Sigma models describing low energy effective actions on D0-brane probes with $\mathcal{N} = 8$ supercharges are studied in detail using a manifestly $d = 1$, $\mathcal{N} = 4$ super-space formalism. Two $0+1$ dimensional $\mathcal{N} = 4$ multiplets together with their general actions are constructed. We derive the condition for these actions to be $\mathcal{N} = 8$ supersymmetric and apply these techniques to various D-brane configurations. We find that if in addition to $\mathcal{N} = 8$ supersymmetry the action must also have Spin(5) invariance, the form of the sigma model metric is uniquely determined by the one-loop result and is not renormalized perturbatively or non-perturbatively.
1. Introduction and summary

Recent developments [1,2,3] have emphasized the crucial role played by D0-branes in probing space-time structure at sub-stringy scales as well as in a non-perturbative definition of eleven dimensional M-theory. The basic feature that enables D-particles to test short distances in string theory is that their low energy dynamics is a quantum mechanics of the lightest open string degrees of freedom. The geometrical background in the sub-stringy domain is reproduced by quantum open string effects while the classical background at distances larger than the string scale is described by supergravity results which are essentially mediated by massless closed strings. As discussed in [1], in the cases with enough supersymmetry the two regimes are continuously connected by factorization of the open string annulus diagram. The general behavior in such cases is that the long distance supergravity results coincide with the one-loop quantum corrections to the probe moduli space. By analogy with higher dimensional field theories it is plausible that higher order perturbative corrections as well as non-perturbative ones vanish, leading to non-renormalization theorems. A similar non-renormalization result for a higher derivative interaction proves to be essential [2] in the Matrix theory formulation of M-theory.

The purpose of the present work is to study the $\mathcal{N} = 8$ quantum mechanics of a D0-brane probe moving in different D4-brane and/or orientifold plane backgrounds. The low energy degrees of freedom in the probe theory are five bosons and eight fermions. A single D0-D4 configuration has $\mathcal{N} = 8$ supersymmetry and a $\text{Spin}(5)$ rotational symmetry in the transverse directions under which the bosons transform as a vector and the fermions as a spinor. We construct two $\mathcal{N} = 4$ multiplets that together have these degrees of freedom, but are not manifestly $\text{Spin}(5)$ symmetric. We call the pair of these multiplets the $d = 1$, $\mathcal{N} = 8$ vector multiplet. Our main result is that the condition for $\text{Spin}(5)$ invariance of the vector multiplet action is compatible with the condition for it to have $\mathcal{N} = 8$ supersymmetry, and that when taken together these invariances uniquely determine the form of the target space metric. The form of the metric we find agrees with the one-loop result of [1] and we conclude that it can not receive perturbative or non perturbative corrections.

The plan of this paper is as follows. In section 2 we develop a manifestly $\mathcal{N} = 4$ superspace formalism in $(0+1)$ dimensions and describe $\mathcal{N} = 4$ chiral and linear multiplets that together contain the right number of bosonic and fermionic degrees of freedom. We then find in section 3 the condition for this action to admit four additional supersymmetries.
and argue that this condition is essentially unique. Requiring also $\text{Spin}(5)$ invariance leads to the non-renormalization theorem. This result is applied in section 4 to various D-brane configurations. Finally, we discuss the range of validity of this theorem in connection with three dimensional analogues and string duality.

2. $\mathcal{N} = 4$ multiplets in one dimension

The $d = 1$, $\mathcal{N} = 4$ superspace is parameterized by one commuting coordinate $t$, and four non-commuting ones arranged as an $SU(2)$ spinor, $\theta_\alpha$ and its complex conjugate $\bar{\theta}^\alpha$. The covariant derivatives and supercharges are given by (our conventions are summarized in appendix A):

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \bar{\theta}_\alpha \partial_0 \quad \bar{D}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - i \theta_\alpha \partial_0 \\
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \bar{\theta}_\alpha \partial_0 \quad \bar{Q}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} + i \theta_\alpha \partial_0 ,
\]

and satisfy the algebra

\[
\{D_\alpha, \bar{D}_\beta\} = 2i \epsilon_{\alpha\beta} \partial_0 \\
\{Q_\alpha, \bar{Q}_\beta\} = -2i \epsilon_{\alpha\beta} \partial_0 , \tag{2.1}
\]

with all the other anticommutators vanishing. The manifest supersymmetry transformations are generated by $\epsilon^\alpha Q_\alpha + \bar{\epsilon}^\alpha \bar{Q}_\alpha$ acting on the various multiplets.

