{4\pi|\bar{\sigma} - \bar{\sigma}_2|} + \frac{K_{ef}(\bar{\sigma}, \bar{\sigma}_2; \tau)}{4\pi|\bar{\sigma} - \bar{\sigma}_1|} \right) + \]

\[ + \int \frac{d^{3}\sigma'}{4\pi} c_{bef} \bar{A}_{f \perp}(\tau, \bar{\sigma}') \cdot \zeta_{em}^{(A_{\perp})}(\bar{\sigma}', \bar{\sigma}; \tau) \]

\[ c_{man} c_{ndk} \left( \frac{K_{kc}(\bar{\sigma}, \bar{\sigma}_1; \tau)}{4\pi|\bar{\sigma} - \bar{\sigma}_2|} + \frac{K_{kc}(\bar{\sigma}, \bar{\sigma}_2; \tau)}{4\pi|\bar{\sigma} - \bar{\sigma}_1|} \right) - \]

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where the identity

\[ P_{\perp}^{rs}(\tilde{\sigma})c_{ab}^{(A_{\perp})s}(\hat{\sigma}, \hat{\sigma}'; \tau) = P_{\perp}^{rs}(\hat{\sigma})D_{ac}^{(A_{\perp})s}(\tau, \tilde{\sigma})K_{cb}(\tilde{\sigma}, \hat{\sigma}'; \tau) \]

was used in Eq. (110) to obtain \( F_{abcd} \).

The equations of motion for the particles are

\[
\frac{d}{d\tau} \left[ \eta_i m_i \frac{\dot{\eta}_i(\tau) + \tilde{g}(\tau)}{\sqrt{1 - (\tilde{\eta}_i(\tau) + \tilde{g}(\tau))^2}} - \sum_a \tilde{Q}_{ia}(\tau) \tilde{A}_{a\perp}(\tau, \tilde{\eta}_i(\tau)) \right] = \]

\[
= - [\tilde{\eta}_i^u(\tau) + \tilde{g}^u(\tau)] \sum_a \tilde{Q}_{ia}(\tau) \frac{\partial \tilde{A}_{a\perp}(\tau, \tilde{\eta}_i(\tau))}{\partial \tilde{\eta}_i} -
\]

\[
- g_s^2 \sum_{a,b} \tilde{Q}_{ia}(\tau) \int d^3 \sigma \frac{\partial K_{ab}(\tilde{\eta}_i(\tau), \tilde{\sigma}; \tau)}{\partial \tilde{\eta}_i} \tilde{\rho}_b(\tau, \tilde{\sigma}),
\]

or

\[
\frac{d}{d\tau} \left[ \eta_i m_i \frac{\dot{\eta}_i(\tau) + \tilde{g}(\tau)}{\sqrt{1 - (\tilde{\eta}_i(\tau) + \tilde{g}(\tau))^2}} \right] = \sum_a \tilde{A}_{a\perp}(\tau, \tilde{\eta}_i(\tau)) c_{ab} \tilde{Q}_{id}(\tau)
\]

\[
[\tilde{\eta}_i(\tau) \cdot \tilde{A}_{c\perp}(\tau, \tilde{\eta}_i(\tau)) - g_s^2 \int d^3 \sigma K_{cb}(\tilde{\eta}_i(\tau), \tilde{\sigma}; \tau) \tilde{\rho}_b(\tau, \tilde{\sigma})] +
\]

\[
+ \sum_a \tilde{Q}_{ia}(\tau) \left[ \frac{\partial \tilde{A}_{a\perp}(\tau, \tilde{\eta}_i(\tau))}{\partial \tau} + \dot{\tilde{\eta}}_i(\tau) \cdot \frac{\partial \tilde{A}_{a\perp}(\tau, \tilde{\eta}_i(\tau))}{\partial \tilde{\eta}_i} \right] -
\]

\[
- [\tilde{\eta}_i^u(\tau) + \tilde{g}^u(\tau)] \sum_a \tilde{Q}_{ia}(\tau) \frac{\partial \tilde{A}_{a\perp}(\tau, \tilde{\eta}_i(\tau))}{\partial \tilde{\eta}_i} -
\]

\[
- g_s^2 \sum_{a,b} \tilde{Q}_{ia}(\tau) \int d^3 \sigma \frac{\partial K_{ab}(\tilde{\eta}_i(\tau), \tilde{\sigma}; \tau)}{\partial \tilde{\eta}_i} \tilde{\rho}_b(\tau, \tilde{\sigma}),
\]

or

48
\[
\frac{d}{d\tau} \left[ \eta_i \dot{\eta}_i (\tau) + \ddot{g}(\tau) \right] = \sum_a \hat{Q}_{ia}(\tau) \left\{ \hat{E}_{a \perp} (\tau, \eta_i (\tau)) + [(\dot{\eta}_i (\tau) + \ddot{g}(\tau)) \times \hat{B}_a (\tau, \eta_i (\tau))] \right\} - \\
- \sum_a \hat{Q}_{ia}(\tau) P^{rs}_a (\eta_i) c_{amn} \hat{A}_s^{\perp}(\tau, \eta_i (\tau)) \\
\int d^3\sigma K_{dc}(\eta_i (\tau), \vec{\sigma}; \tau) c_{emb} \hat{A}_{n \perp}(\tau, \vec{\sigma}) \cdot \vec{E}_{b \perp}(\tau, \vec{\sigma}) + \\
+ g_s^2 \sum_{a,b} \hat{Q}_{ia}(\tau) \sum_{j=1}^N \left[ c_{adc} \hat{A}_{d \perp}^r (\tau, \eta_i (\tau)) K_{cb}(\eta_i (\tau), \eta_j (\tau); \tau) + \\
\frac{\partial K_{ab}(\eta_i (\tau), \eta_j (\tau); \tau)}{\partial \eta^r_i} \right] \hat{Q}_{jb}(\tau) - \\
- \hat{Q}_{ia}(\tau) \int d^3\sigma [ c_{adc} \hat{A}_{d \perp}^r (\tau, \eta_i (\tau)) K_{cb}(\eta_i (\tau), \vec{\sigma}; \tau) + \\
\frac{\partial K_{ab}(\eta_i (\tau), \vec{\sigma}; \tau)}{\partial \eta^r_i} \right] \hat{\rho}_b (Y^M) (\tau, \vec{\sigma}) \right\} - \\
- \sum_a \hat{Q}_{ia}(\tau) \ddot{g}(\tau) \left[ \frac{\partial \hat{A}_{a \perp}^r (\tau, \eta_i (\tau))}{\partial \eta_i} - c_{acb} \hat{A}_{c \perp}^s (\tau, \eta_i (\tau)) \hat{A}_{a \perp}^r (\tau, \eta_i (\tau)) \right], \quad (118)
\]

where Eqs. (108), (109), (114), (115), the definition \( \hat{F}_a^{rs} = \sigma^r \hat{A}_a^{\perp} - \sigma^s \hat{A}_a^r + c_{abc} \hat{A}_{b \perp} \hat{A}_{c \perp} = \epsilon^{rst} \hat{B}_a^t \) and \( \hat{D}_{ac}^{(A_\perp)}(\tau, \vec{\sigma}) P^{sr}_\perp(\vec{\sigma}) \hat{D}_{cb}^{(A_\perp)} r (\tau, \vec{\sigma}) = 0 \) have been used. Let us remark that Eqs.(117) and (118) are a system of integrodifferential equations due to the presence of the transverse projectors and a completely open problem is how to define an initial data problem for them.

Eq.(118) is the non-Abelian version of the particle equations of motion given in Ref. [14], which should produce the Abraham-Lorentz-Dirac equation if we were able to get a non-Abelian Lienard-Wiechert potential from Eq.(117). The last term would be absent in the gauge \( \vec{A}(\tau) = \ddot{g}(\tau) = 0 \). This equation shows explicitly the chromo-electric \( \hat{E}_{a \perp} \) and chromo-magnetic \( \hat{B}_a, \hat{A}_{a \perp} \) forces acting on the particles. Moreover, the explicit interparticle instantaneous color forces are given by

\[
g_s^2 \sum_{a,b} \hat{Q}_{ia}(\tau) \sum_{j=1}^N \hat{Q}_{jb}(\tau) c_{adc} \hat{A}_{d \perp}^r (\tau, \eta_i (\tau)) K_{cb}(\eta_i (\tau), \eta_j (\tau); \tau) + \frac{\partial K_{ab}(\eta_i (\tau), \eta_j (\tau); \tau)}{\partial \eta^r_i} = \\
= g_s^2 \sum_{a,b} \hat{Q}_{ia}(\tau) \sum_{j=1}^N \hat{Q}_{jb}(\tau) \\
\left[ c_{adc} \hat{A}_{d \perp}^r (\tau, \eta_i (\tau)) \right] \int d^3\sigma_3 d^3\sigma_4 \frac{\delta^3(\sigma_3 - \eta_i (\tau)) \delta^3(\sigma_4 - \eta_j (\tau))}{4\pi | \sigma_3 - \sigma_4 |} + \\
49\]
condition the functions $f$ \(V\) and the expression of the constant invariant mass phase space is defined for the case of SU(3) \([in this discussion we will denote \(\theta\) the Berezin-Marinov distribution function \([see in Refs. [16]\)] for the Grassmann variables of with the existence of asymptotic freedom already at this pseudoclassical level.\]

To find the classical theory implied by the pseudoclassical one we follow Ref. [15], where

\[
\frac{\partial}{\partial \eta_i} \int d^3 \sigma d^3 \sigma_4 \{ \frac{\delta_{cb} \delta^3(\bar{\sigma}_3 - \bar{\eta}_i(\tau)) \delta^3(\bar{\sigma}_4 - \bar{\eta}_j(\tau))}{4\pi | \bar{\sigma}_3 - \bar{\sigma}_4 |} + \frac{\delta^3(\bar{\sigma}_4 - \bar{\eta}_j(\tau)) [\tilde{A}_\perp(\tau, \bar{\sigma}_3) \cdot \tilde{\zeta}(\bar{A}_\perp)(\bar{\sigma}_3, \bar{\eta}_i(\tau); \tau)]_{cb} + (\bar{\eta}_i \leftrightarrow \bar{\eta}_j) + \frac{[\tilde{A}_\perp(\tau, \bar{\sigma}_3) \cdot \tilde{\zeta}(\bar{A}_\perp)(\bar{\sigma}_3, \bar{\eta}_i(\tau); \tau)]_{cb}}{4\pi | \bar{\sigma}_3 - \bar{\sigma}_4 |} \}
\]

\begin{equation}
\tag{119}
\end{equation}

It is evident that there are divergences due to the self-energies for \(i=j\). They will be discussed in the case \(N=2\) in the next Section, because their regularization is connected with the existence of asymptotic freedom already at this pseudoclassical level.

