cc

Augmented Temporal Logic Formalism for
Histories-based Generalized Quantum Mechanics

Tulsi Dass^a and Yogesh Joglekar^b

Department of Physics
Indian Institute of Technology, Kanpur, India 208016

22y722

Abstract

An axiomatic development of dynamics of systems in the framework of histories is given which contains the history versions of classical and traditional quantum mechanics as special cases. We consider theories which admit a quasitemporal structure - a generalization of the concept of time using partial semigroups - and whose 'single time' propositions have the mathematical structure of a logic; isomorphism of logics at different 'instants of time' is not assumed. The concept of directed partial semigroup is introduced to incorporate the concept of direction of flow of time in quasitemporal theories. Starting with a few simple axioms, the space of history propositions is explicitly constructed and shown to be an orthoalgebra (as envisaged in the scheme of Isham and Linden); its subspace consisting of history filters ('homogeneous histories') is a meet semilattice. The partial semigroups employed allow semi-infinite irreducible decompositions. Symmetries are defined in this framework and their relationship with traditional formulations discussed. In a subclass of theories, conservation laws are defined and (in a smaller subclass) a Noether type theorem (implying a connection between continuous symmetries of dynamics and conservation laws) is proved. The formalism is applied to (classical and quantum) field theories on general Lorentzian space-times; it is shown that a symmetry, as defined here, implies a conformal isometry of the space-time metric.

I. INTRODUCTION

In the traditional formulation of quantum mechanics, all measurements/observations refer to external (classical) observers; such a formalism obviously cannot serve as a basis for quantum cosmology. The consistent histories approach to quantum mechanics¹⁻⁷ is an outcome of efforts to evolve a quantum mechanical formalism applicable to closed systems, in particular, the Universe.

In traditional formulations, a history is a temporal sequence of events relating to a system where an event means a single time proposition about the system - some observable (or a set of simultaneously measurable observables) taking values in a specified range. (This need not refer to an actual measurement on the system by an external observer.) The term 'consistent' (or decoherent) refers to families of alternative histories among which quantum mechanical interference is absent and (once the probability of a history is suitably defined) classical additivity of probabilities of histories in the family holds.

In traditional quantum mechanics, a formula can be obtained for a temporal sequence of measurements [an actual realization of a history; see eq. (2.5) below]. In the interpretive scheme of Griffiths¹ and Omnes^{2,3} the Hilbert space-based mathematical development of traditional quantum mechanics is retained; however, eq. (2.5) is proposed to be adopted (as an interpretive postulate) as the probability of the relevant history (which is supposedly a member of an appropriate family of consistent histories) for a closed system. Measurement situations are

proposed to be analyzed in this formalism by considering the closed system consisting of the system and the observer.

Gell-Mann and Hartle⁴⁻⁷ (GH) have evolved an axiomatic scheme constituting a generalization of quantum mechanics in which fine-grained and coarse-grained histories are the basic objects (there being no reference to an underlying Hilbert space). An essential ingredient in their scheme is the concept of decoherence functional which associates with a pair of histories a complex number which is a measure of the quantum mechanical interference between the two histories; this functional serves to define the conditions for a family of histories to be decoherent and also the probabilities of individual histories in a decoherent set. History versions of Hilbert space-based or path integral-based quantum mechanics and of classical mechanics appear as special cases of this generalized quantum mechanics.

Isham⁸ (see also Isham and Linden⁹) has proposed that the proper mathematical framework for the axiomatic development of a GH type scheme is that of *temporal quantum logic* - a multitime generalization of traditional quantum logic¹⁰⁻¹⁴. In his scheme a partial semigroup (for definition, see section II F) structure was introduced in a pair of spaces [a space $\mathcal{T}$ of temporal supports (replacing the space of time sequences $t_1, \dots, t_n$ employed in the traditional description of histories) and a space $\mathcal{U}$ of history filters (homogeneous histories); our notations are slightly different from those of ref. 8, 9] to generalize the concept of time and to describe the composition of histories in

appropriate temporal relation to form larger histories. A homomorphism σ of $\mathcal{U}$ into $\mathcal{T}$ associates, with an element α of $\mathcal{U}$, the element $\sigma(\alpha)$ of $\mathcal{T}$ representing the sequence of 'times' at which the propositions of the history α are specified. The space $\mathcal{U}$ was assumed to have, besides the partial semigroup structure, the structure of a meet semilattice with the operations of partial order $\leq$ (coarse-graining) and a meet operation $\wedge$ (simultaneous realization of two histories). To accommodate the operation of negation of a history, the space $\mathcal{U}$ was proposed to be embedded in a larger space Ω (denoted as $\mathcal{UP}$ in ref. 8, 9) of history propositions ('inhomogeneous' histories), the structure of an orthocomplemented lattice was envisaged (but not derived) for Ω . The decoherence functionals (objects associating complex numbers with pairs of elements of Ω) were assumed to have the same properties as in the GH scheme. A formalism of this type has the generality required to accommodate in it history versions of field theories on general space-times (which may not admit foliation in terms of spacelike surfaces).

In ref. (9), a more general scheme was proposed in which the basic objects were the space Ω [having the structure of an orthoalgebra (see section IIG for definition) with operations of partial order $\leq$, disjointness $\perp$, disjoint join operation $\oplus$ and negation $\neg$] and the space $\mathcal{D}$ of decoherence functionals. History theories with quasitemporal structure (i.e. those covered by an Isham type scheme) constitute a subclass of theories covered by this more general scheme.

While histories, in principle, contain information about the structural features and dynamics of a system, the Isham-Linden formalism is not adequately equipped to bring it out in an autonomous framework. In the present work we present a more detailed formalism for quasitemporal theories with a view to fulfil this need. Our framework is defined by three axioms A_1 - A_3 . The first axiom A_1 introduces a quasitemporal structure in terms of a triple $(U, \mathcal{T}, \sigma)$ as in the Isham scheme. The second is causality axiom which forbids 'closed time loops'. Our main new axiom is A_3 which relates to the spaces $u_\tau = \sigma^{-1}(\tau)$ for nuclear elements in $\mathcal{T}$ ('points of time') - the spaces of 'single time propositions'. We assume that these spaces have the structure of a logic as defined in ref. (13) - a natural assumption to make in view of the fact that this structure was arrived at (in the sixties) by very general considerations. (In any case, it is worth studying history theories with this structure assumed for the u_τ 's and generalizing the concept of time as described above.) We, however, do not assume isomorphism of u_τ 's for different τ 's. This generality, we feel, must be kept in order that the formalism is applicable to systems whose structural properties undergo qualitative change with temporal evolution. Using these three axioms, the space Ω of history propositions ('inhomogeneous histories') is explicitly constructed and shown to be an orthoalgebra as envisaged in the Isham-Linden scheme.

An important subclass of the theories covered by these three axioms is the one in which $u_\tau = \mathcal{P}(\mathcal{H}_\tau)$, the lattice of

projection operators in a separable Hilbert space $\mathcal{H}_\tau$, the Hilbert spaces $\mathcal{H}_\tau$ for different τ 's being not necessarily naturally isomorphic. We shall refer to these theories as Hilbert space-based theories. Traditional quantum mechanics (in history-version) obviously belongs to this class of theories.

The space $\mathcal{D}$ of decoherence functionals in both the GH and Isham-Linden schemes is not properly integrated with the basic framework; it is, for example, not made clear as to what it is that distinguishes different elements of $\mathcal{D}$ and when is a particular element of $\mathcal{D}$ relevant. In traditional quantum mechanics, a decoherence functional is determined by the evolution operator and the initial state [see eq. (2.9)]. In the Isham-Linden formalism there is no explicitly defined concept of state or of evolution; information about both of these is contained in the decoherence functionals. It is, however, not clear how to bring out and use this information.

In our formalism, the $\mathcal{U}_\tau$'s being logics, both the concepts can be introduced in the conventional manner. Our axiom A_4 introduces temporal evolution in terms of an appropriate class of mappings between the $\mathcal{U}_\tau$'s. The last axiom A_5 associates a decoherence functional with a given law of temporal evolution and a given initial state. The formalism would be complete if an explicit expression for the decoherence functional (analogous to that in eq. (2.9) in the case of traditional quantum mechanics) is given. We are able to do it only for the Hilbert space-based theories. The decoherence functional proposed for these theories [eq. (3.83)] is an adaptation from eq. (4.19) of ref. (9); it is,

expectedly, a straight-forward generalization of the expression in eq. (2.9) for traditional quantum mechanics.

An innovation introduced in the present work is that, in our axiomatic scheme, a general history is allowed to have countably semi-infinite temporal support; this means that (homogeneous) histories are taken to involve single 'time' propositions at 'times' $\tau_1, \tau_2, \dots$ or $\dots, \tau_{-1}, \tau_0$. Similarly, an inhomogeneous history is taken to include a countably infinity of mutually exclusive homogeneous histories. This, we believe, should be helpful in addressing questions relating to, for example, the long time behaviour of the Universe (under suitable hypotheses) besides facilitating analytical work in the present setting.

After developing the basic formalism along the lines indicated above, we have presented a systematic treatment of symmetries in this formalism. We define a symmetry as a triple $\underline{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ of bijective mappings ($\Phi_1 : \mathcal{T} \rightarrow \mathcal{T}$, $\Phi_2 : \mathcal{U} \rightarrow \mathcal{U}$ and $\Phi_3 : \mathcal{D} \rightarrow \mathcal{D}$) preserving the quasitemporal structure, decoherence condition and probabilities of histories (see section IV C for details). In the case of (history versions of) traditional quantum mechanics and classical mechanics, a two-way connection is established between symmetries defined here and symmetries of the formalism of the traditional sort - Wigner symmetries in quantum mechanics and bijective Borel measurable maps of phase space in classical mechanics (the latter are the natural structure preserving maps in the treatment of classical mechanics in the framework of logics).

The implementation of Lorentz transformations as structure

preserving maps in the present formalism appears somewhat problematic because Lorentz transformations involve mixing of space-time co-ordinates whereas descriptions of histories involve sequences of times. This difficulty is overcome by noting that the t_i 's appearing in a typical sequence $(t_1, \dots, t_n)$ of times appearing in the description of a history, are, in operational terms, time intervals from a reference event.

It is of some interest to treat conservation laws in the framework of histories. As might be intuitively anticipated, conservation laws can be defined only if the spaces of single 'time' propositions (the u_τ 's mentioned above) at various instants of 'time' are mutually isomorphic. We first give a straightforward definition of conservation law in terms of temporal evolution described by mappings between pairs of u_τ 's. It is then translated into a (supposedly equivalent) definition in terms of decoherence functionals. The equivalence of the two definitions is verified in history versions of traditional quantum mechanics and classical mechanics.

In any scheme of dynamics, one expects a general connection between continuous symmetries and conservation laws. We prove, in a subclass of theories admitting conservation laws (the Hilbert space based theories with general quasitemporal structure) a Noether type theorem showing that a continuous symmetry of dynamics implies a conservation law.

One of the motivations for the development of quasitemporal theories by Isham⁸ was to have a formalism applicable to field theories in a general space-time (which may not admit foliation

in spacelike hypersurfaces). We have applied our formalism to the treatment of these theories concentrating mainly on quasitemporal structure and symmetries. In the treatment of quasitemporal structure, we have adopted definitions and conventions relating to basic objects employed in the formalism ('basic regions', 'temporal order', 'temporal support', 'nuclear temporal support' etc.) somewhat different from the previous workers^{8,5,10} so as to be consistent with conventions adopted by us in the rest of the paper. In the treatment of symmetries, an interesting result obtained is that, the mapping Φ_1 appearing in the triple of maps defining a symmetry of a field theory on a general space-time (M,g) induces a bijective mapping of M onto itself which is a conformal isometry of the metric g .

The rest of the paper is organized as follows. To put some developments in proper perspective and to have some results for ready reference for later use, we have devoted the next section to the development of histories approach to quantum theory. We have taken this opportunity to restore some justice to the work of Houtappel, Van Dam and Wigner¹⁵ (which, unfortunately, is almost never mentioned in the literature on the histories approach) in which objects analogous to histories (with, however, explicit reference to observers making measurements) were employed for a general treatment of geometric invariance principles (space-time symmetries) in classical and quantum theory and to correct a couple of mutually compensating mistakes in section II A of ref. (8). Our treatment of partial semigroups in section II F includes some improvements on ref. (8), some

novelties and a few instructive examples. In section III we present an axiomatic development of our formalism. In section IV symmetries are treated in this framework and section V is devoted to fields in general space times.

As should be clear from sections IV and V, the formalism of this paper (section III) has potential for useful applications. Keeping this in view, this paper has been written in a fairly easy style so as to be within the grasp of a reasonably wide class of readership. The only background assumed in most of the paper is familiarity with some basic notions in Hilbert space theory and definition of partial order. Section II is written so as to serve the twin purposes of an introductory review and a stepping stone for section III.

Most of the present work was done in a two-semester theoretical project undertaken by the second-named author as an M.Sc. student at I.I.T. Kanpur with the first-named author as supervisor.

II. HISTORIES APPROACH TO QUANTUM THEORY

A. The Π functions of Houtappel, Van Dam and Wigner

The first paper to employ objects analogous to what are now called histories was probably the one by Houtappel, Van Dam and Wigner¹⁵ (HVW) who gave a formulation for geometric invariance principles (i.e. those invariance principles which correspond to transformations between equivalent reference frames) in terms of observations or measurements and their results. This was done by introducing the conditional probabilities $\Pi(B/A)$ where $A = (\alpha, r_\alpha; \beta, r_\beta; \dots; \epsilon, r_\epsilon)$ represents a set of measurements $\alpha, \beta, \dots, \epsilon$ (at times $t_\alpha, t_\beta, \dots, t_\epsilon$) with outcomes $r_\alpha, \dots, r_\epsilon$ and similarly $B = (\zeta, r_\zeta; \eta, r_\eta; \dots; \nu, r_\nu)$; the quantity $\Pi(B/A)$ represents the probability of the realization of the sequence B given A. [We have changed HVW's notation $\Pi(A/B)$ to $\Pi(B/A)$ to bring it in correspondence with standard usage in probability theory]. The ordering of the times $t_\alpha, \dots, t_\epsilon, t_\zeta, \dots, t_\nu$ is arbitrary. The objects $\Pi(B/A)$ are quite general and can be employed in classical as well as quantum mechanics. Discussion of their general properties and their expression in traditional quantum mechanics [eq. (4.7) of ref. (15)] may be found in ref. (15).

We shall have more to say about the HVW formalism in our discussion of symmetries in section IV.

B. Histories in Traditional Quantum Theory

The objects presently used in histories based (generalized) quantum theory are closely related to the Π functions mentioned above. By a history we typically mean a sequence of measurements/

records for a system in given initial state [See eq. (2.5) below.] We give below an outline of the calculation of the probability of such a history in standard quantum theory.

Let the state of a system S be given at time t_0 by a density operator $\rho(t_0)$. If a measurement represented by a projection operator P_1 is performed at time $t_1 > t_0$, the probability of a positive outcome is

$$\text{Prob}(P_1, t_1 | \rho(t_0)) = \text{Tr}(P_1 \rho(t_1)) = \text{Tr}[P_1(t_1) \rho(t_0)]$$

where (defining, in the Heisenberg picture, $P_1(t_0) = P_1$)

$$\rho(t_1) = U(t_1, t_0) \rho(t_0) U(t_1, t_0)^\dagger; \quad P_1(t_1) = U(t_1, t_0)^\dagger P_1(t_0) U(t_1, t_0) \quad (2.2)$$

and $U(.,.)$ is the time evolution operator. By reduction postulate, immediately after measurement, the state is represented by the reduced density operator

$$\rho_{\text{red}}(t_1) = \frac{P_1 \rho(t_1) P_1}{\text{Tr}[P_1 \rho(t_1)]} = \frac{P_1 \rho(t_1) P_1}{\text{Tr}[P_1(t_1) \rho(t_0)]} \quad (2.3)$$

Now, if another measurement P_2 is made at time $t_2 > t_1$, the conditional probability of positive outcome, given the state $\rho_{\text{red}}(t_1)$, is

$$\begin{aligned} \text{Prob}(P_2, t_2 | \rho_{\text{red}}(t_1)) &= \text{Tr}(P_2 \rho_{\text{red}}(t_2)) \\ &= [\text{Tr} P_1(t_1) | \rho(t_0)]^{-1} \text{Tr}[P_2 U(t_2, t_1) P_1 U(t_1, t_0) \rho(t_0) \\ &\quad \cdot U(t_1, t_0)^\dagger P_1 U(t_2, t_1)^\dagger]. \end{aligned}$$

The joint probability of positive outcomes for (P_1, t_1) and (P_2, t_2) [given the state $\rho(t_0)$] is

$$\text{Prob}(P_1, t_1; P_2, t_2 | \rho(t_0))$$

$$= \text{Prob}(P_2, t_2 | \rho_{\text{red}}(t_1)) \cdot \text{Prob}(P_1, t_1 | \rho(t_0))$$

$$= \text{Tr}[P_2 U(t_2, t_1) P_1 U(t_1, t_0) \rho(t_0) U(t_1, t_0)^\dagger P_1 U(t_2, t_1)^\dagger P_2].$$

Proceeding as above, the joint probability for outcomes for the sequence of measurements

$$(P_1, t_1; P_2, t_2; \dots; P_n, t_n), \quad t_1 < t_2 < \dots < t_n \quad (2.4)$$

is

$$\text{Prob}(P_1, t_1; \dots; P_n, t_n | \rho(t_0)) = \text{Tr}[C_\alpha \rho(t_0) C_\alpha^\dagger] \quad (2.5)$$

where α denotes the ~~sequence~~ ^{history} (2.4) and

$$C_\alpha = P_n U(t_n, t_{n-1}) \dots P_2 U(t_2, t_1) P_1 U(t_1, t_0). \quad (2.6)$$

It should be noted that the formula (2.5) refers to outcomes of measurements on the system by an external observer.

C. Consistent History Interpretation of Quantum Mechanics

Griffiths¹ proposed a modified version of quantum mechanics in which the Hilbert space based mathematical formalism is retained, the reduction postulate is discarded and (2.5) is proposed to be interpreted as the probability for the history (2.4) for the closed system S (no external observers) with the further stipulation that situations involving measurements by external observers are to be dealt with by treating the closed system (system + observer) as above. He showed that this formalism retains all the predictions of traditional quantum mechanics and avoids some conceptual difficulties relating to interpretation of the formalism.

Some restrictions need to be imposed on admissible histories if the formula (2.5) is to be used as a classical probability. This is because the classical addition of probabilities cannot hold in the presence of quantum mechanical interference. Consider, for example, the history β obtained from the history α of eq. (2.4) by replacing the projection operator P_j for some fixed j by another projection operator Q_j and leaving everything else unchanged. If P_j and Q_j are mutually orthogonal (i.e. $P_j Q_j = Q_j P_j = 0$), the two histories α and β represent mutually exclusive alternatives and a history γ representing the history ' α or β ' can be defined; it is obtained from α by replacing P_j by $P_j + Q_j$ leaving the rest unchanged. Note that

$$C_\gamma = C_\alpha + C_\beta. \quad (2.7)$$

We have [using the short notation $\text{Prob}(\alpha)$ for the quantity in eq. (2.5)]

$$\begin{aligned} \text{Prob.}(\gamma) &= \text{Tr}(C_\gamma \rho(t_0) C_\gamma^\dagger) \\ &= \text{Prob}(\alpha) + \text{Prob}(\beta) + d(\alpha, \beta) + d(\beta, \alpha) \end{aligned} \quad (2.8)$$

where we have put

$$d(\alpha, \beta) = \text{Tr}(C_\alpha \rho(t_0) C_\beta^\dagger); \quad (2.9)$$

it is a complex number (to be called 'decoherence functional') satisfying the conditions

$$(i) \quad d(\alpha, \beta) = d(\beta, \alpha)^* \quad (\text{hermiticity}) \quad (2.10)$$

$$(ii) \quad d(\alpha, \alpha) \geq 0 \quad (\text{positivity}). \quad (2.11)$$

Note that

$$\text{Prob}(\alpha) = d(\alpha, \alpha). \quad (2.12)$$

To have the additivity of probabilities in eq. (2.8), we must have

$$\operatorname{Re} [d(\alpha, \beta)] = 0. \quad (2.13)$$

Conditions of the form (2.13) are referred to as 'decoherence conditions' or 'consistency conditions'. Histories α, β satisfying (2.13) are said to (weakly) decohere.

$\mathcal{C}$
curly C) A family $\mathcal{C} = \{\alpha^{(i)}, i \in J\}$ of mutually exclusive histories such that one of the $\alpha^{(i)}$'s is sure to be realized is said to be 'complete'. (Here J is some index set.) Such a complete family is said to (weakly) 'decohere' if

$$\operatorname{Re} d(\alpha^{(i)}, \alpha^{(j)}) = 0 \quad i \neq j. \quad (2.14)$$

For each member of such a decoherent set, we have

$$\operatorname{Prob}(\alpha^{(i)}) = d(\alpha^{(i)}, \alpha^{(i)}). \quad (2.15)$$

The classical probability addition holds :

$$\sum_i \operatorname{Prob}(\alpha^{(i)}) = 1. \quad (2.16)$$

In the consistent histories approach, probability assignments are supposed to be made only to such decoherent or consistent families of histories.

Note that eqs. (2.14), (2.15) and (2.16) give

$$\sum_i \sum_j d(\alpha^{(i)}, \alpha^{(j)}) = 1. \quad (\text{normalization}) \quad (2.17)$$

$\mathcal{C}$
curly C) A subcollection in $\mathcal{C}$ can be jointly considered as a coarse-grained history [in the same sense as γ above is a

coarse-graining of α and β - it provides less precise information than either of α or β individually does]. Given any two subcollections

$$\underline{\alpha} = \{\alpha^{(i)}, i \in J_1\}, \quad \underline{\alpha}' = \{\alpha^{(j)}, j \in J_2\}$$

$\mathcal{C}$
(curly C)

(where J_1 and J_2 are subsets of J) in $\mathcal{C}$, we define [see eqs (2.9) and (2.7)]

$$d(\underline{\alpha}, \underline{\alpha}') = \sum_{i \in J_1} \sum_{j \in J_2} d(\alpha^{(i)}, \alpha^{(j)}) \quad (\text{additivity}). \quad (2.18)$$

Omnes^{2,3} refined and extended Griffiths' history formalism putting the whole development in the framework of formal logic. Introducing an interpretive rule which essentially says that all logical implications following from the system being in a given initial state are contained in the probabilities of consistent histories with that initial state, he gave a logical reformulation of quantum mechanics which permits a reasonably satisfactory understanding of the approximation of quantum logic by classical logic in the macroscopic domain and analysis of experiments entirely in quantum theoretic framework. Wave function collapse and the statement 'possible results of measurement of an observable are its eigenvalues' are derived rather than postulated. A treatment of EPR experiment¹⁶ based on consistent histories was given in ref. (1b) and (2b) showing there is no paradox involved.

D. The Gell-Mann and Hartle Scheme

Gell-Mann and Hartle⁴⁻⁷ (GH) have pursued a program for evolving a framework for the quantum dynamics of the Universe as

a whole. Within that framework they aim at understanding the origin of the 'quasiclassical domain' of familiar experience taking as basic inputs the action of the elementary particles and the initial condition of the Universe. In the 'generalized quantum mechanics' (of a closed system) evolved by them, the basic ingredients are a set of fine-grained histories [usually taken to be the paths $q^i(t)$ in configuration space], allowed coarse-grainings and a space of decoherence functionals $d(\alpha, \beta)$ (complex valued functionals defined on pairs of histories in coarse-grained sets) for which the four conditions of hermiticity (eq. 2.10), positivity (eq. 2.11), normalization (eq. 2.17) and additivity (2.18) are *postulated*. Probabilities are assigned only to sets of alternative coarse-grained histories satisfying a decoherence condition, the weakest version of which is eq. (2.13). [In several applications, they employ the stronger condition $d(\alpha, \beta) = 0$ for $\alpha \neq \beta$.] The probabilities in a decoherent set of histories are given by eq. (2.12).

In most of the concrete applications^{6,7}, their calculation of decoherence functionals is based on suitably augmented versions of the path integral formalism.

E. An HPO (History Projection Operator) Version of Standard Quantum Theory

A useful result obtained by Isham⁸ was the demonstration that history propositions in standard quantum theory can be realized as genuine projection operators on a certain Hilbert space. As some of our constructions in the next section are

motivated by constructions in this formalism, we present here an outline of this development.

Taking the view that histories of a system are (multitime) propositions about the system, it is desirable to have them represented, if possible, by projection operators in a suitable Hilbert space. To construct this space, consider the Cartesian product of the (naturally isomorphic) copies $\mathcal{H}_t$ ($t \in \mathbb{R}$) of the quantum mechanical Hilbert space $\mathcal{H}$ of the system :

$$V = \prod_{t \in \mathbb{R}} \mathcal{H}_t. \quad (2.19)$$

An element v of V is a function $v : \mathbb{R} \rightarrow \mathcal{H}$ taking $t \in \mathbb{R}$ to $v_t \in \mathcal{H}_t \equiv \mathcal{H}$. Let w be a fixed vector in V whose 'component vectors' w_t are unit vectors : $\|w_t\|_{\mathcal{H}_t} = 1$. Let $\mathcal{F}$ be the subspace of V consisting of vectors v such that $v_t = w_t$ for all but a finite set of t -values. The scalar product $(,)$ on $\mathcal{H}$ induces the following scalar product on $\mathcal{F}$:

$$(v', v)_{\mathcal{F}} = \prod_{t \in \mathbb{R}} (v'_t, v_t). \quad (2.20)$$

The completion $\tilde{\mathcal{H}}$ of this inner product space is the desired Hilbert space; it is the infinite tensor product of the $\mathcal{H}_t$'s denoted as

$$\tilde{\mathcal{H}} = \bigotimes_{t \in \mathbb{R}}^W \mathcal{H}_t. \quad (2.21)$$

General linear operators on $\tilde{\mathcal{H}}$ are linear combinations of operators of the form $A = A_{t_1} \otimes \dots \otimes A_{t_n}$ where A_{t_i} is a linear operator on $\mathcal{H}_{t_i}$ (it should be considered as a tensor product $\bigotimes_t A_t$ of operators such that $A_t = I_t$, the identity operator on $\mathcal{H}_t$ for all but a finite number of t 's) or limits of such linear

combinations. Given a set of projection operators $(\alpha_{t_1}, \dots, \alpha_{t_n})$ constituting a history α , the operator

$$\bar{\alpha} = \alpha_{t_1} \otimes \dots \otimes \alpha_{t_n} \quad (2.22)$$

is easily seen to be a projection operator in $\tilde{\mathcal{H}}$; such projection operators are referred to as homogeneous projection operators. Given a set $\alpha^{(1)}, \dots, \alpha^{(k)}$ of homogeneous projection operators in $\tilde{\mathcal{H}}$ which are mutually orthogonal (i.e. $\alpha^{(i)} \alpha^{(j)} = 0$ for $i \neq j$), the sum $\alpha^{(1)} + \dots + \alpha^{(k)}$ is also an (inhomogeneous) projection operator. In the HPO formalism histories are represented by such projection operators in $\tilde{\mathcal{H}}$. One refers to histories as homogeneous or inhomogeneous depending on whether they are represented by homogeneous or inhomogeneous projection operators. If $\bar{\alpha}$ and $\bar{\beta}$ are mutually orthogonal homogeneous projection operators representing the histories (more precisely, history propositions) α and β respectively, then the projection operator $\bar{\alpha} + \bar{\beta}$ represents the history proposition which means that either α or β is realized.