**Chiral Multiplet**

As in the $d = 4$, $\mathcal{N} = 1$ case, the chiral and antichiral multiplets are defined by the constraints $\bar{D}\Phi = D\bar{\Phi} = 0$, which are solved by functions of $y = t - i \theta^\alpha \bar{\theta}_\alpha$ and $\bar{y} = t + i \theta^\alpha \bar{\theta}_\alpha$. In component form they are given by

\[
\Phi(y) = \Phi(y) + 2 \theta^\alpha \psi_\alpha(y) + \theta \theta F(y) \\
= \Phi - i \theta^\alpha \bar{\theta}_\alpha \dot{\Phi} + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \dot{\Phi} + 2 \theta^\alpha \psi_\alpha - i \theta \theta \bar{\theta} \bar{\theta} \dot{\psi}^\alpha + \theta \theta \bar{\psi}^\alpha + \theta \theta F
\]

and

\[
\bar{\Phi}(\bar{y}) = \bar{\Phi}(\bar{y}) - 2 \bar{\theta}_\alpha \bar{\psi}^\alpha(\bar{y}) - \bar{\theta} \bar{\theta} \bar{F}^*(\bar{y}) \\
= \bar{\Phi} + i \theta^\alpha \bar{\theta}_\alpha \dot{\bar{\Phi}} + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \dot{\bar{\Phi}} - 2 \bar{\theta}_\alpha \bar{\psi}^\alpha + i \bar{\theta} \bar{\theta} \bar{\theta} \bar{\psi}^\alpha - \bar{\theta} \bar{\theta} \bar{F}^*
\]

which are the $d = 4$, $\mathcal{N} = 1$ chiral and antichiral multiplets reduced to one dimension. The physical on-shell degrees of freedom arising from these multiplets are two bosons and four fermions.
Linear Multiplet

The $d = 4$, $\mathcal{N} = 1$ vector multiplet dimensionally reduced to $D = 3$ becomes equivalent [4] to the real linear multiplet $G$ defined by the constraints

$$D^2 G = \bar{D}^2 G = 0,$$

where $D$, $\bar{D}$ denote the spinor derivatives of the $d = 3$, $\mathcal{N} = 2$ superspace. They are solved by

$$G = D \bar{D} V$$

with $V$ an arbitrary real superfield. The physical degrees of freedom consist of a real scalar boson, a three dimensional vector field and their fermionic superpartners. The real scalar can be thought of as the fourth component of the four dimensional vector field. Further reduction to two dimensions yields [4,5] the twisted chiral multiplet $\Sigma_{+}$ defined [4] by the constraints

$$\bar{D}_+ \Sigma_{+} = D_- \Sigma_{+} = 0.$$

The solution of these constraints can be expressed similarly in terms of a real superfield

$$\Sigma_{+} = \frac{1}{\sqrt{2}} \bar{D}_+ D_- V,$$

describing the dynamics of the two real scalars obtained by dimensional reduction of the four dimensional vector field plus their superpartners.

In one dimension the closest analogue of the above conditions would be

$$D^2 \Sigma = \bar{D}^2 \Sigma = 0. \quad \text{(2.2)}$$

Proceeding naively we take

$$\Sigma = \bar{D}_\alpha \Theta^\alpha$$

where $\Theta^\alpha$ is a superfield, as the general solution of the second constraint in (2.2). By making use of $[\bar{D}_\alpha, D^2] = 4iD_\alpha \partial_0$ the first condition in (2.2) is satisfied if

$$\bar{D}_\alpha D^2 \Theta^\alpha + 4iD_\alpha \partial_0 \Theta^\alpha = 0$$

which is unacceptable since the time dependence of $\Theta^\alpha$ is restricted. A natural modification would be to consider a triplet of superfields which we denote by $\Sigma_{\alpha \beta}$. More precisely, the linear multiplet $\Sigma_{\alpha \beta}$ is defined by

$$D^\gamma D^\alpha \Sigma_{\alpha \beta} = \bar{D}^\gamma \bar{D}^\alpha \Sigma_{\alpha \beta} = 0, \quad \text{(2.3)}$$

3
and the reality condition
\[ \bar{\Sigma}_{\alpha\beta} \equiv \Sigma^{\alpha\beta} = \epsilon^{\alpha\gamma} \Sigma_{\gamma\delta} \epsilon^{\delta\beta}. \] (2.4)

The unique solution of these constraints with no restriction on the time dependence is given by
\[ \Sigma_{\alpha\beta} = \bar{D}_{(\alpha} D_{\beta)} V, \] (2.5)
where \( V \) is the real superfield of \( d = 4, N = 1 \) reduced to one dimension and \( (\cdot) \) denote symmetrization. The second constraint in (2.3) is identically satisfied, while the first one follows from the algebra (2.1):
\[ D^2 \Sigma_{\alpha\beta} = \frac{1}{2} (D^2 \bar{D}_\alpha D_{\beta} + D^2 \bar{D}_\beta D_{\alpha}) V = i(\epsilon_{\beta\alpha} + \epsilon_{\alpha\beta}) \partial_0 V. \] (2.6)