After an integration by parts, the constraints \(\mathcal{H}_{pr}(\tau) \approx 0\) become

\[
\sum_{i=1}^{N} \left[ \eta_im_i \frac{\tilde{\eta}_i(\tau) + \bar{g}(\tau)}{\sqrt{1 - (\tilde{\eta}_i(\tau) + \bar{g}(\tau))^2}} - \sum_a \tilde{Q}_{ia}(\tau) \tilde{A}_{a\perp}(\tau, \bar{\eta}_i(\tau)) \right] + 
\]

\[
\quad + g_s^2 \int d^3 \sigma [(\tilde{\partial}\tilde{A}^s_{a\perp}) \tilde{E}^s_{a\perp}](\tau, \bar{\sigma}) \circ \equiv 0,
\]

\begin{equation}
\tag{120}
\end{equation}

and the expression of the constant invariant mass \(H_{rel}\) becomes

\[
E_{rel} = \sum_{i=1}^{N} \frac{\eta_im_i}{\sqrt{1 - (\tilde{\eta}_i(\tau) + \bar{g}(\tau))^2}} + 
\]

\[
\quad + \frac{1}{2} V[\tilde{\eta}_i, \tilde{A}_{a\perp}, \tilde{E}_{a\perp}] + \sum_a \frac{1}{2g_s^2} \int d^3 \sigma [\tilde{E}^2_{a\perp} + \tilde{B}^2_{a\perp}](\tau, \bar{\sigma}).
\]

\begin{equation}
\tag{121}
\end{equation}

To find the classical theory implied by the pseudoclassical one we follow Ref. [15], where the Berezin-Marinov distribution function \([see in Refs. [16]\)] for the Grassmann variables of phase space is defined for the case of SU(3) \([in this discussion we will denote \(\theta\) the Dirac observables \(\tilde{\theta}\)]. For each particle \(i\), on the space of analytic functions of the \(\theta_{ia}\)'s, namely of the functions \(f(\theta_i) = \alpha_i + \beta_{ia}\theta_{ia} + \gamma_{ia}\epsilon_{a\beta\gamma}\theta_{ia}\theta_{\beta}\gamma + \delta_{ia}\epsilon_{a\beta\gamma}\theta_{ia}\theta_{ia}\theta_{\beta}\gamma\), the more general density function \(\rho_i(\theta_i, \theta_i^*\) satisfying the normalization condition \(\int \rho_i d\mu_i = 1\) \([d\mu_i = \prod_{a=1}^{3} d\theta_{ia} d\theta_{ia}^*\) is the Berezin integration measure on the Grassmann sector of phase space] and the positivity condition \(\int \rho_i f^* f d\mu_i \geq 0\) for any analytic function \(f\), is \([c_i = c_i(\tau), W_{ia\beta} = W_{ia\beta}(\tau), V_{ia\beta} = V_{ia\beta}(\tau)\]
\[ \rho_i(\theta_i, \theta^*_i) = c_i + \sum_{\alpha\beta} \theta^*_i \alpha W_{i\alpha\beta} \theta_{i\beta} + 2(Tr V_i)(\sum_\alpha \theta^*_i \alpha \theta_{i\alpha})^2 - 4(\sum_{\alpha\beta} \theta^*_i \alpha V_{i\alpha\beta} \theta_{i\beta})(\sum_\gamma \theta^*_i \gamma \theta_{i\gamma}) + \frac{1}{6}(\sum_{\alpha} \theta^*_i \alpha \theta_{i\alpha})^3, \]

\[ c_i = c_i^* > 0, \]
\[ W_i = W_i^\dagger = w_{io} + w_{ia} \frac{1}{2} \lambda_a, \text{ definite positive,} \]
\[ V_i = V_i^\dagger = v_{io} + v_{ia} \frac{1}{2} \lambda_a, \text{ definite positive,} \]

\[ w_{io} = \frac{1}{3} Tr W_i, \quad w_{ia} = Tr (\lambda_a W_i), \]
\[ v_{io} = \frac{1}{3} Tr V_i, \quad v_{ia} = Tr (\lambda_a V_i), \quad (122) \]

since one gets \( \int \rho_i^* f^* d\mu_i = |\alpha|^2 + 36c_i |\delta|^2 + 4 \beta^* V_i \beta + 4 \gamma^8 W_i \gamma. \) Other useful formulas are

\[ \frac{1}{2} d_{abc} = Tr \left\{ \{ \frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b \} \frac{1}{2} \lambda_c \right\} \]

\[ \int \rho_i \theta^*_i \alpha \theta_{i\alpha} d\mu_i = 4V_{i\beta\alpha}, \]
\[ \int \rho_i \theta^*_i \alpha \theta_{i\alpha} \theta^*_{i\beta} \theta_{i\beta} d\mu_i = W_{i\lambda\epsilon \gamma \lambda \epsilon \beta \tau}, \]
\[ \int \rho_i \theta^*_i \alpha \theta_{i\alpha} \theta^*_{i\beta} \theta_{i\gamma} \theta^*_{i\gamma} \theta_{i\tau} d\mu_i = c_i \epsilon \alpha \gamma \lambda \epsilon \beta \tau, \]

\[ < N_i > = < \sum_{\alpha} \theta^*_i \alpha \theta_{i\alpha} > = 12v_{io}, \]
\[ < Q_{ia} > = < \sum_{\alpha\beta} \theta^*_i \alpha (\frac{1}{2} \lambda_a)_{\alpha\beta} \theta_{i\beta} > = 2v_{ia}, \]
\[ < N_i^2 > = < (\sum_{\alpha} \theta^*_i \alpha \theta_{i\alpha})^2 > = 6w_{io}, \]
\[ < (\sum_{\alpha\beta} \theta^*_i \alpha (\frac{1}{2} \lambda_a)_{\alpha\beta} \theta_{i\beta})(\sum_{\gamma} \theta^*_i \gamma \theta_{i\gamma}) > = - \frac{1}{2} w_{ia}, \]
\[ < N_i^3 > = < (\sum_{\alpha} \theta^*_i \alpha \theta_{i\alpha})^3 > = 6c_i, \]
\[ < Q_{ia} Q_{ib} > = < ((\sum_{\alpha\beta} \theta^*_i \alpha (\frac{1}{2} \lambda_a)_{\alpha\beta} \theta_{i\beta})(\sum_{\gamma \delta} \theta^*_i \gamma (\frac{1}{2} \lambda_b)_{\gamma \delta} \theta_{i\delta}) > = \frac{1}{2} (d_{abc} w_{ic} - w_{ia} \delta_{ab}), \]
\[ \langle Q_{ia} Q_{ib} Q_{ic} \rangle = \langle \sum_{\alpha \beta} \theta^*_{ia} \left( \frac{1}{2} \lambda_a \right)_{\alpha \beta} \theta_{i\beta} \rangle \langle \sum_{\gamma \delta} \theta^*_{i\gamma} \left( \frac{1}{2} \lambda_b \right)_{\gamma \delta} \theta_{i\delta} \rangle \langle \sum_{\mu \nu} \theta^*_{i\mu} \left( \frac{1}{2} \lambda_c \right)_{\mu \nu} \theta_{i\nu} \rangle \geq \]
\[ = \frac{1}{2} c_i d_{abc}, \]
\[ \langle Q_{ia} Q_{ib} N_i \rangle = \langle \sum_{\alpha \beta} \theta^*_{ia} \left( \frac{1}{2} \lambda_a \right)_{\alpha \beta} \theta_{i\beta} \rangle \langle \sum_{\gamma \delta} \theta^*_{i\gamma} \left( \frac{1}{2} \lambda_b \right)_{\gamma \delta} \theta_{i\delta} \rangle \langle \sum_{\mu} \theta^*_{i\mu} \theta_{i\mu} \rangle \geq \]
\[ = -\frac{1}{2} c_i \delta_{ab}. \quad (123) \]

Therefore, the constraints \( N_i \approx 0 \) imply at the classical level [at the quantum level in general \( c_i \neq 0 \) due to ordering problems]
\[ c_i = v_{io} = w_{io} = w_{ia} = 0, \]
\[ \Rightarrow \rho_i(\theta_i, \theta^*_i) = -4v_{ia}(\tau) Q_{ia}(\tau) N_i + \frac{1}{6} N_i^3 \approx 0, \]
\[ \langle Q_{ia} \rangle = 2v_{ia}, \quad \langle Q_{ia} Q_{ib} \rangle = \langle Q_{ia} Q_{ib} Q_{ic} \rangle = 0, \]
\[ \langle N_i \rangle = \langle N_i^2 \rangle \approx \langle N_i^3 \rangle \approx \langle Q_{ia} Q_{ib} N_i \rangle = 0. \quad (124) \]

The classical observable associated to an even Grassmann-valued function \( g \) is
\[ < g > = \int g \prod_{i=1}^{N} \rho_i d\mu_i = \int g \rho d\mu, \quad \rho = \prod_{i=1}^{N} \rho_i, \quad d\mu = \prod_{i=1}^{N} d\mu_i. \quad (125) \]

The distribution function \( \rho = \prod_{i=1}^{N} \rho_i \) must satisfy the Liouville equation
\[ \frac{\partial \rho}{\partial \tau} + \{ \rho, \hat{H}_D \} \geq 0, \]
\[ \sum_{k=1}^{N} \rho_1 \cdots \rho_{k-1} [\frac{\partial \rho_k}{\partial \tau} + \{ \rho_k, \hat{H}_D \}] \rho_{k+1} \cdots \rho_N \geq 0, \]
\[ \downarrow \]
\[ \frac{\partial \rho}{\partial \tau} + \{ \rho, \hat{H}_{rel} \} \geq 0, \]
\[ \{ \rho, \hat{\tilde{H}}_{\rho}(\tau) \} \approx 0, \]
\[ \{ \rho, N_i = \sum_{\alpha} \theta^*_{ia} \theta_{i\alpha} \} \approx 0, \quad i = 1, \ldots, N, \quad (126) \]
with $\hat{H}_D$ the Hamiltonian of Eq.(106). The last two equations are identically satisfied. Therefore, by using Eqs.(122) and (108) for $\dot{Q}_{ia}$, we get the equations [valid in a neighbourhood of the region $N_i \approx 0$, where however we will put $N_i^3 \equiv 0$ in the sense of Dirac’s strong equality]

$$
\sum_{k=1}^{N} \rho_1...\rho_{k-1} [-4\{\dot{V}_{k\alpha\beta} + [V_k, T^a]_{\alpha\beta}[\dot{\eta}_k(\tau) \cdot \ddot{A}_{a\perp}(\tau, \eta_k(\tau)) -
- g_s^2 \int d^3\sigma K_{ab}(\bar{\eta}_k(\tau), \bar{\sigma}; \tau) \rho_b^{(YM)}(\tau, \bar{\sigma}) +
+ g_s^2 \sum_{h=1}^{N-1} K_{ab}(\bar{\eta}_h(\tau), \eta_h(\tau); \tau) i\theta_{h\gamma}(T^b)_{\gamma\delta} \theta_{h\delta})] \theta^*_{k\alpha} \theta_{k\beta} N_k] \rho_{k+1}...\rho_{N} =
= \sum_{k=1}^{N} \rho_1...\rho_{k-1} [-4\{\dot{V}_{k\alpha\beta} + [V_k, T^a]_{\alpha\beta}[\dot{\eta}_k(\tau) \cdot \ddot{A}_{a\perp}(\tau, \eta_k(\tau)) -
- g_s^2 \int d^3\sigma K_{ab}(\bar{\eta}_k(\tau), \bar{\sigma}; \tau) \rho_b^{(YM)}(\tau, \bar{\sigma})]\rho_{k+1}...\rho_{N} +
+ (terms containing factors \ N^3_h ) \equiv 0,
$$

(127)

so that, with $N_h^3 \equiv 0$ for every $h$, the equations for $v_{ka}(\tau)$ are

$$
\frac{dv_{ka}(\tau)}{d\tau} - c_{abc} v_{kc}(\tau)[\dot{\eta}_k(\tau) \cdot \ddot{A}_{b\perp}(\tau, \eta_k(\tau)) -
- g_s^2 \int d^3\sigma K_{ab}(\bar{\eta}_k(\tau), \bar{\sigma}; \tau) \rho_b^{(YM)}(\tau, \bar{\sigma})] \equiv 0,
$$

(128)

in accord with the mean value of the last of Eqs.(108) and with Ref. [15] except for the last term which is absent when the field is considered external.

In analogy with the electromagnetic case (see Ref. [15]), we could try to find solutions to the field equations (117) necessarily of the form $\ddot{A}_{a\perp}(\tau, \bar{\sigma}) = \ddot{A}_{a\perp}(\tau, \bar{\sigma}) + (terms \ at \ least \ linear \ in \ the \ \dot{Q}_{ia}(\tau)'s)$ $[\ddot{A}_{a\perp}(\tau, \bar{\sigma})$ satisfies Eq.(117) with $\dot{Q}_{ia}(\tau) = 0$, namely in absence of particle sources], to put them in the particle equations (118) and to take the mean value ($< D > = \int \rho D d\mu$) of the resulting equations to get the real “classical” equations. In the electromagnetic case, this procedure eliminates infinities from self-energies and causal pathologies (runaway solutions or preaccelerations) from the classical equations (see also Ref. [14] for the rest-frame analysis of these problems), which, instead, would be present by taking immediately the mean value of Eqs.(117) and (118). However, it is not possible
to attack the former procedure in the non-Abelian case in absence of an analogue of the Lienard-Wiechert potential [but see in any case next Section for the N=2 case].
VI. THE QUARK MODEL AND THE PSEUDOCLASSICAL ASYMPTOTIC FREEDOM FOR N=2

In the nonrelativistic quark model, in which no SU(3) color Yang-Mills field appears, one assumes that the physical states are color singlets. Since Eq.(107) gives the invariant mass of N colored relativistic scalar particles together with the SU(3) color Yang-Mills field, this is the starting point to try to extract a pseudoclassical basis for the missing relativistic quark model.

