Algebraic approach to the spectral problem for the Schrödinger equation with power potentials

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Abstract

The method reducing the solution of the Schrödinger equation for several types of power potentials to the solution of the eigenvalue problem for the infinite system of algebraic equations is developed. The finite truncation of this system provides high accuracy results for low-lying levels. The proposed approach is appropriate both for analytic calculations and for numerical computations. This method allows also to determine the spectrum of the Schrödinger-like relativistic equations. The heavy quarkonium (charmonium and bottomonium) mass spectra for the Cornell potential and the sum of the Coulomb and oscillator potentials are calculated. The results are in good agreement with experimental data.

1 Introduction

The solution of the spectral problem for the Schrödinger equation with spherically symmetrical potentials is an important problem for the atomic and hadronic spectroscopy. It is well known that potential models give a good description of the heavy quarkonium ($\psi, \Upsilon$) mass spectrum [1]. The interaction potential in such a system is usually assumed to be of the confining type. The example is the Cornell potential which contains two terms, one is responsible for the Coulomb interaction dominating at small distances and the other corresponds to the string-like interaction leading to the confinement.

The problem of calculating the energy spectrum of the Schrödinger equation with different potentials is attracting the attention of physicists for a long time and is being solved by many methods: by the direct numerical solution with fixed boundary conditions for wave functions, by the variational method and by different modifications combining analytical and numerical
approaches. The quasiclassical method works rather well for a wide class of quantum mechanical problems [2, 3]. However, most of these approaches does not allow to determine explicitly the dependence of the solution on the parameters of the potential.

In this paper we develop further the method proposed in Refs. [4, 5] which accounts for the asymptotic behaviour of the wave functions at large distances (or small momenta). This approach, in fact, generalizes the method of integral transformations for the kernels of special type, determining the correct asymptotic behaviour of the wave functions. As it was shown in Refs. [4, 5] such a method is rather efficient for finding the low-lying spectral levels. These low-lying levels are of a special interest for the meson and baryon spectroscopy.

Here we give the exact algorithm for the reduction of the radial Schrödinger equation to the algebraic eigenvalue problem, from which it is possible to obtain the spectrum of the initial equation using successive approximation method. This approach is rather simple and suitable both for analytical calculations and numerical computations and allows to get approximate analytical expressions for the low-lying levels. These expressions are suitable for the qualitative analysis of the dependence of the spectrum on the parameters of the model potential. It is necessary to note that the resulting approximation scheme is rather rapidly convergent. This result is of the independent interest and shows the reliability of developed approach.

In this paper we consider spherically symmetrical confining potentials of the following form

\[ U_k(r) = a^2 r^k - \frac{Z}{r}, \quad a > 0, \quad k = 1, 2, \quad (1) \]

which are important for hadron physics.

The general construction of the fast converging approximation scheme with the account of the algebraic symmetry is given in [4]. Here we apply this method for the potentials of the type (1).

The radial part of the Schrödinger equation can be presented as follows:

\[ \left[ -\frac{d^2}{dr^2} + U_{\text{eff}}(r) \right] R(r) = \lambda R(r). \quad (2) \]

We assume that the total wave function of the solution contains a standard angular part as well:

\[ \psi_{nlm}(r) = \frac{R_n(r)}{r} Y_{lm}(\theta, \varphi). \quad (3) \]

In this case the effective potential in (2) will be of a well-known type:

\[ U_{\text{eff}}(r) = \frac{l(l+1)}{r^2} + U_k(r). \quad (4) \]

If the potential \( U_k(r) \) does not contain terms more singular at the origin than \( r^{-1} \) (as in the case of (1)), the behaviour of the radial part of the wave function \( R(r) \sim r^{l+1} \) at the origin will be determined by the first term in (4). Let us make the substitution \( R(r) \sim r^{l+1} F(r) \), where \( F(0) \neq 0. \) Then the equation can be rewritten in the form convenient for further transformations

\[ F'' + \frac{2}{r} (l + 1) F' + [\lambda - U(r)] F = 0. \quad (5) \]
2 The spectral problem for potential $U_1(r)$