The component form of \( \Sigma_{\alpha\beta} \) is given by
\[ \Sigma_{\alpha\beta} = - \sigma^{i}_{\alpha\beta} x_i + i(\theta_{\alpha} \bar{\lambda}_\beta + \theta_{\beta} \bar{\lambda}_\alpha) + i(\bar{\theta}_{\alpha} \lambda_\beta + \bar{\theta}_{\beta} \lambda_\alpha) - (\bar{\theta}_{\beta} \theta_{\alpha} + \bar{\theta}_{\alpha} \theta_{\beta}) D \]
\[ + \frac{i}{2} (\theta_{\beta} \bar{\theta}_{\gamma} \sigma^{i}_{\gamma\alpha} + \theta_{\alpha} \bar{\theta}_{\gamma} \sigma^{i}_{\gamma\beta} + \bar{\theta}_{\beta} \theta_{\gamma} \sigma^{i}_{\gamma\alpha} + \bar{\theta}_{\alpha} \theta_{\gamma} \sigma^{i}_{\gamma\beta}) \dot{x}_i \]
\[ + \frac{1}{2} \bar{\theta} \theta (\bar{\theta}_{\beta} \dot{\lambda}_\alpha + \bar{\theta}_{\alpha} \dot{\lambda}_\beta) - \frac{1}{2} \theta \bar{\theta} (\theta_{\beta} \dot{\bar{\lambda}}_\alpha + \theta_{\alpha} \dot{\bar{\lambda}}_\beta) + \frac{1}{4} \theta \bar{\theta} \theta \bar{\theta} \sigma^{i}_{\alpha\beta} \dot{x}_i, \] (2.7)
or alternatively by
\[ \Sigma^i = \frac{1}{2} \sigma^{i\alpha\beta} \Sigma_{\alpha\beta} \]
\[ = - x^i + i \theta_{\gamma} \sigma^{i\gamma\delta} \bar{\lambda}_\delta + i \bar{\theta}_{\gamma} \sigma^{i\gamma\delta} \lambda_\delta - \bar{\theta}_{\gamma} \sigma^{i\gamma\delta} \theta_\delta D \]
\[ + \epsilon^{ijk} \bar{\theta}_{\gamma} \sigma^{i\gamma\delta} \theta_\delta \dot{x}^k - \frac{1}{2} \bar{\theta} \theta \theta_{\gamma} \sigma^{i\gamma\delta} \dot{\bar{\lambda}}_\delta + \frac{1}{2} \theta \bar{\theta} \theta_{\gamma} \sigma^{i\gamma\delta} \dot{\lambda}_\delta + \frac{1}{4} \theta \bar{\theta} \theta \bar{\theta} \dot{x}^i, \] (2.8)
which is more convenient for our purpose. A supersymmetric lagrangian will be a general function of the \( \Sigma^i \) superfields\(^1\).

Finally we note that by analogy with the \( d = 4, N = 1 \) case one can define chiral and antichiral field-strength multiplets by
\[ W_\alpha \equiv \bar{D}^\beta \Sigma_{\alpha\beta}, \quad \bar{W}_\alpha \equiv D^\beta \Sigma_{\alpha\beta}. \] (2.9)

Then,
\[ -\frac{1}{6} \left( W_\alpha W_\alpha |_{\theta\theta} + \bar{W}_\alpha \bar{W}_\alpha |_{\theta\theta} \right) \]
yields the same kinetic terms as \( \Sigma^i \Sigma^j |_{\theta\theta} \).

\(^1\) In the original version of the paper we erroneously stated that a general superspace action density must be an \( SO(3) \) invariant function. We thank E. Witten for pointing out this mistake.
3. $\mathcal{N} = 8$ supersymmetry and non-renormalization

We will ultimately be concerned with applications to a D0-brane in D4-branes and orientifold backgrounds. As will be explained in section 4, the low energy degrees of freedom on the D0-brane world-line are precisely described by the pair $(\Phi, \Sigma)$ which is the $d = 1, \mathcal{N} = 8$ multiplet. Such systems have only eight supersymmetries so quadratic terms in the velocities are generally not protected from renormalization. In the regime where the velocity of the D0-brane is small we may restrict our attention to an action which is quadratic in velocities and neglect higher order terms. A general such action with four manifest supersymmetries is given by

$$\int d^2\theta d^2\bar{\theta} \, K(\Phi, \bar{\Phi}, \Sigma^i)$$

where $K$ is an arbitrary real prepotential. It is possible to add a superpotential integrated over half of superspace, but it will not contribute to the metric. This remark actually applies to a wider class of actions. We may think of (3.1) as the first term in an expansion of the form

$$L = \int d^2\theta d^2\bar{\theta} \left(K_2 + K_{4ij} \partial_0 \Sigma^i \partial_0 \Sigma^j + \bar{K}_4 \partial_0 \Phi \partial_0 \bar{\Phi} + \ldots\right),$$

where each successive $K_i$ produces an $i$-th power in velocity term in the lagrangian (we do not have to consider an expansion in covariant derivatives of the multiplets since these lead to cubic terms in the velocities). Again, $K_4, \bar{K}_4, \ldots$ cannot give metric terms, so the non-renormalization result we will prove below applies also to the metric terms in these actions as well.