A first step is to study what happens if we add 8 extra constraints implying the vanishing of the total color charge, so that only global color singlets are allowed for the particles+Yang-Mills field system:

\[ \bar{Q}_a = \bar{Q}^{(YM)}_a + \sum_{i=1}^{N} \bar{Q}_{ia} = 0. \] (129)

Then, we ask that these 8 conditions be fulfilled by the separate vanishing of the particle and field contributions to the color charge

\[ \sum_i \bar{Q}_{ia} = 0 \]
\[ \bar{Q}^{(YM)}_a = 0. \] (130)

The first condition defines the relativistic scalar quark model: the particles by themselves are a color singlet independently from the SU(3) color field. For N=1, Eqs.(130) plus the constraint \( \bar{N} = \sum_\alpha \bar{\theta}^*_{1\alpha} \bar{\theta}_{1\alpha} \approx 0 \) gives 9 conditions on 6 Grassmann variables: therefore a single pseudoclassical scalar quark cannot be a color singlet. For N=2, besides the constraints \( \bar{N}_1 = \sum_\alpha \bar{\theta}^*_{1\alpha} \bar{\theta}_{1\alpha} \approx 0 \) and \( \bar{N}_2 = \sum_\alpha \bar{\theta}^*_{2\alpha} \bar{\theta}_{2\alpha} \approx 0 \), one has

\[ \bar{Q}_{1a} + \bar{Q}_{2a} = 0, \] (131)

namely 10 conditions on 12 Grassmann variables.

The condition \( \bar{Q}^{(YM)}_a = 0 \) replaces the Abelian condition \( \bar{A}_\perp(\tau,\vec{\sigma}) = \bar{\pi}_\perp(\tau,\vec{\sigma}) = 0 \) of absence of radiation [see Ref. [7]]. Since in the non-Abelian case we do not know how to
solve the equations of motion and since the superposition principle does not hold, we can only ask that there is no color flux on the surface at space infinity. This requirement also implies that the pseudoclassical solutions of the SU(3) Yang-Mills equations are restricted to those configurations which are color singlets like the glueball states at the quantum level.

Due to the Gauss laws, the condition \( \ddot{Q}_a(\tau) = 0 \) can be imposed by choosing suitable boundary conditions on the transverse SU(3) fields (see Eqs.(2-40) of Ref. [1]):

\[
\tilde{A}_{a\perp}(\tau, \vec{\sigma}) \rightarrow |\vec{\sigma}| \rightarrow 0 O(|\vec{\sigma}|^{-(2+\epsilon)}), \quad \tilde{\pi}_{a\perp}(\tau, \vec{\sigma}) \rightarrow |\vec{\sigma}| \rightarrow 0 O(|\vec{\sigma}|^{-(2+\epsilon)}) \text{ with } \epsilon > 0 \quad \text{(for } \epsilon \rightarrow 0 \text{ one gets } \ddot{Q}_a \neq 0). \]

With these boundary conditions the requirement of color singlets for the whole theory becomes automatically the same requirement in field-independent quark models, if the equations of motion imply \( \dot{Q}_a^{(YM)}(\tau) \overset{=}{\approx} 0 \).

Eqs.(130), (108) and (109) imply

\[
\begin{align*}
\frac{d}{d\tau} \dot{Q}_a^{(YM)}(\tau) &\overset{=}{\approx} 0, \\
\frac{d}{d\tau}(\ddot{Q}_{1a}(\tau) + \ddot{Q}_{2a}(\tau)) &\overset{=}{\approx} c_{acd}\dot{Q}_{1d}(\tau)\{\tilde{\eta}_1(\tau) \cdot \tilde{A}_{c\perp}(\tau, \tilde{\eta}_1(\tau)) - \tilde{\eta}_2(\tau) \cdot \tilde{A}_{c\perp}(\tau, \tilde{\eta}_2(\tau)) - \}
\end{align*}
\]

\[
-g_s^2 \int d^3\sigma[K_{cb}(\tilde{\eta}_1(\tau), \tilde{\sigma}; \tau) - K_{cb}(\tilde{\eta}_2(\tau), \tilde{\sigma}; \tau)][\dot{\rho}_b^{(YM)}(\tau, \tilde{\sigma}) +
\begin{align*}
+g_s^2[K_{cb}(\tilde{\eta}_1(\tau), \tilde{\eta}_1(\tau); \tau) + K_{cb}(\tilde{\eta}_2(\tau), \tilde{\eta}_2(\tau); \tau) -
\end{align*}
\]

\[
-K_{cb}(\tilde{\eta}_1(\tau), \tilde{\eta}_2(\tau); \tau) - K_{cb}(\tilde{\eta}_2(\tau), \tilde{\eta}_1(\tau); \tau)[\dot{Q}_{1b}(\tau) \overset{=}{\approx} 0].
\]

In the gauge \( \tilde{\lambda}(\tau) = \dot{\tilde{g}}(\tau) = 0 \), the equations of motion for the field and the particles and the equations defining the rest-frame become [in Eq.(133) there is the same F-function of Eq.(117)]

\[
P_{\perp}^{rs}(\tilde{\sigma})\{\delta^{st}[\delta_{ad}(\frac{\partial}{\partial \tau})^2 + \triangle] + c_{acd}\tilde{A}_{c\perp}(\tau, \tilde{\sigma}) \cdot \frac{\partial}{\partial \tilde{\sigma}}\}:
\]

\[
\begin{align*}
\int d^3\tilde{\sigma}\delta^3(\tilde{\sigma} - \tilde{\sigma}) + \frac{\partial}{\partial \tau} \dot{D}_{uv}^{(A_{\perp})t}(\tau, \tilde{\sigma}) \int d^3\tilde{\sigma}
\end{align*}
\]

\[
\begin{align*}
K_{uv}(\tilde{\sigma}, \tilde{\sigma}; \tau)\dot{D}_{uv}^{(A_{\perp})t}(\tau, \tilde{\sigma}) \frac{\partial}{\partial \tau} \} \tilde{A}_{d\perp}(\tau, \tilde{\sigma}) \overset{=}{\approx}
\end{align*}
\]

\[
[\tilde{\rho}_c^{(YM)}(\tau, \tilde{\sigma}) - \tilde{Q}_{1c}(\tau)[\delta^3(\tilde{\sigma} - \tilde{\eta}_1(\tau)) - \delta^3(\tilde{\sigma} - \tilde{\eta}_2(\tau))]] -
\]

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\[-g_4^2 P_{r_2}^k(\tilde{\sigma}) \dot{A}_{d\perp}^s(\tau, \tilde{\sigma}) \int d^3 \sigma_1 d^3 \sigma_2 \]

\[ [\tilde{\rho}_b^{(YM)}(\tau, \tilde{\sigma}_1) - \tilde{Q}_{1b}(\tau) [\delta^3(\tilde{\sigma}_1 - \tilde{\eta}_1(\tau)) - \delta^3(\tilde{\sigma}_1 - \tilde{\eta}_2(\tau))] ] \]

\[ [F_{abc2d}(\tilde{\sigma}, \tilde{\sigma}_1, \tilde{\sigma}_2; \tau) + c_{auec,d} K_{eb}(\tilde{\sigma}, \tilde{\sigma}_1; \tau) K_{uc}(\tilde{\sigma}, \tilde{\sigma}_2; \tau)] \]

\[ [\tilde{\rho}_c^{(YM)}(\tau, \tilde{\sigma}_2) - \tilde{Q}_{1c}(\tau) [\delta^3(\tilde{\sigma}_2 - \tilde{\eta}_1(\tau)) - \delta^3(\tilde{\sigma}_2 - \tilde{\eta}_2(\tau))] ], \] (133)

\[
\frac{d}{d\tau} [\eta_1 m_1 - \dot{\eta}_1(\tau) \frac{1}{\sqrt{1 - \dot{\eta}_1^2(\tau)}}] \overset{=}{=} \sum_a \tilde{Q}_{1a}(\tau) \{ \tilde{E}_{a\perp}^r(\tau, \tilde{\eta}_1(\tau)) \times [\tilde{B}_a(\tau, \tilde{\eta}_1(\tau))] \frac{\partial}{\partial \eta_1} \}
\]

\[-\sum_a \tilde{Q}_{1a}(\tau) P_{s_1}^r(\tilde{\eta}_1) c_{amn} \tilde{A}_{s_m\perp}(\tau, \tilde{\eta}_1(\tau)) \]

\[
\int d^3 \sigma K_{de}(\tilde{\eta}_1(\tau), \tilde{\sigma}; \tau) c_{enb} \tilde{A}_{n\perp}(\tau, \tilde{\eta}_1(\tau)) \cdot \tilde{E}_{b\perp}(\tau, \tilde{\sigma}) +
\]

\[
+ g_2^2 \sum_{a,b} \{ \tilde{Q}_{1a}(\tau) \tilde{Q}_{1b}(\tau) [c_{ad} \tilde{A}_{d\perp}^r(\tau, \tilde{\eta}_1(\tau))(K_{eb}(\tilde{\eta}_1(\tau), \tilde{\eta}_1(\tau); \tau) -
\]

\[-K_{eb}(\tilde{\eta}_1(\tau), \tilde{\eta}_2(\tau); \tau) ) + \frac{\partial}{\partial \sigma} |_{\sigma = \tilde{\eta}_1} (K_{ab}(\tilde{\sigma}, \tilde{\eta}_1(\tau); \tau) - K_{ab}(\tilde{\sigma}, \tilde{\eta}_2(\tau); \tau) ) ] \}
\]

\[-\tilde{Q}_{1a}(\tau) \int d^3 \sigma [c_{ad} \tilde{A}_{d\perp}^r(\tau, \tilde{\eta}_1(\tau)) K_{eb}(\tilde{\eta}_1(\tau), \tilde{\eta}_1(\tau); \tau) +
\]

\[+ \frac{\partial K_{ab}(\tilde{\eta}_1(\tau), \tilde{\sigma}; \tau)}{\partial \eta_1} |_{\tilde{\rho}_b^{(YM)}(\tau, \tilde{\sigma})}, \] (134)

\[
\frac{d}{d\tau} [\eta_2 m_2 - \dot{\eta}_2(\tau) \frac{1}{\sqrt{1 - \dot{\eta}_2^2(\tau)}}] \overset{=}{=} - \sum_a \tilde{Q}_{1a}(\tau) \{ \tilde{E}_{a\perp}^r(\tau, \tilde{\eta}_2(\tau)) + [\tilde{B}_a(\tau, \tilde{\eta}_2(\tau))] \frac{\partial}{\partial \eta_2} \}
\]

\[+ \sum_a \tilde{Q}_{1a}(\tau) P_{s_1}^r(\tilde{\eta}_1) c_{amn} \tilde{A}_{s_m\perp}(\tau, \tilde{\eta}_2(\tau)) \]

\[
\int d^3 \sigma K_{de}(\tilde{\eta}_2(\tau), \tilde{\sigma}; \tau) c_{enb} \tilde{A}_{n\perp}(\tau, \tilde{\eta}_2(\tau)) \cdot \tilde{E}_{b\perp}(\tau, \tilde{\sigma}) -
\]

\[-g_2^2 \sum_{a,b} \{ \tilde{Q}_{1a}(\tau) \tilde{Q}_{1b}(\tau) [c_{ad} \tilde{A}_{d\perp}^r(\tau, \tilde{\eta}_2(\tau))(K_{eb}(\tilde{\eta}_2(\tau), \tilde{\eta}_1(\tau); \tau) -
\]

\[-K_{eb}(\tilde{\eta}_2(\tau), \tilde{\eta}_2(\tau); \tau) ) + \frac{\partial}{\partial \sigma} |_{\sigma = \tilde{\eta}_2} (K_{ab}(\tilde{\sigma}, \tilde{\eta}_1(\tau); \tau) - K_{ab}(\tilde{\sigma}, \tilde{\eta}_2(\tau); \tau) ) ] \}
\]

\[-\tilde{Q}_{1a}(\tau) \int d^3 \sigma [c_{ad} \tilde{A}_{d\perp}^r(\tau, \tilde{\eta}_2(\tau)) K_{eb}(\tilde{\eta}_2(\tau), \tilde{\eta}_1(\tau); \tau) +
\]

\[+ \frac{\partial K_{ab}(\tilde{\eta}_2(\tau), \tilde{\sigma}; \tau)}{\partial \eta_2} |_{\tilde{\rho}_b^{(YM)}(\tau, \tilde{\sigma})}, \] (134)

\[
\eta_1 m_1 \frac{\dot{\eta}_1(\tau)}{\sqrt{1 - \dot{\eta}_1^2(\tau)}} + \eta_2 m_2 \frac{\dot{\eta}_2(\tau)}{\sqrt{1 - \dot{\eta}_2^2(\tau)}} -
\]

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\[ -\sum_a Q_{1a}(\tau)\tilde{A}_{a\perp}(\tau, \tilde{\eta}_1(\tau)) - \tilde{A}_{a\perp}(\tau, \tilde{\eta}_2(\tau)) + g_s^{-2}\int d^3\sigma \{\tilde{\partial} A_{a\perp}^s E_{a\perp}^s\}(\tau, \tilde{\sigma}) = 0. \quad (135) \]

Note that in Eqs. (134) the Coulomb interaction inside the kernel \( K \) does not contribute to the terms \( \tilde{Q}_{1a}(\tau)\tilde{Q}_{1b}(\tau) \) due to \( c_{ab} \tilde{Q}_{1a} \tilde{Q}_{1b} = \delta_{ab} \tilde{Q}_{1a} \tilde{Q}_{1b} = 0 \).