In the case of the potential (1) linearly depending on $r$ the constant $a^2 > 0$ can be removed by the simultaneous change of the variable in eq. (5) $r \rightarrow a^{-2/3}r$ and introduction of the dimensionless combinations of parameters $\lambda^* = \lambda a^{-4/3}$, $z^* = za^{-2/3}$. The solutions of eq. (5) are sought in the form

$$F(r) = \sum_{k=0}^{\infty} \frac{a_k}{f_k} r^k, \quad f_k = \left( \frac{2}{3} \right)^{2k/3} \frac{2k+4l+6}{\Gamma \left( \frac{4l+5}{3} \right)}.$$

(6)

Coefficients $f_k$ are chosen from the considerations of symmetrization of the Jacobi matrix, constructed from the coefficients $a_k$, and receiving of the correct asymptotical behaviour of the wave function ad infinitum $R(r) \sim \exp \left[ -\frac{2}{3} r^{3/2} \right]$, coinciding in this case with the asymptotics of the Airy function. Moreover, such a choice of coefficients allows to adjust the kernel of generalized integral Laplace transformation (Mittag-Leffler transformation). The recursion relation obtained by substitution of (6) in (5) is the following:

$$(k + 2) a_{k+2} - \left( k + 2l + \frac{2}{3} \right) a_{k+1} + \left( \frac{2}{3} \right)^{1/3} \frac{z^*}{\Gamma \left( \frac{1}{3} \right)} B \left( \frac{2k+4l+6}{3}; \frac{1}{3} \right) a_{k+1}$$

$$+ \left( \frac{2}{3} \right)^{-1/3} \frac{\lambda^*}{\Gamma \left( \frac{1}{3} \right)} B \left( \frac{2k+4l+6}{3}; -\frac{1}{3} \right) a_k = 0.$$  

(7)

Introducing generating function $\Phi(w) = \sum_{k=0}^{\infty} a_k w^{2k/3}$ this relation can be transformed to the integro-differential equation:

$$\left[ \left( 1 - w^2 \right) \Phi' - \frac{4l+5}{3} w \Phi \right] + \frac{z^*}{\Gamma \left( \frac{1}{3} \right)} \left( \frac{2}{3} \right)^{4/3} w^{-\frac{4}{3}} I + \frac{\lambda^*}{\Gamma \left( \frac{1}{3} \right)} \left( \frac{2}{3} \right)^{2/3} w^{\frac{4}{3}} I^* = 0,$$

(8)

where

$$I(w) = \int_0^1 dt \, t^{\frac{4l+1}{3}} (1-t)^{-\frac{4}{3}} \Phi(w),$$

$$I^*(w) = \int_0^1 dt \, t^{\frac{4l+3}{3}} (1-t)^{-\frac{4}{3}} \Phi(w).$$

Factoring out explicitly a pole singularity of the function at $w = 1$,

$$\Phi(w) = \left( 1 - w^2 \right)^{-\frac{4l+5}{6}} H(w),$$

and performing the conformal mapping of the $w$ plane in the unit circle $x = \frac{1-w}{1+w}$ for new unknown function $\overline{\Lambda}(x)$, where $H(w) \equiv x^{\frac{4l+5}{6}} (1-x)^{-\frac{4l+2}{3}} \overline{\Lambda}(x)$, we get an equation:

$$x (1-x) \overline{\Lambda}' + \left[ \frac{4l+5}{6} + \frac{4l-9}{6} \right] x \overline{\Lambda} = \frac{2^{2/3} z^*}{3^{4/3} \Gamma \left( \frac{1}{3} \right)} (1-x)^{-2/3} I(x) + \frac{\lambda^*}{2^{2/3} 3^{2/3} \Gamma \left( \frac{1}{3} \right)} I^*(x).$$

(9)
Here the integrals $I, I^*$ are connected to the function $\bar{\Lambda}(x)$ by the transformations:

$$I(x) = \int_x^1 (1-u) u^{-\frac{z}{3}} \left(1 - \frac{x}{u}\right)^{-\frac{4}{3}} \bar{\Lambda}(u) \, du,$$

$$I^*(x) = \int_x^1 (1-u)^{\frac{z}{3}} u^{-\frac{4}{3}} \left(1 - \frac{x}{u}\right)^{-\frac{4}{3}} \bar{\Lambda}(u) \, du.$$