The metric can be read from the kinetic terms arising from the superspace integration,

$$\frac{1}{4} K_{\Sigma^i \Sigma^j} (\dot{x}^j \dot{x}^j + i(\dot{\bar{\lambda}} \dot{\lambda} + \dot{\lambda} \dot{\bar{\lambda}})) - K_{\Phi \Phi} (\dot{\Phi} \dot{\bar{\Phi}} + i(\dot{\bar{\psi}} \dot{\psi} + \dot{\psi} \dot{\bar{\psi}})),$$

and consists of only two undetermined functions, $\frac{1}{4} K_{\Sigma^i \Sigma^j}$ (summation over $i$ is implicit here) and $-K_{\Phi \Phi}$. Note the absence of mixed derivative terms – this will prove crucial for the applications to the $0 - 4$ system.
3.1. Non-manifest supersymmetries

If the action (3.1) admits more supersymmetries their form is severely constrained by the following considerations. First, they must be realized as spinorial derivatives acting on superfields so that the supersymmetry algebra is satisfied. The four manifest supersymmetries of each multiplet are already generated by the supercharges acting on it. Therefore, additional supersymmetries, if they exist, must be generated by spinorial derivatives acting on the other multiplets. This means that $\Sigma_{\alpha\beta}$ will enter the non-manifest transformations of $\Phi, \bar{\Phi}$ and that $\Phi, \bar{\Phi}$ enter symmetrically the non-manifest variation of $\Sigma_{\alpha\beta}$. The form of these variations is further constrained, and in fact determined up to a constant, by requiring the variations to respect the defining constraints of the multiplets. Thus we conclude that if there are four additional supersymmetries their form is

$$
\delta \Phi \propto i \bar{\epsilon}^\beta \bar{D}^\alpha \Sigma_{\alpha\beta} \\
\delta \bar{\Phi} \propto i \epsilon^\beta D^\alpha \Sigma_{\alpha\beta} \\
\delta \Sigma_{\alpha\beta} \propto i (\epsilon_{(\alpha} D_{\beta)} \Phi - \bar{\epsilon}_{(\alpha} \bar{D}_{\beta)} \bar{\Phi})
$$

(3.3)

(The chiral and antichiral constraints of $\delta \Phi$ and $\delta \bar{\Phi}$ follow directly form (2.3). The variation of $\Sigma_{\alpha\beta}$ can be seen to satisfy the conditions (2.4) and (2.3) by using the algebra (2.1). It is also easy to verify that the commutator of two non-manifest variations closes on translations).

A straight-forward (and a little laborious) calculation in components shows that the action (3.1) admits the four non-manifest supersymmetries

$$
\delta \Phi = \frac{-2i}{3} \bar{\epsilon}^\beta \bar{D}^\alpha \Sigma_{\alpha\beta} \\
\delta \bar{\Phi} = \frac{-2i}{3} \epsilon^\beta D^\alpha \Sigma_{\alpha\beta} \\
\delta \Sigma_{\alpha\beta} = i (\epsilon_{(\alpha} D_{\beta)} \Phi - \bar{\epsilon}_{(\alpha} \bar{D}_{\beta)} \bar{\Phi}),
$$

(3.4)

provided that the following condition holds:

$$
K_{\Sigma^i \Sigma^j} + 4 K_{\Phi \bar{\Phi}} = 0.
$$

(3.5)

This is actually also a necessary condition. The explicit form of the action shows that it cannot be invariant under the supersymmetry transformations of the form (3.3) unless (3.5) holds, that is unless the action depends only one arbitrary function. As argued above, the form of the non-manifest variations is unique so we conclude that any $\mathcal{N} = 4$
supersymmetric action is automatically $\mathcal{N} = 8$ supersymmetric if and only if (3.5) holds. The metric, the action and all the supersymmetry transformations are now determined by
\[
f = -\frac{1}{4} K_{\Sigma_i \Sigma_j} = K_{\Phi \bar{\Phi}}
\]
and are given in appendix B ($f$ enters the variation laws once the auxiliary fields are solved for). Differentiating $f$ twice with respect to $\Sigma^i$ and with respect to $\Phi$ and $\bar{\Phi}$, and using (3.5), shows that the metric satisfies
\[
f_{\Sigma_i \Sigma^i} + 4f_{\Phi \bar{\Phi}} = 0
\]
as well.

3.2. Spin(5) invariance and non-renormalization

The five scalars in the vector multiplet can be thought of as local coordinates, $y_1, \ldots, y_5$, on a five dimensional target space manifold by making the change of variables
\[
y_i = x^i \quad i = 1, 2, 3
\]
\[
y_4 = \frac{1}{2}(\Phi + \bar{\Phi})
\]
\[
y_5 = \frac{1}{2i}(\Phi - \bar{\Phi}).
\]
In these coordinates the condition (3.7) satisfied by the metric $f$ is precisely the Spin(5) invariant Laplace equation. Any function of $r^2$, $r$ being the five dimensional radius, must also satisfy this equation, and therefore the condition for a Spin(5) invariant metric is compatible with the condition for $\mathcal{N} = 8$ supersymmetry. This conclusion depends crucially on the relative sign and factor in (3.2) and would not have been valid otherwise. Furthermore, $f$ is now determined up to two constants. The condition (3.7) on a Spin(5) invariant function reduces to
\[
r^2 f'' + \frac{5}{2} f' = 0,
\]
and is solved by
\[
f = C' + \frac{C}{r^3}
\]
where $C, C'$ are arbitrary constants. We conclude that the metric of a general action compatible with the above symmetries is not renormalized either perturbatively or non-perturbatively.
It is also possible to restore manifest $\text{Spin}(5)$ invariance in the full lagrangian. After solving algebraically for the auxiliary fields the superspace lagrangian is given by *