Inclusion of inhomogeneous histories facilitates ^{at} the introduction of the logical operation of negation of a history proposition. Given α , $\bar{\alpha}$ as above, the projection operator representing the negation of the history α is the inhomogeneous projection operator $\tilde{I} - \bar{\alpha}$ where $\tilde{I}$ is the identity operator on $\tilde{\mathcal{H}}$.

It is useful to note that, if α_{t_1} and $\alpha'_{t'_1}$ are mutually orthogonal projection operators in $\mathcal{H}_{t_1}$ and $\beta_{t_2}, \dots, \beta_{t_m}$ (where $t_1 < t_2 < \dots < t_m$) and $\gamma_{t'_2}, \dots, \gamma_{t'_n}$ (where $t_1 < t'_2 < \dots < t'_n$) are arbitrary projection operators on appropriate $\mathcal{H}_t$'s, the objects λ_1 and λ_2 given by

$$\lambda_1 = \alpha_{t_1} \otimes \beta_{t_2} \otimes \dots \otimes \beta_{t_m}, \quad \lambda_2 = \alpha'_{t_1} \otimes \gamma_{t_2} \otimes \dots \otimes \gamma_{t_n} \quad (2.23)$$

are also projection operators in $\tilde{\mathcal{H}}$ which are mutually orthogonal and their sum is a projection operator.

F. Partial Semigroups

Note that, in the developments presented above, the time sequences $(t_1, \dots, t_n)$ were employed only for book keeping; the properties of time t as a real variable were not used. It was Isham's⁸ observation that the relevant mathematical structure employed in the book keeping of histories is that of a partial semigroup. He then used this idea to propose a generalization of the concept of time which will be discussed in the next subsection.

We briefly recall a few definitions. A *semigroup* is a set with an associative composition rule (product) defined on it. A *partial semigroup* (psg) is a set on which a similar composition rule is defined but not necessarily for all pairs of elements. (Such an operation is called a partial operation.) Denoting the composition in a psg K (with elements $s, t, u, \dots$) by $\circ$ we have

$$(s \circ t) \circ u = s \circ (t \circ u) \quad (2.24)$$

whenever the two sides are defined.

A *homomorphism* of a psg K into another psg K' is a mapping $\sigma : K \rightarrow K'$ such that, for all $s, t \in K$ with $s \circ t$ defined, we have $\sigma(s) \circ \sigma(t)$ defined and

$$\sigma(s \circ t) = \sigma(s) \circ \sigma(t). \quad (2.25)$$

We give below a few instructive examples of psg's. It will be useful to have these examples in mind while following the developments in the remaining paper.

(i) $K_1 = \{\text{finite ordered subsets of } R, \text{ the real line}\}.$

A typical element of K_1 is

$$t = \{t_1, t_2, \dots, t_n\}, \quad t_1 < t_2 < \dots < t_n. \quad (2.26)$$

[Recall the time sequences employed above in the treatment of histories; see eq. (2.4).]

If $s = \{s_1, s_2, \dots, s_m\} \in K_1$ such that $s_m < t_1$, we have $s \circ t$ defined and

$$s \circ t = \{s_1, s_2, \dots, s_m, t_1, \dots, t_n\} = s \cup t. \quad (2.27)$$

It is useful to adopt the convention

$$\{t_1\} \circ \{t_1\} = \{t_1\}. \quad (2.28)$$

With this convention we have $s \circ t$ defined for $s_m \leq t_1$.

[When $s_m = t_1$, we replace the pair s_m, t_1 in eq. (2.27) by the single entry $s_m = t_1$.]

We shall generally follow such a convention in the formalism to be developed in the next section.

(ii) $K_2 = \{\text{ordered subsets of } R \text{ which are at most countably semi-infinite}\}.$

A typical element of K_2 is of the form

$$\begin{aligned} s &= \{s_1, s_2, \dots, s_m\} && \text{(finite)} \\ \text{or } s &= \{s_1, s_2, \dots\} && \text{(right-infinite)} \\ \text{or } s &= \{\dots, s_{-1}, s_0\} && \text{(left-infinite)}. \end{aligned} \quad (2.29)$$

The product $s \circ t$ is defined if at least one of s and t is finite, s is right-finite t is left-finite and the largest entry

in s is not greater than the smallest entry in t ; then $s \circ t = s \cup t$.

(iii) $\mathcal{K}_3 = \{\text{intervals in } \mathbb{R}\}$. We allow all intervals : finite, semi-infinite, doubly infinite (i.e. $\mathbb{R}$ itself), open, closed or semi-closed. Denoting the elements of $\mathcal{K}_3$ by $E, F, \dots$ we define the composition in $\mathcal{K}_3$ as follows :

(i) If E is right-bounded, F is left-bounded and $\text{lub}(E) \leq \text{glb}(F)$, then

$$E \circ F = E \cup F ; \quad 2.30a$$

(ii) $E \circ \mathbb{R} = \mathbb{R} \circ E = \mathbb{R}$ for all $E \in \mathcal{K}_3$. 2.30b

(iv) $\mathcal{K}_4 = \{ \text{homogeneous projection operators of the form (2.22) on } \tilde{H} \}$.

With $\bar{\alpha}$ as in eq. (2.22) and

$\bar{\beta} = \beta_{s_1} \otimes \dots \otimes \beta_{s_m}$, we have $\alpha \circ \beta$ defined if $t_n < s_1$; then

$$\alpha \circ \beta = \alpha_{t_1} \otimes \dots \otimes \alpha_{t_n} \otimes \beta_{s_1} \otimes \dots \otimes \beta_{s_m}. \quad (2.31)$$

This can be extended to the case $t_n = s_1$, $\alpha_{t_n} = \beta_{s_1}$; then $\alpha \circ \beta$ is given by the rhs of eq. 2.31 with $\alpha_{t_n} \otimes \beta_{s_1}$ replaced by α_{t_n} or β_{s_1} . The convention (2.28) has the analogue here given by

$$\alpha_{t_1} \circ \alpha_{t_1} = \alpha_{t_1}. \quad (2.32)$$

There is a homomorphism $\sigma : \mathcal{K}_4 \rightarrow \mathcal{K}_1$ given by

$$\sigma \left(\alpha_{t_1} \otimes \dots \otimes \alpha_{t_n} \right) = \{t_1, \dots, t_n\}. \quad (2.33)$$

(v) Let M be space-time with a Lorentzian metric g . We take

$$\mathcal{K}_5 = \{\text{subsets of } M\}. \quad (2.34)$$

Subsets of M will be denoted as $A, B, \dots$. As usual, we denote by $J^+(A)$ the causal future of A . We say that $A \triangleleft B$ (A precedes B) if⁸

$$J^+(A) \cap B \neq \phi, \quad J^+(B) \cap A = \phi. \quad (2.35)$$

The composition in K_5 is defined as follows :

(a) If $A \triangleleft B$, we define

$$A \circ B = A \cup B; \quad (2.36a)$$

(b) $A \circ M = M \circ A = M$ for all $A \in K_5$. (2.36b)

Note that $A \triangleleft B$ and $B \triangleleft C$ do not, in general, imply $A \triangleleft C$. (This is an example of a non-transitive temporal **order**⁸.) This psg is a crude relative of the psg employed by Isham⁸ to define a quasitemporal structure on space-time.

A *unit element* in a psg K is an element e such that

$$e \circ s = s \circ e = s \quad \text{for all } s \in K. \quad (2.37)$$

A psg may or may not have a unit element; if it has, it is unique. Indeed, if e' is another element satisfying (2.37), then $e' = e' \circ e = e$.

The empty set (when formally considered as an element of the psg in question) serves as unit element in K_1, K_2, K_3 and (with a suitable redefinition of $\triangleleft$) K_5 . For K_4 , the unit operator on $\mathcal{H}$ serves as the unit element.

An *absorbing element* in a psg K is an element a such that

$$a \circ s = s \circ a = a \quad \text{for all } s \in K. \quad (2.38)$$

Such an element, if it exists, is unique. Indeed, if a' is another element satisfying (2.38), we have $a' = a' \circ a = a$.

The psg's K_1 and K_2 above have no absorbing elements; K_3 and

K_5 have, respectively, R and M as absorbing elements [see eqs. (2.30b) and 2.36b)] and K_4 has the null operator on $\tilde{K}$ as the absorbing element.

Let K be a psg with elements e, a as above and σ is homomorphism of K into K' . We must have, for all $s \in K$,

$$\sigma(e) \circ \sigma(s) = \sigma(s) \circ \sigma(e) = \sigma(e \circ s) = \sigma(s)$$

$$\sigma(a) \circ \sigma(s) = \sigma(s) \circ \sigma(a) = \sigma(a \circ s) = \sigma(a).$$

It follows that

- (i) If K has a unit element e and/or an absorbing element a and σ is onto, then K' must correspondingly have a unit element e' and/or an absorbing element a' and we must have

$$\sigma(e) = e' \quad ; \quad \sigma(a) = a'. \quad (2.39)$$

- (ii) More generally, $\sigma(e)$ and $\sigma(a)$ act, respectively, as unit and absorbing elements in the homomorphic image $\sigma(K)$ of K in K' . In a psg, elements other than the unit or the absorbing element will be called *typical*.

A psg K will be called *directed* if, for any two different typical elements s, t in K , if $s \circ t$ is defined, then $t \circ s$ is not defined. Note that, in this definition, the composition $s \circ s$ is allowed. All the five psg's considered above are directed. For the use of this property in defining the direction of flow of time, see ^{the} remark at the end of section IIG.

A useful concept relating to partial semigroups which serves to introduce the concept of 'point of time' in the framework of histories-based theories is that of irreducible decompositions of typical elements of a psg. This means expressing a typical element of a psg as a composition of '*nuclear*' elements which

cannot be expressed as nontrivial compositions. For example, general elements of K_1 , K_2 and K_4 have irreducible decompositions of the form

$$K_1 : \quad \{ t_1, t_2, \dots, t_n \} = \{ t_1 \} \circ \{ t_2 \} \circ \dots \circ \{ t_n \} \quad (2.40a)$$

$$K_2 : \quad \{ t_1, t_2, \dots \} = \{ t_1 \} \circ \{ t_2 \} \circ \dots \quad (2.40b)$$

$$K_4 : \quad \bar{\alpha} = \alpha_{t_1} \circ \alpha_{t_2} \circ \dots \circ \alpha_{t_n} \quad (2.40c)$$

In K_1 and K_2 , the nuclear elements correspond to points of time and, in K_4 , to single time history propositions. The set of nuclear elements of a psg K will be denoted as $N(K)$.

The term 'nontrivial' employed in the definition of a nuclear element refers to the possibility of trivial compositions of the form $s \circ e = e \circ s = s$ [see eqs. (2.30b) and (2.36b) above] or of the form (2.28).

Not all psg's admit irreducible decompositions. For example, K_3 and K_5 do not admit such decompositions.

Some more specialized material on psg's used in our work will be presented in section IIIA.

G. The Temporal Logic Formalism of Isham and Linden

Isham⁸ made the key observation that the statement that a certain history is realized is itself a proposition and explored the possibility of a lattice structure in the set of history propositions of a (closed) system analogous to the lattice structure of single time propositions in standard quantum theory. He evolved a temporal logic formalism for the axiomatic development of histories-based theories which was sharpened and developed further in ref. (9).

In its broadest formulation⁹, the fundamental ingredients of a histories-based theory were taken to be a pair $(\Omega, \mathcal{D})$ where Ω (denoted as $\mathcal{UP}$ in ref. 8, 9) is the space of history propositions and $\mathcal{D}$ the space of decoherence functionals (maps $d : \Omega \times \Omega \rightarrow \mathbb{C}$); these are analogues of the pair $(\mathcal{L}, \mathcal{S})$ in the traditional quantum logic where $\mathcal{L}$ is the space of single time propositions of a system and $\mathcal{S}$ the space of states (essentially generalized probability measures on $\mathcal{L}$). The minimal structure envisaged for Ω (whose elements will be denoted as $\underline{\alpha}, \underline{\beta}, \dots$) was that of an *orthoalgebra*. This is a structure incorporating partial order $\leq$ (coarse graining), disjointness $\perp$ (mutual exclusion) and disjoint join operation $\oplus$ ($\underline{\alpha} \oplus \underline{\beta}$, defined only when $\underline{\alpha} \perp \underline{\beta}$, is to mean ' $\underline{\alpha}$ or $\underline{\beta}$ ') with the stipulation that $\underline{\alpha} \leq \underline{\beta}$ if and only if there exists a (unique) $\underline{\gamma} \in \Omega$ such that $\underline{\beta} = \underline{\alpha} \oplus \underline{\gamma}$. In Ω , there are two distinguished elements : the null history proposition $\underline{0}$ (which is impossible to realize) and the unit history proposition $\underline{1}$ (which is sure to be realized); one has $\underline{0} \leq \underline{\alpha} \leq \underline{1}$ for all $\underline{\alpha} \in \Omega$. For every $\underline{\alpha} \in \Omega$, the relation $\underline{\alpha} \leq \underline{1}$ implies the existence of a unique element $\neg \underline{\alpha}$ (the negation of $\underline{\alpha}$) of Ω such that $\underline{\alpha} \oplus \neg \underline{\alpha} = \underline{1}$. A set $\underline{\alpha}_1, \dots, \underline{\alpha}_n$ of pairwise disjoint history propositions is said to be *complete* if $\underline{\alpha}_1 \oplus \dots \oplus \underline{\alpha}_n = \underline{1}$. The four axioms about decoherence functionals are essentially the same as in GH. The normalization condition (2.17) now takes the form

$$d(\underline{1}, \underline{1}) = 1. \quad (2.41)$$

Note that, in this general formulation, elements of Ω are not defined as some sort of temporal sequences of 'single time

propositions' (or as some objects constructed from such sequences). The motivation for taking such a broad view of histories is to make allowance for construction of histories-based theories of the Universe or of quantum gravity in which the concept of time appears only at an appropriate coarse-grained level.

An important subclass of theories (to which ref. 8 was largely devoted) of the above sort consists of those which admit a quasitemporal structure (a generalization of the concept of time employing partial semigroups as exemplified by K_1 , K_2 , K_3 and K_5 of section IIF). In these theories Ω itself appears as a secondary object, the primary objects being a triple $(U, \mathcal{T}, \sigma)$ where U and $\mathcal{T}$ (the latter denoted as $\mathcal{S}$ in ref. 8 and as Sup in ref. 9) are psg's admitting irreducible decompositions and $\sigma : U \rightarrow \mathcal{T}$ is a psg homomorphism. In traditional quantum mechanics, this triple is (K_4, K_1, σ) where σ is the homomorphism of eq. (2.33). Elements of U are called *history filters* and those of $\mathcal{T}$ *temporal supports*; correspondingly U and $\mathcal{T}$ are referred to as the 'space of history filters' and the 'space of temporal supports' respectively. (The term 'history filter' is intended to capture the idea that realization of each 'single time' proposition α_j in a typical element $\alpha = \alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_n$ of U is a filtering operation on the system.) The quasitemporal structure mentioned above is represented by the psg $\mathcal{T}$.

In quasitemporal theories, the space Ω of history propositions is supposed to be a larger space in which U is supposed to be embedded and whose general elements are obtained

from those of $\mathcal{U}$ by application of the operations $\oplus$ and $\neg$ defined in Ω .

For example, in the HPO formulation of traditional quantum mechanics given in section II E, the operation $\oplus$ is simply the addition of mutually orthogonal homogeneous projection operators in $\mathcal{H}$ and $\neg$ is the operation of subtraction from the unit operation $\tilde{I}$; Ω itself is the collection $\mathcal{P}$ of inhomogeneous projection operators generated from the homogeneous projectors (elements of $\mathcal{U}$) by these operations.

In ref. (8), the space $\mathcal{U}$ was assumed to have, besides the structure of a psg, the operations of partial order $\leq$ (coarse graining) and meet $\wedge$ (meaning 'and') defined in it such that $(\mathcal{U}, \leq, \wedge)$ is a meet semilattice. This requirement was, however, dropped from the characterization of quasitemporal theories in ref. (9).

In our work, presented in the next three sections, we shall be concerned with quasitemporal theories only. Starting with a triple $(\mathcal{U}, \mathcal{T}, \sigma)$ as above and assuming the structure of a logic (in the sense of ref. 13) for 'single time' history propositions (nuclear elements of $\mathcal{U}$), we shall explicitly construct, in the next section, the space Ω and show it to be an orthoalgebra. We do not assume isomorphism of logics of propositions at different 'instants of time' (nuclear elements of $\mathcal{T}$). When these logics are those of projection operators in a Hilbert space, this construction reduces essentially to (an augmented version of) the HPO formalism of section II E. Our psg's $\mathcal{U}$ and $\mathcal{T}$ are of the type

of K_2 of section II F admitting semi-infinite irreducible decompositions.

At an intermediate stage in our construction, a space $\tilde{U}$, obtained from U by removing some redundancies, will be seen to be a meet semi-lattice of the sort mentioned above.

Before closing the section, a remark is in order about the definition of direction of flow of time in the present formalism. It appears that the concept of a directed psg introduced in section II F is quite appropriate for this. This should be clear from the examples of section II F. In our work, the axiom A_2 (see section III B), which essentially excludes histories corresponding to closed time loops, guarantees that our partial semigroups U and $\mathcal{T}$ are directed.

III. THE AUGMENTED TEMPORAL LOGIC FORMALISM

In this section we shall present an axiomatic development of the dynamics of systems in the framework of histories. This, as we shall see, contains the history versions of classical mechanics and traditional quantum mechanics as special cases. As mentioned earlier, we shall be concerned with theories having quasitemporal structure as described in section II G.

Before presenting the axioms, we introduce a few concepts relating to partial semigroups not covered in section II F.

A. More about Partial Semigroups

A *topological partial semigroup* is a set K which is a psg as well as a topological space such that the composition rule, considered as a mapping of the appropriate subset of $K \times K$ into K , $[(s,t) \rightarrow s \circ t]$ is continuous.

This concept will, in fact, be used only at one place in section V where we shall need to invoke a continuity argument. In the rest of the paper, topological considerations will be ignored.

We shall employ psg's of the type of K_2 of section IIF which admit semi-infinite irreducible decompositions and in which conventions of the type of eq. (2.28) are incorporated. We shall call such psg's *special*. To formalize it, we shall call a psg K (which may or may not contain unit and/or absorbing elements) *special* if there exists a subset $N(K)$ of typical elements of K (whose elements will be called nuclear elements of K) such that

- (i) for every $s \in N(K)$, we have $s \circ s = s$;
- (ii) every non-nuclear typical element s of K admits a unique irreducible decomposition into a composition of elements of $N(K)$ which is at most semi-infinite, i.e.

$s = s_1 \circ \dots \circ s_n$ or $s = s_1 \circ s_2 \circ \dots$ or $s = \dots s_{-1} \circ s_0$
 where $s_j \in N(K)$. The uniqueness of these decompositions must be understood as modulo trivial compositions of the form $s_k \circ s_k = s_k$.

Among the five psg's of section II F, K_1 , K_2 and K_4 are special whereas K_3 and K_5 are not. A variant of K_3 which is special is mentioned in section IIIH; a similar variant of K_5 can be constructed using appropriate space-time cells.

A subset A (which is at most countable and whose elements are listed in some definite order) of a psg K will be called *orientable* if there exists a bijection π of A such that, in $\pi(A)$, the composition of every pair of consecutive elements is defined. An ordering of elements of A permitting such a composition [such as that given by $\pi(A)$ above] will be called an *orientation* of A . In a semigroup every countable subset is orientable and every ordering of such a subset gives an orientation. It is clear from the example of semigroups that, in a psg, orientations of orientable subsets need not be unique.

One might expect that, in a directed psg, the orientations of orientable subsets are unique. This need not be true. As a

every element in E other than $s_{i_1}^{(1)}$ is either a potential successor or a potential predecessor of $s_{i_1}^{(1)}$ (but not both because K does not admit ^{nontrivial} closed chains).

It is adequate to show how the collection F of potential successors of $s_{i_1}^{(1)}$, if nonempty, is to be used up. For this, it is adequate to show that, in F (assumed non-empty), there is an 'earliest' element s^1 (by this we mean that every other element in F is a potential successor of s^1).

In every E_i , the collection F_i of potential successors of $s_{i_1}^{(1)}$ is either empty or has an earliest element. In E_1 , it is $s_{i_1+1}^{(1)}$ (if it exists); in E_j ($j \neq 1$), it is the first element of E_j on the right of $s_{i_1}^{(1)}$ in the ordering showing the orientation of $E_1 \cup E_j$. The possible successors of $s_{i_1}^{(1)}$ (in the construction of s_E) are, therefore, in the (unique) set

$$[F] = \{s_{i_1+1}^{(1)}, s_{j_1}^{(2)}, \dots, s_{k_1}^{(n)}\}, \text{ say}$$

where some of the entries on the right may be vacuous. By assumption, $[F]$ is nonempty. If it has only one element, it is the unique successor of $s_{i_1}^{(1)}$. If there are more than one, note that every pair in $[F]$ has a definite orientation which implies that $[F]$ has an earliest element. This element is the successor of $s_{i_1}^{(1)}$. Similarly for the predecessor. QED.

B. The Basic Ingredients

We now proceed with the axioms. We shall denote our axioms as $A_1, \dots, A_5$. In this subsection we take up the first three of them which define the basic structure.

counter-example, consider the three element directed psg $K = \{a, b, c\}$ with $a \circ b$, $b \circ c$ and $c \circ a$ defined. One can have $a \circ b \circ c$ and $b \circ c \circ a$ and $c \circ a \circ b$ as possible orientations of K .

The following lemma will be used in section III D.

Lemma (3.1) Let K be a special psg which does not admit elements expressible as nontrivial closed chains of the form $s \circ \dots \circ s$ for any $s \in N(K)$. [Such a psg is directed and its orientable subsets have unique orientations; see propositions (3.6) and (3.7) below.]. Let $E_1, E_2, \dots, E_n$ be orientable subsets of $N(K)$ such that $E_i \cup E_j$ is orientable for every pair (i, j) .

Then $E = \bigcup_i E_i$ is orientable.

Proof. Let $E_i = \left\{ s_k^{(i)}, k \in A_i \right\}$, $i = 1, \dots, n$

where A_i are some countable index sets. To show that E is orientable, we must show that we can construct, by composition of all elements of E , a single element, say s_E , of K .

Let us pick up some element of E , say $s_{i_1}^{(1)}$. We must show that, starting with $s_{i_1}^{(1)}$, one can go on adjoining element of E on the right and/or on the left (to construct an element of K) till all members of E are used up. An element s in E will be called a potential successor of $s_{i_1}^{(1)}$ if there exists an orientable subset of E containing both $s_{i_1}^{(1)}$ and s such that its (unique) orientation is given by an ordering of the form $\{ \dots, s_{i_1}^{(1)}, \dots, s, \dots \}$; similarly we define a potential predecessor. Since $E_i \cup E_j$ is orientable for every pair (i, j) ,

A_1 (Quasitemporal Structure Axiom) : Associated with every dynamical system is a *history system* which, by definition, is a triple $(\mathcal{U}, \mathcal{T}, \sigma)$ where $\mathcal{U}$ (called 'the space of history filters') and $\mathcal{T}$ (called 'the space of temporal supports') are special psg's and σ is a homomorphism of $\mathcal{U}$ onto $\mathcal{T}$ such that

$$\sigma[N(\mathcal{U})] = N(\mathcal{T}). \quad (3.1)$$

Elements of $\mathcal{U}$ (to be called history filters) will be denoted as $\alpha, \beta, \dots$ and those of $\mathcal{T}$ (to be called temporal supports) as $\tau, \tau', \dots$. If $\alpha \circ \beta$ and $\tau \circ \tau'$ are defined, we shall write $\alpha \triangleleft \beta$ (which means α precedes β) and $\tau \triangleleft \tau'$ (τ precedes τ'). We shall often refer to 'temporally ordered' sequences of the form $\alpha \triangleleft \beta \triangleleft \dots \triangleleft \gamma$ and $\tau \triangleleft \tau' \triangleleft \dots \triangleleft \tau''$ as *chains*.

A_2 (Causality Axiom) : If $\alpha, \beta, \dots, \gamma \in N(\mathcal{U})$ are such that

$$\alpha \triangleleft \beta \triangleleft \dots \triangleleft \gamma \quad \text{with} \quad \sigma(\alpha) = \sigma(\gamma) \quad (3.2)$$

then we must have

$$\alpha = \beta = \dots = \gamma. \quad (3.3)$$

In essence, this axiom forbids histories corresponding to 'closed time loops'. This is one of the weakest formulations of causality.

Prop. (3.2) If $\tau, \tau', \dots, \tau'' \in N(\mathcal{T})$ are such that

$$\tau \triangleleft \tau' \triangleleft \dots \triangleleft \tau'' \triangleleft \tau \quad (3.4)$$

then we must have

$$\tau = \tau' = \dots = \tau''. \quad (3.5)$$

Proof. Since σ is onto, there exist $\alpha, \gamma \in \sigma^{-1}(\tau)$, $\beta \in \sigma^{-1}(\tau')$, $\dots$ such that (3.2) holds which gives (3.3). Applying σ to (3.3), we get (3.5). QED

Corollary (3.3) (a) If $\tau, \tau' \in N(\mathcal{T})$ are such that $\tau \triangleleft \tau'$ and $\tau' \triangleleft \tau$, then we must have $\tau = \tau'$.