According to general theory [4] the necessary asymptotic behaviour of the wave function is satisfied if $x = 0$ is a regular point of the function $\bar{\Lambda}(x)$. This allows to search for such solutions in the form of the nonnegative power series:

$$\bar{\Lambda}(x) = \sum_{k=0}^{\infty} A_k x^k. \quad (10)$$

Performing the integrals $I, I^*$ with the help of the direct and inverse integral Mellin transformation, it is easy to find the recursion relation for the coefficients $A_k$:

$$\left(n + \frac{2}{3}l + \frac{5}{6}\right) A_n - \left(n - \frac{2}{3}l + \frac{1}{2}\right) A_{n-1} = \bar{\zeta} \sum_{k=0}^{\infty} F_{nk} A_k + \bar{\lambda} \sum_{k=0}^{\infty} G_{nk} A_k. \quad (11)$$

Here

$$F_{nk} = \frac{1}{n!} \sum_{m=0}^{n} C_n^m \left[ \frac{\Gamma\left(n - m + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \right] \left[ \frac{\Gamma\left(m + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \right] \left[ \frac{\Gamma\left(k - m + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \right] \left[ \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k - m + \frac{7}{3}\right)} \right],$$

$$G_{nk} = \frac{1}{n!} \left[ \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k - n + \frac{7}{3}\right)} \right] \left[ \frac{\Gamma\left(n + \frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}\right)} \right] \left[ \frac{\Gamma\left(k - n - \frac{1}{3}\right)}{\Gamma\left(-\frac{1}{3}\right)} \right],$$

are semi-infinite matrices and $\bar{\lambda} = 5 \cdot 2^{-5/3} 3^{-2/3} a^{-4/3} \lambda \Gamma\left(\frac{2}{3}\right)/\Gamma\left(\frac{1}{3}\right)$, $\bar{\zeta} = 2^{-4/3} 3^{2/3} a^{-2/3} z / \Gamma\left(\frac{1}{3}\right)$ are parameters. The further procedure of getting approximate energy eigenvalues $\lambda_n$ consists in the finite truncation of the matrices $\hat{F}, \hat{G}$ at the dimensions $(N+1) \times (N+1)$, where $N = 0, 1, 2, \ldots$. The resulting $N+1$ eigenvalues $\lambda_n, n = 0, 1, \ldots, N$ correspond to the position of the first levels of the spectral problem. The higher is the value of $N$, the lower is the dispersion value. The accuracy of the lowest levels is the highest. In Appendix we illustrate the convergence of the proposed approximation by the exactly solvable examples, which are the limits of the potential (1). There it is shown that the proposed approach exactly reproduces the solutions of the Schrödinger equation for the oscillator and the Coulomb potentials. Our perturbation results for $S$ wave linear potential rapidly converge to the Airy function zeros. For the ground state the 1% accuracy is achieved already for $N = 1$.

For the nonzero $z$ values it is necessary to take higher approximations in $N$. However, the dependence of the solution on the potential parameters $a$ and $z$ is qualitatively reproduced even for $N = 0, 1, 2$. For example, for $N = 0$ the ground state in this potential is

$$\lambda_{0,l}^{(N=0)} = 1.742a^{4/3} \left(l + \frac{5}{4}\right) - 2.415a^{2/3} z \quad (12)$$
For \( N = 1 \) we get two series of levels corresponding to the ground and first exited radial states
\[
\bar{\lambda}^{(N=1)}_{0,l} = q - \sqrt{\frac{D}{4}}, \quad \bar{\lambda}^{(N=1)}_{1,l} = q + \sqrt{\frac{D}{4}},
\]
where
\[
q = \frac{13}{18} l - \frac{23}{24} z + \frac{43}{36},
\]
and
\[
\frac{D}{4} = q^2 - \left( \frac{31}{36} l + \frac{113}{144} - \frac{13 z}{12} \right) \left( \frac{7}{12} l + \frac{77}{48} - \frac{5 z}{6} \right) + \left( \frac{1}{12} l + \frac{11}{48} - \frac{11 z}{42} \right) \left( \frac{49}{36} l - \frac{49}{144} - \frac{7 z}{12} \right).
\]

For \( N = 2 \) the solutions of cubic equation correspond to three radial quantum numbers 0,1,2.