$$- f \left( \dot{x}^i \dot{x}^i + i (\bar{\eta} \gamma^i \eta + \bar{\eta} \gamma^i \eta) \right) + \dot{x}^i f_{,i j} (\eta \gamma^{i j} \bar{\eta}) + \frac{1}{2} \left( f_{,i j} - \frac{1}{2} f_{,i} f_{,j} \right) \left( \eta \gamma^i \bar{\eta} \eta \gamma^j \bar{\eta} + \eta \gamma^i \bar{\eta} \gamma^j \bar{\eta} \right),$$

and has a natural geometric interpretation. Specifically the bilinear fermion term, taking into account the fünfbein factors, is the pull back of the minimal spin connection made of the metric $f$. With a little algebra, the quadratic fermion term can also be seen to be

$$R_{\alpha \beta \gamma \delta} \bar{\eta}^\alpha \bar{\eta}^\beta \eta^\gamma \eta^\delta,$$

with the curvature computed from the minimal connection. The manifest and non-manifest SUSY transformations also match up in a nice way. The manifest ones can be recovered if in the five dimensional SUSY transformations (B.6), the parameter $\epsilon_5^\alpha$ is taken as

$$\epsilon_5^\alpha = \left( \epsilon_\alpha \right),$$

and the non-manifest ones if we take

$$\epsilon_5^\alpha = \left( 0 \epsilon_\alpha \right).$$

Since the target space is odd dimensional these restrictions can not be made in an invariant way, but together they combine into an $\mathcal{N} = 8$ SUSY parameter. This is again due to the consistency of the $\text{Spin}(5)$ invariance and $\mathcal{N} = 8$ conditions.

4. D0-D4 system

The formalism developed in the previous sections can be applied to the study of low energy effective actions of D0-brane probes in different Type IIA backgrounds. Extending the analysis of [1] we consider D0-brane probes in Type I' theory realized as an orientifold of the Type IIA theory compactified on a five torus $T^5$ [6,7]. More precisely, one starts with Type I theory on $T^5$ and performs T-duality on all the five circles of the torus. The resulting theory is Type IIA on $T^5/Z_2 \Omega$ with sixteen pairs of D4-branes in the background.

* To avoid clutter, the same tangent space indices are used to denote the flat space carried by the $\gamma$ matrices.
to cancel the charge of the 32 orientifold fixed planes. In the normalization of [6,7], the RR charge of a fixed plane is \(-1\) while the charge of a four-brane is 1 such that cancellation holds globally. Local cancellation occurs in a configuration with a four-brane at each orientifold plane. The supersymmetric probes for this background are D0-branes whose world-line effective action is expected to reproduce the string background [1]. We will consider two distinct configurations:

- \(n\) D4-branes coalesce away from an orientifold fixed plane. In \(d = 1\), \(\mathcal{N} = 8\) language, the degrees of freedom on the D0-brane world-line consist of an Abelian vector multiplet and an adjoint hypermultiplet arising from 0-0 strings and \(n\) hypermultiplets in the fundamental of the \(U(1)\) gauge group arising from 0-4 strings. The space-time positions of the four-branes correspond to bare masses \(\vec{m}_i\) of the charged multiplets in the gauge theory on the probe. When the branes come together, one obtains \(SU(n)\) gauge symmetry enhancement in space-time corresponding to \(SU(n)\) global symmetry enhancement in the probe theory.

When the 0-brane is away from the 4-brane the massive 0-4 string states can be integrated out. The surviving low energy degrees of freedom in the world-line theory are the \(\mathcal{N} = 8\) vector multiplet and neutral hypermultiplet. The later decouples so the low energy effective action is the theory of an interacting \(\mathcal{N} = 8\) vector multiplet. If the positions of the four-branes coincide the system is rotationally invariant in the five transverse directions so the theory has \(Spin(5) \simeq Sp(2)\) symmetry under which the bosons transform in the 5 and fermions in the 4.

The result of the previous section applies to this configuration and it remains to determine the constants in (3.9). In the present case,

\[
C' = \frac{1}{g_s}
\]

is the asymptotic value of the dilaton far from the four-branes and at the same time the classical coupling constant of the gauge theory on the probe. The second constant \(C\) is determined by the one-loop effects of the \(n\) charged hypermultiplets [1] to be

\[
C = n.
\]

Therefore we conclude that the one-loop results of [1] are exact already in this order and do not receive further corrections. This statement is true as long as the theory is described
in terms of the multiplets introduced above but it may break down in a description in
terms of different variables. A similar phenomenon is encountered in three dimensional
gauge theories [8,9,10] where the monopole corrections become visible only after dualizing
the photon. As we will see latter, string duality suggests that this happens in the present
case as well.