(b) If $\alpha, \beta \in N(\mathcal{U})$ are such that $\alpha \triangleleft \beta$ and $\beta \triangleleft \alpha$, then we must have $\alpha = \beta$.

Prop. (3.4) (a) If $\tau', \tau'' \in N(\mathcal{T})$ are such that (i) and/or (ii) below holds :

- (i) $\tau' \triangleleft \tau$ for any $\tau \in N(\mathcal{T})$ if and only if $\tau'' \triangleleft \tau$
 - (ii) $\tau \triangleleft \tau'$ for any $\tau \in N(\mathcal{T})$ if and only if $\tau \triangleleft \tau''$,
- then we must have $\tau' = \tau''$.

(b) If $\alpha, \beta \in N(\mathcal{U})$ are such that (i) and/or (ii) below holds :

- (i) $\alpha \triangleleft \gamma$ for any $\gamma \in N(\mathcal{U})$ if and only if $\beta \triangleleft \gamma$
- (ii) $\gamma \triangleleft \alpha$ for any $\gamma \in N(\mathcal{U})$ if and only if $\gamma \triangleleft \beta$

then we must have $\alpha = \beta$.

Proof. We prove only (a) for the case (i). By eq. (2.28), we have $\tau' \triangleleft \tau'$ and $\tau'' \triangleleft \tau''$. Putting $\tau = \tau'$ and $\tau = \tau''$ in the given condition (i), we get, respectively, $\tau'' \triangleleft \tau'$ and $\tau' \triangleleft \tau''$ which gives, by coroll. (3.3), $\tau' = \tau''$. QED

Prop. (3.5) (a) If $\tau \in \mathcal{T}$ is non-nuclear, then $\tau \circ \tau$ is not allowed.

(b) If $\alpha \in \mathcal{U}$ is non-nuclear then $\alpha \circ \alpha$ is not allowed.

Proof. We prove only (a). We can have $\tau \circ \tau$ defined only if τ has a finite irreducible decomposition, say, $\tau = \tau_1 \circ \dots \circ \tau_n$. Then $\tau \circ \tau$ contains the element $\tau_1 \circ \tau_2 \circ \dots \circ \tau_n \circ \tau_1$ which implies $\tau_1 = \tau_2 = \dots = \tau_n$. Eq. (2.28) now gives $\tau = \tau_1 \in N(\mathcal{T})$ (contradiction). QED

Prop. (3.6) The partial semigroups $\mathcal{T}$ and $\mathcal{U}$ are directed.

Proof. We prove it only for $\mathcal{T}$. Suppose it is not directed. Then there are typical elements $\tau, \tau' \in \mathcal{T}$, $\tau \neq \tau'$ such that

$\tau \triangleleft \tau'$ and $\tau' \triangleleft \tau$. If τ, τ' are both nuclear then the chain $\tau \triangleleft \tau' \triangleleft \tau$ gives $\tau = \tau'$ (contradiction). If one or both of τ, τ' are non-nuclear, $\tau \circ \tau'$ and $\tau' \circ \tau$ both can be defined only if both have finite irreducible decompositions, say

$$\tau = \tau_1 \circ \tau_2 \circ \dots \circ \tau_m, \tau' = \tau'_1 \circ \tau'_2 \circ \dots \circ \tau'_n \quad (n \geq 1, m > 1).$$

The element $\tau \circ \tau' \circ \tau$ contains, in its irreducible decomposition, the element

$$\tau_1 \circ \dots \circ \tau_m \circ \tau'_1 \circ \dots \circ \tau'_n \circ \tau_1$$

giving $\tau_1 = \tau_2 = \dots = \tau_m = \tau'_1 = \tau'_2 = \dots = \tau'_n = \tau_1$ which, in turn, gives $\tau' = \tau = \tau_1 \in \mathcal{N}(\mathcal{T})$ (contradiction). QED

Prop. (3.7) Orientations of orientable subsets of $\mathcal{N}(\mathcal{T})$ and $\mathcal{N}(\mathcal{U})$ are unique.

Proof. We prove it for $\mathcal{N}(\mathcal{T})$; analogous proof works for $\mathcal{N}(\mathcal{U})$. Suppose the proposition is not true. Then there exists an orientable subset $A = \{\tau_1, \tau_2, \dots\}$ of $\mathcal{N}(\mathcal{T})$ having (at least) two different orientations. This implies that for some pair (j, k) , $j \neq k$, we have τ_j appearing earlier than τ_k in one orientation and later in the other so that we can form the two chains

$$\tau_j \triangleleft \dots \triangleleft \tau_k \quad \text{and} \quad \tau_k \triangleleft \dots \triangleleft \tau_j$$

giving, in turn, the chain

$$\tau_j \triangleleft \dots \triangleleft \tau_k \triangleleft \dots \triangleleft \tau_j$$

which implies $\tau_j = \tau_k$ (contradiction). QED

We next introduce an axiom relating to the spaces $u_\tau = \sigma^{-1}(\tau)$ ($\tau \in \mathcal{N}(\mathcal{T})$) of 'single-time propositions'; we give them the same structure as one has for the analogous objects in the traditional proposition calculus for quantum mechanics. Formulations of proposition calculus by different authors differ

in some details; we find the one of Varadarajan¹³ best suited for our needs.

A_3 (Logic Structure Axiom). Every U_τ (for τ nuclear) has the structure of a *logic* as defined in ref. 13. This means that the following operations are defined in U_τ with the indicated interpretations. [Some (harmless) redundancy in the following description can be ignored.]

- (a) Partial order ($\leq$). The relation $\alpha \leq \beta$ for $\alpha, \beta \in U_\tau$ is to be interpreted as implying that the proposition α implies β (i.e. whenever one can predict α with certainty, one can predict β with certainty) or α is finer than β (i.e. it gives more precise information about the system than β) or β is coarser than α .
- (b) In each U_τ , there is a unique null proposition 0_τ (to be interpreted as the absurd proposition which can never be true) and a unique unit proposition 1_τ (to be interpreted as the trivial proposition which is always true) such that

$$0_\tau \leq \alpha \leq 1_\tau \quad \text{for all } \alpha \in U_\tau. \quad (3.6)$$

- (c) For each subset $A = \{\alpha_1, \alpha_2, \dots\}$ of U_τ which is at most countable, the greatest lower bound (glb) of A denoted as $\bigwedge_i \alpha_i$ (it is the coarsest proposition finer than all propositions in A) and the least upper bound of A denoted as $\bigvee_i \alpha_i$ (it is the finest proposition coarser than all propositions in A) exist in U_τ . For $\alpha, \beta \in U_\tau$ their glb (or 'meet') $\alpha \wedge \beta$ stands for ' α and β ' and lub (or 'join') $\alpha \vee \beta$ for ' α or β '.

(d) Negation ($\neg$) or Orthocomplementation. It is a bijection $\neg : \mathcal{U}_\tau \rightarrow \mathcal{U}_\tau$ such that

$\mathcal{U}_\tau \rightarrow \mathcal{U}_\tau$ such that

$$(i) \quad \neg(\neg\alpha) = \alpha \quad \text{for all } \alpha \in \mathcal{U}_\tau \quad (3.7a)$$

$$(ii) \quad \alpha \leq \beta \text{ if and only if } \neg\beta \leq \neg\alpha \quad (3.7b)$$

$$(iii) \quad \alpha \wedge (\neg\alpha) = 0_\tau, \quad \alpha \vee (\neg\alpha) = 1_\tau \quad \text{for all } \alpha \in \mathcal{U}_\tau \quad (3.4c,d)$$

The proposition $\neg\alpha$ is to be interpreted as the negation of α i.e. α is realized if and only if $\neg\alpha$ is not realized. We have

$$\neg(0_\tau) = 1_\tau, \quad \neg(1_\tau) = 0_\tau. \quad (3.8)$$

(e) If $\alpha, \beta \in \mathcal{U}_\tau$, then $\alpha \leq \beta$ if and only if there exists an element γ in $\mathcal{U}_\tau$ such that

$$\gamma \leq (\neg\alpha) \quad \text{and} \quad \gamma \vee \alpha = \beta. \quad (3.9)$$

In fact,¹³ γ is unique and given by

$$\gamma = (\neg\alpha) \wedge \beta; \quad (3.10)$$

it is often denoted as $\beta - \alpha$. Denoting the disjoint join operation in (3.9) by $\oplus$, we have

$$\beta = \alpha \oplus [(\neg\alpha) \wedge \beta]. \quad (3.11)$$

We do not assume isomorphism of $\mathcal{U}_\tau$'s (as logics) for different τ 's; this is in contrast to the situation in the proposition calculus for traditional quantum mechanics where one assumes the logics at different instants of time to be isomorphic.

Given $\alpha, \beta \in \mathcal{U}_\tau$, we shall say that $\alpha \perp \beta$ if $\alpha \leq \neg\beta$ (which is equivalent to $\beta \leq \neg\alpha$). This is a disjointness relation implying that if α is realized, then β is ruled out and vice versa. It implies the ~~condition~~ ^{relation} $\alpha \wedge \beta = 0_\tau$ (simultaneous realization of α and β is impossible).

Recalling eq. (3.1) and the definition of $\mathcal{U}_\tau$, we have

$$\mathcal{N}(\mathcal{U}) = \bigcup_{\tau \in \mathcal{N}(\mathcal{T})} \mathcal{U}_\tau. \quad (3.12)$$

For a typical $\alpha \in \mathcal{U}$, we define $\text{supp}(\alpha)$ (called the *temporal support* of α) as the unique collection of elements of $\mathcal{N}(\mathcal{T})$ appearing in the irreducible decomposition of $\sigma(\alpha)$. Clearly $\text{supp}(\alpha)$ is an orientable subset of $\mathcal{N}(\mathcal{T})$ with the unique orientation provided by the unique irreducible decomposition of $\sigma(\alpha)$. For any $\tau \in \text{supp}(\alpha)$, the nuclear element in the irreducible decomposition of α projecting to τ under σ will be denoted as α_τ .

An $\alpha \in \mathcal{U}$ will be called

- (i) a *null history filter* if $\alpha_\tau = 0_\tau$ for at least one $\tau \in \text{supp}(\alpha)$;
- (ii) a *unit (or trivial) history filter* if $\alpha_\tau = 1_\tau$ for all $\tau \in \text{supp}(\alpha)$.

By the *reduced form* of a nontrivial non-null history filter α we mean the structure obtained by deleting the unit elements (1_τ 's) in the irreducible decomposition of α . Such a reduced form may or may not itself represent a history filter. (Recall that $\triangleleft$ need not be a transitive relation in $\mathcal{U}$ or $\mathcal{T}$.)

C. The Spaces $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{T}}$

Our next step is to remove some redundancies in the description of history filters. We do this by introducing an equivalence relation $\sim$ on $\mathcal{U}$. We say that $\alpha \sim \beta$ if

- (i) both α and β are null history filters, or
- (ii) both α and β are unit (trivial) history filters, or
- (iii) both are nontrivial non-null history filters (we shall

henceforth call them *generic* history filters) with identical reduced forms.

The set of equivalence classes in $\mathcal{U}$ will be denoted as $\tilde{\mathcal{U}}$. Conditions (i) and (ii) define, respectively, the *null element* $\tilde{0}$ and *unit element* $\tilde{1}$ of $\tilde{\mathcal{U}}$. The equivalence class of $\alpha \in \mathcal{U}$ will be denoted as $[\alpha]$ or $\tilde{\alpha}$ depending on convenience.

The psg structure on $\mathcal{U}$ induces a psg structure on $\tilde{\mathcal{U}}$ given by

$$[\alpha] \circ [\beta] = [\alpha \circ \beta]. \quad (*)$$

It is easily verified that the right hand side of (*) is the same for all $\alpha \in [\alpha]$ and $\beta \in [\beta]$ and that associativity holds. Note that, for all $[\alpha]$ in $\tilde{\mathcal{U}}$,

$$\tilde{0} \circ \tilde{\alpha} = \tilde{\alpha} \circ \tilde{0} = \tilde{0}$$

$$\tilde{1} \circ \tilde{\alpha} = \tilde{\alpha} \circ \tilde{1} = \tilde{\alpha}$$

implying that $\tilde{0}$ and $\tilde{1}$ are, respectively, the absorbing and unit elements of $\tilde{\mathcal{U}}$. There is a one-to-one correspondence between typical elements of $\tilde{\mathcal{U}}$ and reduced forms of generic elements of $\mathcal{U}$. The irreducible decomposition of a generic $\alpha \in \mathcal{U}$ defines, by restriction to the reduced form of α , the (unique) irreducible decomposition of $\tilde{\alpha}$. It can now be easily seen that $(\tilde{\mathcal{U}}, \circ)$ is a directed special psg [with $N(\tilde{\mathcal{U}}) = N(\mathcal{U})$] with unit and absorbing elements.

We now construct a psg $\tilde{\mathcal{T}}$ which is in the same relation to $\tilde{\mathcal{U}}$ as $\mathcal{T}$ is to $\mathcal{U}$. The typical elements of the space $\tilde{\mathcal{T}}$ and the homomorphism $\tilde{\sigma} : \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{T}}$ (restricted to typical elements) can be

obtained in a single step by defining (recalling that $\sigma : \mathcal{U} \rightarrow \mathcal{I}$ is onto) $\tilde{\tau} = \tilde{\sigma}(\tilde{\alpha})$ to be the reduced object obtained from $\tau = \sigma(\alpha)$ (for any representative α of $\tilde{\alpha}$) by deleting the images under σ of those elements in the irreducible decomposition of α which are deleted to obtain the reduced form of α ; the psg structure on $\tilde{\mathcal{I}}$ is defined in the obvious way to make $\tilde{\sigma}$ a homomorphism.

We ^{now} ~~must~~ augment $\tilde{\mathcal{I}}$ by introducing a unit element e and an absorbing element a . We have, in the augmented $\tilde{\mathcal{I}}$, the relations

$$e \circ \tilde{\tau} = \tilde{\tau} \circ e = \tilde{\tau} \quad \text{for all } \tilde{\tau} \in \tilde{\mathcal{I}} \quad (3.13a)$$

$$a \circ \tilde{\tau} = \tilde{\tau} \circ a = a \quad \text{for all } \tilde{\tau} \in \tilde{\mathcal{I}}. \quad (3.13b)$$

The homomorphism $\tilde{\sigma}$ can now be extended to include the relations [recall eq. (2.39)]

$$\tilde{\sigma}(\tilde{1}) = e, \quad \tilde{\sigma}(\tilde{0}) = a. \quad (3.14a,b)$$

The question naturally arises : can we identify e and a with familiar objects? To find an answer, it is useful to note that typical elements of $\tilde{\mathcal{I}}$ are essentially (ordered) subsets of $\mathcal{N}(\tilde{\mathcal{I}})$; for example, $\tilde{\tau} = \tau_1 \circ \tau_2 \circ \tau_3$ may be identified with the ordered subset $\{\tau_1, \tau_2, \tau_3\}$ of $\mathcal{N}(\tilde{\mathcal{I}}) = \mathcal{N}(\mathcal{I})$. The subset corresponding to $\tilde{\tau} \circ \tilde{\tau}'$ is just the union of the subsets corresponding to $\tilde{\tau}$ and $\tilde{\tau}'$. Extension of this to include e and a and demanding consistency with eqs. (3.13a,b) leads to the identification

$$e = \phi \text{ (the empty set)}, \quad a = \mathcal{N}(\tilde{\mathcal{I}}). \quad (3.15a,b)$$

The situation here is similar to the psg $\mathcal{K}_5$ of section II F [See eq. (2.36b).]

Defining, for any $\tilde{\alpha} \in \tilde{\mathcal{U}}$, $\text{supp}(\tilde{\alpha}) = \tilde{\sigma}(\tilde{\alpha})$ [considered as a subset of $\mathcal{N}(\tilde{\mathcal{T}})$] we have, from eqs. (3.14) and (3.15),

$$\text{supp}(\tilde{1}) = \emptyset, \quad \text{supp}(\tilde{0}) = \mathcal{N}(\tilde{\mathcal{T}}). \quad (3.16a,b)$$

An alternative way of understanding eq. (3.16a) is obtained by noting that if, following Isham⁸, we try to describe a typical element $\tilde{\alpha}$ of $\tilde{\mathcal{U}}$ in terms of a map $\hat{\alpha} : \mathcal{N}(\tilde{\mathcal{T}}) \rightarrow Y$ where Y is the Cartesian product

$$Y = \prod_{\tau \in \mathcal{N}(\tilde{\mathcal{T}})} \mathcal{U}_{\tau}$$

given by

$$\hat{\alpha}(\tau) = \begin{cases} \tilde{\alpha}_{\tau} & \text{for } \tau \in \text{supp}(\tilde{\alpha}) \\ 1_{\tau} & \text{for } \tau \in \mathcal{N}(\tilde{\mathcal{T}}) - \text{supp}(\tilde{\alpha}), \end{cases} \quad (**)$$

its extension to $\tilde{\alpha} = \tilde{1}$ would yield the correct description if and only if eq. (3.16a) is assumed. Eq. (3.16b), however, is not on the same footing in this respect. Temporal supports of null history filters in $\mathcal{U}$ can be all possible orientable subsets of $\mathcal{N}(\mathcal{T})$. All the null history filters have been put together in a single element $\tilde{0}$ of $\tilde{\mathcal{U}}$. Eq. (3.16b) may be understood in the sense that the universal absorber/annihilator $\tilde{0}$ may have its support 'anywhere it pleases' and, therefore, 'everywhere'. (Indeed, the support of a null history filter can be chosen to contain the support of any non-null history filter; this will be adequate for all practical purposes.) Eq. (3.16b) just formalizes this. In any situation where any of the equations (3.13b), (3.14b) or (3.16b) is likely to create problems of consistency and/or interpretation, one must go back to the

description of the relevant situation in terms of the null history filters in $\mathcal{U}$.

Temporal supports of typical elements of $\mathcal{U}$ are orientable subsets of $\mathcal{N}(\tilde{\mathcal{T}})$; however, since $\tilde{0}$ is only a formal ^{construction} ~~device~~, there is no compelling need to assume the whole of $\mathcal{N}(\tilde{\mathcal{T}})$ [= $\text{supp } (\tilde{0})$] to be orientable. Indeed, as we shall see below, the formalism can be consistently developed without this assumption.

We next consider some operations/relations in $\tilde{\mathcal{U}}$ which the rich structure on $\mathcal{U}_\tau$'s permits us to define.

a. Partial Order ($\leq$). For $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{U}}$, we say that $\tilde{\alpha} \leq \tilde{\beta}$ (to mean that $\tilde{\alpha}$ is finer than $\tilde{\beta}$) if $\text{supp } \tilde{\beta} \subset \text{supp } \tilde{\alpha}$ and (if $\text{supp } \tilde{\beta}$ is nonempty) there exist representatives α and β of $\tilde{\alpha}$ and $\tilde{\beta}$ in $\mathcal{U}$ (with $\text{supp } \beta \subset \text{supp } \alpha$) such that $\alpha_\tau \leq \beta_\tau$ for all $\tau \in \text{supp } \beta$.

Prop. (3.8) The relation $\leq$ defined above is a partial order.

Proof. Clearly $\tilde{\alpha} \leq \tilde{\alpha}$. Transitivity is easily seen from the definition and transitivity of the partial order on $\mathcal{U}_\tau$'s. Now, if $\tilde{\alpha} \leq \tilde{\beta}$ and $\tilde{\beta} \leq \tilde{\alpha}$, we have $\text{supp } \tilde{\alpha} = \text{supp } \tilde{\beta}$ and (if these supports are nonempty) $\alpha_\tau \leq \beta_\tau$ and $\beta_\tau \leq \alpha_\tau$ for each τ implying $\tilde{\alpha} = \tilde{\beta}$. When $\text{supp } \tilde{\alpha} = \text{supp } \tilde{\beta} = \varnothing$, we trivially have $\tilde{\alpha} = \tilde{\beta}$ ($= \tilde{1}$). QED

It immediately follows from the above definition that

$$\tilde{0} \leq \tilde{\alpha} \leq \tilde{1} \quad \text{for all } \tilde{\alpha} \in \tilde{\mathcal{U}}. \quad (3.17)$$

b. Disjointness ($\perp$). For $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{U}}$, we say that $\tilde{\alpha} \perp \tilde{\beta}$ (i.e. $\tilde{\alpha}$ and $\tilde{\beta}$ are *disjoint* which is to mean that if one is realized, the other is ruled out) if they have representatives α and β in $\mathcal{U}$

such that $S_{\alpha\beta} \equiv (\text{supp } \alpha) \cap (\text{supp } \beta)$ is nonempty and $\alpha_\tau \perp \beta_\tau$ for at least one $\tau \in S_{\alpha\beta}$.

This definition has the following desirable implications :

- (i) The null element $\tilde{0}$ is disjoint with all $\tilde{\alpha}$ in $\tilde{U}$.
- (ii) The only element of $\tilde{U}$ disjoint with $\tilde{1}$ is $\tilde{0}$.

c. Disjoint Joint Operation ($\odot$). This operation is defined between two typical elements $\tilde{\alpha}$ and $\tilde{\beta}$ of $\tilde{U}$ if

- (i) $\text{supp } \tilde{\alpha} = \text{supp } \tilde{\beta}$
- (ii) $\tilde{\alpha}_{\tau_0} \perp \tilde{\beta}_{\tau_0}$ for one element $\tau_0 \in \text{supp}(\tilde{\alpha})$
- (iii) $\tilde{\alpha}_\tau = \tilde{\beta}_\tau$ for $\tau \in \text{supp}(\tilde{\alpha}) - \{\tau_0\}$.

Given these conditions (which imply $\tilde{\alpha} \perp \tilde{\beta}$; this special case of disjointness will be referred to as *weak disjointness*), we define $\tilde{\alpha} \odot \tilde{\beta}$ as follows :

$$\text{supp}(\tilde{\alpha} \odot \tilde{\beta}) = \text{supp } \tilde{\alpha} = \text{supp } \tilde{\beta}$$

$$(\tilde{\alpha} \odot \tilde{\beta})_\tau = \begin{cases} \tilde{\alpha}_{\tau_0} \vee \tilde{\beta}_{\tau_0} & \text{for } \tau = \tau_0 \\ \tilde{\alpha}_\tau (= \tilde{\beta}_\tau) & \text{for } \tau \in \text{supp}(\tilde{\alpha}) - \{\tau_0\}. \end{cases} \quad (3.18)$$

$$\text{Prop. (3.9)} \quad (i) \quad \tilde{\alpha} \odot \tilde{\beta} = \tilde{\beta} \odot \tilde{\alpha} \quad (3.19a)$$

$$(ii) \quad (\tilde{\alpha} \odot \tilde{\beta}) \odot \tilde{\gamma} = \tilde{\alpha} \odot (\tilde{\beta} \odot \tilde{\gamma}). \quad (3.19b)$$

These equations should be understood to mean that whenever one of the two sides in these equations is defined, the other is also defined and the two are equal.

Proof. Straightforward.

d. Meet Operation ($\wedge$). For general elements $\tilde{\alpha}, \tilde{\beta} \in \tilde{U}$, we define $\tilde{\alpha} \wedge \tilde{\beta}$ as follows :

- (i) If $\text{supp}(\tilde{\alpha}) \cup \text{supp}(\tilde{\beta})$ is not orientable, we put $\tilde{\alpha} \wedge \tilde{\beta} = \tilde{0}$.
- (ii) If $\text{supp } \tilde{\alpha} \cup \text{supp } \tilde{\beta}$ is orientable, we define

$$\text{supp } (\tilde{\alpha} \wedge \tilde{\beta}) = \text{supp } (\tilde{\alpha}) \cup \text{supp } (\tilde{\beta}) \quad (3.20a)$$

$$(\tilde{\alpha} \wedge \tilde{\beta})_{\tau} = \begin{cases} \tilde{\alpha}_{\tau} & \text{for } \tau \in \text{supp } \tilde{\alpha} - \text{supp } \tilde{\beta} \\ \tilde{\beta}_{\tau} & \text{for } \tau \in \text{supp } \tilde{\beta} - \text{supp } \tilde{\alpha} \\ \tilde{\alpha}_{\tau} \wedge \tilde{\beta}_{\tau} & \text{for } \tau \in \text{supp } \tilde{\alpha} \cap \text{supp } \tilde{\beta}. \end{cases} \quad (3.20b)$$

The history filter $\tilde{\alpha} \wedge \tilde{\beta}$ is to mean ' $\tilde{\alpha}$ and $\tilde{\beta}$ ' (i.e. it is realized if and only if both $\tilde{\alpha}$ and $\tilde{\beta}$ are realized).

Prop. (3.10) The following relations hold in $\tilde{U}$ ($\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ denote general elements of $\tilde{U}$) :

$$(i) \quad \tilde{\alpha} \wedge \tilde{\alpha} = \tilde{\alpha} \quad (ii) \quad \tilde{\alpha} \wedge \tilde{\beta} = \tilde{\beta} \wedge \tilde{\alpha} \quad (3.21a,b)$$

$$(iii) \quad \tilde{\alpha} \wedge \tilde{1} = \tilde{\alpha} \quad (iv) \quad \tilde{\alpha} \wedge \tilde{0} = \tilde{0} \quad (3.22a,b)$$

$$(v) \quad \tilde{\alpha} \perp \tilde{\beta} \text{ implies } \tilde{\alpha} \wedge \tilde{\beta} = \tilde{0}. \text{ [(iv) is a special case of this.]}$$

(vi) When $\tilde{\alpha} \circ \tilde{\beta}$ is defined, we have

$$\tilde{\alpha} \wedge \tilde{\beta} = \tilde{\alpha} \circ \tilde{\beta}. \quad (3.23)$$

$$(vii) \quad \tilde{\alpha} \wedge \tilde{\beta} \leq \tilde{\alpha} \quad (3.24)$$

(viii) $\tilde{\alpha} \leq \tilde{\alpha} \wedge \tilde{\beta}$ if and only if $\tilde{\alpha} \leq \tilde{\beta}$. This, with (vii), implies that $\tilde{\alpha} = \tilde{\alpha} \wedge \tilde{\beta}$ if and only if $\tilde{\alpha} \leq \tilde{\beta}$.