It is necessary to point out that the reasonable accuracy of these analytical expressions can be achieved only in the limited region of parameters \( z < z_{\text{max}}(N), a < a_{\text{max}}(N) \). The boundary values of parameters are higher for higher values of the matrix truncation parameter \( N \). The estimates of the boundary parameter values can be obtained during numerical computations. The convergence of this method is guaranteed by the compact nature of the initial differential operator and its speed is rather high for carrying numerical computations which do not need the extensive computer resources.

### 3 The spectral problem for potential \( U_2(r) \)

As in the previous case, for the potential \( U_2(r) \) the substitution \( r \rightarrow r/a, z^* = z/\sqrt{a}, \lambda^* = \lambda/a \) removes the coefficient of the highest power of \( r \). Carrying out the similar consideration it is easy to get an equation corresponding to (9) in the form:
\[
x (1-x) \bar{\lambda} + \left[ \left( \frac{1}{2} + \frac{3}{4} - \frac{\lambda}{4} \right) + \left( \frac{1}{2} - \frac{3}{4} - \frac{\lambda}{4} \right) x \right] \bar{\lambda} = \frac{1}{2} Z I(x),
\]
and
\[
I(x) = \int_x^1 u^{-\frac{1}{2}} \left( 1 - \frac{x}{u} \right)^{-\frac{3}{4}} \bar{\lambda}(u) du, \quad z = \frac{Z^*}{\sqrt{2} \Gamma \left( \frac{1}{2} \right)}.
\]

The recursion relation for the coefficients of the decomposition \( \bar{\lambda}(x) = \sum_{k=0}^{\infty} A_k x^k \) is
\[
\left( n + \frac{l}{2} + \frac{3}{4} - \frac{\lambda}{4} \right) A_n - \left( n - \frac{l}{2} - \frac{1}{4} - \frac{\lambda}{4} \right) A_{n+1} = \frac{z}{2} \sum_{k=0}^{\infty} F_{nk} A_k,
\]
where
\[
F_{nk} = \frac{1}{2n!} \left[ \frac{\Gamma \left( n + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \right] \left[ \frac{1}{k - n + \frac{1}{2}} \right].
\]

The zero order approximation gives:
\[
\lambda^{(N=0)}_{0,l} = a (2l + 3) - \sqrt{\frac{8a}{\pi}} z.
\]
Note that although this approximation for \( Z = 0 \) gives the exact result, its accuracy with the growth of \( Z \) is decreased. Thus for nonzero \( Z \) it is necessary to take higher approximations.

The limiting exactly solvable cases \( Z \to 0, a \to 0 \) are considered in Appendix.

4 Heavy quarkonia mass spectrum

In this section we apply the developed method of the solution of the Schrödinger equation to the calculation of the mass spectra of heavy quarkonia – mesons consisting from heavy quark and antiquark (\( \Upsilon(b\bar{b}), \psi(c\bar{c}) \)). The main assumption of the nonrelativistic quark model [1], which is widely used for the heavy hadron description, is the application of the nonrelativistic approximation. This approximation works better with the increase of the constituent quark mass. The calculations in the framework of relativistic quark models [6] indicate that nonrelativistic approximation gives a good description of the static properties of heavy quarkonia (such as mass spectrum, radius etc.), while for dynamical properties (such as decays) the account of relativistic corrections is considerably more important. In the nonrelativistic quark model quarkonium is described by the Schrödinger equation with local spherically symmetrical potential, which ensures quark confinement:

\[
-\frac{1}{2\mu} \Delta \Psi(r) + [V(r) - E] \Psi(r) = 0,
\]

(17)

where \( \mu = m_1 m_2 / (m_1 + m_2) \) is the reduced mass for the bound system consisting from the quark and atiquark with masses \( m_1 \) and \( m_2 \).

One of the most popular is the Cornell potential [7] which accounts for asymptotic freedom in QCD at small distances and the linear rise of the potential with the increasing distance between quark and antiquark:

\[
V(r) = -\frac{4}{3} \alpha_s r + Ar + B,
\]

(18)

where \( \alpha_s \) is a strong interaction constant. Sometimes the quadratic confinement is used:

\[
V(r) = -\frac{4}{3} \alpha_s \frac{1}{r} + Ar^2 + B.
\]

(19)

The equation (17) with potentials (18), (19) reduces to the radial equation (2) with the potential (4) after substitution:

\[
\lambda = 2\mu (E - B), \quad a^2 = 2\mu A, \quad Z = \frac{8}{3} \alpha_s \mu.
\]