If the four-branes are localized at different points of coordinates $\vec{m}_i$ in the transverse
space, the $SU(n)$ global symmetry on the probe is broken since the hypermultiplets have
different masses. In this case one-loop metric is given by

$$f(\vec{x}) = \frac{1}{g_s} + \frac{1}{|\vec{x} - \vec{m}_1|^3} + \cdots + \frac{1}{|\vec{x} - \vec{m}_n|^3}. \quad (4.1)$$

This configuration is no longer $Spin(5)$ symmetric in the transverse directions, but the
non-renormalization result still holds. The system is still $\mathcal{N} = 8$ supersymmetric so the
exact $f$ must still satisfy the five dimensional Laplace equation. The boundary conditions
on the exact metric close to $\vec{m}_i$ are given by

$$f = \frac{1}{|\vec{x} - \vec{m}_i|^3},$$

since near any of the $n$ D4 branes the remaining $n - 1$ hypermultiplets (corresponding to
the the rest of the D4-branes) are very massive and can be neglected. As $f$ is uniquely
determined by the boundary conditions the one-loop result (4.1) is exact.

- $n$ D4-branes coalesce at an orientifold fixed plane. The degrees of freedom on the D0-
brane consist now of a non-Abelian $SU(2)$ vector multiplet plus an adjoint hypermultiplet
arising from 0-0 strings and $n$ hypermultiplets in the fundamental of the gauge group. The
$n$ coalescing four-branes can be viewed as a collection of $2n$ branes pairwise identified by
the $Z_2$ projection. Therefore they are localized at points $\vec{m}_i, -\vec{m}_i$ in the transverse space.
The space-time gauge symmetry is enhanced to $SO(2n)$ when the branes coincide with
the orientifold plane. As before this corresponds to $SO(2n)$ global symmetry enhancement
on the probe. The $SU(2)$ gauge group on the world-line is spontaneously broken to $U(1)$
by expectation values of the five scalars in the vector multiplet which parameterize the
Coulomb branch of the theory. Strictly speaking, this terminology is inappropriate as
there is no real moduli space in quantum mechanics. Nevertheless one can still refer to
a quantum mechanical moduli space in the Born-Oppenheimer approximation [8]. In this
sense the low energy effective action on the probe is a $U(1)$ gauge theory with $\mathcal{N} = 8$
supersymmetry. When the four-branes coalesce at the fixed plane there is a global $Spin(5)$
symmetry rotating the five scalars in the Abelian vector multiplet. Therefore the general action is exactly of the form (3.10). The only difference with respect to the previous case is reflected in the value of the constant $C$,

$$C = 2n - 1$$

where the negative term represents the one-loop contribution of the non-Abelian vector multiplet to the effective action. As above there are no quantum corrections beyond one-loop. When the four-branes are in general positions the one-loop metric is

$$f(\vec{x}) = \frac{1}{g_s} + \frac{1}{|\vec{x} - \vec{m}_1|^3} + \frac{1}{|\vec{x} + \vec{m}_1|^3} + \ldots + \frac{1}{|\vec{x} - \vec{m}_n|^3} + \frac{1}{|\vec{x} + \vec{m}_n|^3} - \frac{1}{|\vec{x}|^3},$$

and by the same argument used for (4.1) does not get renormalized beyond this order.

5. Discussion

We have shown above that the one dimensional action describing five bosons and eight fermions in the $5$ and $4$ of $Spin(2)$ is, up to two constants, uniquely determined by requiring $\mathcal{N} = 8$ supersymmetry and $Spin(5)$ invariance. Since the form of the action is fixed by a solution of a differential equation, we can not determine in our formalism the constants that appear in the metric. Indeed, if the probe is near an orientifold fixed plane with all four-branes far away the metric becomes negative definite at a finite distance in moduli space. The description of the physics in terms of the $\mathcal{N} = 8$ vector multiplet degrees of freedom breaks down and one has to look for another set of variables. Similar phenomena occur in three dimensional gauge theories where the equivalent description involves dualizing the photon [8,9,10]. In the new variables the three dimensional non-renormalization theorem is violated by an infinite series of monopole corrections [9].