(ix) If $\tilde{\gamma} \leq \tilde{\alpha}$ and $\tilde{\gamma} \leq \tilde{\beta}$, then $\tilde{\gamma} \leq (\tilde{\alpha} \wedge \tilde{\beta})$. (3.25)

(x) If $\tilde{\alpha} \perp \tilde{\beta}$, then $(\tilde{\alpha} \wedge \tilde{\gamma}) \perp (\tilde{\beta} \wedge \tilde{\gamma})$ for all $\tilde{\gamma} \in \tilde{U}$.

Proof. Items (i) - (iv) follow immediately from the definition given above. Item (v) which, in view of item (iv) and the result that $\tilde{\alpha} \perp \tilde{1}$ only if $\tilde{\alpha} = \tilde{0}$, need be proved only for generic $\tilde{\alpha}$ and $\tilde{\beta}$, follows easily from the definitions of $\perp$ and $\wedge$.

Item (vi) follows immediately by noting that, when $\tilde{\alpha} \circ \tilde{\beta}$ is defined, we have $\text{supp } \tilde{\alpha} \cap \text{supp } \tilde{\beta} = \emptyset$.

Item (vii): When $\tilde{\alpha}$ and/or $\tilde{\beta}$ equals $\tilde{0}$ or $\tilde{1}$, it is easily seen to follow from items (iii), (iv), (i) and eq. (3.17). For generic $\tilde{\alpha}$ and $\tilde{\beta}$, the required support condition is trivially satisfied and the required relation follows from the definition (3.20) and the relation $\tilde{\alpha}_\tau \wedge \tilde{\beta}_\tau \leq \tilde{\alpha}_\tau$.

Item (viii): Cases when $\tilde{\alpha}$ and/or $\tilde{\beta}$ equals $\tilde{0}$ or $\tilde{1}$ are easily handled. We, therefore, take $\tilde{\alpha}$ and $\tilde{\beta}$ to be generic.

Given $\tilde{\alpha} \leq \tilde{\alpha} \wedge \tilde{\beta}$ we have

$$\text{supp } \tilde{\alpha} \cup \text{supp } \tilde{\beta} = \text{supp } (\tilde{\alpha} \wedge \tilde{\beta}) \subset \text{supp } \tilde{\alpha} \quad (\text{A})$$

which implies

$$\text{supp } \tilde{\beta} \subset \text{supp } \tilde{\alpha} ; \quad (\text{B})$$

moreover,

$$\tilde{\alpha}_\tau \leq (\tilde{\alpha} \wedge \tilde{\beta})_\tau \text{ for all } \tau \in \text{supp } \tilde{\alpha} \cup \text{supp } \tilde{\beta}. \quad (C)$$

Now

$$(\tilde{\alpha} \wedge \tilde{\beta})_\tau = \tilde{\alpha}_\tau \wedge \tilde{\beta}_\tau \text{ for } \tau \in \text{supp } \tilde{\alpha} \cap \text{supp } \tilde{\beta} = \text{supp } \tilde{\beta}. \quad (D)$$

Eqs. (C) and (D) give

$$\tilde{\alpha}_\tau \leq \tilde{\alpha}_\tau \wedge \tilde{\beta}_\tau \text{ for all } \tau \in \text{supp } \tilde{\beta}. \quad (E)$$

But $\tilde{\alpha}_\tau \wedge \tilde{\beta}_\tau \leq \tilde{\beta}_\tau$. Hence (E) gives

$$\tilde{\alpha}_\tau \leq \tilde{\beta}_\tau \text{ for all } \tau \in \text{supp } \tilde{\beta}. \quad (F)$$

The relations (B) and (F) imply $\tilde{\alpha} \leq \tilde{\beta}$.

Conversely, given $\tilde{\alpha} \leq \tilde{\beta}$, we have (B) and (F). The relation (B) implies (A) whereas (F), along with the trivial condition $\tilde{\alpha}_\tau \leq \tilde{\alpha}_\tau$ implies (E) which, in turn, implies, with eq. (3.20) and (B), the relation (C). The relations (A) and (C) then imply $\tilde{\alpha} \leq \tilde{\alpha} \wedge \tilde{\beta}$.

Item (ix): The case $\tilde{\gamma} = \tilde{0}$ is trivial; $\tilde{\gamma} = \tilde{1}$ is possible only if $\tilde{\alpha} = \tilde{\beta} = \tilde{1}$ which, with (iii) above, reduces the required relation to triviality. We can, therefore, take $\tilde{\gamma}$ to be generic. The cases when $\tilde{\alpha}$ and/or $\tilde{\beta}$ equals $\tilde{0}$ or $\tilde{1}$ are trivial. We can, therefore, take $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ all three to be generic.

The given conditions imply the following relations :

$$\text{supp } \tilde{\alpha} \subset \text{supp } \tilde{\gamma}, \quad \tilde{\gamma}_\tau \leq \tilde{\alpha}_\tau \text{ for all } \tau \in \text{supp } \tilde{\alpha};$$

$$\text{supp } \tilde{\beta} \subset \text{supp } \tilde{\gamma}, \quad \tilde{\gamma}_\tau \leq \tilde{\beta}_\tau \text{ for all } \tau \in \text{supp } \tilde{\beta}.$$

This implies that $\text{supp } \tilde{\alpha} \cup \text{supp } \tilde{\beta} \subset \text{supp } \tilde{\gamma}$ and

$$\tilde{\gamma}_\tau \leq \tilde{\alpha}_\tau \wedge \tilde{\beta}_\tau \text{ for all } \tau \in \text{supp } \tilde{\alpha} \cap \text{supp } \tilde{\beta}.$$

These results, along with the given conditions and eq. (3.20), imply $\tilde{\gamma} \leq \tilde{\alpha} \wedge \tilde{\beta}$.

Item (x): The given condition $\tilde{\alpha} \perp \tilde{\beta}$ implies that

$$\tilde{S}_{\alpha\beta} \equiv (\text{supp } \tilde{\alpha}) \cap (\text{supp } \tilde{\beta}) \neq \emptyset \text{ and } \tilde{\alpha}_{\tau_0} \perp \tilde{\beta}_{\tau_0} \text{ for some } \tau_0 \in \tilde{S}_{\alpha\beta}.$$

Clearly, $\tau_0 \in \text{supp}(\tilde{\alpha} \wedge \tilde{\gamma}) \cap \text{supp}(\tilde{\beta} \wedge \tilde{\gamma})$.

Now, if $\tau_0 \notin \text{supp } \tilde{\gamma}$, we have $(\tilde{\alpha} \wedge \tilde{\gamma})_{\tau_0} = \tilde{\alpha}_{\tau_0}$ and $(\tilde{\beta} \wedge \tilde{\gamma})_{\tau_0} = \tilde{\beta}_{\tau_0}$

which are disjoint.

If $\tau_0 \in \text{supp } \tilde{\gamma}$, we have $(\tilde{\alpha} \wedge \tilde{\gamma})_{\tau_0} = \tilde{\alpha}_{\tau_0} \wedge \tilde{\gamma}_{\tau_0}$ and $(\tilde{\beta} \wedge \tilde{\gamma})_{\tau_0} = \tilde{\beta}_{\tau_0} \wedge \tilde{\gamma}_{\tau_0}$.

We have

$$\tilde{\alpha}_{\tau_0} \wedge \tilde{\gamma}_{\tau_0} \leq \tilde{\alpha}_{\tau_0} \leq (\neg \tilde{\beta}_{\tau_0}) \leq (\neg \tilde{\beta}_{\tau_0}) \vee (\neg \tilde{\gamma}_{\tau_0}) = \neg (\tilde{\beta}_{\tau_0} \wedge \tilde{\gamma}_{\tau_0})$$

proving the desired disjointness. QED.

This proposition gives relations between various operations in $\tilde{\mathcal{U}}$. In particular, the collection $(\tilde{\mathcal{U}}, \leq, \wedge, \tilde{0}, \tilde{1})$ can now be seen to be a meet-semilattice. [For a well-organized treatment of lattices and semilattices see, for example, ref. (17).]. As mentioned in section IIG, a meet-semilattice structure was envisaged for the space of history filters in the scheme of Isham⁸ for quasitemporal theories.

D. The Space $\Omega = \Omega(\tilde{\mathcal{U}}, \tilde{\mathcal{T}})$

As was noted in ref. (8), the space $\mathcal{U}$ (or its refined version $\tilde{\mathcal{U}}$) of history filters ('homogeneous histories') is not adequate for a histories-based formalism of (generalized) quantum mechanics. An important ingredient which is present in $\mathcal{U}_\tau$'s (families of 'single time' propositions) but absent in $\mathcal{U}$ or $\tilde{\mathcal{U}}$ is the concept of negation. To incorporate this, we shall now embed $\tilde{\mathcal{U}}$ in a larger space Ω of history propositions ('inhomogeneous histories'). To motivate the following constructions, it is useful to point out here that the spaces $\tilde{\mathcal{U}}$ and Ω are the analogues of, respectively, the spaces of homogeneous and inhomogeneous projection operators in the HPO space $\tilde{\mathcal{H}}$ (see section II E).

The elements of Ω (to be called history propositions), will be taken to be of the form

$$\underline{\alpha} = \{\tilde{\alpha}^{(1)}, \tilde{\alpha}^{(2)}, \dots\} \quad (3.26)$$

where the entries on the right are an at most countable collection of elements of $\tilde{\mathcal{U}}$ such that

- (i) the order of the entries on the right is immaterial;
- (ii) $\tilde{\alpha}^{(j)} \perp \tilde{\alpha}^{(k)}$ for $j \neq k$;
- (iii) inclusion of any number of $\tilde{0}$'s inside the braces leaves $\underline{\alpha}$ unchanged;
- (iv) the union of temporal supports of every finite subcollection of non-null $\tilde{\alpha}^{(j)}$'s is orientable;
- (v) if $\tilde{\alpha}$ and $\tilde{\beta}$ are weakly disjoint (as defined in section III C) with $\tilde{\alpha} \circ \tilde{\beta} = \tilde{\gamma}$, then

$$\{\tilde{\alpha}, \tilde{\beta}, \dots\} = \{\tilde{\gamma}, \dots\} \quad (3.27)$$

where entries represented by dots are identical on the two sides.

For consistency, we must check that, given $\tilde{\delta} \in \tilde{U}$ in the collection represented by ... in eq. (3.27), we have $\tilde{\delta} \perp \tilde{\gamma}$ if and only if $\tilde{\delta} \perp \tilde{\alpha}$ and $\tilde{\delta} \perp \tilde{\beta}$. To verify this, let τ_0 be the point of weak disjointness of $\tilde{\alpha}$ and $\tilde{\beta}$ and τ' a point of disjointness of $\tilde{\gamma}$ and $\tilde{\delta}$. Since $\tilde{\alpha}_\tau = \tilde{\beta}_\tau = \tilde{\gamma}_\tau$ for all $\tau \neq \tau_0$, the required consistency condition is trivially satisfied if $\tau' \neq \tau_0$. We need, therefore, only consider the case when the disjointness of $\tilde{\gamma}$ with $\tilde{\delta}$ holds only at $\tau' = \tau_0$. We have

$$\tilde{\alpha}_{\tau_0} \perp \tilde{\beta}_{\tau_0}, \quad \tilde{\gamma}_{\tau_0} = \tilde{\alpha}_{\tau_0} \vee \tilde{\beta}_{\tau_0}, \quad \neg \tilde{\gamma}_{\tau_0} = (\neg \tilde{\alpha}_{\tau_0}) \wedge (\neg \tilde{\beta}_{\tau_0}).$$

Now (i) given $\tilde{\gamma}_{\tau_0} \perp \tilde{\delta}_{\tau_0}$ we have

$$\tilde{\delta}_{\tau_0} \leq \neg \tilde{\gamma}_{\tau_0} \leq \neg \tilde{\alpha}_{\tau_0} \quad \text{implying that } \tilde{\delta} \perp \tilde{\alpha} \text{ and similarly } \tilde{\delta} \perp \tilde{\beta}.$$

(ii) Given $\tilde{\delta}_{\tau_0} \perp \tilde{\alpha}_{\tau_0}$ and $\tilde{\delta}_{\tau_0} \perp \tilde{\beta}_{\tau_0}$, we have $\tilde{\delta}_{\tau_0} \leq \neg \tilde{\alpha}_{\tau_0}$ and

$$\tilde{\delta}_{\tau_0} \leq \neg \tilde{\beta}_{\tau_0} \quad \text{which gives}$$

$$\tilde{\delta}_{\tau_0} \leq (\neg \tilde{\alpha}_{\tau_0}) \wedge \neg \tilde{\beta}_{\tau_0} = \neg (\tilde{\alpha}_{\tau_0} \vee \tilde{\beta}_{\tau_0}) = \neg \tilde{\gamma}_{\tau_0}$$

implying $\tilde{\delta} \perp \tilde{\gamma}$. This completes the verification.

The object $\underline{\alpha}$ of eq. (3.26) represents a history proposition whose realization means realization of any of the mutually exclusive $\tilde{\alpha}^{(j)}$'s [$\tilde{\alpha}^{(1)}$ or $\tilde{\alpha}^{(2)}$ or $\tilde{\alpha}^{(3)}$ or ...]; it is analogous

to a sum of mutually orthogonal homogeneous projection operators in the HPO space $\tilde{\mathcal{H}}$. [See eq. (3.36) below.]

The mapping $\tilde{\alpha} \rightarrow \{\tilde{\alpha}\}$ defines an embedding of $\tilde{\mathcal{U}}$ in Ω .

An intuitively appealing representation of $\underline{\alpha}$, equivalent to that in (3.26), is as a *formal* disjoint join :

$$\underline{\alpha} = \tilde{\alpha}^{(1)} \circ \tilde{\alpha}^{(2)} \circ \dots \quad (3.28)$$

[with stipulations analogous to those in (i)-(v) above]. Eq. (3.27) then appears more natural in the form

$$\tilde{\alpha} \circ \tilde{\beta} \circ \dots = \tilde{\gamma} \circ \dots \quad (3.29)$$

We shall, in fact, introduce a $\circ$ operation in Ω below and a representation of the form (3.28) will appear [see eq. (3.36)] with $\tilde{\alpha}^{(j)}$'s replaced by $\{\tilde{\alpha}^{(j)}\}$'s and $\circ$ as the genuine disjoint join operation between the entries. The representation (3.26), however, has its own conveniences and we shall retain it.

We define the null element $\underline{0}$ and the unit element $\underline{1}$ in Ω as

$$\underline{0} = \{\tilde{0}\}, \quad \underline{1} = \{\tilde{1}\}. \quad (3.30)$$

Elements of Ω other than these two will be called *generic*.

Note. In the HPO space $\tilde{\mathcal{H}}$, the analogue of our Ω is the family $\mathcal{P}'$ of projection operators which are expressible as countable sums of mutually orthogonal homogeneous projection operators (taking the null and the unit operator as homogeneous projection operators). As we shall see below (see Coroll. 3.15), Ω is closed under a negation operation which is analogous, in $\mathcal{P}'$, to subtraction from the unit operator. An adaptation of the proof of Coroll. (3.15) shows that $\mathcal{P}'$ is closed under negation.

It must be noted that representation of the form (3.26) for an $\underline{\alpha} \in \Omega$ is not unique - even after disposing off the redundant $\bar{0}$'s and the weakly disjoint pairs. For example,

$$\underline{\alpha} = \left\{ \neg \alpha_{\tau_1} \circ \alpha_{\tau_2}, \alpha_{\tau_1} \circ \neg \alpha_{\tau_2}, \neg \alpha_{\tau_1} \circ \neg \alpha_{\tau_2} \right\}$$

can be represented as $\left\{ \neg \alpha_{\tau_1}, \alpha_{\tau_1} \circ \neg \alpha_{\tau_2} \right\}$ and also as $\left\{ \neg \alpha_{\tau_1} \circ \alpha_{\tau_2}, \neg \alpha_{\tau_2} \right\}$. This point must be taken care of to ensure that the definitions of various operations/relations in Ω are unambiguous.

We now consider operations and relations in Ω . The interpretation of these objects must be obvious from the discussion so far.

a. Partial Order ($\leq$). Given $\underline{\alpha}, \underline{\beta} \in \Omega$, we say that $\underline{\alpha} \leq \underline{\beta}$ if they admit representations $\underline{\alpha} = \left\{ \tilde{\alpha}^{(j)}; j \in A \right\}, \underline{\beta} = \left\{ \tilde{\beta}^{(k)}; k \in B \right\}$ where A and B are suitable index sets, such that, for every $j \in A$, there is a $k \in B$ such that $\tilde{\alpha}^{(j)} \leq \tilde{\beta}^{(k)}$.

Remark. In the definition given above, we may have $\tilde{\alpha}^{(j)}$ for more than one j satisfying $\tilde{\alpha}^{(j)} \leq \tilde{\beta}^{(k)}$ for a single k . For example, let

$$\underline{\alpha} = \left\{ \tilde{\alpha}^{(1)}, \tilde{\alpha}^{(2)} \right\}, \underline{\beta} = \left\{ \tilde{\beta}^{(1)}, \tilde{\beta}^{(2)} \right\} \quad (3.31a)$$

where

$$\tilde{\alpha}^{(1)} = \alpha_{\tau_1} \circ \alpha_{\tau_2} \circ \alpha_{\tau_3}, \quad \tilde{\alpha}^{(2)} = \alpha'_{\tau_1} \circ \alpha'_{\tau_2},$$

$$\tilde{\beta}^{(1)} = \beta_{\tau_1} \circ \beta_{\tau_2}, \quad \tilde{\beta}^{(2)} = \beta'_{\tau_2} \circ \beta'_{\tau_3} \circ \beta'_{\tau_4}$$

with

$$\alpha_{\tau_i} \leq \beta_{\tau_i}, \quad \alpha'_{\tau_i} \leq \beta_{\tau_i} \quad i = 1, 2 \quad (3.31b)$$

and

$$\alpha_{\tau_1} \perp \alpha'_{\tau_1} \quad \text{i.e.} \quad \alpha_{\tau_1} \leq \neg \alpha'_{\tau_1} \quad \text{and} \quad \beta_{\tau_2} \perp \beta'_{\tau_2}. \quad (3.31c)$$

Eq. (3.31b) implies $\tilde{\alpha}^{(i)} \leq \tilde{\beta}^{(1)}$ for $i = 1, 2$ and we have $\underline{\alpha} \leq \underline{\beta}$. We shall come back to this example just before giving the proof of theorem (3.13) below and show explicitly that $\underline{\beta} = \underline{\alpha} \odot \underline{\gamma}$ with a unique $\underline{\gamma}$.

Recalling eq. (3.17), we have

$$\underline{0} \leq \underline{\alpha} \leq \underline{1} \quad \text{for all } \underline{\alpha} \in \Omega. \quad (3.32)$$

Before introducing other operations in Ω , it is useful to introduce the concept of support-compatibility. We shall say that $\underline{\alpha}, \underline{\beta} \in \Omega$ are support-compatible if

- (i) at least one of them is $\underline{0}$ or $\underline{1}$, or
- (ii) both are generic satisfying the condition that, for any representation $\underline{\alpha} = \{\tilde{\alpha}^{(i)}; i \in A\}$, $\underline{\beta} = \{\tilde{\beta}^{(j)}; j \in B\}$, the joint collection $\{\tilde{\alpha}^{(i)}, \tilde{\beta}^{(j)}; i \in A, j \in B\}$ is such that the union of temporal supports of any finite subset is orientable.

b. Disjointness ($\perp$). Given $\underline{\alpha} = \{\tilde{\alpha}^{(1)}, \tilde{\alpha}^{(2)}, \dots\}$ and $\underline{\beta} = \{\tilde{\beta}^{(1)}, \tilde{\beta}^{(2)}, \dots\}$ we shall say that $\underline{\alpha} \perp \underline{\beta}$ if they are support-compatible and if $\tilde{\alpha}^{(i)} \perp \tilde{\beta}^{(j)}$ for all i, j . In view of the discussion after eq. (3.27), this definition suffers no ambiguity due to nonuniqueness of representation of elements of Ω .

It immediately follows from this definition that

$$(i) \quad \underline{0} \perp \underline{\alpha} \text{ for all } \underline{\alpha} \text{ in } \Omega \quad (3.33a)$$

$$(ii) \quad \underline{\alpha} \perp \underline{1} \text{ if and only if } \underline{\alpha} = \underline{0}. \quad (3.33b)$$

c. Disjoint Join Operation ($\odot$) Given two disjoint elements $\underline{\alpha} = \{\tilde{\alpha}^{(i)}; i \in A\}$ and $\underline{\beta} = \{\tilde{\beta}^{(j)}; j \in B\}$ in Ω we define

$$\underline{\alpha} \odot \underline{\beta} = \{\tilde{\alpha}^{(i)}, \tilde{\beta}^{(j)}; i \in A, j \in B\}. \quad (3.34)$$

Lemma (3.1) permits the construction of disjoint sums of a countable number of pairwise disjoint elements of Ω (note that only the union of temporal supports of any *finite* subset of the joint collection of history filters needs to be orientable).

Prop. (3.11) We have

$$(i) \quad \underline{\alpha} \odot \underline{\beta} = \underline{\beta} \odot \underline{\alpha}$$

$$(\underline{\alpha} \odot \underline{\beta}) \odot \underline{\gamma} = \underline{\alpha} \odot (\underline{\beta} \odot \underline{\gamma})$$

with the same qualifications as for eqs. (3.18) and (3.19);

(ii) moreover

$$\underline{\alpha} \odot \underline{0} = \underline{\alpha} \quad \text{for all } \underline{\alpha} \in \Omega. \quad (3.35)$$

Proof. Straightforward.

The $\underline{\alpha}$ of (3.26) can now be represented as

$$\underline{\alpha} = \{\tilde{\alpha}^{(1)}\} \odot \{\tilde{\alpha}^{(2)}\} \odot \dots \quad (3.36)$$

This representation makes clear the correspondence between Ω and the family $\mathcal{P}'$ of inhomogeneous projectors in the HPO space $\tilde{\mathcal{H}}$ mentioned above.

The $\odot$ operations in $\tilde{\mathcal{U}}$ and Ω are compatible. Indeed, if $\tilde{\alpha}$ and $\tilde{\beta}$ are weakly disjoint with $\tilde{\alpha} \odot \tilde{\beta} = \tilde{\gamma}$ we have

$$\{\tilde{\alpha}\} \odot \{\tilde{\beta}\} = \{\tilde{\alpha}, \tilde{\beta}\} = \{\tilde{\alpha} \odot \tilde{\beta}\} = \{\tilde{\gamma}\}.$$

An (at most countable) set of mutually disjoint history propositions $\underline{\alpha}, \underline{\beta}, \dots$ will be called complete (or exhaustive) if

$$\underline{\alpha} \odot \underline{\beta} \odot \dots = \underline{1}. \quad (3.37)$$

Prop. (3.12) Given $\underline{\alpha}, \underline{\beta} \in \Omega$ which are mutually disjoint, we have

$$\underline{\alpha} \leq (\underline{\alpha} \odot \underline{\beta}). \quad (3.38)$$

Proof. Follows directly from the relevant definitions.

Prop. (3.13) If $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ are elements of Ω , then

$$(i) \quad \underline{\alpha} \odot \underline{\beta} \leq \underline{\alpha} \odot \underline{\gamma} \quad \text{implies} \quad \underline{\beta} \leq \underline{\gamma};$$

$$(ii) \quad \underline{\alpha} \odot \underline{\beta} = \underline{\alpha} \odot \underline{\gamma} \quad \text{implies} \quad \underline{\beta} = \underline{\gamma}.$$

Proof. (i) Let $\underline{\alpha} = \{\tilde{\alpha}^{(i)}; i \in A\}$, $\underline{\beta} = \{\tilde{\beta}^{(j)}; j \in B\}$ and $\underline{\gamma} = \{\tilde{\gamma}^{(k)}; k \in C\}$. We have $\underline{\alpha} \odot \underline{\beta} = \{\tilde{\alpha}^{(i)}, \tilde{\beta}^{(j)}; i \in A, j \in B\}$ and $\underline{\alpha} \odot \underline{\gamma} = \{\tilde{\alpha}^{(i)}, \tilde{\gamma}^{(k)}; i \in A, k \in C\}$. The given condition $\underline{\alpha} \odot \underline{\beta} \leq \underline{\alpha} \odot \underline{\gamma}$ implies that, for every $\tilde{\beta}^{(j)}$, there exists either an $\tilde{\alpha}^{(i)}$ such that $\tilde{\beta}^{(j)} \leq \tilde{\alpha}^{(i)}$ or a $\tilde{\gamma}^{(k)}$ such that $\tilde{\beta}^{(j)} \leq \tilde{\gamma}^{(k)}$. The first possibility is ruled out because $\tilde{\alpha}^{(i)} \perp \tilde{\beta}^{(j)}$ for all $i \in A, j \in B$. The second condition implies $\underline{\beta} \leq \underline{\gamma}$.

(ii) The given condition $\underline{\alpha} \odot \underline{\beta} = \underline{\alpha} \odot \underline{\gamma}$ implies

$$\underline{\alpha} \odot \underline{\beta} \leq \underline{\alpha} \odot \underline{\gamma} \quad \text{and} \quad \underline{\alpha} \odot \underline{\gamma} \leq \underline{\alpha} \odot \underline{\beta}$$

which implies, respectively, $\underline{\beta} \leq \underline{\gamma}$ and $\underline{\gamma} \leq \underline{\beta}$ giving $\underline{\beta} = \underline{\gamma}$. QED.

We now move towards introducing a negation operation in Ω . The following theorem will play a crucial role.

Theorem (3.14) Given $\underline{\alpha}, \underline{\beta} \in \Omega$, we have $\underline{\alpha} \leq \underline{\beta}$ if and only if there exists a $\underline{\gamma} \in \Omega$ such that $\underline{\beta} = \underline{\alpha} \circ \underline{\gamma}$; the element $\underline{\gamma}$ is unique.

Before giving the proof, it is instructive to verify the claim of the theorem for the example of eq. (3.31); it will serve to make clear the idea of the proof.