The invariant mass of the system is [we choose \( \eta_1 = \eta_2 = +1 \), i.e. the quark and antiquark have positive energies and the antiquark is distinguished by the opposite color charge as a classical antiparticle moving forward in \( \tau \) (see Ref. [20–22] and its bibliography)]

\[
H_{rel} = \sqrt{m_1^2 + (\tilde{\kappa}_1(\tau) + \sum_a \tilde{Q}_{1a}(\tau) \tilde{A}_{a\perp}(\tau, \tilde{\eta}_1(\tau)))^2 + \sum_a \tilde{Q}_{1a}(\tau) \tilde{A}_{a\perp}(\tau, \tilde{\eta}_2(\tau)))^2 + \frac{1}{2} \sum_a \int d^3\sigma [g_s^2 \tilde{\pi}_{a\perp}^2 + g_s^{-2}\tilde{\pi}_{a\perp}^2]((\tau, \tilde{\sigma}). \quad (136) \]

In the case \( N=2 \), which should correspond at the quantum level to a meson configuration formed from a scalar quark and a scalar antiquark, plus glue [there are not sea-quarks, because there is no pair production at this pseudoclassical level], if \( \tilde{N}_1 \rightarrow \sum_{\alpha=0}^3 b_{1a}^\perp b_{1a} - 1 \) and \( \tilde{N}_2 \rightarrow \sum_{\alpha=0}^3 b_{2a}^\perp b_{2a} - 2 \), the first condition in Eq. (130) and the observation that the SU(3) Yang-Mills fields, solutions of the field equations (133), will depend on the Grassmann variables of the particles only through the color charges \( \tilde{Q}_{1a}(\tau) \), allow us to write the following developments [from now on we shall not write \( \sum_a \) for repeated color indices]

\[
\tilde{A}_{a\perp}(\tau, \tilde{\sigma}) = \tilde{A}_{a\perp}^{(0)}(\tau, \tilde{\sigma}) + \tilde{Q}_{1u}(\tau)\tilde{A}_{a\perp}^{(1)}(\tau, \tilde{\sigma}) + \tilde{Q}_{1v}(\tau)\tilde{A}_{a\perp}^{(2)}(\tau, \tilde{\sigma}), \\
\tilde{\pi}_{a\perp}(\tau, \tilde{\sigma}) = \tilde{\pi}_{a\perp}^{(0)}(\tau, \tilde{\sigma}) + \tilde{Q}_{1u}(\tau)\tilde{\pi}_{a\perp}^{(1)}(\tau, \tilde{\sigma}) + \tilde{Q}_{1v}(\tau)\tilde{\pi}_{a\perp}^{(2)}(\tau, \tilde{\sigma}). \quad (137) \]

As a consequence we have [we suppress the \( \tau \)-dependence and also the \( \tilde{\sigma} \)-dependence when possible with the replacement \( \tilde{\sigma}_i \rightarrow i \)]

\[
\tilde{\rho}_a(\tilde{\sigma}) = \tilde{\rho}_a^{(YM)}(\tilde{\sigma}) + \sum_{i=1}^2 \tilde{\rho}_{ia}(\tilde{\sigma}) = \\
= \tilde{\rho}_a^{(YM)(o)}(\tilde{\sigma}) + \tilde{Q}_{1u}\tilde{\rho}_a^{(YM)(1)}(\tilde{\sigma}) + \tilde{Q}_{1v}\tilde{\rho}_a^{(YM)(2)}(\tilde{\sigma}) + \sum_{i=1}^2 \tilde{\rho}_{ia}(\tilde{\sigma}) = \\
= c_{abc}(\tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(\tilde{\sigma}) + \tilde{Q}_{1u}[c_{abc}(\tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}^{(1)} + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}^{(2)}) - \tilde{\pi}_{c\perp}](\tilde{\sigma}) - \\
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\[
\rho_a(\tilde{\sigma}_1) \rho_b(\tilde{\sigma}_2) = c_{amn} \tilde{A}_{m\perp}(1) \cdot \tilde{\pi}_{c\perp}(1) c_{brs} \tilde{A}_{r\perp}(2) \cdot \tilde{\pi}_{s\perp}(2) + \\
\tilde{Q}_{1u} \tilde{Q}_{1v} c_{abc} [\tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1)](\tilde{\sigma}),
\]

\[
\rho(\sigma) = \rho_a(\tilde{\sigma}_1) \rho_b(\tilde{\sigma}_2) = c_{amn} \tilde{A}_{m\perp}(1) \cdot \tilde{\pi}_{c\perp}(1) c_{brs} \tilde{A}_{r\perp}(2) \cdot \tilde{\pi}_{s\perp}(2) + \\
\tilde{Q}_{1u} \tilde{Q}_{1v} c_{abc} \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) - \\
- \delta_{au}(\delta^3(\tilde{\sigma}_1 - \tilde{\eta}_1) - \delta^3(\tilde{\sigma}_2 - \tilde{\eta}_2)) \frac{\tilde{r}_{(1)}}{\tilde{r}_{(2)}} + \\
\tilde{Q}_{1u} \tilde{Q}_{1v} c_{abc} [\tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{b\perp} \cdot \tilde{\pi}_{c\perp}(1)](\tilde{\sigma}),
\]

\[
R_{(1,2)ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau) = \tilde{Q}_{1a} \tilde{Q}_{1b} [\delta^3(\tilde{\sigma}_1 - \tilde{\eta}_1) - \delta^3(\tilde{\sigma}_2 - \tilde{\eta}_2)][\delta^3(\tilde{\sigma}_2 - \tilde{\eta}_1) - \delta^3(\tilde{\sigma}_1 - \tilde{\eta}_2)],
\]

\[
R_{(i)ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau) = (-)^i \tilde{Q}_{1u} [\delta_{au} \delta^3(\tilde{\sigma}_1 - \tilde{\eta}_1) c_{brs} \tilde{A}_{r\perp}(2) \cdot \tilde{\pi}_{s\perp}(2) + \\
+ \delta_{bu} \delta^3(\tilde{\sigma}_2 - \tilde{\eta}_b) c_{amn} \tilde{A}_{m\perp}(1) \cdot \tilde{\pi}_{n\perp}(1) = \tilde{Q}_{1u}(\tau) R_{(i)ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau),
\]

\[
R_{ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau) = R_{ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau) + \tilde{Q}_{1u} R_{ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau) + \tilde{Q}_{1u} \tilde{Q}_{1v} R_{ab}(\tilde{\sigma}_1, \tilde{\sigma}_2; \tau) = \\
= c_{amn} \tilde{A}_{m\perp}(1) \cdot \tilde{\pi}_{c\perp}(1) c_{brs} \tilde{A}_{r\perp}(2) \cdot \tilde{\pi}_{s\perp}(2) + \\
+ \tilde{Q}_{1u} [c_{amn} (\tilde{A}_{m\perp}(1) \cdot \tilde{\pi}_{c\perp}(1) + \tilde{A}_{m\perp}(1) \cdot \tilde{\pi}_{c\perp}(1))] c_{brs} \tilde{A}_{r\perp}(2) \cdot \tilde{\pi}_{s\perp}(2) +
\]

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In the last lines we separated the particle-particle $R_{(1,2)ab}$, the particle-field $[R_{(i)ab}]$ and field-field $[R_{ab}]$ terms.

If we write

\[
\zeta_{ab}^{(A_\perp)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = \zeta_{ab}^{(A_\perp)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) + \varrho_{1u} \zeta_{ab}^{(A_\perp, A_\perp)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) + \varrho_{1u} \Phi_{1u} \zeta_{ab}^{(A_\perp, A_\perp)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau),
\]

(139)

where $\zeta_{ab}^{(A_\perp)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = \zeta(\vec{\sigma}_1 - \vec{\sigma}_2)(Pe^{-\int d^4x \cdot \cdot \cdot})_{ab}$ and $\zeta_{ab}^{(A_\perp, A_\perp)}$ and $\zeta_{ab}^{(A_\perp, A_\perp)}$ are functions which could be evaluated by using the definition of Wilson path-ordering, then we can write the following decompositions of the interaction kernel $K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)$

\[
K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = K_{ab}^{(A_\perp)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) + \varrho_{1u} K_{ab}^{(1)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) + \varrho_{1u} \varrho_{1u} K_{ab}^{(2)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau),
\]

(138)
and of the potential $V$ in particle-particle $[V_{PP}]$, particle-field $[V_{iPF}]$ and field-field $[V_{FF}]$ contributions

$$
\frac{1}{2} V[\vec{\eta}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}](\tau) = V_{PP}[\vec{\eta}_i, \vec{\eta}_j, \vec{A}_{a\perp}](\tau) + \sum_{i=1}^{2} V_{i\cdot PF}[\vec{\eta}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}, \vec{A}_{au\perp}](\tau) + \sum_{i=1}^{2} V_{\cdot FF}[\vec{\eta}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}](\tau),
$$

$$
V_{PP} = g_s^2 \int d^3\sigma_1 d^3\sigma_2 \sum_{a,b} [R_{(1,2)ab} K_{ab}](\vec{\sigma}_1, \vec{\sigma}_2; \tau) =
$$

$$
g_s^2 \int d^3\sigma_1 d^3\sigma_2 \sum_{a,b} \tilde{Q}_{1a} \tilde{Q}_{1b} [\delta^3(\vec{\sigma}_1 - \vec{\eta}_1) - \delta^3(\vec{\sigma}_1 - \vec{\eta}_2)]
$$

$$
[\delta^3(\vec{\sigma}_2 - \vec{\eta}_1) - \delta^3(\vec{\sigma}_2 - \vec{\eta}_2)] K_{ab}^{(o)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau),
$$

$$
V_{i\cdot PF} = g_s^2 \int d^3\sigma_1 d^3\sigma_2 \sum_{a,b} [R_{i\cdot ab} K_{ab}](\vec{\sigma}_1, \vec{\sigma}_2; \tau) =
$$

$$
g_s^2 \int d^3\sigma_1 d^3\sigma_2 \sum_{a,b} \tilde{Q}_{1a} R_{i\cdot ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)[K^{(o)}_{ab} + \tilde{Q}_{1b} K^{(1)}_{ab}](\vec{\sigma}_1, \vec{\sigma}_2; \tau),
$$

$$
\frac{1}{2} V[\vec{\eta}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}](\tau) = V_{PP}[\vec{\eta}_i, \vec{\eta}_j, \vec{A}_{a\perp}](\tau) + \sum_{i=1}^{2} V_{i\cdot PF}[\vec{\eta}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}, \vec{A}_{au\perp}](\tau) + \sum_{i=1}^{2} V_{\cdot FF}[\vec{\eta}_i, \vec{A}_{a\perp}, \vec{\pi}_{a\perp}](\tau),
$$

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\[ V_{FF} = g_s^2 \int d^3 \sigma_1 d^3 \sigma_2 \bar{\rho}_a^{(YM)}(\tau, \vec{\sigma}_1) K_{ab}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) \bar{\rho}_b^{(YM)}(\tau, \vec{\sigma}_2) = \]
\[ = g_s^2 \int d^3 \sigma_1 d^3 \sigma_2 \sum_{a,b} \left[ R_{ab}^{(o)} K_{ab}^{(o)} + \tilde{Q}_{1a} (R_{ab}^{(o)} R_{ab}^{(1)} + R_{ab}^{(1)} K_{ab}^{(1)}) + \right. \]
\[ \left. + \tilde{Q}_{1a} \tilde{Q}_{1b} (R_{ab}^{(o)} K_{ab}^{(2)} + R_{ab}^{(1)} K_{ab}^{(1)} + R_{ab}^{(2)} K_{ab}^{(o)}) \right][\vec{\sigma}_1, \vec{\sigma}_2; \tau]. \tag{141} \]