This is also likely to be the case here since a dual set of variables will not necessarily have a $Spin(5)$ symmetry. Further evidence for this conclusion can be inferred from string duality arguments analogous to those presented in [8,9] for the three dimensional case. The Type I' orientifold studied above is T-dual to Type I theory on a five torus $T^5$ which is in turn S-dual to Heterotic string theory on the same $T^5$. This is further dual to Type IIA theory on $K3 \times S^1$ and after T-duality on the $S^1$ factor to Type IIB theory on $K3 \times \tilde{S}^1$. The D0-brane probe is mapped by the first duality in the chain to a Type I D5-brane wrapped on $T^5$ while the D4-branes in the background are mapped to the 32 D9-branes...
of Type I theory. According to the analysis of [11] the zero modes of the Type I D5-brane wrapped on a four torus correspond to the world-sheet degrees of freedom of the Type IIB string in static gauge. In our case the five-brane is wrapped on an extra circle thus it maps to a fundamental Type IIB string wrapped on the extra circle $S^1$ which is a particle in the five non-compact dimensions. This is the image of the initial D0-brane probe through the above chain of dualities. The resulting sigma model is very different from the one we started with. It represents the motion of the particle on $K3 \times S^1$, thus the target space metric is the product of a hyper-Kähler metric on $K3$ and a trivial metric on $S^1$. The fermions are target space vectors and the symmetry is reduced to a product $U(1) \times G$ where $G$ is the isometry group of the hyper-Kähler metric. The orientifold background is mapped to a non-compact hyper-Kähler manifold asymptotic to an $S^1$ bundle over the projective space $RP^2$. In particular the metric is smooth and positive definite due to non-perturbative corrections [9]. The singularities at infinite distance corresponding four-brane backgrounds away are mapped to orbifold singularities in the complex structure of the hyper-Kähler surface.

While in the three dimensional analysis of [8,9] the string duality picture is entirely reproduced by electric-magnetic duality on a D2-brane probe, the present situation is less clear. One could try to define an analogue of higher dimensional duality transformations for the linear multiplet but we leave this for further study.

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Appendix A. Spinor Conventions

A.1. $Sp(1)$ spinors

The anticommuting coordinates $\theta_\alpha$ and $\bar{\theta}^\alpha \equiv (\theta_\alpha)^*$ are spinors of $SU(2) \simeq Sp(1)$. Raising and lowering indices is done with the $Sp(1)$ invariant metric as

$$\theta_\alpha = \epsilon_{\alpha\beta} \theta^\beta \quad \theta^\alpha = \epsilon^{\alpha\beta} \theta_\beta$$

$$\bar{\theta}_\alpha = \epsilon_{\alpha\beta} \bar{\theta}^\beta \quad \bar{\theta}^\alpha = \epsilon^{\beta\alpha} \bar{\theta}_\beta.$$  \hspace{1cm} (A.1)

* The isometry group of a generic K3 surface is trivial. However the moduli spaces of the probe theories are usually non-compact pieces of the entire surface. In this case the hyper-Kähler metric can have non-trivial isometry group.
Complex conjugation of anticommuting numbers is defined by

$$(\eta_\alpha \xi_\beta)^* = \bar{\xi_\beta} \bar{\eta_\alpha},$$  \hspace{1cm} (A.2)$$

which imply that $\partial^*_\alpha = -\bar{\partial_\alpha}$ and $\partial^{\alpha*} = -\bar{\partial_\alpha}$. (This is necessary for the solution $\Sigma_{\alpha\beta}$ of (2.3) to be consistent with the reality condition (2.4)). We also use

$$\psi \psi \equiv \psi^\alpha \psi_{\alpha}, \quad \bar{\psi} \bar{\psi} \equiv \bar{\psi}_{\alpha} \bar{\psi}^\alpha, \quad \psi \bar{\psi} \equiv \psi_{\alpha} \bar{\psi}^\alpha = \psi^\alpha \bar{\psi}_{\alpha}.$$ 

The symmetric $\gamma$ matrices are

$$\sigma^1_{\alpha\beta} = i \mathbf{1}, \quad \sigma^2_{\alpha\beta} = \tau^3, \quad \sigma^3_{\alpha\beta} = \tau^1,$$  \hspace{1cm} (A.3)$$

where $\tau$ are the Pauli matrices. They satisfy the algebra

$$(\sigma^i \sigma^j)_{\alpha\beta} = \delta^{ij} \epsilon_{\alpha\beta} + i \epsilon^{ijk} \sigma^k_{\alpha\beta},$$  \hspace{1cm} (A.4)$$

and the reality condition

$$(\sigma^i_{\alpha\beta})^* \equiv \sigma^{i\alpha\beta} = \epsilon^{\alpha\gamma} \sigma^i_{\gamma\delta} \epsilon^\delta_{\beta}. \hspace{1cm} (A.5)$$

A.2. $Sp(2)$ spinors

We give the decomposition of $Sp(2)$ spinors and $\gamma$ matrices in terms of the corresponding $Sp(1)$ quantities which is used to write the action and supersymmetry variations in a $Spin(5)$ form. Unless otherwise noted, all conventions are similar to those used above. Written in terms of $Sp(1)$ spinors, the $Sp(2)$ ones are

$$\eta_\alpha = \left( \lambda_\alpha \bar{\psi}_\alpha \right), \quad \bar{\eta}_\alpha = \left( \bar{\lambda}_\alpha \psi_\alpha \right).$$  \hspace{1cm} (A.6)$$