According to our method, the solution of (17) is reduced to the solution of the systems of algebraic equations (11) for the Cornell potential (18) and (15) for potential (19). Bottomonium (\( \Upsilon(b\bar{b}) \)) and charmonium (\( \psi(c\bar{c}) \)) mass spectra, obtained by the final truncation of the matrices in (11) and (15) at the size \( 9 \times 9 \) \((N = 8)\) are given in comparison with experimental data [8] in Tables 1, 2. We use the standard notations for the level centers of gravity: \((n + 1)L\), where \( n \) is the radial quantum number.
Table 1: Bottomonium ($\Upsilon(b\bar{b})$) mass spectrum calculated for the sum of linear confining potential with the Coulomb potential and for the sum of oscillator potential with the Coulomb potential (for $N = 8$). The parameters of the Coulomb+linear potential are $A = 0.18$ GeV$^2$, $B = -0.29$ GeV, $\alpha_s = 0.39$, $m_b = 4.93$ GeV. The parameters of the Coulomb+oscillator potential are $A = 0.174$ GeV$^3$, $B = -0.05$ GeV, $\alpha_s = 0.345$, $m_b = 4.95$ GeV.

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* $^3S_1$ state.

We can see from Tables 1, 2 that by fitting the parameters of potentials (18) and (19) it is possible to get a good agreement with the low-lying level masses of charmonium and bottomonium for both potentials. Nevertheless, for higher radial excitations the discrepancy becomes more pronounced. Unfortunately, these excited levels are near the threshold of the open charm and bottom production, which makes the comparison with experimental data more complicated.

The important advantage of the developed method is the possibility to explicitly determine the dependence of the spectrum on the parameters of the interquark interaction potential. On Figs. 1-3 we plot the dependence of the binding energy (in GeV) on the quark mass $m_Q$ (in GeV) and the Coulomb potential parameter $\alpha_s$ for the Cornell potential (18). On Figs. 4-6 we present the similar dependences for the sum of the Coulomb and oscillator potentials (19).

5 Mass spectrum of the relativistic Schrödinger-like equation

There are several different equations for relativistic two-particle bound states [9]-[13]. Most of these relativistic equations are the nonlocal integro-differential equations. However, some of them can be rationalized and reduced to a local Schrödinger-like equation [14, 15], which in the center of mass frame can be written as follows:

$$\left(\frac{P^2}{2\mu_R(M)} + V(M,r)\right)\Psi(r) = E(M)\Psi(r), \quad (20)$$
Table 2: Charmonium ($\psi(c\bar{c})$) mass spectrum calculated for the sum of linear confining potential with the Coulomb potential and for the sum of oscillator potential with the Coulomb potential (for $N = 8$). The parameters of the Coulomb+linear potential are $A = 0.18$ GeV$^2$, $B = -0.29$ GeV, $\alpha_s = 0.47$, $m_c = 1.56$ GeV. The parameters of the Coulomb+oscillator potential are $A = 0.174$ GeV$^3$, $B = -0.05$ GeV, $\alpha_s = 0.345$, $m_b = 1.55$ GeV.

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where $\mathbf{p}^2 = -\nabla^2 = -\Delta$; $E(M)$ and $\mu_R(M)$ are some functions of the bound state mass $M$; potential $V(M, r)$ can also depend on $M$.

Equation (20) differs from the ordinary Schrödinger equation (17) by the dependence of the reduced mass $\mu_R$ and potential on energy eigenvalues, which are nontrivial functions of the bound state mass. In the case of heavy quarkonia the problem can be simplified by taking the nonrelativistic limit of potential in eq. (20) as the initial approximation. Solving the obtained equation one exactly accounts for relativistic kinematics. The relativistic dynamical effects can be taken into account by calculating corrections to the potential using perturbation theory.

The explicit dependence of the spectrum of the Schrödinger equation (17) on parameters obtained here also allows the calculation in the case when the reduced mass $\mu_R$ and energy eigenvalue $E$ depend in the complicated way on the bound state mass $M$. Thus it is possible to solve the relativistic equation of the type (20) with the Cornell potential (18). Substituting obtained eigenvalues in the initial equation it is possible to reconstruct the wave function and to take into account dynamical relativistic corrections using perturbation theory.