From the first inequality in (3.31b) we have

$$\beta_{\tau_i} = \alpha_{\tau_i} \circ \lambda_{\tau_i} \quad \text{with} \quad \lambda_{\tau_i} = (\neg \alpha_{\tau_i}) \wedge \beta_{\tau_i}, \quad i = 1, 2.$$

This gives

$$\tilde{\beta}^{(1)} = \beta_{\tau_1} \circ \beta_{\tau_2} = (\alpha_{\tau_1} \circ \lambda_{\tau_1}) \circ (\alpha_{\tau_2} \circ \lambda_{\tau_2})$$

which implies

$$\begin{aligned} \left\{ \tilde{\beta}^{(1)} \right\} &= \left\{ \alpha_{\tau_1} \circ \alpha_{\tau_2} \right\} \odot \left\{ \alpha_{\tau_1} \circ \lambda_{\tau_2} \right\} \odot \left\{ \lambda_{\tau_1} \circ \alpha_{\tau_2} \right\} \odot \left\{ \lambda_{\tau_1} \circ \lambda_{\tau_2} \right\} \\ &= \left\{ \alpha_{\tau_1} \circ \alpha_{\tau_2} \circ \left(\alpha_{\tau_3} \circ \neg \alpha_{\tau_3} \right) \right\} \odot \dots \\ &= \left\{ \alpha_{\tau_1} \circ \alpha_{\tau_2} \circ \alpha_{\tau_3} \right\} \odot \underline{\epsilon} = \left\{ \tilde{\alpha}^{(1)} \right\} \odot \underline{\epsilon} \end{aligned} \quad (3.39)$$

where

$$\begin{aligned} \underline{\epsilon} &= \left\{ \alpha_{\tau_1} \circ \lambda_{\tau_2} \right\} \odot \left\{ \lambda_{\tau_1} \circ \alpha_{\tau_2} \right\} \odot \left\{ \lambda_{\tau_1} \circ \lambda_{\tau_2} \right\} \odot \left\{ \alpha_{\tau_1} \circ \alpha_{\tau_2} \circ \neg \alpha_{\tau_3} \right\} \\ &= \left\{ \lambda_{\tau_1} \circ \beta_{\tau_2} \right\} \odot \dots \end{aligned} \quad (3.40)$$

Now, from eqs. (3.31c) and (3.31b), we have

$$\alpha'_{\tau_1} \leq \lambda_{\tau_1} \quad \text{and} \quad \alpha'_{\tau_2} \leq \beta_{\tau_2}$$

which implies, recalling (3.40) and (3.38),

$$\left\{ \tilde{\alpha}^{(2)} \right\} = \left\{ \alpha'_{\tau_1} \circ \alpha'_{\tau_2} \right\} \leq \left\{ \lambda_{\tau_1} \circ \beta_{\tau_2} \right\} \leq \underline{\epsilon}. \quad (3.41a)$$

Proceeding as above [see eq. (3.39)], we have

$$\underline{\epsilon} = \left\{ \tilde{\alpha}^{(2)} \right\} \odot \dots$$

Eq. (3.39) now gives

$$\{\tilde{\beta}^{(1)}\} = \{\tilde{\alpha}^{(1)}\} \odot \{\tilde{\alpha}^{(2)}\} \odot \dots = \underline{\alpha} \odot \dots \quad (3.41b)$$

and finally

$$\underline{\beta} = \{\tilde{\beta}^{(1)}\} \odot \{\tilde{\beta}^{(2)}\} = \underline{\alpha} \odot \dots$$

verifying the claim.

Proof of Theorem (3.14). As the 'if' part of the theorem is already implied in prop. (3.12), we have only to consider the 'only if' part.

In the main part of the proof below, we shall ignore possible complications arising from support considerations. Appropriate remarks relating to this will be made at the end of the proof.

For $\underline{\alpha} = 0$, the desired result follows trivially from the first of the order relations in (3.32) and eq. (3.35). If $\underline{\alpha} = \underline{1}$, the second relation in (3.32) implies that $\underline{\alpha} \leq \underline{\beta}$ if and only if $\underline{\beta} = \underline{1}$; the desired result now holds trivially with $\underline{\gamma} = \underline{0}$. We, therefore, need consider only the case when $\underline{\alpha}$ is a generic element of Ω represented as in eq. (3.26). In this case $\underline{\beta}$ is either generic or equals $\underline{1}$.

Let

$$\underline{\alpha} = \{\tilde{\alpha}^{(1)}\} \odot \{\tilde{\alpha}^{(2)}\} \odot \dots ; \underline{\beta} = \{\tilde{\beta}^{(1)}\} \odot \{\tilde{\beta}^{(2)}\} \odot \dots$$

When $\underline{\beta} = \underline{1}$, we have, say, $\tilde{\beta}^{(1)} = \bar{1}$ and $\tilde{\beta}^{(k)} = \bar{0}$ for $k > 1$. As a representative of $\tilde{\beta}^{(1)}$, we may choose any $\beta \in \mathcal{U}$ with $\text{supp}(\beta)$ any orientable subset of $N(\mathcal{U})$ and $\beta_\tau = 1_\tau$ for all $\tau \in \text{supp}(\beta)$.

The relation $\underline{\alpha} \leq \underline{\beta}$ means that, for every $\tilde{\alpha}^{(j)}$, there is a $\tilde{\beta}^{(k)}$ such that

$$\tilde{\alpha}^{(j)} \leq \tilde{\beta}^{(k)}. \quad (3.42a)$$

In general, there may be more than one $\tilde{\alpha}^{(j)}$'s satisfying (3.42a) with the same $\tilde{\beta}^{(k)}$ (for example, $\tilde{\alpha}^{(1)}$ and $\tilde{\alpha}^{(2)}$ in the example above). There may be some $\tilde{\beta}^{(k)}$'s not involved in any relation of the form (3.42a) (for example $\tilde{\beta}^{(2)}$ in the example above); these will be called 'spectators' and the remaining ones 'active'. The spectator $\tilde{\beta}^{(k)}$'s clearly go as 'summands' in $\underline{\gamma}$ as in the above example.

To prove the theorem, it is clearly adequate to consider a general active $\tilde{\beta}^{(k)}$ such that

$$\{\tilde{\alpha}^{(p)}\} \leq \{\tilde{\beta}^{(k)}\} \quad \text{for } p = j, j', j'', \dots \quad (3.42b)$$

and show that

$$\{\tilde{\beta}^{(k)}\} = \left(\{\tilde{\alpha}^{(j)}\} \odot \{\tilde{\alpha}^{(j')}\} \odot \dots \right) \odot \underline{\rho}^{(k)} \quad (3.43)$$

where the entries in the parentheses are in one-to-one correspondence with the values of p in (3.42b). [See eq. (3.41b) above.] To prove (3.43), it is, in turn, adequate to prove that

$$\{\tilde{\beta}^{(k)}\} = \{\tilde{\alpha}^{(j)}\} \odot \underline{\lambda}^{(\dots)} \quad (3.44)$$

$$\underline{\lambda}^{(\dots)} = \{\tilde{\alpha}^{(q)}\} \odot \dots \quad \text{for } q = j', j'', \dots \quad (3.45)$$

We shall prove (3.44) and (3.45) by following steps parallel to those in the example above.

When $\underline{\beta}$ is generic, the relation (3.42b) implies that $\text{supp}(\tilde{\beta}^{(k)}) \subset \text{supp}(\tilde{\alpha}^{(p)})$ and

$$\tilde{\alpha}_{\tau}^{(p)} \leq \tilde{\beta}_{\tau}^{(k)} \quad \text{for all } \tau \in \text{supp}(\tilde{\beta}^{(k)}). \quad (3.46)$$

Eq. (3.11) now gives

$$\tilde{\beta}_{\tau}^{(k)} = \tilde{\alpha}_{\tau}^{(p)} \otimes \tilde{\epsilon}_{\tau}^{(k,p)} \text{ for all } \tau \in \text{supp } \tilde{\beta}^{(k)}. \quad (3.47)$$

Now, given an element $\mu \in \mathcal{U}$ and an orientable subset A of $\mathcal{N}(\mathcal{T})$ containing $\text{supp}(\mu)$, we can always extend μ to another element $\mu' \in \mathcal{U}$ with $\text{supp}(\mu') = A$ by taking

$$\mu'_{\tau} = \begin{cases} \mu_{\tau} & \text{for } \tau \in \text{supp}(\mu) \\ 1_{\tau} & \text{for } \tau \in A - \text{supp}(\mu) \end{cases}. \quad (3.48)$$

We shall call such an extension *trivial* (for the obvious reason that the corresponding element of $\tilde{\mathcal{U}}$ is left unchanged by such an extension).

Using such a trivial extension, if necessary, we can choose representatives $\alpha^{(p)}$ and $\beta^{(k)}$ of $\tilde{\alpha}^{(p)}$ ^{and} $\tilde{\beta}^{(k)}$ respectively such that $\text{supp}(\alpha^{(p)}) = \text{supp}(\beta^{(k)})$. Eq. (3.47) now gets replaced by

$$\beta_{\tau}^{(k)} = \alpha_{\tau}^{(p)} \otimes \epsilon_{\tau}^{(k,p)} \text{ for all } \tau \in \text{supp}(\alpha^{(p)}). \quad (3.49)$$

When $\underline{\beta} = \underline{1}$ we have, with $\beta \in \mathcal{U}$ introduced as above, the analogue of eq. (3.49) directly :

$$\beta_{\tau} (= 1_{\tau}) = \alpha_{\tau}^{(p)} \otimes \neg \alpha_{\tau}^{(p)} \text{ for all } \tau \in \text{supp}(\alpha^{(p)}). \quad (3.50)$$

Now consider eq. (3.49) for $p = j$ and q where q is any of the values $j', j'', \dots$ of p in (3.42b). Since $\alpha^{(j)} \perp \alpha^{(q)}$, we have

$$\alpha_{\tau_0}^{(q)} \leq \epsilon_{\tau_0}^{(k,j)} \text{ for some } \tau_0 \in (\text{supp } \alpha^{(j)}) \cap (\text{supp } \alpha^{(q)}). \quad (3.51)$$

Appealing to eq. (3.11) once again, we have

$$\epsilon_{\tau_0}^{(k,j)} = \alpha_{\tau_0}^{(q)} \oplus \dots$$

Eq. (3.49) now gives

$$\beta_{\tau_0}^{(k)} = \alpha_{\tau_0}^{(j)} \otimes \alpha_{\tau_0}^{(q)} \otimes \dots \quad (3.52)$$

Let

$$\beta^{(k)} = \lambda' \circ \beta_{\tau_0}^{(k)} \circ \lambda'' \quad (3.53)$$

where λ' and λ'' are elements of $\mathcal{U}$ representing entries in the irreducible decomposition of $\beta^{(k)}$ 'before' and 'after' τ_0 respectively. From eqs. (3.53) and (3.52), we have

$$\{\beta^{(k)}\} = \{\lambda' \circ \alpha_{\tau_0}^{(j)} \circ \lambda''\} \otimes \{\lambda' \circ \alpha_{\tau_0}^{(q)} \circ \lambda''\} \otimes \dots \quad (3.54)$$

Recalling eq. (3.49) (with $p = j, q$) we have

$$\{\lambda' \circ \alpha_{\tau_0}^{(j)} \circ \lambda''\} = \{\alpha^{(j)}\} \otimes \dots \quad (3.55a)$$

and

$$\{\lambda' \circ \alpha_{\tau_0}^{(q)} \circ \lambda''\} = \{\alpha^{(q)}\} \otimes \dots \quad (3.55b)$$

Eqs. (3.54) and (3.55a,b) together complete the proof of eqs. (3.44) and (3.45) and hence that of the existence of the promised $\underline{\gamma}$ modulo possible support problems (arising from support requirements in the definition of elements of Ω).

All support problems are taken care of by the following two observations :

(i) Since $\underline{\alpha} \in \Omega$, the union of supports of any finite subcollection of $\tilde{\alpha}^{(j)}$'s is given as orientable.

(ii) The right hand sides of all equations in the proof involving possibly infinite $\otimes$ operations are such that the union of

supports of any finite number of 'summands' equals the union of supports of a finite number of $\tilde{\alpha}^{(j)}$'s.

The uniqueness of $\underline{\gamma}$ follows immediately from prop. 3.13(ii). QED.

Corol. (3.15) For every $\underline{\alpha} \in \Omega$, there is a unique $\underline{\gamma} \in \Omega$ (which is $\perp$ to $\underline{\alpha}$) such that $\underline{\alpha} \odot \underline{\gamma} = \underline{1}$.

Proof. Take $\underline{\beta} = \underline{1}$ in theorem (3.14). QED.

We denote the history proposition $\underline{\gamma}$ given by corol. (3.15) as $\neg \underline{\alpha}$ and call it the *negation* of $\underline{\alpha}$. It is the unique history proposition which is realized if and only if $\underline{\alpha}$ is not realized.

Clearly, the mapping $\neg : \Omega \rightarrow \Omega$ taking $\underline{\alpha}$ to $\neg \underline{\alpha}$ is a bijection satisfying the conditions

$$(i) \quad \neg(\neg \underline{\alpha}) = \underline{\alpha} \quad (ii) \quad \underline{\alpha} \odot \neg \underline{\alpha} = \underline{1} \quad (iii) \quad \neg \underline{0} = \underline{1} ; \neg \underline{1} = \underline{0}.$$

These are the traditional defining properties of negation (or *orthocomplementation*).

Prop. (3.16) Given $\underline{\alpha}, \underline{\beta} \in \Omega$ which are support-compatible, we have $\underline{\alpha} \perp \underline{\beta}$ if and only if $\underline{\alpha} \leq \neg \underline{\beta}$.

Proof. (i) Given $\underline{\alpha} \perp \underline{\beta}$. Let $\underline{\gamma} = \underline{\alpha} \odot \underline{\beta}$. We have

$$\underline{1} = \underline{\gamma} \odot \neg \underline{\gamma} = \underline{\alpha} \odot \underline{\beta} \odot (\neg \underline{\gamma}) = \underline{\beta} \odot \neg \underline{\beta}$$

giving

$$\underline{\alpha} \odot (\neg \underline{\gamma}) = \neg \underline{\beta}$$

which implies, by eq. (3.38), $\underline{\alpha} \leq (\neg \underline{\beta})$.

(ii) Given $\underline{\alpha} \leq \neg \underline{\beta}$. We have $\neg \underline{\beta} = \underline{\alpha} \odot \underline{\delta}$, say. Now $(\neg \underline{\beta}) \perp \underline{\beta}$ implies $\underline{\alpha} \perp \underline{\beta}$ (recall the definition of $\odot$ in Ω). QED.

It is clear from the developments in this section that Ω is an orthoalgebra as envisaged in the scheme of Isham and Linden⁹.

E. States and Observables; Temporal Evolution

As the propositions in each $\mathcal{U}_\tau$ have the structure of a logic, we can define observables, states and probability distributions of observables as in the traditional proposition calculus. Of course, since $\mathcal{U}_\tau$'s for different τ 's are generally not isomorphic, a similar non-isomorphism of spaces of observables and states at different τ 's will hold in the present formalism.

A state at 'time' $\tau \in \mathcal{N}(\mathcal{T})$ is a kind of generalized probability on $\mathcal{U}_\tau$; more precisely, it is a map $p_\tau : \mathcal{U}_\tau \rightarrow \mathbb{R}$ such that

- (i) $0 \leq p_\tau(\alpha) \leq 1$ for each $\alpha \in \mathcal{U}_\tau$;
- (ii) $p_\tau(0_\tau) = 0$, $p_\tau(1_\tau) = 1$;
- (iii) given a sequence $\alpha_1, \alpha_2, \dots$ in $\mathcal{U}_\tau$ which is pairwise disjoint (i.e. $\alpha_i \leq \neg \alpha_j$ for $i \neq j$), we have

$$p_\tau \left(\bigvee_i \alpha_i \right) = \sum_i p_\tau(\alpha_i). \quad (3.57)$$

We shall denote the collection of states at 'time' τ by $\mathcal{S}(\tau)$.

An *observable* at 'time' τ is a map $A_\tau : B(\mathbb{R}) \rightarrow \mathcal{U}_\tau$ [where $B(\mathbb{R})$ is the σ -algebra of Borel sets in $\mathbb{R}$ - the smallest σ -algebra containing all the open intervals] such that

- (i) $A_\tau(\emptyset) = 0_\tau$, $A_\tau(\mathbb{R}) = 1_\tau$;
- (ii) given disjoint sets E, F in $B(\mathbb{R})$, we have $A_\tau(E)$ and $A_\tau(F)$ disjoint;
- (iii) if $E_1, E_2, \dots$ is an at most countable collection of pairwise disjoint sets in $B(\mathbb{R})$, we have

$$A_\tau \left(\bigcup_k E_k \right) = \bigvee_k A_\tau(E_k). \quad (3.58)$$

We shall denote the collection of observables at 'time' τ by $\mathcal{O}(\tau)$.

The probability distribution of the observable A_τ in the state p_τ is given by the measure $\mu^{(A_\tau, p_\tau)}$ on the measurable space $(R, B(R))$ given by

$$\mu^{(A_\tau, p_\tau)}(E) = p_\tau(A_\tau(E)) \quad \text{for all } E \in B(R). \quad (3.59)$$

It is worth remarking here that the 'time'-dependence of our states and observables corresponds neither to the Schrödinger picture nor Heisenberg picture (our states and observables both have τ -dependence). This is because we are not operating in the framework of a fixed logic; we have different (generally nonisomorphic) logics at different τ 's. When all the $\mathcal{U}_\tau$'s are isomorphic, we can recover the (analogue of) Heisenberg picture by referring all the states p_τ 's in various $\mathcal{P}(\tau)$'s to a single $\mathcal{P}(\tau)$, say $\mathcal{P}(\tau_0)$; similarly, a Schrödinger type picture is realized by referring all the observables to $\mathcal{O}(\tau_0)$.

It appears natural to assume that the temporal evolution of the system is given by homomorphisms (structure preserving mappings) of logics; we do this in our next axiom.

A_4 (Temporal Evolution Axiom): The temporal evolution of the system is given by a family of mappings $V(\tau', \tau) : \mathcal{U}_\tau \rightarrow \mathcal{U}_{\tau'}$, [defined for each pair $\tau, \tau' \in \mathcal{N}(\mathcal{T})$ such that $\tau \triangleleft \tau'$] such that

- (i) $V(\tau', \tau)$ is onto ;
- (ii) $V(\tau', \tau)(0_\tau) = 0_{\tau'}$, $V(\tau', \tau)(1_\tau) = 1_{\tau'}$;

(iii) given $\alpha^{(1)}, \alpha^{(2)}, \dots \in \mathcal{U}_\tau$, we have

$$V(\tau', \tau) \left(\bigvee_j \alpha^{(j)} \right) = \bigvee_j V(\tau', \tau) (\alpha^{(j)})$$

$$V(\tau', \tau) \left(\bigwedge_j \alpha^{(j)} \right) = \bigwedge_j V(\tau', \tau) (\alpha^{(j)})$$

(iv) $V(\tau', \tau) (\neg \alpha) = \neg [V(\tau', \tau) (\alpha)]$ for all $\alpha \in \mathcal{U}_\tau$;

(v) if $\tau \triangleleft \tau'$, $\tau' \triangleleft \tau''$ and $\tau \triangleleft \tau''$, we have

$$V(\tau'', \tau') \cdot V(\tau', \tau) = V(\tau'', \tau)$$

where the dot between the V's indicates composition of mappings. (Note: These requirements have not been stated in minimal terms; there is some redundancy.) It is easily seen by employing eqs. (3.11) and (iii) and (iv) above that the evolution mappings $V(\dots)$ preserve partial order, i.e. given $\alpha \leq \beta$ in $\mathcal{U}_\tau$, we have $V(\tau', \tau) (\alpha) \leq V(\tau', \tau) (\beta)$ in $\mathcal{U}_{\tau'}$.

A comment is in order about the condition (i) above. It reflects the view that everything that is there in the system at a 'later time' must have evolved from something in the system (supposedly closed) at an 'earlier time'. An evolution of a closed system (through structure preserving mapping) which is not onto does not appear to make physical sense.

What about the property of $V(\tau', \tau)$ being injective (one-one)? In fact, structure preserving mappings of logics are generally taken to be injective¹³. Let us say that an evolution is *nondegenerate* if the evolution mapping is injective; otherwise it is *degenerate*. For a nondegenerate evolution, the mapping $V(\tau', \tau)$ will be bijective and therefore an isomorphism of the

logics $\mathcal{U}_\tau$ and $\mathcal{U}_{\tau'}$. It follows that, if the generality of non-isomorphic $\mathcal{U}_\tau$'s is to be retained, the possibility of degenerate evolution must be admitted.

In situations where the $\mathcal{U}_\tau$'s are mutually isomorphic, we shall generally take the evolution maps $V(\tau', \tau)$ to be bijective.

The evolution described by $V(\tau', \tau)$ induces a mapping from $\mathcal{O}(\tau)$ to $\mathcal{O}(\tau')$ which maps $A_\tau \in \mathcal{O}(\tau)$ to $A_{\tau'} \in \mathcal{O}(\tau')$ given by

$$A_{\tau'}(E) = V(\tau', \tau)(A_\tau(E)) \text{ for all } E \in B(R). \quad (3.60)$$

If the mapping $V(\tau', \tau)$ is invertible, we also have a mapping on states taking $p_\tau \in \mathcal{P}(\tau)$ to $p_{\tau'} \in \mathcal{P}(\tau')$ given by

$$p_{\tau'}(\beta) = p_\tau(V(\tau', \tau)^{-1}(\beta)) \text{ for all } \beta \in \mathcal{U}_{\tau'}. \quad (3.61)$$

The probability distribution $\mu^{(A_{\tau'}, p_{\tau'})}$ of $A_{\tau'}$ in the state $p_{\tau'}$ is mapped to the probability distribution $\mu^{(A_\tau, p_\tau)}$ of A_τ in the state p_τ , given by, for every $E \in B(R)$,

$$\begin{aligned} \mu^{(A_{\tau'}, p_{\tau'})}(E) &= p_{\tau'}(A_{\tau'}(E)) \\ &= p_{\tau'}(V(\tau', \tau)(A_\tau(E))) \\ &= p_\tau(A_\tau(E)) \\ &= \mu^{(A_\tau, p_\tau)}(E). \end{aligned} \quad (3.62)$$

The invariance of the probability measure μ reflected in eq. (3.62) is essentially a consequence of the 'picture' we are employing (see the remark after eq. 3.59 above). To see this, we

consider the corresponding situation in traditional quantum mechanics. Let $\hat{A}$ be a self-adjoint operator and E_λ its spectral family. In any state Ψ , we have

$$(\Psi, \hat{A}\Psi) = \int_{\mathbb{R}} \lambda d(\Psi, E_\lambda \Psi). \quad (3.63)$$

The probability distribution $\mu^{(\hat{A}, \Psi)}$ is given by

$$\mu^{(\hat{A}, \Psi)}(E) = \int_E d(\Psi, E_\lambda \Psi) \text{ for all } E \in \mathcal{B}(\mathbb{R}). \quad (3.64)$$

Now consider time dependence of states and observables (in our composite picture) given by the unitary evolution operator $U(t', t)$:

$$\begin{aligned} \Psi(t') &= U(t', t) \Psi(t) \\ \hat{A}(t') &= U(t', t) \hat{A}(t) U(t', t)^{-1} \end{aligned} \quad (3.65)$$

This gives

$$(\Psi(t'), \hat{A}(t') \Psi(t')) = (\Psi(t), \hat{A}(t) \Psi(t)) \quad (3.66)$$

which implies

$$\mu^{(\hat{A}_{t'}, \Psi(t'))} = \mu^{(\hat{A}_t, \Psi(t))} \quad (3.67)$$

as claimed.

F. Decoherence Functionals

Our fifth axiom relates to decoherence functionals. It has two parts :

$A_5(a)$ (Decoherence Functionals): Given a state $p_0 \in \mathcal{P}(\tau_0)$ for some $\tau_0 \in \mathcal{N}(\mathcal{I})$ and a law of evolution $V(\tau', \tau)$, we have a decoherence functional $d = d_{p_0, V}$ which is a mapping $d : \Omega \times \Omega \rightarrow \mathbb{C}$

such that

- (i) $d(\underline{\alpha}, \underline{\beta})^* = d(\underline{\beta}, \underline{\alpha})$ (hermiticity)
- (ii) $d(\underline{\alpha}, \underline{\alpha}) \geq 0$ (positivity)
- (iii) If $\underline{\alpha}, \underline{\beta}, \dots$ is an at most countable collection of pairwise disjoint elements of Ω , we have

$$d(\underline{\alpha} \circ \underline{\beta} \circ \dots, \underline{\gamma}) = d(\underline{\alpha}, \underline{\gamma}) + d(\underline{\beta}, \underline{\gamma}) + \dots \quad (3.68)$$

(countable additivity)

- (iv) $d(\underline{1}, \underline{1}) = 1$ (normalization).

The space of decoherence functionals will be denoted as $\mathcal{D}$.

A complete set $\mathcal{C}$ of history propositions is said to be (weakly) *decoherent* (or *consistent*) with respect to a decoherence functional d if

$$\operatorname{Re} d(\underline{\alpha}, \underline{\beta}) = 0, \quad \underline{\alpha}, \underline{\beta} \in \mathcal{C}, \quad \underline{\alpha} \neq \underline{\beta}. \quad (3.69)$$

$A_5(b)$ (Probability Interpretation): The probability that a history $\underline{\alpha}$ in a complete set $\mathcal{C}$ which is decoherent with respect to a decoherence functional d is realized is given by

$$P(\underline{\alpha}) = d(\underline{\alpha}, \underline{\alpha}). \quad (3.70)$$

For $\underline{\alpha}, \underline{\beta} \in \mathcal{C}$, we have the classical probability sum rule :

$$\begin{aligned} P(\underline{\alpha} \circ \underline{\beta}) &= d(\underline{\alpha} \circ \underline{\beta}, \underline{\alpha} \circ \underline{\beta}) \\ &= d(\underline{\alpha}, \underline{\alpha}) + d(\underline{\beta}, \underline{\beta}) + 2 \operatorname{Re} d(\underline{\alpha}, \underline{\beta}) \\ &= P(\underline{\alpha}) + P(\underline{\beta}). \end{aligned} \quad (3.71)$$

Finally

$$\begin{aligned} 1 = d(\underline{1}, \underline{1}) &= d(\underline{\alpha} \circ \underline{\beta} \circ \dots, \underline{\alpha} \circ \underline{\beta} \circ \dots) \\ &= \sum_{\underline{\alpha} \in \mathcal{C}} P(\underline{\alpha}). \end{aligned} \quad (3.72)$$

We shall give an explicit construction of $d_{p_0, V}$ in a subclass of theories in subsection G below.