The explicit form of the particle-particle potential is

\[ V_{PP} = \frac{1}{2} g_s^2 \sum_{a,b} \tilde{Q}_{1a} \tilde{Q}_{1b} \int d^3 \sigma_1 \int d^3 \sigma_2 [\delta^3(\vec{\sigma}_1 - \vec{\eta}_1(\tau)) \delta^3(\vec{\sigma}_2 - \vec{\eta}_1(\tau)) + \]
\[ + \frac{1}{4\pi} \int d^3 \sigma_3 \int d^3 \sigma_4 \left\{ \delta_{ab} \delta^3(\vec{\sigma}_3 - \vec{\sigma}_1) \delta^3(\vec{\sigma}_4 - \vec{\sigma}_2) \right\} + \]
\[ + \frac{c_{fav}}{4\pi} \left( \frac{\delta^3(\vec{\sigma}_3 - \vec{\sigma}_1)(\vec{\sigma}_4 - \vec{\sigma}_2) \cdot \vec{A}_{\perp}^{(0)}(\tau, \vec{\sigma}_4) \cdot \vec{A}_{\parallel}^{(o)}(\vec{\sigma}_2, \vec{\sigma}_4; \tau)}{|| \vec{\sigma}_3 - \vec{\sigma}_4 || || \vec{\sigma}_3 - \vec{\sigma}_4 ||^3} \right) + \]
\[ + \frac{c_{arb} c_{bst}}{(4\pi)^2} \]
\[ \times \left( \delta^3(\vec{\sigma}_4 - \vec{\sigma}_2)(\vec{\sigma}_3 - \vec{\sigma}_1) \cdot \vec{A}_{\perp}^{(0)}(\tau, \vec{\sigma}_3) \cdot \vec{A}_{\parallel}^{(o)}(\vec{\sigma}_1, \vec{\sigma}_3; \tau) \right) \]
\[ = \frac{1}{2} g_s^2 \sum_{a,b} \tilde{Q}_{1a} \tilde{Q}_{1b} \int d^3 \sigma_1 \int d^3 \sigma_2 [\delta^3(\vec{\sigma}_1 - \vec{\eta}_1(\tau)) \delta^3(\vec{\sigma}_2 - \vec{\eta}_1(\tau)) + \]
\[ + \frac{1}{4\pi} \int d^3 \sigma_3 \int d^3 \sigma_4 \left\{ \delta_{ab} \delta^3(\vec{\sigma}_3 - \vec{\sigma}_1) \delta^3(\vec{\sigma}_4 - \vec{\sigma}_2) \right\} + \]
\[ + \frac{c_{fav}}{4\pi} \left( \frac{\delta^3(\vec{\sigma}_3 - \vec{\sigma}_1)(\vec{\sigma}_4 - \vec{\sigma}_2) \cdot \vec{A}_{\perp}^{(0)}(\tau, \vec{\sigma}_4) \cdot \vec{A}_{\parallel}^{(o)}(\vec{\sigma}_2, \vec{\sigma}_4; \tau)}{|| \vec{\sigma}_3 - \vec{\sigma}_4 || || \vec{\sigma}_3 - \vec{\sigma}_4 ||^3} \right) + \]
\[ + \frac{c_{arb} c_{bst}}{(4\pi)^2} \]
\[ \times \left( \delta^3(\vec{\sigma}_4 - \vec{\sigma}_2)(\vec{\sigma}_3 - \vec{\sigma}_1) \cdot \vec{A}_{\perp}^{(0)}(\tau, \vec{\sigma}_3) \cdot \vec{A}_{\parallel}^{(o)}(\vec{\sigma}_1, \vec{\sigma}_3; \tau) \right). \tag{142} \]

The first term, with the instantaneous Coulomb interaction, does not contribute, because for every N we have \( \sum_a \tilde{Q}_{1a} \tilde{Q}_{1a} = 0 \), due to Eq.(23).

With regard to the other two terms, let us remark that in the limit \( \vec{\eta}_1 = \vec{\eta}_2 \) [and thus also in the points of maximal divergence, i.e. \( \vec{\sigma}_1 = \vec{\sigma}_2 = \vec{\sigma} \) for the first two terms and \( \vec{\sigma}_1 = \vec{\sigma}_2 = \vec{\sigma} = \vec{\sigma}' \) for the third one] the potential vanishes due to the multiplicative term containing the Dirac functions.
With the substitutions $\vec{\sigma} = \vec{\sigma}_1 + \vec{\xi}$ and $\vec{\sigma}' = \vec{\sigma}_2 + \vec{\xi}'$, the particle-particle potential can be put in the form

\[
V_{PP} = g_s^2 \sum_{a,b} \hat{Q}_{1a} \hat{Q}_{1b} \{ c_{\text{ave}} \int \frac{d^3 \xi}{(4\pi)^2 |\xi|^2} \cdot
\sum_{i=1}^{2} \vec{\xi} \cdot \vec{A}^{(0)}_{v\perp}(\tau, \vec{\eta}_i(\tau) + \vec{\xi}) \zeta^{(A^{(o)})}_{sb}(\vec{\eta}_i(\tau), \vec{\eta}_i(\tau) + \vec{\xi}; \tau) -
\vec{\xi} \cdot \vec{A}^{(0)}_{u\perp}(\tau, \vec{\eta}_i(\tau) + \vec{\xi}) \zeta^{(A^{(o)})}_{sb}(\vec{\eta}_i(\tau), \vec{\eta}_i(\tau) + \vec{\xi}; \tau) -
\vec{\xi} \cdot \vec{A}^{(0)}_{v\perp}(\tau, \vec{\eta}_2(\tau) + \vec{\xi}) \zeta^{(A^{(o)})}_{sb}(\vec{\eta}_2(\tau), \vec{\eta}_2(\tau) + \vec{\xi}; \tau) +
\vec{\xi} \cdot \vec{A}^{(0)}_{u\perp}(\tau, \vec{\eta}_2(\tau) + \vec{\xi}) \zeta^{(A^{(o)})}_{sb}(\vec{\eta}_2(\tau), \vec{\eta}_2(\tau) + \vec{\xi}; \tau)
\} +
\frac{1}{2} c_{\text{ave}} c_{\text{bost}} \int \frac{d^3 \xi d^3 \xi'}{(4\pi)^3 |\xi'|^3 |\xi|^3} \cdot
\sum_{i=1}^{2} \vec{\xi} \cdot \vec{A}^{(0)}_{v\perp}(\tau, \vec{\eta}_i(\tau) + \vec{\xi}) \vec{\xi}' \cdot \vec{A}^{(0)}_{u\perp}(\tau, \vec{\eta}_i(\tau) + \vec{\xi}')
\cdot \zeta^{(A^{(o)})}_{ru}(\vec{\eta}_i(\tau), \vec{\eta}_i(\tau) + \vec{\xi}; \tau) \zeta^{(A^{(o)})}_{ru}(\vec{\eta}_2(\tau), \vec{\eta}_2(\tau) + \vec{\xi}; \tau) +
- 2 \cdot \vec{\xi} \cdot \vec{A}^{(0)}_{u\perp}(\tau, \vec{\eta}_1(\tau) + \vec{\xi}) \vec{\xi}' \cdot \vec{A}^{(0)}_{v\perp}(\tau, \vec{\eta}_2(\tau) + \vec{\xi})
\cdot \zeta^{(A^{(o)})}_{ru}(\vec{\eta}_1(\tau), \vec{\eta}_1(\tau) + \vec{\xi}; \tau) \zeta^{(A^{(o)})}_{ru}(\vec{\eta}_2(\tau), \vec{\eta}_2(\tau) + \vec{\xi}; \tau)
=\frac{g_s^2[\vec{\eta}_1 - \vec{\eta}_2; \vec{\eta}_1; \vec{A}_{\perp}(\tau)](\tau)}{|\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)|} \rightarrow_{\vec{\eta}_1 \rightarrow \vec{\eta}_2} 0,
\]

\[
\Rightarrow \quad g_s^2[\vec{\eta}_1 - \vec{\eta}_2; \vec{\eta}_1; \vec{A}_{\perp}(\tau)](\tau) \rightarrow_{\vec{\eta}_1 \rightarrow \vec{\eta}_2} 0.
\]

This is the pseudoclassical statement of asymptotic freedom for N=2.

By using the developments (137), the equation of motion for the color charge $\hat{Q}_{1a}(\tau) = -\hat{Q}_2(\tau)$ becomes

\[
\frac{d}{d\tau} \hat{Q}_{1a}(\tau) = c_{\text{ave}} c_{\text{ave}} \int d^3 \sigma K^{(o)}_{cb}(\vec{\eta}_1(\tau), \vec{\sigma}; \tau) c_{\text{ave}} c_{\text{ave}} \hat{A}_{v\perp}(\tau, \vec{\sigma}) \cdot \hat{A}_{v\perp}(\tau, \vec{\sigma}) -
\]

\[
- g_s^2 \int d^3 \sigma K^{(o)}_{cb}(\vec{\eta}_1(\tau), \vec{\sigma}; \tau) c_{\text{ave}} c_{\text{ave}} \hat{A}_{v\perp}(\tau, \vec{\sigma}) \cdot \hat{A}_{v\perp}(\tau, \vec{\sigma}) +
\]

\[
+ c_{\text{ave}} c_{\text{ave}} \hat{Q}_{1d}(\tau) \hat{Q}_{1d}(\tau)[\vec{\eta}_1(\tau), \vec{\eta}_1(\tau) -
\]

\[
+ g_s^2 \int d^3 \sigma \dot{K}^{(o)}_{cb}(\vec{\eta}_1(\tau), \vec{\sigma}; \tau) c_{\text{ave}} c_{\text{ave}} \hat{A}_{v\perp}(\tau, \vec{\sigma}) \cdot \hat{A}_{v\perp}(\tau, \vec{\sigma}) -
\]

\[
- g_s^2 \int d^3 \sigma \{ K^{(o)}_{cb}(\vec{\eta}_1(\tau), \vec{\sigma}; \tau) c_{\text{ave}} c_{\text{ave}} \hat{A}_{v\perp}(\tau, \vec{\sigma}) \cdot \hat{A}_{v\perp}(\tau, \vec{\sigma}) -
\]

\[
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\]
while the particle equations become

\[
\frac{d}{d\tau} \left[ m_1 \frac{\dot{\eta}_1(\tau)}{\sqrt{1 - \dot{\eta}_1^2(\tau)}} \right] = \\
\sum_a \dot{Q}_{1a}(\tau) \left\{ \dot{E}_{a_\perp}^{(o)}(\tau, \eta_1(\tau)) + [\ddot{\eta}_1(\tau) \times \dot{B}_{a}^{(o)}(\tau, \eta_1(\tau))]^r \right\} - \\
-P_{\perp}^{a}(\eta_1) c_{\text{amd}} \tilde{A}_{m_{\perp}}^{(o)}(\tau, \eta_1(\tau)) \int d^3 \sigma K_{de}^{(a)}(\eta_1(\tau), \vec{\sigma}; \tau) c_{\text{emb}} A_{n_{\perp}}^{(a)}(\tau, \vec{\sigma}) \cdot \dot{E}_{b_{\perp}}^{(a)}(\tau, \vec{\sigma}) - \\
-g_s^2 \int d^3 \sigma \left[ c_{\text{ad}} A_{d_{\perp}}^{(a)}(\tau, \eta_1(\tau)) K_{cb}^{(a)}(\eta_1(\tau), \vec{\sigma}; \tau) + \\
\frac{\partial K_{cb}^{(a)}(\eta_1(\tau), \vec{\sigma}; \tau)}{\partial \eta_1^r} \dot{\rho}_b^{(Y M)(a)}(\tau, \vec{\sigma}) \right] + \\
\sum_a \dot{Q}_{1a}(\tau) \dot{Q}_{1u}(\tau) \left\{ \dot{E}_{a_{\perp}}^{(o)}(\tau, \eta_1(\tau)) + [\ddot{\eta}_1(\tau) \times \dot{B}_{a}^{(o)}(\tau, \eta_1(\tau))]^r \right\} - \\
-P_{\perp}^{a}(\eta_1) c_{\text{amd}} \int d^3 \sigma \left[ \tilde{A}_{m_{\perp}}^{(o)}(\tau, \eta_1(\tau)) K_{de}^{(a)}(\eta_1(\tau), \vec{\sigma}; \tau) + \right. \\
\left. (\tilde{A}_{m_{\perp}}^{(a)}(\tau, \eta_1(\tau)) K_{de}^{(a)}(\eta_1(\tau), \vec{\sigma}; \tau) \right] c_{\text{emb}} \left( A_{n_{\perp}}^{(a)} \cdot \dot{E}_{b_{\perp}}^{(a)}(\tau, \vec{\sigma}) \right) + \\
+g_s^2 \left[ c_{\text{ad}} A_{d_{\perp}}^{(a)}(\tau, \eta_1(\tau)) (\dot{K}_{cb}^{(a)}(\eta_1(\tau), \eta_2(\tau); \tau) - K_{cb}^{(a)}(\eta_1(\tau), \eta_2(\tau); \tau)) + \\
\frac{\partial}{\partial \sigma} \dot{\rho}_b^{(Y M)(a)}(\tau, \vec{\sigma}) \right] + \\
\frac{d}{d\tau} \left[ m_2 \frac{\dot{\eta}_2(\tau)}{\sqrt{1 - \dot{\eta}_2^2(\tau)}} \right] = \\
\sum_a \dot{Q}_{1a}(\tau) \left\{ \dot{E}_{a_{\perp}}^{(o)}(\tau, \eta_2(\tau)) + [\ddot{\eta}_2(\tau) \times \dot{B}_{a}^{(o)}(\tau, \eta_2(\tau))]^r \right\} - \\
-P_{\perp}^{a}(\eta_2) c_{\text{amd}} \tilde{A}_{m_{\perp}}^{(o)}(\tau, \eta_2(\tau)) \int d^3 \sigma K_{de}^{(a)}(\eta_2(\tau), \vec{\sigma}; \tau) c_{\text{emb}} A_{n_{\perp}}^{(a)}(\tau, \vec{\sigma}) \cdot \dot{E}_{b_{\perp}}^{(a)}(\tau, \vec{\sigma}) - \\
-g_s^2 \int d^3 \sigma \left[ c_{\text{ad}} A_{d_{\perp}}^{(a)}(\tau, \eta_2(\tau)) K_{cb}^{(a)}(\eta_2(\tau), \vec{\sigma}; \tau) + \\
\frac{\partial K_{cb}^{(a)}(\eta_2(\tau), \vec{\sigma}; \tau)}{\partial \eta_2^r} \dot{\rho}_b^{(Y M)(a)}(\tau, \vec{\sigma}) \right],
\]