The quasipotential equation [10], which is widely used for the description of bound states in quantum field theory (for application to meson properties see [16]), can be rationalized and transformed to the local equation [15] of the type (20). This equation has the form:

\[
\left(\frac{b^2(M)}{2\mu_R} - \frac{\mathbf{p}^2}{2\mu_R}\right)\Psi_M(\mathbf{p}) = \int \frac{d^3q}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}; M)\Psi_M(\mathbf{q}),
\]

where the relativistic reduced mass

\[
\mu_R = \frac{M^4 - (m_1^2 - m_2^2)^2}{4M^3};
\]
and the square of relative momentum on the mass shell

\[ b^2(M) = \frac{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{4M^2}, \quad (23) \]

\[ V(p, q; M) \text{ is a quasipotential operator. Converting to the coordinate representation it is easy} \]
to see that quasipotential equation is of the type (20) with

\[ E(M) = \frac{b^2(M)}{2\mu_R} = \frac{M[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{2M[M^4 - (m_1^2 - m_2^2)^2]}. \quad (24) \]

The solution of quasipotential equation (21) with the Cornell potential can be found substituting relativistic reduced mass (22) and \( E \) from (24) in equation (11). Note that for fixed \( l \) \( \frac{l}{2} \) the number of determinant zeros, corresponding to the bound state masses, coincides with the matrix order. This is a nontrivial result\(^1\) showing that our approach gives the correct number of states even for the complicated mass dependence of energy and reduced mass. The calculated bottomonium and charmonium masses are given in Tables 3, 4 together with the results of numerical solution of the quasipotential equation with the account of spin-independent relativistic corrections of order \( v^2/c^2 \) [17] and in comparison with experimental data [8]. From these tables we see that our results for level centers of gravity, calculated using quasipotential equation, differ from experimental data by no more than 1% for bottomonium and 3% for charmonium. The account of spin-independent relativistic corrections of order \( v^2/c^2 \) [17] further improves the agreement with experiment.

Table 3: Bottomonium (\( \Upsilon(bb) \)) mass spectrum calculated with the quasipotential equation for the sum of linear confining potential with the Coulomb potential (for \( N = 8 \)). NR – without account of relativistic corrections to the quasipotential. SI – with the account of spin-independent relativistic corrections of order \( v^2/c^2 \) to the quasipotential. Potential parameters are \( A = 0.18 \text{ GeV}^2, B = -0.30 \text{ GeV}, \alpha_s = 0.26, m_b = 4.88 \text{ GeV} \).

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</tr>
<tr>
<td>1D</td>
<td>10.102</td>
<td>10.157</td>
<td></td>
</tr>
<tr>
<td>2D</td>
<td>10.386</td>
<td>10.448</td>
<td></td>
</tr>
</tbody>
</table>

\(^*\) = \(^3\)S\(_1\) state.

\(^1\)The resulting equation for bound state masses contains fractional powers.
Table 4: Charmonium ($\psi(c\bar{c})$) mass spectrum calculated with the quasipotential equation for the sum of linear confining potential with the Coulomb potential (for $N = 8$). NR – without account of relativistic corrections to the quasipotential. SI – with the account of spin-independent relativistic corrections of order $v^2/c^2$ to the quasipotential. Potential parameters are $A = 0.18$ GeV$^2$, $B = -0.30$ GeV, $\alpha_s = 0.32$, $m_b = 1.55$ GeV.

<table>
<thead>
<tr>
<th>State</th>
<th>NR</th>
<th>SI [17]</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1S$</td>
<td>3.155</td>
<td>3.065</td>
<td>3.0675</td>
</tr>
<tr>
<td>$2S$</td>
<td>3.718</td>
<td>3.669</td>
<td>3.663</td>
</tr>
<tr>
<td>$1P$</td>
<td>3.546</td>
<td>3.517</td>
<td>3.525</td>
</tr>
<tr>
<td>$1D$</td>
<td>3.833</td>
<td>3.808</td>
<td>3.770*</td>
</tr>
</tbody>
</table>

* $^3D_1$ state.

6 Conclusions

In this paper the approach reducing the solution of the Schrödinger equation for power potentials to the eigenvalue problem for infinite system of algebraic equations is proposed. Such potentials are widely used for heavy quarkonium mass spectrum calculations. The finite truncation (even for small $N$) of this system provides a high accuracy values for the low-lying levels. The explicit dependence of the determinant of the system on potential parameters allows easy determination of the solution spectrum dependence on these parameters. It is necessary to note that by substituting found eigenvalues in the initial equation it is possible to reconstruct the wave function, which has the number of zeros coinciding with the radial excitation number (this is in agreement with the known theorem on the wave function zero number).