G. Hilbert space-based Theories

An important subclass of the class of theories covered by the formalism presented above are those in which, corresponding to each $\tau \in \mathcal{N}(\mathcal{T})$, we have a separable Hilbert space $\mathcal{H}_\tau$ (there being generally no natural isomorphism between $\mathcal{H}_\tau$'s for different τ 's) and $\mathcal{U}_\tau = \mathcal{P}(\mathcal{H}_\tau)$, the lattice of projection operators in $\mathcal{H}_\tau$. Here $\mathcal{T}$ is a general space of temporal supports satisfying the conditions of axioms A_1 and A_2 . The object $\mathcal{U}$ in such a theory is a psg whose elements α have irreducible decompositions of the form

$$\alpha = \left[\dots \alpha_{\tau_1} \circ \alpha_{\tau_2} \circ \alpha_{\tau_3} \circ \dots \right] \quad (3.73)$$

where

$$\dots \triangleleft \tau_1 \triangleleft \tau_2 \triangleleft \tau_3 \triangleleft \dots \quad (3.74)$$

and the bracket in eq. (3.73) is to serve as reminder that the decompositions can be at most semi-infinite.

In a straightforward extension of the HPO formalism of section II E, elements of $\mathcal{U}$ (or, more appropriately, of $\tilde{\mathcal{U}}$) are homogeneous projection operators in the Hilbert space

$$\tilde{\mathcal{H}} = \bigotimes_{\tau \in \mathcal{N}(\mathcal{T})}^W \mathcal{H}_\tau. \quad (3.75)$$

Corresponding to α of eq. (3.73), we have a projection operation $\bar{\alpha}$ in $\tilde{\mathcal{H}}$ given by [compare eq. (2.22)]

$$\bar{\alpha} = \left[\dots \otimes \alpha_{\tau_1} \otimes \alpha_{\tau_2} \otimes \alpha_{\tau_3} \otimes \dots \right]. \quad (3.76)$$

Objects like $\bar{\alpha}$ of eq. (3.76), in which the number of nontrivial

α_{τ_1} 's is not restricted to be finite, should be understood as elements of the space

$$\mathcal{B} = \bigotimes_{\tau \in \mathcal{N}(\mathcal{I})} \mathcal{B}(\mathcal{H}_{\tau}). \quad (3.77)$$

Here $\mathcal{B}(\mathcal{H}_{\tau})$ denotes the algebra of bounded linear operators on $\mathcal{H}_{\tau}$. The construction⁸ of $\mathcal{B}$ is similar to that of $\tilde{\mathcal{H}}$. One first defines a space $\mathcal{A}$ (analogous to the space $\mathcal{F}$ of section IIE) whose general element A is a map from $\mathcal{N}(\mathcal{I})$ to the Cartesian product

$$X = \prod_{\tau \in \mathcal{N}(\mathcal{I})} \mathcal{B}(\mathcal{H}_{\tau}) \quad (3.78)$$

such that $A(\tau) = I_{\tau}$ (the identity operator on $\mathcal{H}_{\tau}$) for all but a finite set of values of τ . Clearly $\mathcal{A}$ is a subset of X [identifying $A \in \mathcal{A}$ with the set of values $A(\tau)$]. Algebraic operations on $\mathcal{A}$ are defined in the obvious manner :

$$(aA + bB)(\tau) = aA(\tau) + bB(\tau) ; \quad AB(\tau) = A(\tau)B(\tau)$$

for all $a, b \in \mathbb{C}$ and $\tau \in \mathcal{N}(\mathcal{I})$.

Weak operator topologies on $\mathcal{B}(\mathcal{H}_{\tau})$'s define the corresponding product topology π_w on X . The space $\mathcal{B}$ is defined as the closure of $\mathcal{A}$ in the topological space (X, π_w) .

The process of association of homogeneous projection operators in $\mathcal{B}$ with elements of $\mathcal{U}$ takes care of the formality of going from the triple $(\mathcal{U}, \mathcal{I}, \sigma)$ to $(\tilde{\mathcal{U}}, \tilde{\mathcal{I}}, \tilde{\sigma})$. This is because the superfluous I_{τ} 's get automatically ignored in passing from α to $\bar{\alpha}$ above and presence of a 0_{τ} anywhere in α makes $\bar{\alpha} = 0$.

The operations in $\tilde{\mathcal{U}}$ described in section IIIC get translated into those on homogeneous projection operators in $\tilde{\mathcal{H}}$.

Partial order ($\leq$) translates to the partial order among projectors. Disjointness gets translated to orthogonality of projectors. The operation $\oplus$ becomes orthogonal sum. Finally, the meet operation is easily seen to correspond to the meet operation on projectors (which is equivalent to that on the subspaces of the Hilbert space).

Elements of Ω are inhomogeneous projection operators in $\tilde{\mathcal{H}}$ expressible as countable sums of mutually orthogonal homogeneous projection operators. Statements in the previous para apply, with due alteration of details, also to the operations in Ω described in section III D. The negation operation $\neg$ translates into subtraction from the unit operator $\tilde{I}$.

For each $\tau \in \mathcal{N}(\mathcal{T})$, the state space $\mathcal{P}(\tau)$ is the family of density operators in $\mathcal{H}_\tau$ and the space $\mathcal{O}(\tau)$ of observables is the family of projectors of the form [see eq. (3.64)]

$$P^{(\hat{A})}(E) = \int_E dE_\lambda \quad (3.79)$$

where $\hat{A}$ is a self adjoint operator in $\mathcal{H}_\tau$ and E is a Borel set in $\mathbb{R}$. The maps $P^{(\cdot)} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{H}_\tau)$ are in one to one correspondence with selfadjoint operators in $\mathcal{H}_\tau$.

The histories version of traditional quantum mechanics presented in section II is clearly a special case of these theories with $\mathcal{T} = \mathcal{K}_1$ of section II F [which implies $\mathcal{N}(\mathcal{T}) = \mathbb{R}$, the traditional time] and, for each $\tau \in \mathbb{R}$, $\mathcal{H}_\tau$ an isomorphic copy of the quantum mechanical Hilbert space $\mathcal{H}$.

The decoherence functional (2.9) can be easily extended to finite histories in general Hilbert space-based theories. The

one constructed below [eq. (3.83)] is essentially a generalization of eq. (4.19) of ref. (9) to the case of general temporal support space $\mathcal{T}$. Let

$$\alpha = \alpha_{\tau_1} \circ \dots \circ \alpha_{\tau_n}, \quad \beta = \beta_{\tau'_1} \circ \dots \circ \beta_{\tau'_m}$$

be two histories with the same initial state $\rho(\tau_0)$. It is understood here that $\tau_0 \triangleleft \tau_1$ and $\tau_0 \triangleleft \tau'_1$. For every pair $\tau, \tau' \in \mathcal{N}(\mathcal{T})$ with $\tau \triangleleft \tau'$, let $K(\tau', \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_{\tau'}$ be the linear evolution maps satisfying the condition

$$K(\tau'', \tau') K(\tau', \tau) = K(\tau'', \tau) \quad (3.80)$$

for all triples τ, τ', τ'' such that $\tau \triangleleft \tau' \triangleleft \tau''$ and $\tau \triangleleft \tau''$. Let us assume that, there exists $\tau_f \in \mathcal{N}(\mathcal{T})$ such that

$$\tau_n \triangleleft \tau_f \quad \text{and} \quad \tau'_m \triangleleft \tau_f. \quad (3.81)$$

[The existence of such a τ_f (which is trivial when $\mathcal{N}(\mathcal{T}) = \mathbb{R}$) is not guaranteed by our axioms and is a new assumption.] Let [compare eq. (2.6)]

$$C'_\alpha = K(\tau_f, \tau_n) \alpha_{\tau_n} K(\tau_n, \tau_{n-1}) \dots \alpha_{\tau_2} K(\tau_2, \tau_1) \alpha_{\tau_1} K(\tau_1, \tau_0) \quad (3.82)$$

with a similar expression for C'_β . We now define

$$d(\alpha, \beta) = N \text{Tr} [C'_\alpha \rho(\tau_0) C'^\dagger_\beta] \quad (3.83a)$$

with

$$N^{-1} = \text{Tr} [A \rho(\tau_0) B^+] \quad (3.83b)$$

where A and B are operators obtained from C'_α and C'_β respectively

by putting each of the α_{τ_i} and $\alpha_{\tau'_i}$ equal to the unit operator. It is implicitly assumed here that the right hand side of eq. (3.83b) does not vanish. If the temporal relations among the temporal supports of α and β are such that compositions of the form (3.80) are permitted for all successive pairs, we have $A = K(\tau_f, \tau_o)$ and $B = K(\tau_f, \tau_o)^\dagger$. Now, if the maps $K(\dots)$ satisfy the condition

$$K(\tau', \tau)^\dagger K(\tau', \tau) = I_\tau, \quad (3.84)$$

we have $N = 1$.

To be consistent, we must connect the maps $K(\tau', \tau)$ to the evolution maps $V(\tau', \tau)$ postulated in axiom A_4 . In the present case the latter are maps $V(\tau', \tau) : \mathcal{P}(\mathcal{H}_\tau) \rightarrow \mathcal{P}(\mathcal{H}_{\tau'})$ satisfying the conditions of axiom A_4 (i.e. they are onto and logic homomorphisms). To have this evolution implemented by maps $K(\tau', \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_{\tau'}$, employed above, it appears necessary to assume that $V(\dots)$'s are injective (one-one) as well. In this case they are logic isomorphisms; Wigner's theorem¹¹ then implies the existence of unitary or antiunitary maps $K(\tau', \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_{\tau'}$, such that, for any $P \in \mathcal{P}(\mathcal{H}_\tau)$,

$$V(\tau', \tau)(P) = K(\tau', \tau) P K(\tau', \tau)^\dagger. \quad (3.85)$$

Eq.(3.80) shows that $K(\tau'', \tau)$ is unitary when $K(\tau', \tau)$ and $K(\tau'', \tau')$ are both unitary or both antiunitary. The simplest possibility, therefore, is that all $K(\tau', \tau)$ are unitary.

Conversely, if linear maps $K(\tau', \tau)$ are given and are used to define $V(\tau', \tau)$ through eq. (3.85), we must have the relation

(3.84) if the condition $P^2 = P$ is to be preserved. The condition that the map $V(\tau', \tau)$ maps the identity operator I_τ to $I_{\tau'}$, demands that

$$K(\tau', \tau) K(\tau', \tau)^+ = I_{\tau'}, \quad (3.86)$$

implying that $K(.,.)$ be unitary.

The subclass of theories considered in this subsection is general enough to include history version of Hilbert space-based quantum field theory on general (i.e. not necessarily globally hyperbolic) spacetimes. (See section V for more details on this.)

H. Classical Mechanics

As promised earlier, we shall now present history version of classical mechanics in the framework of theories described in sections III B-F.

We consider classical mechanics of a system with phase space Γ . For simplicity we take $\mathcal{T} = \mathcal{K}_1$ of section II F which gives $\mathcal{N}(\mathcal{T}) = \mathbb{R}$, the traditional space for the flow of time. For each $t \in \mathbb{R}$, the space $\mathcal{U}_t$ of 'single time propositions' at time t is the collection $B(\Gamma)$ of all Borel subsets of Γ which is a Boolean logic. A general element of $\mathcal{U}$ is a collection

$$\alpha = \left\{ \alpha(t_1), \dots, \alpha(t_n); t_i \in \mathbb{R}, t_1 < \dots < t_n; \alpha(t_i) \in \mathcal{U}_{t_i} \right\} \quad (3.87a)$$

which is represented, in the notation of section III B, as

$$\alpha = \alpha(t_1) \circ \alpha(t_2) \circ \dots \circ \alpha(t_n). \quad (3.87b)$$

The transition from $\mathcal{U}$ to $\tilde{\mathcal{U}}$ is made by the stipulation that the

null history $\tilde{0}$ and the unit history $\tilde{1}$ are to be taken as a single element of $\tilde{U}$ [Corresponding, respectively, to situations where $\alpha(t) = \phi$ for at least one $t \in \text{supp}(\alpha)$ and $\alpha(t) = \Gamma$ for all $t \in \text{supp}(\alpha)$] and all other elements of $\tilde{U}$ are $\alpha(t)$'s of the form (3.87) (now called $\tilde{\alpha}(t)$) such that $\tilde{\alpha}(t) \neq \phi$ or Γ for any $t \in \text{supp}(\tilde{\alpha})$.

Given $\tilde{\alpha}, \tilde{\beta} \in \tilde{U}$ with $\text{supp } \tilde{\alpha} = A$ and $\text{supp } \tilde{\beta} = B$, the various operations in $\tilde{U}$ are to be interpreted as follows :

- (i) $\tilde{\alpha} \leq \tilde{\beta}$ if $B \subset A$ and $\tilde{\alpha}(t) \subset \tilde{\beta}(t)$ for all $t \in B$.
- (ii) $\tilde{\alpha} \perp \tilde{\beta}$ if $A \cap B \neq \phi$ and $\tilde{\alpha}(t) \cap \tilde{\beta}(t) = \phi$ for at least one $t \in A \cap B$.
- (iii) Given $\tilde{\alpha}$ and $\tilde{\beta}$ weakly disjoint [i.e. $\tilde{\alpha}, \tilde{\beta}$ are disjoint, $\tilde{\alpha}(t_0) \cap \tilde{\beta}(t_0) = \phi$ where $t_0 \in A = B$ and $\tilde{\alpha}(t) = \tilde{\beta}(t)$ for $t \in A - \{t_0\}$] we have $\tilde{\alpha} \circ \tilde{\beta} \in \tilde{U}$ given by
$$(\tilde{\alpha} \circ \tilde{\beta})(t) = \begin{cases} \tilde{\alpha}(t_0) \cup \tilde{\beta}(t_0) & \text{for } t = t_0 \\ \tilde{\alpha}(t) (= \tilde{\beta}(t)) & \text{for } t \in A - \{t_0\} \end{cases} \quad (3.88)$$
- (iv) $\tilde{\alpha} \wedge \tilde{\beta} = \tilde{0}$ if $S \equiv \text{supp } \tilde{\alpha} \cup \text{supp } \tilde{\beta}$ is not orientable. When S is orientable, we have [see eq. (3.20)]

$$\tilde{\alpha} \wedge \tilde{\beta}(t) = \begin{cases} \tilde{\alpha}(t) & \text{for } t \in A - B \\ \tilde{\beta}(t) & \text{for } t \in B - A \\ \tilde{\alpha}(t) \cap \tilde{\beta}(t) & \text{for } t \in A \cap B. \end{cases} \quad (3.89)$$

Elements of Ω are, of course, at most countable collections of mutually disjoint elements of $\tilde{U}$. The operations $\leq$, $\perp$ and $\circ$ in Ω have straightforward description which is left to the reader. We only describe $\neg \underline{\alpha}$ for $\underline{\alpha} = \{\tilde{\alpha}^{(1)}, \tilde{\alpha}^{(2)}, \dots\}$.

First we define $\neg\{\tilde{\alpha}\}$ for any $\tilde{\alpha} \in \tilde{U}$. Let $\text{supp}(\tilde{\alpha}) = A =$

$\{t_1, \dots, t_n\}$ and let $h(\tilde{\alpha})$ be the collection of all elements of $\mathcal{U}$ of the form $\tilde{\alpha}'(t_1) \circ \dots \circ \tilde{\alpha}'(t_n)$ where $\tilde{\alpha}'(t_i) = \tilde{\alpha}(t_i)$ or $\Gamma - \tilde{\alpha}(t_i)$ ($= \neg \tilde{\alpha}(t_i)$ in the notation of section III B). The collection $h(\tilde{\alpha})$ includes $\tilde{\alpha}$ itself and history filters disjoint with $\tilde{\alpha}$; it is an element of Ω . In fact

$$h(\tilde{\alpha}) = \underline{1} ; \quad (3.90)$$

this is because $h(\tilde{\alpha})$ contains all the possible 2^n history propositions corresponding to the possible choices of the system point being in the subset $\tilde{\alpha}(t_i)$ or $\Gamma - \tilde{\alpha}(t_i)$ at time t_i for $i = 1, \dots, n$. It, therefore, represents a history proposition which is sure to be realized. We now have

$$\neg \{\tilde{\alpha}\} = h(\tilde{\alpha}) - \{\tilde{\alpha}\}. \quad (3.91)$$

Now let $\underline{\alpha} = \{\tilde{\alpha}^{(1)}, \tilde{\alpha}^{(2)}, \dots\}$. Put

$$h(\underline{\alpha}) = \bigcup_i h(\tilde{\alpha}^{(i)}) \quad (3.92)$$

where the index i labels the constituents $\tilde{\alpha}^{(i)}$ of $\underline{\alpha}$. By the argument given above, we have again

$$h(\underline{\alpha}) = \underline{1} \quad (3.93)$$

and finally

$$\neg \underline{\alpha} = h(\underline{\alpha}) - \underline{\alpha}. \quad (3.94)$$

States at any time are probability measures on the measurable space $(\Gamma, \mathcal{B}(\Gamma))$. Observables at any time t are measurable maps $A_t : \mathcal{B}(R) \rightarrow \mathcal{B}(\Gamma)$. For $E \in \mathcal{B}(R)$, $A_t(E)$ represents the subset of the phase space corresponding to the observable (represented by the mapping) A_t having values in E . The probability that, at time t , the observable A_t has values in

E , the state being p_t , is

$$P\{A_t \in E \mid p_t\} = p_t(A_t(E)). \quad (3.95)$$

Temporal evolution, according to the scheme of section III E, is given, for $t < t'$, by [identifying $u_t = u_{t'} = B(\Gamma)$] a bijective measurable map $V(t', t) : B(\Gamma) \rightarrow B(\Gamma)$. This, of course, includes the traditional Hamiltonian evolution as a special case.

Given an initial state p_{t_0} and the family of temporal evolution maps $V(\dots)$, we construct the decoherence functional $d = d_{p_0, V}$ taking clue from the D_{cl} of ref. (6) [eq. IV. 5.3] constructed for Hamiltonian evolution with a given Hamiltonian. We do it for two histories of the form $\alpha = (\alpha_{t_1}, \dots, \alpha_{t_n})$ and $\beta = (\beta_{t_1}, \dots, \beta_{t_n})$ such that $t_0 < t_1 < \dots < t_n$. (The general case can be reduced to this by trivially extending the temporal supports of α and β to the union of their temporal supports.) We have [writing $V_{t', t}$ for $V(t', t)$]

$$d(\alpha, \beta) = \int_{\Gamma} d p_{t_0}(\xi_0) K^{(\alpha, \beta)}(\xi_0) \quad (3.96a)$$

where [using the notation $\xi(t; \xi_0) = V_{t, t_0}(\xi_0)$]

$$K^{(\alpha, \beta)}(\xi_0) = \begin{cases} 1 & \text{if, for every } t_j (j=1, \dots, n), \xi(t_j; \xi_0) \in \alpha_{t_j} \cap \beta_{t_j} \\ 0 & \text{otherwise.} \end{cases} \quad (3.96b)$$

Explicitly

$$K^{(\alpha, \beta)}(\xi_0) = \int_{\alpha_{t_1}} d\xi_1 \int_{\beta_{t_1}} d\eta_1 \dots \int_{\alpha_{t_n}} d\xi_n \int_{\beta_{t_n}} d\eta_n \delta(\xi_1 - \eta_1) \dots \delta(\xi_n - \eta_n).$$

$$\delta(\xi_1 - v_{t_1 t_0}(\xi_0)) \dots \delta(\xi_n - v_{t_n t_{n-1}}(\xi_{n-1})). \quad (3.97)$$

Eq. (3.96b) shows that $K^{(\alpha, \beta)}$ is ^{the} characteristic function of the subset $E_{\alpha\beta}$ of Γ consisting of those points ξ_0 for which $\xi(t_j; \xi_0)$ lies in $\alpha_{t_j} \cap \beta_{t_j}$ for each $j = 1, \dots, n$. We can, therefore write

$$d(\alpha, \beta) = p_{t_0}(E_{\alpha\beta}). \quad (3.98)$$

In most practical situations, the probability measure p_{t_0} admits a density and we have $dp_{t_0}(\xi_0) = \rho_{t_0}(\xi_0) d\xi_0$. The δ -functions $\delta(\xi_i - \eta_i)$ in eq. (3.97) ensure that $d(\alpha, \beta) = 0$ for $\alpha \neq \beta$ (in any complete set of histories) showing that the decoherence condition (3.69) is trivially satisfied. Finally, the probability $P(\alpha) = d(\alpha, \alpha)$ that the history α is realized, given by eq. (3.96), is an expected result.

Note. In the histories-^{based treatment} ~~version~~ of classical mechanics given in ref. (6), fine-grained histories [paths $\xi(t)$ in Γ] are also included; general histories are coarse-grainings of these (i.e. families of fine grained histories). Fine-grained histories can be accommodated by relaxing the requirement on $\mathcal{T}$ (and, therefore, $\mathcal{U}$) to be a special psg and take, for example, $\mathcal{T} = \mathcal{K}_3$ of section III. An element of $\mathcal{U}$ is then an assignment

$$t \rightarrow \alpha(t) \in B(\Gamma) \quad \text{for } t \in I \text{ (some interval)} \quad (3.99)$$

Most of the developments outlined above proceed in this case along parallel lines. The proof of the theorem (3.14), however, does not go through and the construction of negation of a history becomes problematic.

As a finer point, we would like to mention that, in a

serious attempt to accommodate fine-grained histories, the choice $\mathcal{T} = \mathcal{K}_3$ mentioned above, turns out, in fact, to be inadequate; one needs to replace it by an 'elder brother' of $\mathcal{K}_3$ whose elements are disjoint collections of intervals and the composition rule is defined so that the composition of two such collections is another such collection. The need for this is felt at the stage of defining $\tilde{\alpha} \wedge \tilde{\beta}$. We leave the verification of this as an interesting exercise for the reader.

Note. Just as we considered, in section III G, Hilbert space based theories with general temporal supports, we could have done the same here thereby obtaining a formalism general enough to include histories ^{- based treatments} ~~versions~~ of classical field theories on general spacetimes. As no new technicality is involved in this, we leave it as an exercise to the reader.

IV. SYMMETRIES AND CONSERVATION LAWS IN THE AUGMENTED TEMPORAL LOGIC FORMALISM

Before we take up the treatment of symmetries in the formalism of section III, it is instructive to have a look at the description of symmetries in quantum mechanics in the HPO formalism.

A. Symmetries in Traditional Quantum Mechanics in the HPO Formalism

We denote by $\mathcal{P}_1(\mathcal{H})$ the space of one-dimensional projection operators (i.e. objects of the form $P_\Psi = |\Psi\rangle \langle\Psi|$) on the quantum mechanical Hilbert space $\mathcal{H}$. In terms of these objects the transition probability formula reads

$$P(|\Psi\rangle \rightarrow |\Phi\rangle) = |\langle\Phi|\Psi\rangle|^2 = \text{Tr}(P_\Psi P_\Phi). \quad (4.1)$$

According to Wigner's theorem, given a bijection $\mathcal{P}_1(\mathcal{H}) \rightarrow \mathcal{P}_1(\mathcal{H})$ ($P \rightarrow P'$) such that

$$\text{Tr}(P'_1 P'_2) = \text{Tr}(P_1 P_2), \quad (4.2)$$

there exists a unitary or antiunitary operator U on $\mathcal{H}$ such that

$$P' = U P U^{-1} \quad \text{for all } P \in \mathcal{P}_1(\mathcal{H}). \quad (4.3)$$

Recall that, the action of such a U on $\mathcal{B}(\mathcal{H})$ is given by

$$A \rightarrow A' = U A U^{-1} \quad \text{for unitary } U \quad (4.4a)$$

$$A \rightarrow A' = U A^* U^{-1} \quad \text{for antiunitary } U. \quad (4.4b)$$

If A is selfadjoint, the expressions on the rhs in eqs (4.4a) and (4.4b) are the same; in particular, this is the case for a

(general, i.e. not necessarily one-dimensional) projection operator or a density operator.

The unitary/antiunitary operator U on $\mathcal{H}$ defines a unitary/antiunitary operator $\tilde{U}$ on $\tilde{\mathcal{H}}$ such that

$$\tilde{U} \left(\Psi_{t_1} \otimes \Psi_{t_2} \otimes \dots \otimes \Psi_{t_n} \right) = U \Psi_{t_1} \otimes U \Psi_{t_2} \otimes \dots \otimes U \Psi_{t_n}. \quad (4.5)$$

The homogeneous projection operator $\bar{\alpha}$ of eq. (2.22) transforms into

$$\bar{\alpha}' = \tilde{U} \bar{\alpha} \tilde{U}^{-1} = \alpha'_{t_1} \otimes \dots \otimes \alpha'_{t_n} \quad (4.6)$$

where

$$\alpha'_{t_j} = U \alpha_{t_j} U^{-1}. \quad (4.7)$$

This action trivially extends to inhomogeneous projectors :

$$\bar{\alpha} + \bar{\beta} \rightarrow \bar{\alpha}' + \bar{\beta}' = \tilde{U}(\bar{\alpha} + \bar{\beta}) \tilde{U}^{-1}. \quad (4.8)$$

Given $\bar{\alpha}$ of eq. (2.22) and $\bar{\beta} = \beta_{s_1} \otimes \dots \otimes \beta_{s_m}$ such that

$t_n < s_1$, the composite homogeneous history projector

$$\bar{\alpha} \circ \bar{\beta} = \alpha_{t_1} \otimes \dots \otimes \alpha_{t_n} \otimes \beta_{s_1} \otimes \dots \otimes \beta_{s_m},$$

being a homogeneous projector, transforms as in eq. (4.6) giving

$$\bar{\alpha} \circ \bar{\beta} \rightarrow \tilde{U}(\bar{\alpha} \circ \bar{\beta}) \tilde{U}^{-1} = (\tilde{U} \bar{\alpha} \tilde{U}^{-1}) \circ (\tilde{U} \bar{\beta} \tilde{U}^{-1}). \quad (4.9)$$

The transformation law (4.6), therefore, defines an automorphism of the psg $\mathcal{K}_4$ of section II F (which is nothing but the space $\mathcal{U}$ of section IIIB in the present case).