\[ (144) \]
\[ + \frac{\partial K_{ab}^{(o)}(\vec{g}_2(\tau), \vec{\sigma}; \tau)}{\partial \eta_2} \rho_b^{(YM)(o)}(\tau, \vec{\sigma}) \} - \\
- \sum_{ab} \tilde{Q}_{1a}(\tau) \tilde{Q}_{1b}(\tau) \{ \tilde{E}_{a \perp}^{(1)r}(\tau, \vec{\eta}_2(\tau)) + [\tilde{\eta}_2(\tau) \times \tilde{B}_a^{(1)}(\tau, \vec{\eta}_2(\tau))] \}^r - \\
- P_{\perp}(\vec{\eta}_2) \cim \int d^3 \sigma [\tilde{A}_{m \perp}^{(o)s}(\tau, \vec{\eta}_2(\tau)) K_{de}^{(o)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) + \\
c_{emb}(\tilde{A}_{n \perp} \cdot \vec{E}_{bu} + \tilde{A}_{nu \perp} \cdot \vec{E}_{b \perp})(\tau, \vec{\sigma}) + (\tilde{A}_{m \perp}^{(o)s}(\tau, \vec{\eta}_2(\tau)) K_{deu}^{(1)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) + \\
+ \tilde{A}_{m \perp}^{(1s)}(\tau, \vec{\eta}_2(\tau)) K_{de}^{(o)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) c_{emb}(\tilde{A}_{n \perp} \cdot \vec{E}_{b \perp})(\tau, \vec{\sigma})] + \\
+ g_s^2 [c_{ad}\tilde{A}_{d \perp}^{(o)r}(\tau, \vec{\eta}_2(\tau))(K_{cu}^{(o)}(\vec{\eta}_2(\tau), \vec{\eta}_1(\tau); \tau) - K_{cu}^{(o)}(\vec{\eta}_2(\tau), \vec{\eta}_2(\tau); \tau)) + \\
+ \frac{\partial}{\partial \sigma^r}[\vec{g}_{\tilde{\eta}_2}(K_{aa}(\vec{\sigma}, \vec{\eta}_1(\tau); \tau) - K_{aa}(\vec{\sigma}, \vec{\eta}_2(\tau); \tau))] - \\
- g_s^2 \int d^3 \sigma \{ c_{ad}\tilde{A}_{d \perp}^{(o)r}(\tau, \vec{\eta}_2(\tau)) K_{cb}^{(1)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) + \tilde{A}_{d \perp}^{(1r)}(\tau, \vec{\eta}_2(\tau)) K_{cb}^{(o)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) + \\
+ \frac{\partial K_{ab}^{(1)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau)}{\partial \eta_2^r} \} \rho_b^{(YM)(o)}(\tau, \vec{\sigma}) \}, \tag{145} \]

and the equations defining the rest-frame are

\[
\frac{m_1}{\sqrt{1 - \vec{\eta}_1^2(\tau)}} + \frac{m_2}{\sqrt{1 - \vec{\eta}_2^2(\tau)}} - \\
- \sum_{a} \tilde{Q}_{1a}(\tau) [\tilde{A}_{a \perp}^{(o)}(\tau, \vec{\eta}_1(\tau)) - \tilde{A}_{a \perp}^{(1)}(\tau, \vec{\eta}_2(\tau))] + \tilde{Q}_{1a}(\tau) (\tilde{A}_{a \perp}^{(o)}(\tau, \vec{\eta}_1(\tau)) - \tilde{A}_{a \perp}^{(1)}(\tau, \vec{\eta}_2(\tau))) + \\
+ g_s^2 \int d^3 \sigma \{ \tilde{\sigma} \tilde{\eta}_2 E_{a \perp}^{(o)s} + \tilde{Q}_{1a}(\tau) (\tilde{\sigma} \tilde{\eta}_2 E_{a \perp}^{(1)s} + \tilde{\sigma} \tilde{\eta}_2 E_{a \perp}^{(1)s}) \} \tau, \vec{\sigma} \leq 0. \tag{146} \]

The invariant mass (the relative Hamiltonian) of the pseudoclassical scalar quark model takes the form

\[
H_{rel} = \sqrt{M_1^2[m_1, \vec{\eta}_1, \vec{\kappa}_1, \tilde{A}_{a \perp}, \tilde{A}_{a \perp}^{(1)}](\tau) + \tilde{\kappa}_1^2(\tau)} + \sqrt{M_2^2[m_2, \vec{\eta}_2, \vec{\kappa}_2, \tilde{A}_{a \perp}, \tilde{A}_{a \perp}^{(1)}](\tau) + \tilde{\kappa}_2^2(\tau)} + \\
+ V_{FP}[\vec{\eta}_1 - \vec{\eta}_2; \vec{\eta}_1; \tilde{A}_{a \perp}(\tau)] + \sum_{i=1}^2 V_{(i)PF}[(\vec{\eta}_i; \tilde{A}_{a \perp; \vec{\sigma} \perp}; \tilde{A}_{a \perp; \vec{\sigma} \perp}^{(1)})(\tau) + \\
+ V_{FP}[\tilde{A}_{a \perp; \vec{\sigma} \perp; \vec{\sigma} \perp}(\tau)] + \frac{1}{2} \int d^3 \sigma \sum_a [g_s^2 \tilde{\pi}_a^2 + g_s^2 \tilde{B}_a^2](\tau, \vec{\sigma}) \tag{147} \]

with
\[ M_i^2 = m_i^2 + (-)^{i+1}2\tilde{\kappa}_i(\tau) \cdot \sum_a \tilde{Q}_{1a}(\tau)[\tilde{A}_{a\perp}^{(0)} + \sum_u \tilde{Q}_{1u}(\tau)\tilde{A}_{au\perp}^{(1)}](\tau, \tilde{\eta}_i(\tau)) + \]
\[ + \sum_{a,b} \tilde{Q}_{1a}(\tau)\tilde{Q}_{1b}(\tau)[\tilde{A}_{a\perp}^{(0)} \cdot \tilde{A}_{b\perp}^{(0)}](\tau, \tilde{\eta}_i(\tau)). \] (148)

The first three terms of \( H_{rel} \), after a suitable average over the field degrees of freedom, should give the effective rest-frame Hamiltonian for the pseudoclassical relativistic scalar quark model: while \( m_i \) are the quark current masses, the suitable average of \( M_i \) should give the quark constituent masses. The terms \( V_{(i)PF} \) describe quark-field interactions. The last two terms should describe the pseudoclassical glueball degrees of freedom.

All the previous results can be reexpressed in terms of rescaled Yang-Mills fields by putting \( \tilde{A}_{a\perp} = g_s\tilde{A}_{a\perp}, \tilde{\pi}_{a\perp} = g_s^{-1}\tilde{\pi}_{a\perp}, \tilde{B}_a = g_s^{-1}\tilde{B}_a. \)

Finally, the Berezin-Marinov distribution function is

\[ \rho = \rho_1\rho_2, \]
\[ \rho_i = -2 < \tilde{Q}_{1a}(\tau) > \tilde{Q}_{1a}(\tau)N_i + \frac{1}{6}N_i^3 \approx 0, \] (149)

and, by taking the mean value of equations with it, the “classical equations” are [see Refs. [15,23,24] for the case of an external Yang-Mills field]

\[ \frac{d}{d\tau} < \tilde{Q}_{1a}(\tau) > -c_{abc} < \tilde{Q}_{1b}(\tau) > [\tilde{\eta}_1(\tau) \cdot \tilde{A}_{b\perp}^{(0)}(\tau, \tilde{\eta}_1(\tau)) - \]
\[ -g_s^2 \int d^3\sigma K_{ab}^{(o)}(\tilde{\eta}_1(\tau), \tilde{\sigma}; \tau)\tilde{\rho}_b^{(YM)(o)}(\tau, \tilde{\sigma})] \approx 0 \] (150)

\[ \frac{d}{d\tau} [m_1 - \frac{\tilde{\eta}_1(\tau)}{1 - \tilde{\eta}_2(\tau)}] = \]
\[ \sum_a < \tilde{Q}_{1a}(\tau) \{ \tilde{E}_{a\perp}^{(o)r}(\tau, \tilde{\eta}_1(\tau)) + \tilde{E}_{a\perp}^{(o)}(\tau, \tilde{\eta}_1(\tau)) \} > - \]
\[ -P_{r\perp}^{(o)}(\tilde{\eta}_1)_{[c_{amd}\tilde{A}_{m\perp}^{(o)d}(\tau, \tilde{\eta}_1(\tau))] - \int d^3\sigma K_{de}^{(o)}(\tilde{\eta}_1(\tau), \tilde{\sigma}; \tau)\tilde{\rho}_d^{(YM)(o)}(\tau, \tilde{\sigma}) \] + \[ \frac{\partial K_{da}^{(o)}(\tilde{\eta}_1(\tau), \tilde{\sigma}; \tau)}{\partial\tilde{\eta}_1} \tilde{\rho}_d^{(YM)(o)}(\tau, \tilde{\sigma}) \].
\[
\frac{d}{d\tau} \left[ m_2 \frac{\dot{\vec{\eta}}_2(\tau)}{\sqrt{1 - \dot{\vec{\eta}}_2^2(\tau)}} \right] = \circ - \sum_{a} \left< \dot{Q}_{1a}(\tau) \right> \left\{ \tilde{E}_{a\perp}^{(o)r}(\tau, \vec{\eta}_2(\tau)) + [\dot{\vec{\eta}}_2(\tau) \times \tilde{B}_{a\perp}^{(o)}(\tau, \vec{\eta}_2(\tau))]^r \right\} - \\
- P_{1\perp}^{rs}(\vec{\eta}_2) c_{amd} \tilde{\tilde{A}}_{m\perp}^{(o)s}(\tau, \vec{\eta}_2(\tau)) \int d^3\sigma K_{de}^{(o)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) c_{emb} \tilde{\tilde{A}}_{n\perp}^{(o)}(\tau, \vec{\sigma}) \cdot \tilde{E}_{b\perp}^{(o)}(\tau, \vec{\sigma}) - \\
g_s^2 \int d^3\sigma \left[ c_{adc} \tilde{\tilde{A}}_{d\perp}^{(o)r}(\tau, \vec{\eta}_2(\tau)) K_{cb}^{(o)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau) + \\
\frac{\partial K_{ab}^{(o)}(\vec{\eta}_2(\tau), \vec{\sigma}; \tau)}{\partial \eta_2^r} j_{YM}^{(o)}(\tau, \vec{\sigma}) \right],
\]

(151)

\[
m_1 \frac{\dot{\vec{\eta}}_1(\tau)}{\sqrt{1 - \dot{\vec{\eta}}_1^2(\tau)}} + m_2 \frac{\dot{\vec{\eta}}_2(\tau)}{\sqrt{1 - \dot{\vec{\eta}}_2^2(\tau)}} + g_s^{-2} \int d^3\sigma (\partial \tilde{\tilde{A}}_{a\perp}^{(o)s} \tilde{E}_{a\perp}^{(o)s})(\tau, \vec{\sigma}) - \\
- \sum_{a} \left< \dot{Q}_{1a}(\tau) \right> \left[ \tilde{A}_{a\perp}^{(o)}(\tau, \vec{\eta}_1(\tau)) - \tilde{A}_{a\perp}^{(o)}(\tau, \vec{\eta}_2(\tau)) \right] + \\
g_s^{-2} \sum_{a,u} \left< \dot{Q}_{1a}(\tau) \right> \int d^3\sigma (\partial \tilde{\tilde{A}}_{a\perp}^{(o)s} \tilde{E}_{a\perp}^{(1)s} + \tilde{\tilde{A}}_{a\perp}^{(1)s} \tilde{E}_{a\perp}^{(o)s})(\tau, \vec{\sigma}) \circ = 0.
\]

(152)

The particle “classical equations” are expected to have the same causal pathologies (runaway solutions or preaccelerations) as the Abraham-Lorentz-Dirac ones in the electromagnetic case. In that case, see Refs. [15,14], one can follow a different procedure: i) solve the field equations (the solutions are incoming free linear waves plus the Lienard-Wiechert potential if retarded solutions are selected); ii) put the solution in the particle equations; iii) take the mean value of the new equations. This procedure gives different “classical equations” without pathologies, because one has \( Q_i^2 = 0 \) for the Grassmann-valued electric charges: in this way, at the pseudoclassical level, one regularizes the Coulomb self-energies [7], has no radiation produced by the single Lienard-Wiechert potentials but has radiation from the superposition of them [14] (terms in \( Q_i Q_j \) with \( i \neq j \)).