We have investigated the particular cases of the studied potentials for which the Schrödinger equation has exact solutions. It is shown that for the Coulomb and oscillator potentials the developed method reproduces the exact spectrum. For the linear rising potential, in case $l = 0$ ($S$ states), our solutions rapidly converge to the Airy function zeros.

The mass spectra of charmonium and bottomonium for the Cornell potential (18) and the sum of the Coulomb and oscillator potentials (19) are calculated on the basis of the developed approach. The dependence of spectra on the quark mass $m_Q$ and strong interaction coupling constant $\alpha_s$ is investigated. It is shown that the proposed method allows to determine mass spectra of relativistic equations which can be presented in Schrödinger-like form. The mass spectra of charmonium and bottomonium for the Cornell potential are calculated on the basis of the relativistic quasipotential equation. The obtained results are in accord with experimental data.

Acknowledgments

We are grateful to A.I. Aptekarev, A.M. Badalyan, A.V. Borisov, V.Ch. Zhukovsky, N.N. Nekhoroshev, A.Yu. Simonov, V.N. Sorokin and E.T. Shavgulidze for useful discussions. The work of R.N.F. and V.O.G. was supported in part by Russian Foundation for Fundamental Research under Grant No. 96-02-17171.
A The limiting cases of the potential (1) for which the Schrödinger equation has exact solutions

In this appendix we use our method to calculate the spectrum of the Schrödinger equation for the limiting cases \( Z \to 0 \) or \( a \to 0 \) of the potential (1) and compare the obtained results with the exact ones.

A.1 The spectral problem for oscillator potential

The oscillator potential is the limiting case \( Z = 0 \) of \( U_2(r) \) potential. Setting \( \vec{z} = 0 \) in (15) we can find the exact solution of this equation, which reproduce the spectrum of the spherically symmetrical oscillator

\[
\lambda_n = (4n + 2l + 3)a = 2a \left( N_m + \frac{3}{2} \right),
\]

where \( N_m \) is the main quantum number.

A.2 The spectral problem for linear potential

The linear potential is the limiting case \( Z = 0 \) of \( U_1(r) \) potential. Setting also \( l = 0 \) (S states) in (11) we get the eigenvalues for the spherical linear potential \( U_{eff} = r \). As it is well known, these solutions should coincide with the zeros of the Airy function [18]. In Table 5 we present the numerical results for the lowest eigenvalues of eq. (11) with \( Z = 0 \), \( l = 0 \) versus the truncation index \( N \). We see that for \( N \) values greater than the radial quantum number \( n \) by 3 the calculated eigenvalues agree with the exact ones within 1% accuracy. This allows to conclude that the convergence rate of proposed procedure is rather high.

A.3 The spectral problem for the Coulomb potential

The Coulomb potential is a limiting case \( a = 0 \) of potential (1). It is necessary to note that the Coulomb potential in Schrödinger equation with potential (1) does not give the leading asymptotics for \( r \to \infty \), which is determined by the confining potential. As a result, we cannot get the solution for the Coulomb problem just setting \( a = 0 \) in eq. (11) or eq. (15). Therefore, an additional analysis is necessary. The effective potential in the equation for the radial part of the wave function (2) consists in this case of two terms of different sign:

\[
U_{eff} = \frac{l(l + 1)}{r^2} - \frac{Z}{r}.
\]

The bound states in the potential (A.1) have a negative eigenvalues. Taking this into account, we make in eq. (5) the substitution \( r \to r/\sqrt{\vert \lambda \vert} \), which makes this equation dimensionless, and introduce new spectral parameter \( \tilde{\lambda} = Z/\sqrt{\vert \lambda \vert} \). After these transformations we get:

\[
F'' + \frac{2}{r}(l + 1)F' + \left(-1 + \frac{\tilde{\lambda}}{r}\right)F = 0.
\]
We seek a solution of eq. (A.2) in the form of a series in the powers of $r$:

$$F = \sum_{n=0}^{\infty} \frac{A_n}{(n+2l+1)!} r^n.$$ 

Substituting this decomposition in eq. (A.2) we find the recursion relationship:

$$(n+1)A_{n+1} - (n+2l+1)A_{n-1} = -\bar{\lambda} A_n. \tag{A.3}$$

We have made the asymptotic symmetrization of the Jacobi matrix in eq. (A.3). Introducing the generating function as $\Phi(w) = \sum_n A_n w^n$, we get the following equation for this function:

$$\left(1 - w^2\right) \Phi' + \left(\bar{\lambda} - 2(l+1)w\right) \Phi = 0. \tag{A.4}$$