In writing eq. (4.9), we have implicitly assumed that there is no transformation of time involved. There may, in general, be a transformation of the time variable also involved :

$$t \rightarrow t' = f(t). \quad (4.10)$$

If t' is to serve as a time variable, the function f must be monotone. There are two possibilities :

- (i) f is monotone increasing. In this case $t_i < t_j$ implies $t'_i < t'_j$. Given temporal supports $A = (t_1, \dots, t_n)$ and $B = (s_1, \dots, s_m)$ such that $A \triangleleft B$ (corresponding to $t_n < s_1$) we have

$$A' = (t'_1, \dots, t'_n) \quad B' = (s'_1, \dots, s'_m) \quad (4.11a)$$

and

$$A' \triangleleft B', \quad (A \circ B)' = A' \circ B' \quad (4.11b)$$

giving an auto-morphism of the psg K_1 (which is the space $\mathcal{T}$ in the present case).

- (ii) f is monotone decreasing (time reversal $t' = -t$ is an important special case of this). In this case $t_i < t_j$ implies $t'_j < t'_i$ and we have

$$B' \triangleleft A', \quad (A \circ B)' = B' \circ A' \quad (4.12a,b)$$

giving an *antiautomorphism* of K_1 .

In this case eq. (4.12a) implies that eq. (4.9) must be replaced by an antiautomorphism of $\mathcal{U} = K_4$:

$$\bar{\alpha} \circ \bar{\beta} \rightarrow \tilde{U}(\bar{\alpha} \circ \bar{\beta}) \tilde{U}^{-1} = (\tilde{U} \bar{\beta} \tilde{U}^{-1}) \circ (\tilde{U} \bar{\alpha} \tilde{U}^{-1}). \quad (4.13)$$

Summarising, we have the Wigner symmetry implemented as in eq. (4.3) along with the transformation (4.10) of time implying, for the history system $(\mathcal{U}, \mathcal{T}, \sigma) = (\mathcal{K}_4, \mathcal{K}_1, \sigma)$ (with the homomorphism σ given by eq. 2.33)], the following :

- (i) an automorphism or antiautomorphism Φ_1 of $\mathcal{T}$ [given by $\Phi_1(A) = A'$];
- (ii) an automorphism or antiautomorphism Φ_2 of $\mathcal{U}$ (in accordance with (i) [given by $\Phi_2(\bar{\alpha}) = \tilde{U} \bar{\alpha} \tilde{U}^{-1}$];
- (iii) projections (from $\mathcal{U}$ on $\mathcal{T}$) implied by σ are preserved by the mappings Φ_1 and Φ_2 making the following diagram commutative

$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{\Phi_2} & \mathcal{U} \\
 \sigma \downarrow & & \downarrow \sigma \\
 \mathcal{T} & \xrightarrow{\Phi_1} & \mathcal{T}
 \end{array}
 \quad \text{i.e.} \quad \Phi_1 \circ \sigma = \sigma \circ \Phi_2 \quad (4.14)$$

where $\circ$ in eq. (4.14) indicates composition of mappings. Eq.(4.14) implies that, for any $t \in R [=N(\mathcal{T})$ in the present case], Φ_2 maps $\mathcal{U}_t = \mathcal{P}(\mathcal{K}_t)$ into $\mathcal{U}_{t'} = \mathcal{P}(\mathcal{K}_{t'})$ where $t' = \Phi_1(t) = f(t)$.

- (iv) For each $t \in R = N(\mathcal{T})$, the mapping $\Phi_{2t} \equiv \Phi_2|_{\mathcal{U}_t}$ is an isomorphism of the logic $\mathcal{U}_t = \mathcal{P}(\mathcal{K}_t)$ onto $\mathcal{U}_{t'} = \mathcal{P}(\mathcal{K}_{t'})$.
- (v) Eq. (4.8) and (iv) above imply that the mapping Φ_2 on $\mathcal{U} = \mathcal{K}_4$ extends to a bijective mapping $\underline{\Phi}_2$ on the space Ω of inhomogeneous projectors onto itself preserving the lattice operations in it.

Note. Whether Φ_1 and Φ_2 are automorphisms or antiautomorphisms depends only on whether f in eq. (4.10) is monotone increasing or decreasing and not on whether U (and

correspondingly $\tilde{U}$ is unitary or antiunitary. We can very well have a situation where, for example, U is antiunitary, Φ_1 is trivial and Φ_2 is an automorphism.

The transformation law of the decoherence functional $d(\alpha, \beta)$ of eq. (2.9) following from that of the initial state $\rho(t_0)$ and of the evolution operator $U(t', t)$ given by eqs. (4.4a,b) gives

$$\rho(t_0) \rightarrow U \rho(t_0) U^{-1}$$

$$C_\alpha \rightarrow \begin{cases} U C_\alpha U^{-1} & \text{for } U \text{ unitary} \\ U C_\alpha^+ U^{-1} & \text{for } U \text{ antiunitary} \end{cases} \quad (4.15a)$$

which implies that

$$d(\alpha, \beta) \rightarrow \begin{cases} d(\alpha, \beta) & \text{for } U \text{ unitary} \\ d(\alpha, \beta)^* & \text{for } U \text{ antiunitary.} \end{cases} \quad (4.15b)$$

This subsection was meant to help the reader form an idea as to what kind of mappings to introduce in the treatment of symmetries in the temporal logic formalism. Before proceeding to take up the latter, we shall have a quick look at the HVW paper¹⁵ which, as was mentioned in section II, is a work where objects closely related to histories were employed in the treatment of symmetries.

B. Symmetries in the HVW Approach

In ref. (15), HVW were mainly concerned with the formulation of invariance principles directly in terms of primitive concepts of physical theory, i.e. in terms of observations/measurements and their results. Employing the objects $\Pi(B|A)$ introduced in section IIA for this purpose, they defined an invariance transformation to be a bijection

$$\alpha \longleftrightarrow \bar{\alpha}, \quad \beta \longleftrightarrow \bar{\beta}, \quad \dots \quad (4.16)$$

[recall that the symbol α implicitly includes the time t_α of the measurement α] which leaves the Π function invariant :

$$\begin{aligned} & \Pi \left(\bar{\zeta}, r_{\bar{\zeta}}; \dots; \bar{\nu}, r_{\bar{\nu}} \mid \bar{\alpha}, r_{\bar{\alpha}}; \dots; \bar{\epsilon}, r_{\bar{\epsilon}} \right) \\ &= \Pi \left(\zeta, r_{\zeta}; \dots; \nu, r_{\nu} \mid \alpha, r_{\alpha}; \dots; \epsilon, r_{\epsilon} \right). \end{aligned} \quad (4.17)$$

[Compare eqs (4.2), (4.15).] Geometric invariances (see section II A) are the subclass of these bijections which correspond to transformations between reference frames.

HVW explored some consequences of this definition in the classical (Newtonian) mechanics of point particles, relativistic mechanics of point particles and in quantum theory. From our point of view, the main result is the generalization of Wigner's theorem given below. In the statement of this theorem, the symbol α of eq. (4.16) is taken to represent a decision measurement (Yes-No experiment) represented by a pair (P_α, t_α) where P_α is a 1-dimensional projection operator and t_α the time of measurement.

Theorem (4.1) (Generalized Wigner's Theorem) A mapping of decision measurements onto decision measurements $(P_\alpha, t_\alpha \rightarrow \bar{P}_\alpha, \bar{t}_\alpha)$ will leave the Π function invariant if and only if the following two conditions apply :

- (a) $\bar{P}_\alpha = U P_\alpha U^{-1}$ where U is a unitary or antiunitary operator mapping bijectively coherent subspaces (of the quantum mechanical Hilbert space $\mathcal{H}$ of the system in question) onto coherent subspaces.

(b) The time order of measurements is either preserved by a mapping or reversed.

We draw two conclusions from the foregoing :

- (i) Invariance of the diagonal elements $d(\alpha, \alpha)$ [see eqs. (4.15), (4.17)] of decoherence functionals must be apart of our definition of symmetry *(or an implication of it)*.
- (ii) One should generally expect a two-way connection between Wigner type symmetries and symmetries defined in the language of histories.

It is of some relevance here to note the distinction between symmetries of the formalism (unitary/antiunitary transformations in quantum mechanics, canonical transformations in classical mechanics) and symmetries of dynamics (subclass of symmetries of the formalism leaving the Hamiltonian invariant). Not every symmetry of the formalism need be a symmetry of some given Hamiltonian. For example, the parity operator P given by $(Pf)(x) = f(-x)$ is a unitary operator in $L^2(\mathbb{R})$; however, $PHP^{-1} \neq H$ for $H = -\frac{\partial^2}{\partial x^2} + x$.

An interesting point to note in theorem (4.1) is that the invariance condition (4.17) does not imply a symmetry of dynamics contrary to what one might intuitively expect. (After all histories or Π functions are supposed to contain information about dynamics.)

This can also be seen explicitly by having a close look at (2.5) for $d(\alpha, \alpha)$. A common unitary transformation on all ingredients — $\rho(t_0)$, the projectors P_j and the evolution operators $U(t_j, t_k)$ — leaves $d(\alpha, \alpha)$ invariant [in fact, it leaves $d(\alpha, \beta)$ of (2.9) invariant (see eq. 4.15)]; it does not have to leave

the Hamiltonian invariant to achieve this.

For observations on a similar situation in the context of general history theories, see remark (ii) after eq. (4.24) below.

C. Definition of Symmetry in the Augmented Temporal Logic Formalism

We shall use the following notations :

$$S = (\mathcal{U}, \mathcal{T}, \sigma) \text{ (History System)}$$

$$\tilde{S} = (\tilde{\mathcal{U}}, \tilde{\mathcal{T}}, \tilde{\sigma}) \text{ (Standardized History System)}$$

$$(S) = (\mathcal{U}, \mathcal{T}, \sigma, \Omega, D) \text{ (Augmented History System)}$$

A *morphism* from S into S' is a pair $\Phi = (\Phi_1, \Phi_2)$ of mappings such that

(i) $\Phi_1 : \mathcal{T} \rightarrow \mathcal{T}'$ is a psg homomorphism or antihomomorphism, i.e. it satisfies either (a) or (b) below.

(a) $\tau_1 \triangleleft \tau_2$ implies $\Phi_1(\tau_1) \triangleleft \Phi_1(\tau_2)$ and

$$\Phi_1(\tau_1 \circ \tau_2) = \Phi_1(\tau_1) \circ \Phi_1(\tau_2). \quad (4.18a)$$

(b) $\tau_1 \triangleleft \tau_2$ implies $\Phi_1(\tau_2) \triangleleft \Phi_1(\tau_1)$ and

$$\Phi_1(\tau_1 \circ \tau_2) = \Phi_1(\tau_2) \circ \Phi_1(\tau_1). \quad (4.18b)$$

(ii) $\Phi_2 : \mathcal{U} \rightarrow \mathcal{U}'$ is a psg homomorphism or antihomomorphism in accordance with (i) (i.e. Φ_1 and Φ_2 are either both homomorphisms or both antihomomorphisms).

(iii) The following diagram is commutative

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Phi_2} & \mathcal{U}' \\ \sigma \downarrow & & \downarrow \sigma' \\ \mathcal{T} & \xrightarrow{\Phi_1} & \mathcal{T}' \end{array} \quad \text{i.e.} \quad \Phi_1 \circ \sigma = \sigma' \circ \Phi_2. \quad (4.19)$$

Writing $\Phi_1(\tau) = \tau'$ and $\Phi_2|_{\mathcal{U}_\tau} = \Phi_{2\tau}$, eq. (4.19) implies that $\Phi_{2\tau}$ maps $\mathcal{U}_\tau$ into $\mathcal{U}'_{\tau'}$.

(iv) Each mapping $\Phi_{2\tau} : \mathcal{U}_\tau \rightarrow \mathcal{U}'_\tau$ is a morphism of logics¹³ (i.e. it is injective and preserves partial order, meet, join, orthocomplementation and null and unit elements).

A morphism is called an *isomorphism* if the mappings Φ_1 and Φ_2 are bijective. An isomorphism of S onto itself is called an *automorphism* of S . The family of all automorphisms of S forms a group called $\text{Aut}(S)$.

The next step is to obtain, from Φ , an induced morphism $\tilde{\Phi} = (\tilde{\Phi}_1, \tilde{\Phi}_2)$ of $\tilde{S}$ into $\tilde{S}'$. This appears to go through smoothly only if Φ is an isomorphism which we henceforth assume it to be.

Since each $\Phi_{2\tau}$ is a bijection preserving the logic structure, in particular the null and unit elements, it is clear that Φ_2 maps null history filters onto null history filters (and vice versa) and unit history filters onto unit history filters (and vice versa). It follows that Φ_2 induces a bijective mapping $\tilde{\Phi}_2$ of $\tilde{S}$ onto $\tilde{S}'$ which maps $\tilde{0}$ to $\tilde{0}'$, $\tilde{1}$ to $\tilde{1}'$ and generic elements to generic elements (and vice versa); in fact, it is a psg isomorphism.

It should also be clear that Φ_1 induces a psg isomorphism $\tilde{\Phi}_1$ of $\tilde{\mathcal{T}}$ onto $\tilde{\mathcal{T}}'$ and that the pair $\tilde{\Phi} = (\tilde{\Phi}_1, \tilde{\Phi}_2)$ is an isomorphism of $\tilde{S}$ onto $\tilde{S}'$.

The isomorphism $\tilde{\Phi}_2$ preserves the operations $\leq$, $\perp$, $\otimes$ and $\wedge$ defined on $\tilde{\mathcal{U}}$. The condition of weak disjointness is also preserved. It is now not difficult to see that we have an induced mapping $\underline{\Phi}_2 : \Omega \rightarrow \Omega'$ mapping $\underline{0}$ to $\underline{0}'$, $\underline{1}$ to $\underline{1}'$ and generic elements to generic elements (and vice versa).

All the structural properties going into the definition of various operations in Ω ($\leq$, $\perp$, $\otimes$) are preserved by $\underline{\Phi}_2$. In

particular

$$(a) \quad \underline{\alpha} \perp \underline{\beta} \quad \text{if and only if} \quad \Phi_2(\underline{\alpha}) \perp \Phi_2(\underline{\beta})$$

$$(b) \quad \Phi_2(\underline{\alpha} \odot \underline{\beta}) = \Phi_2(\underline{\alpha}) \odot \Phi_2(\underline{\beta})$$

$$(c) \quad \Phi_2(\neg \underline{\alpha}) = \neg \Phi_2(\underline{\alpha}).$$

To obtain ^{the} transformation law of decoherence functionals [when an explicit expression for $d(\underline{\alpha}, \underline{\beta})$ in terms of initial state, evolution maps and $\underline{\alpha}, \underline{\beta}$ is given], we need transformation laws of states and evolution maps.

A state $p_\tau \in \mathcal{P}(\tau)$ transforms under Φ to a state $p'_{\tau'} \in \mathcal{P}(\tau')$ [where $\tau' = \Phi_1(\tau)$] given by

$$p'_{\tau'}(\beta) = p_\tau[\Phi_{2\tau}^{-1}(\beta)] \quad \text{for all } \beta \in \mathcal{U}'_{\tau'}. \quad (4.20)$$

It is easily seen from eq. (4.20) that the mapping $p_\tau \rightarrow p'_{\tau'}$ preserves convex combinations; in particular, it transforms pure states to pure states (and vice versa).

An observable $A_\tau \in \mathcal{O}(\tau)$ transforms under Φ to $A'_{\tau'} \in \mathcal{O}(\tau')$ given by

$$A'_{\tau'}(E) = \Phi_{2\tau}(A_\tau(E)) \quad \text{for all } E \in B(R). \quad (4.21)$$

Given $\tau_1 \triangleleft \tau_2$ in $\mathcal{N}(\mathcal{T})$ and an evolution map $V(\tau_2, \tau_1) : \mathcal{U}_{\tau_1} \rightarrow \mathcal{U}_{\tau_2}$,

$\mathcal{U}_{\tau_1} \rightarrow \mathcal{U}_{\tau_2}$, Φ induces an evolution map $V'(\tau'_2, \tau'_1) : \mathcal{U}_{\tau'_1} \rightarrow \mathcal{U}_{\tau'_2}$

such that the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{U}_{\tau_1} & \xrightarrow{V(\tau_2, \tau_1)} & \mathcal{U}_{\tau_2} \\ \Phi_{2\tau_1} \downarrow & & \downarrow \Phi_{2\tau_2} \\ \mathcal{U}_{\tau'_1} & \xrightarrow{V'(\tau'_2, \tau'_1)} & \mathcal{U}_{\tau'_2} \end{array} \quad \text{i.e. } V'(\tau'_2, \tau'_1) \circ \Phi_{2\tau_1} = \Phi_{2\tau_2} \circ V(\tau_2, \tau_1). \quad (4.22)$$

Given an expression for $d = d_{p_0, V} : \Omega \times \Omega \rightarrow \mathbb{C}$, the pair $\Phi = (\Phi_1, \Phi_2)$ induces a map $\Phi_3 : \mathcal{D} \rightarrow \mathcal{D}$, given by

$$\Phi_3(d_{p_0, V}) = d_{p'_0, V'} \quad (4.23)$$

where p'_0, V' are given by (4.20) and (4.22). If such an expression not given, Φ_3 may formally be treated as an independent mapping for the purpose of the definition of symmetry given below.

A *symmetry operation* for the augmented history system (S) (see notation above) is a triple $\underline{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ such that

- (i) $\Phi = (\Phi_1, \Phi_2)$ is an automorphism of S;
- (ii) $\Phi_3 : \mathcal{D} \rightarrow \mathcal{D}$ satisfies the condition

$$\text{Re} \left[\Phi_3(d) \left(\Phi_2(\underline{\alpha}), \Phi_2(\underline{\beta}) \right) \right] = \text{Re } d(\underline{\alpha}, \underline{\beta}). \quad (4.24)$$

Note that the condition (4.24) implies preservation of the decoherence condition (3.69) as well as of the probability expression of eq. (3.70).

Remarks (i) The symmetry operation defined above should be understood to be one in the sense of a symmetry of the formalism (see section IV B). A symmetry of dynamics would be a member of the subclass of the family of symmetry operations satisfying the additional condition

$$V'(\dots) = V(\dots). \quad (4.25)$$

This equation can be meaningful only if the two sides define mappings between the same spaces; this implies $u_{\tau'_1} \approx u_{\tau_1}$ and $u_{\tau'_2} \approx u_{\tau_2}$ where $\approx$ indicates isomorphism (of logics). Since τ_1 and τ_2 are fairly arbitrary (subject only to the condition $\tau_1 \triangleleft \tau_2$), it appears that, in the context of the class of

theories discussed in this paper, symmetry of dynamics is definable only for the subclass in which all u_τ 's are isomorphic.

(ii) In the present setting, it is easy to see why it is natural for the condition of invariance of $d(\alpha, \alpha)$ not to imply invariance of dynamics; indeed, in eq. (4.24) V' appearing on the left via eq. (4.23) need not equal V .

D. Traditional vs Temporal Logic Descriptions of Symmetries in Nonrelativistic Quantum and Classical Mechanics.

In this subsection we shall establish a two-way connection (i.e. each implying the other) between traditional descriptions of symmetry operations in nonrelativistic quantum and classical mechanics and those in the present formalism. Lorentz transformations will be treated in the next subsection.

a. Quantum Mechanics

Transition from traditional Wigner symmetries to those in the present formalism was described in section IV A. The former are recovered from the latter by appealing to theor. (4.27) of ref (13) which shows that an automorphism of the logic $\mathcal{P}(\mathcal{H})$ implies the existence of a unitary/antiunitary mapping on $\mathcal{H}$. It is interesting to note that the condition (4.24) in the definition of symmetry above is redundant in the present case.

A pedagogically simpler route to recover Wigner symmetries from symmetries in the present formalism is to obtain the condition of invariance of transition probabilities [eq. (4.2)] and then appeal to Wigner's Theorem. To this end (noting that u , $\mathcal{T}$ and Ω here are as in section IV A), we apply eq. (4.24) to the

case $\underline{\beta} = \underline{\alpha} = \{\alpha\}$ where $\alpha = \alpha_{t_1}$ is a single-time history (i.e. a projection operator on $\mathcal{H}$). From eqs. (2.6) and (2.9) we have $C_\alpha = \alpha_{t_1} U(t_1, t_0)$ and

$$\begin{aligned} d(\alpha, \alpha) &= \text{Tr} [C_\alpha \rho(t_0) C_\alpha^+] \\ &= \text{Tr} [\alpha_{t_1} U(t_1, t_0) \rho(t_0) U(t_1, t_0)^{-1}]. \end{aligned} \quad (4.26)$$

Writing $\Phi_1(t_j) = t'_j$ ($j = 0, 1$), $\Phi_2(\alpha) = \alpha' = \alpha'_{t'_1}$, the condition $d'(\alpha', \alpha') = d(\alpha, \alpha)$ gives

$$\begin{aligned} &\text{Tr} [\alpha'_{t'_1} U'(t'_1, t'_0) \rho'(t'_0) U'(t'_1, t'_0)^{-1}] \\ &= \text{Tr} [\alpha_{t_1} U(t_1, t_0) \rho(t_0) U(t_1, t_0)^{-1}]. \end{aligned} \quad (4.27)$$

Putting $t_1 = t_0$ in eq. (4.27), we get

$$\text{Tr} [\alpha'_{t'_0} \rho'(t'_0)] = \text{Tr} [\alpha_{t_0} \rho(t_0)]. \quad (4.28)$$

Taking $\rho(t_0)$ to be a pure state (one dimensional projection operator), $\rho'(t'_0)$ must also be a pure state. [See the statement after eq. (4.20).] Similarly, taking α_{t_0} to be one-dimensional projector, $\alpha'_{t'_0}$ must also be a one-dimensional projector because the mapping Φ_{2t_0} preserves the logic structure of $\mathcal{U}_{t_0} = \mathcal{P}(\mathcal{H})$. With these choices, eq. (4.28) reduces to an equation of the form (4.2).

b. Classical Mechanics

In the traditional formalism of Hamiltonian dynamics (in the setting of symplectic manifolds), symmetries of the formalism are

the canonical transformations (diffeomorphisms of Γ preserving the symplectic form). The histories-based formalism, however, has nothing to do with smooth structures and infinitesimal version of dynamics (Hamilton's equations). It operates in the more general framework of topological spaces and employs objects like Borel sets and Borel measurable evolution maps. The symmetries of the formalism in the present context, therefore, are bijective Borel measurable maps of Γ onto itself. With this in mind, we now consider the two-way connection between symmetries as mentioned above.

(1) Given an invertible transformation of the time variable [eq. (4.10)] and a bijective Borel measurable mapping $F : \Gamma \rightarrow \Gamma$ [$\xi \rightarrow \xi' = F(\xi)$] we construct a symmetry operation $\underline{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ as follows :

(i) An automorphism/antiautomorphism Φ_1 of K_1 is constructed as in section IV A.

(ii) The mapping $\Phi_{2t} : B(\Gamma) \rightarrow B(\Gamma)$ defined by

$$\Phi_{2t}(A) = F(A) \quad \text{for all } A \in B(\Gamma) \quad (4.29)$$

is an automorphism¹³ of the logic $B(\Gamma)$. Note that the mapping Φ_{2t} is the same for all t .

(iii) Given $\alpha = (\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_n}) \in \mathcal{U}$, we have

$$\Phi_2(\alpha) = \left(\Phi_{2t_1}(\alpha_{t_1}), \dots, \Phi_{2t_n}(\alpha_{t_n}) \right). \quad (4.30)$$

(iv) The verification that the pair $\Phi = (\Phi_1, \Phi_2)$ is an automorphism of the history system $S = (\mathcal{U}, \mathcal{I}, \sigma)$ is straightforward.

(v) The transformation laws of the objects p_{t_0} and $V_{t,s}$ appearing in the classical decoherence functional (3.96) are obtained using eqs. (4.20), (4.22) and (4.29). This gives, for the former,

$$p'_{t_0}(\beta) = p_{t_0}[\Phi_{2t_0}^{-1}(\beta)] = p_{t_0}[F^{-1}(\beta)] \quad (4.31)$$

and, for the latter,

$$V'_{t_2 t_1}(\xi) = F[V_{t_2 t_1}(F^{-1}(\xi))]. \quad (4.32)$$

Using eq. (4.23) and noting that

$$\begin{aligned} \int_{\alpha'_{t_j}} d\xi'_j \delta[\xi'_j - V'_{t_j t_{j-1}}(\xi'_{j-1})] \\ = \int_{\alpha_{t_j}} d\xi_j \delta[\xi_j - V_{t_j t_{j-1}}(\xi_{j-1})] \end{aligned} \quad (4.33)$$

$$\begin{aligned} \int_{\beta'_{t_j}} \delta(\eta'_j - \xi'_j) d\eta'_j &= \int_{\beta_{t_j}} \delta(\eta_j - \xi_j) d\eta_j \\ j &= 1, \dots, n \end{aligned}$$

we get [recalling eq. (3.96)]

$$\begin{aligned} d'(\alpha', \beta') &\equiv (\Phi_3(d))(\alpha', \beta') = \int_{\Gamma} dp_{t_0}(\xi'_0) F^{(\alpha, \beta)}[F^{-1}(\xi'_0)] \\ &= \int_{\Gamma} dp_{t_0}(\xi_0) F^{(\alpha, \beta)}(\xi_0) \\ &= d(\alpha, \beta) \end{aligned} \quad (4.34)$$

which verifies eq. (4.24).

(2) Given a symmetry operation $\underline{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ we recover f and F as follows :

- (i) The restriction of Φ_1 to $N(\mathcal{T}) = \mathbb{R}$ fixes f .
- (ii) Φ_2 determines, for each $t \in N(\mathcal{T}) = \mathbb{R}$, $\Phi_{2t}: B(\Gamma) \rightarrow B(\Gamma)$ which, when restricted to single point sets, determines a mapping $F: \Gamma \rightarrow \Gamma$ which is bijective and Borel measurable.

Here again, as in the quantum mechanical case, eq. (4.24) is redundant. [See eq. (4.34) above.] A result analogous to this was obtained in ref. (15) (in the setting of Π functions of section II A and not considering measurability of mappings).