In the non-Abelian case, we do not know solutions of Eqs.(133) and the superposition principle does not hold, so that we do not know how to apply the second procedure. A first needed step would be to find the Green function of the operator on the left side of Eq.(133).
VII. DISCUSSION AND CONCLUSIONS.

We have obtained the description of the isolated system of N scalar particles with Grassmann-valued color charges plus the color SU(3) Yang-Mills field on spacelike hypersurfaces and then its canonical reduction to the Wigner hyperplane with only physical degrees of freedom (generalized Coulomb gauge in the Wigner-covariant rest-frame instant form of dynamics) following the same steps of Refs. [7,14] for the case of N scalar particles with Grassmann-valued electric charges plus the electromagnetic field.

In particular, we obtained the reduced Hamilton and Euler-Lagrange equations for the reduced transverse Yang-Mills field and for the particles in this rest-frame generalized Coulomb gauge, generalizing those of the electromagnetic case. Four kinds of interactions between two color charge densities (associated either with a particle or with the SU(3) field) were identified in the resulting potential extracted from the gauge part of the Yang-mills field: one of them is a Coulomb interaction, while the others involve Wilson lines 

\[ \zeta(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = P e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}_2} d\vec{\sigma} \cdot \hat{\mathbf{A}}_{\perp}(\tau, \vec{\sigma}) T_a} \]  

along straightlines between the two (simultaneous in the rest frame) points where the two color densities are located. In the electromagnetic case [7,14], where the field is not charged, there is only the interparticle Coulomb potential. Due to the high nonlinearity of the field equations, it is extremely difficult to try to define a non-Abelian analogue of the Lienard-Wiechert potentials of the Abelian case [14]. Moreover, we have found the Berezin-Marinov distribution function for the Grassmann sector [15,16] and, then, the “classical equations of motion” as the mean value of the previous equations. In absence of a Lienard-Wiechert potential, solution of the field equations, to be put in the particle equations of motion (which, otherwise, are going to have causal pathologies like in the electromagnetic Abraham-Lorentz-Dirac equations), we cannot take the mean value of these new equations and check that the resulting equations do not have causal pathologies as it happens in the electromagnetic case [14].

We then studied the N=2 case, which is the pseudoclassical basis of the relativistic scalar-quark model but with the reduced transverse color field present, which describes the
pseudoclassical glueball degrees of freedom. With suitable constraints on the Grassmann variables we obtain the description of an isolated system corresponding to a meson: a quark-antiquark pair plus the transverse SU(3) field. The reduced Hamiltonian of the system is its invariant mass expressed in the intrinsic rest frame. It contains the relativistic kinetic terms of the two quarks minimally coupled to the transverse SU(3) field, an interparticle field-dependent (but quark-mass-independent) potential, a single particle-field potential and the kinetic terms plus a self-interaction potential for the transverse SU(3) field. If one would know the non-Abelian Lienard-Wiechert potentials of the quarks (if this concept makes sense due to the nonlinear self-coupling of the color field, i.e. due to the glueball degrees of freedom), one could obtain a field-independent expression for the invariant mass of the meson and therefore an explicit formulation of a field-independent relativistic scalar-quark model starting from the pseudoclassical Lagrangian for QCD with scalar quarks. This would fill the gap, at least at the pseudoclassical level, between QCD and quark models [see Ref. [13] for the nonrelativistic one]. Moreover, a field-independent (but quark-velocity-dependent) interparticle potential would appear, whose static \[m_i \to \infty \Rightarrow \dot{\eta}_i(\tau) \to 0\] part should have connections with the static potential in QCD [25]. Note that at the pseudoclassical level there is no notion of gluons (or gluon exchange): only SU(3) fields are present and only a Lienard-Wiechert potential (if it exists in some sense) would create a bridge towards the Wilson loop expectation value form of the static potential between heavy quarks [in the quenched approximation in which quark loops from pair production in vacuum (absent in the pseudoclassical theory) are neglected (sea-quarks of infinite mass) and glueballs (closed color loops) are unambiguously defined] re-expressed in terms of perturbative QCD.

When the study of Dirac fields and spinning particles on spacelike hypersurfaces will be finished [12], one will get analogous results for spinning particles (with Grassmann-valued spin), namely one will introduce the quark spin structure in the pseudoclassical relativistic quark model.

In the N=2 (meson) case, we have explored the implications of the imposition of the condition that the isolated system (quarks+transverse SU(3) field) is a color singlet. The
condition $\hat{Q}_a = \hat{Q}_a^{(YM)}(\tau) + \hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) = 0$ is imposed by hand and, moreover, it is asked to be fulfilled by asking separately the two conditions $\hat{Q}_a^{(YM)}(\tau) = 0$ [no color flux of the transverse SU(3) field at space infinity; it replaces the Abelian condition of no radiation field, i.e. $\vec{A}_\perp(\tau, \vec{\sigma}) = 0$ [7,14]] and $\hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) = 0$ [it is the color singlet condition of field-independent quark models, which work very well phenomenologically [13]]. Now, the condition $\hat{Q}_a(\tau) = 0$ can be imposed by choosing suitable Hamiltonian boundary conditions at fixed $\tau$ on the transverse SU(3) field. Therefore, if confinement exists, the condition $\hat{Q}_a^{(YM)}(\tau) = \hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) = 0$ should emerge from the solution of the Hamilton equations [at least approximately seen the phenomenological soundness of the quark model]. Moreover, the condition $\hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) = 0$, together with the other constraints on Grassmann variables, implies the disappearance of the Coulomb term in the interparticle potential [this is a statement stronger than the regularization of the Coulomb self-energies in the Abelian case [7,14]] and that this potential tends to zero when the two quarks tend to the same spatial position in the rest frame $|\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)| \to 0$. This is the pseudoclassical statement of asymptotic freedom: there is an antiscreening even stronger than in QCD, because the reducing (screening) effect of pair production is here absent. This kind of asymptotic freedom is an algebraic consequence of the request of color singlets in the quark model oriented form $\hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) = 0$.

It would be interesting to study in this way $\hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) + \hat{Q}_{3a}(\tau) = 0$ the N=3 (baryon) case of 3 quarks or 3 antiquarks. What happens when either only two quarks or all three quarks tend to the same spatial position?

Coming back to the N=2 case, the main unsolved problem, connected with the color singlet requirement, is confinement. Even if the boundary conditions for the transverse SU(3) fields are chosen so to imply $\hat{Q}_a(\tau) = 0$, we do not know how to do either analytical or numerical calculations to check whether the interparticle potential implies confinement and, if yes, whether confinement implies $\hat{Q}_{1a}(\tau) + \hat{Q}_{2a}(\tau) = 0$, not to speak of the possible glueball degrees of freedom. The obstruction is the lack of control on the Wilson line.
\[
\zeta(\vec{\sigma}_1, \vec{\sigma}_2; \tau) = Pe^{\int_{\vec{\sigma}_2}^{\vec{\sigma}_1} d\vec{\sigma} \cdot \tilde{\vec{A}}_{\perp}(\tau, \vec{\sigma}) T^a_a}
\]
given the function space for the transverse SU(3) potential \( \tilde{\vec{A}}_{\perp}(\tau, \vec{\sigma}) \), which is the behaviour in \( \vec{\sigma}_1 \) and \( \vec{\sigma}_2 \) of \( \zeta(\vec{\sigma}_1, \vec{\sigma}_2; \tau) \) as an element of the group SU(3) in the adjoint representation? Does it belong to the same function space as the transverse SU(3) potential? How to simulate it on a lattice having eliminated all the gauge degrees of freedom in favour of a transverse potential? Which is the pseudoclassical analogue of the quantum Wilson criterion of confinement \([26,27]\) in this Hamiltonian generalized Coulomb gauge? Moreover, an aspect of the confinement problem which has to be understood at the pseudoclassical level is the role of the center \( Z_3 \) of SU(3), namely the zero triality condition [fermions know SU(3), but the Yang-Mills field feels only the not simply connected group \( SU(3)/Z_3 \) \([1]\), relevant in the approaches of Refs. \([28]\). Following Ref. \([29]\), this condition is probably hidden in the fact that the Wilson line \( \zeta(\vec{\sigma}_1, \vec{\sigma}_2; \tau) \) must take values in \( SU(3)/Z_3 \) and not in SU(3).

Let us also note that the results of Ref. \([1]\) on Yang-Mills fields plus Grassmann-valued Dirac fields, once the description of Dirac fields in the rest-frame instant form will be terminated \([12]\), suggest that the potential \( V \) will remain unchanged except for the replacement \( \sum_i \tilde{\rho}_{ia}(\tau, \vec{\sigma}) \rightarrow \tilde{\psi}^\dagger(\tau, \vec{\sigma}) T^a \tilde{\psi}(\tau, \vec{\sigma}) \).

Another yet unsolved problem (also in the electromagnetic case) is how to eliminate the 3 constraints \( \vec{H}_p(\tau) \approx 0 \) defining the intrinsic rest frame. This requires the introduction of 3 gauge-fixings identifying the Wigner 3-vector describing the intrinsic 3-center of mass on the Wigner hyperplane. However, till now these gauge-fixings are known only in the case of an isolated system containing only particles. When the center of mass canonical decomposition of linear classical field theories will be available (see Ref. \([30]\) for the Klein-Gordon field), its reformulation on spacelike hypersurfaces will allow the determination of these gauge-fixings also when fields are present and a Hamiltonian description with only Wigner-covariant relative variables with an explicit control on the action-reaction balance between fields and particles or between two types of fields.

As said in Ref. \([14]\), the quantization of this relativistic scalar-quark model has to over-
come two problems. On the particle side, the complication is the quantization of the square
roots associated with the relativistic kinetic energy terms. On the field side, the obstacle
is the absence (notwithstanding there is no no-go theorem) of a complete regularization
and renormalization procedure of electrodynamics in the Coulomb gauge: see Refs. [31,32]
for the existing results for QED. However, as shown in Refs. [7,1,6], the rest-frame instant
form of dynamics automatically gives a physical ultraviolet cutoff: it is the Möller radius
\( \rho = \sqrt{-W^2 c/P^2} = |\vec{S}| c/\sqrt{P^2} \) (\( W^2 = -P^2 \vec{S}^2 \) is the Pauli-Lubanski Casimir), namely the
classical intrinsic radius of the worldtube, around the covariant noncanonical Fokker-Price
center of inertia, inside which the noncovariance of the canonical center of mass \( \vec{x}^\mu \) is con-
centrated. At the quantum level \( \rho \) becomes the Compton wavelength of the isolated system
multiplied its spin eigenvalue \( \sqrt{s(s+1)} \), \( \rho \mapsto \hat{\rho} = \sqrt{s(s+1)} \hbar/M = \sqrt{s(s+1)} \lambda_M \)
with \( M = \sqrt{P^2} \) the invariant mass and \( \lambda_M = \hbar/M \) its Compton wavelength.