The exact solution of eq. (A.4) is

$$\Phi(w) = (1 - w)^{\frac{5}{2} - l - 1}(1 + w)^{-\frac{5}{2} - l - 1}.$$

As it follows from general theory, the behaviour of the generating function in the point $w = 1$ for given $\bar{\lambda}$ determines the asymptotics of the initial problem wave function. Thus only for regular function $\Phi(w)$ in this point, the wave functions are decreasing with the increase of $r$, and hence, they can be normalized in the semi-infinite region. For this purpose the exponent should be equal to the integer number $n$. This condition determines the eigenvalue spectrum:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2.17747</td>
</tr>
<tr>
<td>2</td>
<td>2.33762</td>
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<td>3</td>
<td>2.33298</td>
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<td>5</td>
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<td>8</td>
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</tr>
<tr>
<td>9</td>
<td>2.3730</td>
</tr>
<tr>
<td>10</td>
<td>2.3810</td>
</tr>
</tbody>
</table>

Table 5: The comparison of the $S$ state solutions of eq. (11) for the potential $V(r) = r$ with the exact ones determined by the zeros of the Airy ($Ai$) function. $n$ is a radial quantum number, $(N + 1)$ is the truncation index of matrices $\hat{F}$ and $\hat{G}$ in eq. (11).
\[ \lambda = 2(n + l + 1). \] Substituting the explicit expression for the spectral parameter \( \lambda \) and taking into account the negative sign of the binding energy, it is easy to obtain the well-known Bohr formula for the energy of a particle in the Coulomb field:

\[ \lambda_N = -\left( \frac{Z}{2N_m} \right)^2, \quad N_m = n + l + 1. \] (A.5)

Here \( N_m \) is the main quantum number.

In this appendix we have shown that our method of the Schrödinger equation solution reproduces the exact results for the Coulomb and oscillator potentials. For \( S \) states in linear potential our solutions rapidly converge to the zeros of the Airy function.

References


Figure Captions

**Fig. 1** The binding energy as a function of the quark mass and the strong coupling constant $\alpha_s$ for the Cornell potential. The parameters of the potential are $A = 0.18 \text{ GeV}^2$, $B = -0.29 \text{ GeV}$. Grey patches correspond to $S$ states, patches with grid correspond to $P$ states.

**Fig. 2** The binding energy as a function of the strong coupling constant $\alpha_s$ for fixed quark mass ($m_Q = 4.9 \text{ GeV}$) for the Cornell potential. $S$ states are plotted by solid lines, $P$ states – by crosses, $D$ states – by diamonds.

**Fig. 3** The binding energy as a function of the quark mass for fixed strong coupling constant ($\alpha_s = 0.39$) for the Cornell potential. $S$ states are plotted by solid lines, $P$ states – by crosses, $D$ states – by diamonds.

**Fig. 4** The binding energy as a function of the quark mass and the strong coupling constant $\alpha_s$ for the sum of the Coulomb and oscillator potentials. The parameters of the potential are $A = 0.174 \text{ GeV}^2$, $B = -0.05 \text{ GeV}$. Grey patches correspond to $S$ states, patches with grid correspond to $P$ states.

**Fig. 5** The binding energy as a function of the strong coupling constant $\alpha_s$ for fixed quark mass ($m_Q = 4.9 \text{ GeV}$) for the sum of the Coulomb and oscillator potentials. $S$ states are plotted by solid lines, $P$ states – by crosses, $D$ states – by diamonds.

**Fig. 6** The binding energy as a function of the quark mass for fixed strong coupling constant ($\alpha_s = 0.39$) for the sum of the Coulomb and oscillator potentials. $S$ states are plotted by solid lines, $P$ states – by crosses, $D$ states – by diamonds.