E. Lorentz Transformations

The presence of the mapping Φ_1 in our definition of symmetry might give the impression that the formalism is not applicable to relativistic situations. We shall see presently that this is not the case. Indeed, a history in one Lorentz frame L must map to a history in another frame L' ; the Lorentz transformation Λ connecting L and L' , when properly implemented, must connect the descriptions of histories in the two frames. This implementation will be discussed here in the context of relativistic mechanics (classical and quantum) of particles; field theories (in general space-times) will be considered in the next section.

We wish to associate, with an element Λ of the Lorentz group, an automorphism $\Phi(\Lambda) = \left[\Phi_1(\Lambda), \Phi_2(\Lambda) \right]$ of the history system $S = (\mathcal{U}, \mathcal{T}, \sigma)$ associated with a relativistic (classical or quantum) system. It is convenient to introduce the notation

$s^{(L)} = (u^{(L)}, g^{(L)}, \sigma^{(L)})$ for the version of S corresponding to the description of histories in the frame L . In a given frame L , we have a well-defined time axis which serves as $N(g^{(L)})$; $g^{(L)}$ is the psg K_1 constructed from finite sequences of points in $N(g^{(L)})$.

The quantum mechanical Hilbert space $\mathcal{H}$ of the system carries a projective representation of the Lorentz group; a Lorentz transformation is represented (upto a phase factor) by an operator $U(\Lambda)$ which is unitary or anti-unitary according as the transformation Λ is orthochronous or non-orthochronous. A familiar example is the Lorentz transformation of a Dirac wave function $\psi^{(L)}(x)$ which is an element of a Hilbert space $\mathcal{H}$ with scalar product

$$(\psi, \phi) = \int \psi(x)^\dagger \phi(x) d^3x. \quad (4.35)$$

The transformation is

$$\psi^{(L')}(x) = (U(\Lambda) \psi^{(L)})(x) = S(\Lambda) \psi^{(L)}(\Lambda^{-1}x) \quad (4.36)$$

where $S(\Lambda)$ is a 4×4 matrix satisfying the familiar relation $S^{-1} \gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu$ (γ^μ are Dirac matrices). Coming back to the general case, the operator $U(\Lambda)$ can be used to define transformations of states (density operators), observables and projectors related to observables as in eq. (4.4); the mapping $\Phi_2(\Lambda)$ can then be constructed as in section IV A.

In classical mechanics, a Lorentz transformation connects positions, momenta and trajectories in different frames and construction of $\Phi_2(\Lambda)$ can proceed as in section IV D.

We now consider $\Phi_1(\Lambda)$ whose treatment is the same in classical and quantum mechanics. The Lorentz transformation of

the time variable x^0 given by

$$x^{0'} = \Lambda^0_{\nu} x^{\nu} = \Lambda^0_0 x^0 + \Lambda^0_j x^j \quad (4.37)$$

involves the spatial coordinates x^j as well. To get rid of the latter, we note that the time values $\tau_1, \tau_2, \dots$ appearing in the temporal supports must be interpreted as time intervals between some reference event and the events corresponding to single time histories occurring in the irreducible decompositions of a history. Let x^0_{ref} and $x^{0'}_{\text{ref}}$ be the time coordinates of the reference even in the two frames. We have

$$x^{0'}_{\text{ref}} = \Lambda^0_0 x^0_{\text{ref}} + \Lambda^0_j x^j \quad (4.38)$$

where we have assumed that all times in the frame L refer to events with fixed spatial coordinates x^j . Defining

$$t^{(L)} = x^0 - x^0_{\text{ref}}, \quad t^{(L')} = x^{0'} - x^{0'}_{\text{ref}} \quad (4.39)$$

we have

$$t^{(L')} = \Lambda^0_0 t^{(L)} \equiv \Phi_1(\Lambda)(t^{(L)}) \quad (4.40)$$

which defines Φ_1 on $\mathcal{N}(\mathcal{T}^{(L)})$; it is trivially extended to $\mathcal{T}^{(L)}$. As expected, Φ_1 of eq. (4.40) is monotone increasing for orthochronous Lorentz transformations and monotone decreasing for the non-orthochronous ones.

Construction of $\Phi_3(\Lambda)$ and verification of the invariance condition for $d(\alpha, \beta)$ [eq. (4.15b) in quantum mechanics and eq. (4.34) in classical mechanics] proceed along familiar lines; we shall skip the details.

F. Conservation Laws

We next consider conservation laws in the present formalism. Since we have the concept of evolution defined in the formalism, it is natural to define conservation of an observable in terms of its preservation under evolution. To do this, however, we shall need to compare elements of $\mathcal{U}_\tau$'s for different τ 's. It follows that a primary requirement for the definition of conservation laws is that $\mathcal{U}_\tau$'s for different τ 's be isomorphic. We shall henceforth assume this in this section and identify $\mathcal{U}_\tau$'s for all $\tau \in \mathcal{N}(\mathcal{T})$.

Given $\tau \triangleleft \tau'$, we say that an observable $A \in \mathcal{O}(\tau)$ is conserved under the evolution $V(\tau', \tau) : \mathcal{U}_\tau \rightarrow \mathcal{U}_{\tau'}$, if

$$V(\tau', \tau)(A(E)) = A(E) \quad \text{for all } E \in B(R). \quad (4.41)$$

We shall now reexpress this definition in terms of equality of probabilities of appropriate single time histories. Let $\tau_0 \triangleleft \tau \triangleleft \tau'$ and suppose a prescription is given to construct a decoherence functional $d_{p_0, V}$ in terms of an initial state p_0 at 'time' τ_0 and the evolution map $V(\dots)$. Let α and β be the single 'time' histories given by

$$\alpha = \alpha_\tau = A(E) ; \quad \beta = \beta_{\tau'} = A(E). \quad (4.42)$$

We expect the following alternative definition of conservation law to be equivalent to the one given above : We say that the observable $A \in \mathcal{O}(\tau)$ is conserved under the evolution $V(\tau', \tau) : \mathcal{U}_\tau \rightarrow \mathcal{U}_{\tau'}$, if

$$d_{p_0, V}(\alpha, \alpha) = d_{p_0, V}(\beta, \beta) \quad (4.43)$$

for all $p_0 \in \mathcal{P}(\tau_0)$ and all $E \in B(R)$.

Let us verify the equivalence of these definitions in the history versions of traditional nonrelativistic quantum mechanics and classical mechanics.

(i) Quantum Mechanics. Let $t_0 < t < t'$. We have $u_{t_0} \approx u_t \approx u_{t'} = \mathcal{P}(\mathcal{H})$. With the initial state $\rho(t_0) = \rho_0$, $\alpha = \alpha_t \in \mathcal{P}(\mathcal{H})$ and $\beta = \beta_{t'} = \alpha_{t'}$, we have (recalling equations 2.6 and 2.9) $C_\alpha = \alpha_t U(t, t_0)$, $C_\beta = \beta_{t'} U(t', t_0)$ and

$$d(\alpha, \alpha) = \text{Tr}(C_\alpha \rho_0 C_\alpha^\dagger) = \text{Tr}[\alpha_t U(t, t_0) \rho_0 U(t, t_0)^{-1}] \quad (4.44a)$$

$$d(\beta, \beta) = \text{Tr}[\beta_{t'} U(t', t_0) \overset{U}{\rho_0(t', t_0)^{-1}}] \quad (4.44b)$$

The equality $d(\alpha, \alpha) = d(\beta, \beta)$ for arbitrary ρ_0 gives

$$U(t', t_0)^{-1} \beta_{t'} U(t', t_0) = U(t, t_0)^{-1} \alpha_t U(t, t_0) \quad (4.45)$$

which implies (recalling that $\beta_{t'} = \alpha_{t'}$)

$$U(t', t)^{-1} \alpha_{t'} U(t', t) = \alpha_t. \quad (4.46)$$

This equation represents the preservation of spectral projectors of the observable A under Schrödinger picture evolution. Consider, for example,

$$\alpha_t = |\Psi_t\rangle \langle \Psi_t| \in u_t$$

where $|\Psi_t\rangle$ is an eigenstate of A corresponding to some eigenvalue λ . Under temporal evolution $|\Psi_t\rangle \rightarrow U(t', t) |\Psi_t\rangle$ and

$$|\Psi_t\rangle \langle \Psi_t| \rightarrow U(t', t) |\Psi_t\rangle \langle \Psi_t| U(t', t)^{-1} \in u_{t'}, \quad (4.47)$$

etc. A general spectral projector of A (which is a sum or

integral of projectors of the form $|\Psi_t\rangle \langle \Psi_t|$ has the same transformation law under temporal evolution. The content of eq. (4.46), therefore, is the same as that of eq. (4.41) in the present case. The condition (4.43), therefore, implies the condition (4.41). Conversely, given eq. (4.46), the equality (4.43) is easily obtained.

(ii) Classical Mechanics. Again, let $t_0 < t < t'$. We have $u_{t_0} \approx u_t \approx u_{t'} = B(\Gamma)$. With $p_0 \in \mathcal{P}(t_0)$,

$$\alpha = \alpha_t = A(E) \in B(\Gamma); \quad \beta = \beta_{t'} = \alpha_t$$

we have

$$\begin{aligned} d(\alpha, \alpha) &= \int_{\Gamma} dp_0(\xi_0) \int_{\alpha_t} \delta[\xi - V(t, t_0)(\xi_0)] \\ &= p_0[V(t, t_0)^{-1}(\alpha_t)] \end{aligned} \quad (4.48a)$$

$$d(\beta, \beta) = p_0[V(t', t_0)^{-1}(\beta_{t'})]. \quad (4.48b)$$

The equality $d(\beta, \beta) = d(\alpha, \alpha)$ for all p_0 gives

$$V(t, t_0)^{-1}(\alpha_t) = V(t', t_0)^{-1}(\beta_{t'}) \quad (4.49)$$

which [with $\alpha_t = \beta_{t'} = A(E)$ and assumed to be valid for all $E \in B(R)$] is easily seen to be equivalent to eq. (4.41) in the present case. Conversely, given eq. (4.48), eq. (4.43) is easily deduced.

In certain forms of dynamics one can obtain a general relation between continuous symmetries (i.e. symmetries labelled by continuously varying parameters) of dynamics (generally expressed in terms invariance of an action integral or of a Hamiltonian) and conservation laws (Noether type theorems).

Examples are Lagrangian and Hamiltonian formulation of classical mechanics and Hilbert space quantum mechanics in the Heisenberg picture. The ~~observed~~^{conserved} quantities are infinitesimal generators of symmetry transformations (in the appropriate implementation of the symmetry) interpreted as observables. In the present formalism, it appears difficult to see such a general connection in the case of general u_τ 's. However, in the subclass of theories obtained from those considered in section IIIG by stipulating that all the $\mathcal{H}_\tau$'s are naturally isomorphic (and can, therefore, be identified with a single $\mathcal{H}$) and that the evolution maps $V(\tau', \tau)$ are unitary (this means one has the usual Hilbert space quantum mechanics with unitary temporal evolution except that the time points $t, t', \dots$ are replaced by elements of a psg), one can show the above mentioned general connection as follows.

Let $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ be a continuous one-parameter symmetry of dynamics of a history system in the above mentioned class. (We shall be mainly concerned with Φ_2 .) According to section IVD, such a symmetry can be taken to be unitarily implemented in the Hilbert space $\mathcal{H}$; let the corresponding infinitesimal generator be the self adjoint operator A . The invariance condition (4.25) [with the transformation law (4.22) for $V(.,.)$] implies that the mappings $K(.,.)$ of section IIIG commute with A and therefore with the spectral projectors of A .

Now let $\tau_0 \triangleleft \tau \triangleleft \tau' \triangleleft \tau_f$, $\tau \triangleleft \tau_f$ ^(and $\tau_0 \triangleleft \tau_f$) and consider the quantities $d(\alpha, \alpha)$ and $d(\beta, \beta)$ with $d(.,.)$ of eq. (3.83) (with $N=1$) and α, β single time histories given by $\alpha_\tau = \beta_{\tau'} = P$ a spectral projector of A . We have

$$\begin{aligned}
C'_\alpha &= K(\tau_f, \tau) P K(\tau, \tau_0) \\
&= K(\tau_f, \tau) K(\tau, \tau_0) P = K(\tau_f, \tau_0) P.
\end{aligned} \tag{4.50}$$

This gives

$$\begin{aligned}
d(\alpha, \alpha) &= \text{Tr}(C'_\alpha \rho_0 C'^{\dagger}_\alpha) \\
&= \text{Tr} \left[K(\tau_f, \tau_0) P \rho_0 P K(\tau_f, \tau_0)^\dagger \right] \\
&= \text{Tr}(P \rho_0).
\end{aligned} \tag{4.51}$$

Similarly, we have $d(\beta, \beta) = \text{Tr}(P \rho_0)$ giving the desired conservation law in the form of eq. (4.43).

V. HISTORIES OF FIELDS IN LORENTZIAN MANIFOLDS

In this section we shall extend the formalism of sections III and IV to (classical and quantum) field theories on Lorentzian manifolds. Basic concepts relating to the treatment of histories in such field theories were developed by Hartle⁵ and Sorkin¹⁸ who proposed to employ averages of fields over space-time regions as basic observables. A partial temporal order can be defined on a family of space-time regions which is causally consistent (which means that the family has no pair of regions in which one region intersects both the future and the past of the other). Appropriate families of such regions serve as 'time points' in the description of field histories.

Isham⁸ translated these ideas in the language of partial semigroups. The scheme proposed below is similar to that of Isham but differing from it mainly in that we take our basic regions to be three dimensional (pieces of spacelike surfaces) or even lower dimensional objects. This is with a view to ensure that spacelike surfaces appear as special cases of nuclear temporal supports. (This means continuing with the idealization of sharp single time measurements employed in the preceding sections.) In this respect our treatment is closer to Blencowe¹⁹ who considered histories of scalar fields in globally hyperbolic spacetimes (which admit foliation into spacelike surfaces). We, however, do not assume the space-time to be globally hyperbolic nor do we employ C^* -algebras as done by Blencowe.

We limit ourselves to, apart from presenting an outline of the augmented temporal logic formalism for relativistic field

theories (section V A), some simple considerations about symmetries in such theories (V B).

A. Augmented Temporal Logic Formalism for Relativistic Field Theories

The space-time (M, g) is assumed to be a 4-dimensional manifold M equipped with a metric g of Lorentzian signature $(- + + +)$. Given two subsets A and B of M , we shall say that A temporally precedes B ($A \triangleleft B$) if

$$J^+(A) \cap B \neq \emptyset, [J^+(B) - B] \cap A = \emptyset. \quad (5.1)$$

Here $J^+(A)$ is the causal future of A (i.e. the set of points in M that can be reached from A by future-directed non-spacelike curves).

Note. We have modified Isham's definition to allow the possibility $A = B$ or, more generally, $A \cap B \neq \emptyset$. This is in keeping with our convention of allowing, for example, $s_m = t_1$ in eq. (2.2).

By a *basic region* we shall mean a connected subset of M such that every pair of points in it are spacelike separated. Given a family $\mathcal{B} = \{O_1, O_2, \dots\}$ of basic regions we shall say that $\mathcal{B}$ has an *earliest* member O_p if there is no O_k ($k \neq p$) in $\mathcal{B}$ temporally preceding O_p . Similarly we define a *latest* member.

A *temporal support* $\tau = \{O_1, O_2, \dots\}$ is an at most countable collection of basic regions which has either an earliest member or a latest member or both. A temporal support will be called *nuclear* if its basic regions are pairwise spacelike separated.

The collection of all temporal supports will be denoted as $\mathcal{T}$ and the nuclear supports as $N(\mathcal{T})$.

Given temporal supports

$$\tau = \{\dots, O_1, O_2, \dots\}, \tau' = \{\dots, O'_1, O'_2, \dots\}$$

we shall say that $\tau \triangleleft \tau'$ if τ has a latest member O_m and τ' has an earliest member O'_1 such that $O_m \triangleleft O'_1$. In this case we define the composition $\tau \circ \tau'$ as the union of the two collections :

$$\tau \circ \tau' = \{\dots, O_{m-1}, O_m, O'_1, O'_2, \dots\} \quad (5.2)$$

(where it is understood that the collection of basic regions on the right is at most semi-infinite so that $\tau \circ \tau'$ qualifies as a temporal support); this makes $\mathcal{T}$ a directed special psg.

With each basic region O , we associate a logic u_O . The logics associated with two spacelike separated basic regions are assumed to be isomorphic. This allows us to associate, with each nuclear temporal support $\tau = \{O_1, O_2, \dots\}$, a logic u_τ which, by definition, is one isomorphic to any of the u_{O_i} 's.

A history filter is an (at most semi-infinite) collection $\alpha = \{\dots, \alpha_{\tau_1}, \alpha_{\tau_2}, \dots\}$ where $\{\dots, \tau_1, \tau_2, \dots\}$ is a family of nuclear temporal supports such that

$$\dots \triangleleft \tau_1 \triangleleft \tau_2 \triangleleft \dots \quad (5.3)$$

and $\alpha_{\tau_j} \in u_{\tau_j}$ for each τ_j . The temporal support of α is the element of $\mathcal{T}$ given by

$$\sigma(\alpha) = \tau = \dots \tau_1 \circ \tau_2 \circ \dots \quad (5.4)$$

A composition rule $\circ$ giving $\mathcal{U}$ (the collection of all history

filters) the structure of a (directed, special) psg is defined in an obvious manner such that $\sigma : \mathcal{U} \rightarrow \mathcal{T}$ is a psg homomorphism. We have

$$\alpha = \dots \circ \alpha_{\tau_1} \circ \alpha_{\tau_2} \circ \dots \quad (5.5)$$

The triple $(\mathcal{U}, \mathcal{T}, \sigma)$ is a history system as defined in section III. One can now proceed as in section III to construct the space Ω of history propositions.

States, observables, temporal evolution and the space $\mathcal{D}$ of decoherence functionals can also be introduced as in section III and an augmented history system $(S) = (\mathcal{U}, \mathcal{T}, \sigma, \Omega, \mathcal{D})$ obtained.

Note that no fields have been introduced so far. We, therefore, have here a framework for more general theories than (history versions of) field theories.

In the special case when the space-time M is globally hyperbolic, we can have a foliation of M into a family $\{\Sigma_t, t \in \mathbb{R}\}$ of spacelike surfaces. In this case we can take Σ_t 's themselves (or subsets thereof) to be basic regions which can then serve as nuclear supports. Replacing Σ_t by t for book keeping, we can now have, for example $\mathcal{T} = \mathcal{K}_1$ of section II F.

An important subclass of the class of theories described above is the history version of the class of relativistic quantum field theories admitting a consistent Fock space (or, more generally, a single Hilbert space) description. Taking $\mathcal{H}$ to be the Hilbert space, one can employ the HPO space $\tilde{\mathcal{H}}$ to construct $\mathcal{U}$ and Ω as in section IV A. The decoherence functional of eq. (2.9) works here as well.

History version of a classical relativistic field theory can be constructed by going to the Hamiltonian formalism where we

now have an infinite dimensional phase space. Constructions parallel to those in section III H can proceed (with some additional technicalities); we shall, however, skip the details.

We close this subsection with the remark that the formalism of this section can be employed to obtain a straightforward extension of Blencowe's formalism¹⁹ to the non-globally hyperbolic case. The only thing needed to be done is to replace the objects $\mathcal{C}_t$ (the Cauchy hypersurfaces constituting the leaves of the foliation of M) in that paper by nuclear temporal supports. The analogue of the algebra $\mathcal{A}(O_t^3)$ is then trivially constructed as the intersection of appropriate quasilocal algebras. The equations (2.2) and (2.3) of ref. (19) giving the expression for the probability of a history use the labels $t_1, t_2, \dots$ only as labels and do not use them as real numbers in any special way. They can, therefore, be used as such in the general case.

B. Symmetries

Recall that, in section IV C, the definition of a symmetry of an augmented history system $(S) = (\mathcal{U}, \mathcal{T}, \sigma, \Omega, \mathcal{D})$ was given in terms of a triple $\underline{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ of bijective mappings $\Phi_1: \mathcal{T} \rightarrow \mathcal{T}$, $\Phi_2: \mathcal{U} \rightarrow \mathcal{U}$ and $\Phi_3: \mathcal{D} \rightarrow \mathcal{D}$ satisfying certain conditions. The same definition, of course, applies here. The mapping which has some special features in the class of theories being considered is Φ_1 . We shall, therefore, concentrate on Φ_1 only.

The mapping Φ_1 is supposed to be an (anti-) automorphism of the psg $\mathcal{T}$. It maps (bijectively) nuclear supports to nuclear supports. Note that our definition of basic region allows points

and spacelike curves in M to be considered as basic regions and all basic regions are nuclear supports. A mapping preserves inclusion relations; hence Φ_1 , being a bijective mapping of $\mathcal{T}$ onto itself, maps (bijectively) points to points, spacelike curves to spacelike curves and three dimensional basic regions to three dimensional basic regions. It follows, in particular, that Φ_1 implies a bijection of M onto itself (which we continue to denote as Φ_1).

We shall prove below that $\Phi_1 : M \rightarrow M$ is a conformal isometry. In the proof we shall need continuity of this mapping. To ensure this, we shall give $\mathcal{T}$ a topology so as to make it a topological psg and require $\Phi_1 : \mathcal{T} \rightarrow \mathcal{T}$ to be continuous. [Similar continuity requirements should be understood on Φ_2 and Φ_3 (with appropriate topologies on the relevant spaces); they will, however, not be discussed.]

We shall give $\mathcal{T}$ the most natural topology induced on it by the manifold topology of M . Since M is a locally Euclidean space, we can define ϵ -neighbourhoods of points, curves and two-dimensional and three dimensional ^(hyper-)~~hyper~~surfaces in M ; it follows that we can define ϵ -neighbourhoods of basic regions in M . Now let $\tau = \{O_1, O_2, \dots\}$ be an element of $\mathcal{T}$ where O_i 's are basic regions. We define an ϵ -neighbourhood of τ to be the collection of elements $\tau' \in \mathcal{T}$ such that

(i) $\tau' = \{O'_1, O'_2, \dots\}$ where O'_i 's are in one-to-one correspondence with O_i 's ;

(ii) for each i , O'_i is in an ϵ_i -neighbourhood of O_i with $\epsilon_i < \epsilon$.

Now let $\tau^{(1)}, \tau^{(2)} \in \mathcal{T}$ such that $\tau^{(1)} \triangleleft \tau^{(2)}$ and $\tau^{(1)'} \in \epsilon_i$ -

neighbourhood of $\tau^{(i)}$ ($i = 1, 2$) such that $\tau^{(1)'} \Delta \tau^{(2)'}$; then [see eq. (5.2)] $\tau^{(1)'} \circ \tau^{(2)'}$ is in an ε -neighbourhood of $\tau^{(1)} \circ \tau^{(2)}$ where $\varepsilon \geq \max(\varepsilon_1, \varepsilon_2)$. This makes $\mathcal{T}$ a topological psg. Requirement of continuity of $\Phi_1 : \mathcal{T} \rightarrow \mathcal{T}$ in this topology implies continuity of the mapping $\Phi_1 : M \rightarrow M$ in the manifold topology.

We, therefore, have a bijective continuous mapping $\Phi_1 : M \rightarrow M$ which maps (bijectively) space-like curves onto spacelike curves. It then follows that it maps (bijectively) non-spacelike curves onto non-spacelike curves. Now, null curves can be realized as limits of spacelike curves. Being continuous, Φ_1 must, therefore, map (bijectively) null curves to null curves and hence (bijectively) timelike curves to timelike curves. It follows that $\Phi_1 : M \rightarrow M$ must be a conformal isometry.

- a) Email address: tulsi@iitk.ernet.in
- b) Present address: Physics Department, University of Indiana, Bloomington, Indiana 47405, USA.
Email address: vojoglek@hamlet.ucs.indiana.edu

References

1. R.B. Griffiths (a) J. Stat. Phys. 36, 219-272 (1984);
(b) Am. J. Phys. 55, 11-17 (1986).
2. R. Omnes (a) J. Stat. Phys. 53, 893-932 (1988); (b) *ibid* 933-955 (1988); (c) *ibid* 957-975 (1988); (d) J. Stat. Phys. 57, 357-382 (1989); (e) Ann. Phys. (NY) 201, 354-447 (1990).
3. R. Omnes, Rev. Mod. Phys. 64, 339-382 (1992).
4. M. Gell-Mann and J. Hartle in *Complexity, Entropy and the Physics of Information, SFI Studies in the Science of Complexity*, edited by W. Zurek (Addison Wesley, Reading, MA, 1990), vol. VIII pp 425-458.
5. J. Hartle, in *Quantum Cosmology and Baby Universes* edited by

- S. Coleman, J. Hartle, T. Piran and S. Weinberg (World Scientific, Singapore, 1991).
6. J. Hartle, in Proceedings of the 1992 Les Houches School, Gravitation and Quantization, 1993.
 7. M. Gell-Mann and J. Hartle, Phys. Rev. D 47, 3345-3382 (1993).
 8. C.J. Isham, J. Math. Phys. 35, 2157-2185 (1994).
 9. C.J. Isham and N. Linden, J. Math. Phys. 35, 5452-5476 (1994).
 10. G. Birkhoff and J. von Neumann, Ann. Math. 37, 823 (1936).
 11. G.W. Mackey : *Mathematical Foundations of Quantum Mechanics*, Benjamin, New York (1963).
 12. J.M. Jauch, *Foundations of Quantum Mechanics*, (Addison-Wesley, London, 1973).
 13. V.S. Varadarajan, *Geometry of Quantum Theory*, second edition, Springer-Verlag, New York (1985).
 14. E.G. Beltrametti and G. Cassinelli : *The Logic of Quantum Mechanics*, Addison Wesley, Reading, Mass. (1981).
 15. R.M.F. Houtappel, H. Van Dam and E.P. Wigner, *The Conceptual Basis and Use of the Geometric Invariance Principles*, Rev. Mod. Phys. 37, 595-632 (1965).
 16. A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).
 17. G. Grätzer : *'General Lattice Theory'*, Birkhäuser Verlag, Basel (1978).
 18. R.D. Sorkin in *'Directions in General Relativity, vol II : A Collection of Essays in honour of Dieter Brill's Sixtieth Birthday'* edited by B.S. Hu and T.A. Jacobson (Cambridge University, Cambridge, 1993).
 19. M. Blencowe, Ann. Phys. (NY) 211, 87-111 (1991).