Let us remark that in the electromagnetic case all the dressings with Coulomb clouds [of
the scalar particles and of charged Klein-Gordon fields in Ref. [14] and of Grassmann-valued
Dirac fields in Ref. [1]] are done with the Dirac phase \( \eta_{em} = -\frac{1}{2} \vec{\partial} \cdot \vec{A} \) [5]. The same phase is
used in Ref. [32] to dress fermions in QED. In these papers there is a definition of dressing
of fermion fields in the non-Abelian quantum case, which is quite similar to the one of Ref.
[1] (used in this paper) even if implemented only perturbatively. Essentially, one looks for a
matrix \( h \in SU(3) \) such that under a gauge transformation \( U \) one has \( h \mapsto hU = U^{-1} h \); then
one has \( \psi = h \psi_{\text{PHYS}} \) and \( \tilde{A}_a \tilde{T}^a = h \tilde{A}_a \tilde{T}^a \psi_{\text{PHYS}} h^{-1} - \vec{\partial} h h^{-1} \) with \( \psi_{\text{PHYS}} \) and \( \tilde{A}_a \tilde{T}^a \) gauge
invariant. Comparison with Eq.(73) and with Ref. [1] shows that at the classical level one
has \( h = P e^{i \Omega_{\text{em}}(\gamma^A) \tilde{T}^a} \).

Also in Ref. [29] the solution of the quantum Gauss law constraint on Schroedinger
functional \( \Psi[A] \) in the case of two static particles of opposite charges, is able to reproduce
the Coulomb potential and the Coulomb self-energy with the same mechanism as in Refs.
[1,14] [namely with the Abelian form of Eq.(93)] only if \( \Psi[A] = e^{i \eta_{em}} \Phi[A] \) with \( \Phi[A] \) gauge
invariant and not with \( \Psi'[A] = e^{i \int_{x_1} x_0} d^2 x \tilde{A}(\vec{x}) \Phi'[A] \) with a phase factor resembling the Wilson
loop operator [one has \( \Phi[A] = e^{i \int_{x_1} x_0} d^2 x \tilde{A}(\vec{x}) \Phi'[A], \) namely the Wilson line operator has been
broken in the gauge part plus the gauge invariant part using \( \vec{A} = \vec{\partial}_\eta \eta_{em} + \vec{A}_\perp \). However, it is difficult to see a connection between the phase of the Schroedinger functional \( \Psi[A] \) proposed in Ref. [29] as a solution of the non-Abelian Gauss laws and our potential \( V \), which is a consequence of Eqs. (73), (75)-(77), (83), (93) [now the Wilson line \( e^{i \int_{x_o}^{x_1} d\vec{x} \cdot \vec{A}_a(x^a, \vec{x}) \hat{T}^a} \) cannot be broken in a gauge part and in a gauge invariant part due to Eq.(73); in Ref. [29] the gauge-dependent part is a product \( U(x_1)U^\dagger(x_o) \) of two gauge transformations].

Let us finish with some heuristic considerations about the Møller radius. In QCD, due to asymptotic freedom and to the renormalization group equations, the strong coupling constant \( \alpha_s = g_s^2/4\pi \) is replaced by the effective running coupling constant (see for instance Ref. [34]) \( \alpha_s(Q^2) = 12\pi/(33 - 2N_F)\ln\frac{Q^2}{\Lambda_{QCD}^2} \) at high \( Q^2 \) (\( N_F \) is the number of flavors, giving the screening contribution of the fermions to the vacuum polarization). Therefore, the adimensional coupling constant \( \alpha_s \) may be replaced with the fundamental QCD scale \( (\hbar = c = 1) \Lambda_{QCD} \approx 300\text{Mev} \approx 10^{-13} \text{cm} = 1 \text{fm} \), which is now usually replaced by \( \alpha_s(m_Z^2) \approx 0.116 \) [33] [dimensional transmutation, connected with the breaking of scale invariance at high energies and with the scale anomaly; the physical mechanism for generating the scale at low energies is unclear (a candidate is the chiral symmetry breaking phase transition of QCD which generates a constituent mass of order 300 Mev for the light quarks)]. This implies [13] that in the nonrelativistic quark model with confinement, one may choose (among the many possible phenomenological potentials) the simple potential \( V(r) = -\frac{4}{3} \alpha_s(r) + \kappa r \) with the short distance behaviour \( \alpha_s(r) = 12\pi/(33 - 2N_F)\ln\frac{r^2}{\Lambda_{QCD}^2} \), i.e. for \( r < r_o = \frac{1}{\Lambda_{QCD}^2} \approx 10^{-13} \text{cm} = 1 \text{fm} \) [in Ref. [25] it is shown that QCD perturbative results cannot be trusted for \( r < 0.07\Lambda_{QCD}^{-1} = 0.07\text{fm} \) for \( \Lambda_{QCD} \approx 210\text{Mev} \), i.e. \( \alpha_s(m_Z^2) = 0.118 \)]. One can consider \( r_o \) as an effective radius of confinement for quarks and glueballs [the proton Compton wavelength is \( \lambda_p = \hbar/m_pc \approx 10^{-13} \text{cm} = 1 \text{fm} < r_o \)]. For \( r \approx r_o \) the expressions of \( \alpha_s(Q^2) \) and \( \alpha_s(r) \) break down due to confinement, which is described by the linear term in \( V(r) [\kappa \approx 0.2\text{Gev}^2 \) is the “string tension”, which turns out to be determined numerically as a function of \( \Lambda_{QCD} \) in lattice gauge theory [27] at least for heavy quarks, for which a string-like (chromoelectric
flux tube) structure emerges; in the theoretical approach based on the analogy with type II superconductors (see the reviews in Refs. [34,13]), where the vacuum is a color-dielectric medium and a $\bar{q}q$ state is a confined color-electric flux tube (anti-Meissner effect), $\kappa$ is determined by the gluon condensate $\frac{\alpha_s}{\pi} \sum a F_{a\mu\nu} F_{a\mu\nu}^\ast > 0$ (as confirmed in strong coupling expansion of lattice QCD [26]), which is present besides the $\langle \bar{q}q \rangle$ condensate responsible for chiral symmetry breaking. The crucial point for the pseudoclassical relativistic quark model would be to see whether Eq.(143) implies $V_{PP} \rightarrow |\vec{\eta}_1 - \vec{\eta}_2| \rightarrow \infty \kappa |\vec{\eta}_1 - \vec{\eta}_2| + \ldots$.

Let us also note that in the MIT bag model (see Ref. [35] for a review), the bag constant $B$ is connected with the gluon condensate and the length of the chromoelectric flux tube for heavy quarks and the string tension $\kappa$ are determined by $B$ and $\alpha_s$.

The Møller radius $\rho = |\vec{S}|/M$ is going to play the role of a ultraviolet cutoff $\hat{\rho} = \sqrt{s(s+1)}\lambda_M$ [\lambda_M is the Compton wavelength of the isolated system with invariant mass $M = \sqrt{P^2} = H_{rel}$] at the quantum level (like the lattice spacing in lattice QCD). Since $\rho$ describes a nontestable classical short distance region [impossibility of frame-independent determination of the location of the relativistic canonical center of mass (also named Pryce center of mass and having the same covariance of the Newton-Wigner position operator [36]); its connection with the Mach’s principle according to which only relative motions are measurable], it sounds reasonable that for a confined system of effective radius $r_o = 1/\Lambda_{QCD}$ one has $\hat{\rho} \approx r_o$. However, this is not correct because it implies a mass-spin relation $|\vec{S}| \approx r_o M$, while the phenomenological Regge trajectories are $|\vec{S}| = \alpha_s' M^2 + \alpha_o$ [$\alpha_s' = 1 Gev^{-2}$], implying, at least for heavy quarks, an effective string theory inside QCD. Now, in string theory [37] the relevant dimensional quantity is the tension $T_s = 1/2\pi\alpha_s'$ (the energy per unit length), which, at the quantum level, determines a minimal length $L_s = \sqrt{\hbar/T_s} = \sqrt{2\pi\hbar\alpha_s'} = \frac{\sqrt{2\pi\alpha_s'}}{\lambda_M}$.

For a classical string one has $|\vec{S}| \leq \alpha_s' M^2$, so that its Møller radius is $\rho \leq \alpha_s' M$; at the quantum level one has $\hat{\rho} = \sqrt{s(s+1)}\lambda_M \leq \alpha_s' M = L_s^2/\lambda_M$. Therefore, if the QCD string has $L_s = \sqrt{2\pi\alpha_s'} \approx 10^{-13} cm = 1 fm \leq r_o = 1/\Lambda_{QCD} \approx 1 fm$, one gets that the Møller radius of a confined system must be of the order $\rho \leq r_o^2 M = M/\Lambda_{QCD}^2$ [$\hat{\rho} \leq r_o^2/\lambda_M$]. This corresponds
to a QCD string with $2\pi\alpha'_s \leq \gamma_o^2 = \Lambda_{QCD}^{-2}$: see Ref. [38] for a long glue string giving rise to an effective Nambu-Goto string with this $\alpha'_s$ and more generally see Ref. [39].

This effective QCD string theory (whose final formulation has still to be found) must not be confused with string cosmology [37,40], in which, at the quantum level, the string tension $T_{cs} = 1/2\pi\alpha'_s = L_{cs}^2/h$ gives rise to a minimal length $L_{cs} = \sqrt{2\pi\alpha'_s} \geq L_P$ [$L_P = 1.6 \times 10^{-33}cm$ is the Planck length] and is determined by the vacuum expectation value of the background metric of the vacuum (if the ground state is flat Minkowski spacetime), while the grand unified coupling constant $\alpha_{GUT}$ (replacing $\alpha_s$ of QCD) is determined by the vacuum expectation value of the background dilaton field. This minimal length $L_{cs} \geq L_P$ (suppressing the gravitational corrections) could be a lower bound for the Møller radius of an asymptotically flat universe, built with the Poincaré Casimirs of the asymptotic ADM Poincaré charges (see Ref. [41] for the canonical reduction of tetrad gravity). The upper bound on $\rho$ (namely a physical infrared cutoff) could be the Hubble distance $cH_o^{-1} \approx 10^{28}cm$ considered as an effective radius of the universe. Therefore, it seems reasonable that our physical ultraviolet cutoff $\rho$ is meaningful in the range $L_P \leq L_{cs} < \rho < cH_o^{-1}$.

Let us remember that $\rho$ is also a remnant in flat Minkowski spacetime of the energy conditions of general relativity [7]: since the Møller noncanonical, noncovariant center of energy has its noncovariance localized inside the same worldtube with radius $\rho$ (it was discovered in this way) [42], it turns out that an extended relativistic system with the material radius smaller of its intrinsic radius $\rho$ has: i) the peripheral rotation velocity can exceed the velocity of light; ii) its classical energy density cannot be positive definite everywhere in every frame.

Moreover, the extended Heisenberg relations of string theory [37], i.e. $\Delta x = \hbar / \Delta p + \dot{\rho}_{cs}$ (see Ref. [37] for the meaning of $\Delta p$ in string theory) implying the lower bound $\Delta x > L_{cs} = \sqrt{\hbar/T_{cs}}$, have a counterpart in the quantization of the Møller radius [7]: if we ask that, also at the quantum level, one cannot test the inside of the worldtube, we must ask $\Delta x > \hat{\rho}$ which is the lower bound implied by the modified uncertainty relation $\Delta x = \hbar / \Delta p + \dot{\rho}_{cs}$

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\( \frac{h \Delta p}{\rho^2} \). This would imply that the center-of-mass canonical noncovariant (Pryce) 3-coordinate \( \vec{z} = \sqrt{P^2}(\vec{x} - \frac{\vec{p}}{m}\vec{x}^o) \) [7] cannot become a self-adjoint operator. See Hegerfeldt’s theorems [43], his interpretation pointing at the impossibility of a good localization of relativistic particles (experimentally one determines only a worldtube in spacetime emerging from the interaction region) and also the comments of Ref. [44] against this interpretation. Since the eigenfunctions of the canonical center-of-mass operator are playing the role of the wave function of the universe, one could also say that the center-of-mass variable has not to be quantized, because it lies on the classical macroscopic side of Copenhagen’s interpretation and, moreover, because, in the spirit of Mach’s principle that only relative motions can be observed, no one can observe it. On the other hand, if one rejects the canonical noncovariant center of mass in favor of the covariant noncanonical Fokker-Pryce center of inertia [45,7,46] \( Y^\mu, \{Y^\mu, Y^\nu\} \neq 0 \), one could invoke the philosophy of quantum groups to quantize \( Y^\mu \) to get some kind of quantum plane for the center-of-mass description.
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