Indices are raised lowered and contracted using the metric

$$J_{\alpha\beta} = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix},$$  \hspace{1cm} (A.7)$$

with $\epsilon$ being the $Sp(1)$ metric. The antisymmetric $\gamma$ matrices are taken to be

$$\gamma^i_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad i = 1, 2, 3$$
$$\gamma^4_{\alpha\beta} = \begin{pmatrix} i\epsilon & 0 \\ 0 & -i\epsilon \end{pmatrix}$$
$$\gamma^5_{\alpha\beta} = \begin{pmatrix} -\epsilon & 0 \\ 0 & -\epsilon \end{pmatrix},$$  \hspace{1cm} (A.8)$$

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with the reality condition being

\[
(\gamma^1_{\alpha\beta})^* \equiv \gamma^i_{\alpha\beta} = -J^\alpha \gamma^i J^\beta. \tag{A.9}
\]

Appendix B. Lagrangian and SUSY variations

- **N = 4** lagrangian

\[
\frac{1}{4} K_{\Sigma^i \Sigma^i} \left( \dot{x}^j \dot{x}^j + i \left( \ddot{\lambda} \dot{\lambda} + \ddot{\lambda} \dot{\lambda} \right) + D^2 \right) - K_{\Phi \Phi} \left( \dot{\Phi} \dot{\Phi} + i \left( \bar{\psi} \dot{\psi} + \dot{\psi} \bar{\psi} \right) + FF^* \right) + \frac{1}{2} \dot{x}^j \left( K_{\Sigma^i \Sigma^i} \phi \psi^j \lambda + K_{\Sigma^i \Sigma^i} \bar{\phi} \sigma^j \bar{\lambda} \right) + \dot{x}^i \epsilon^{ijk} \left( -\frac{1}{4} K_{\Sigma^i \Sigma^j \Sigma^k} \lambda \sigma^j \bar{\lambda} + K_{\Phi \Phi \Sigma^i \psi \sigma^j \bar{\psi}} \right) + \frac{1}{2} \dot{x}^j \left( \frac{1}{4} K_{\Sigma^i \Sigma^i} \lambda \lambda - K_{\Phi \Phi \Phi} \bar{\psi} \bar{\psi} - iK_{\Phi \Phi \Sigma^i} \psi \sigma^i \bar{\lambda} \right)
\]

\[
+ D \left( \frac{1}{4} K_{\Sigma^i \Sigma^i} \lambda \sigma^i \bar{\lambda} + K_{\Phi \Phi \Sigma^i} \psi \sigma^i \bar{\psi} + \frac{i}{2} \left( K_{\Sigma^i \Sigma^i} \phi \lambda + K_{\Sigma^i \Sigma^i} \bar{\phi} \bar{\lambda} \right) \right) + F \left( -\frac{1}{4} K_{\Sigma^i \Sigma^i} \lambda \lambda - K_{\Phi \Phi \Phi} \bar{\psi} \bar{\psi} + iK_{\Phi \Phi \Sigma^i} \lambda \sigma^i \bar{\psi} \right) + F^* \left( \frac{1}{4} K_{\Sigma^i \Sigma^i} \bar{\lambda} \bar{\lambda} + K_{\Phi \Phi \Phi} \psi \psi - iK_{\Phi \Phi \Sigma^i} \psi \sigma^i \bar{\lambda} \right) \tag{B.1}
\]

+ \frac{1}{4} \left( K_{\Sigma^i \Sigma^i \Phi \Phi} \bar{\lambda} \bar{\lambda} \bar{\psi} \psi + K_{\Sigma^i \Sigma^i \Phi \Phi} \lambda \lambda \psi \psi \right) + K_{\Phi \Phi \Phi} \bar{\psi} \bar{\psi} \psi \psi

+ \frac{1}{16} K_{\Sigma^i \Sigma^j \Sigma^i \Sigma^j} \lambda \lambda \lambda \lambda - K_{\Phi \Phi \Sigma^i \Sigma^j} \psi \sigma^i \bar{\lambda} \lambda \sigma^j \bar{\psi}

- \frac{i}{4} \left( K_{\Sigma^i \Sigma^j \Sigma^i \Phi} \lambda \sigma^j \bar{\psi} \bar{\lambda} + K_{\Sigma^i \Sigma^j \Sigma^i \Phi} \psi \sigma^j \bar{\lambda} \lambda \lambda \right)

- i \left( K_{\Phi \Phi \Sigma^i \Phi} \lambda \sigma^i \bar{\psi} \psi + K_{\Phi \Phi \Sigma^i \Phi} \psi \sigma^i \bar{\lambda} \bar{\psi} \right)
• $\mathcal{N} = 8$ lagrangian* – Follows form the $\mathcal{N} = 4$ one by making use of (3.7).

$$\mathcal{L} = - f \left( \dot{x}^i \dot{x}^i + \dot{\Phi} \dot{\Phi} + i(\bar{\lambda} \dot{\lambda} + \dot{\psi} \bar{\psi} + \dot{\psi} \bar{\psi}) \right)$$
$$+ \dot{x}^i f_k \epsilon^{ijk} (\psi \sigma^j \bar{\psi} + \lambda \sigma^j \bar{\lambda}) - 2 \dot{\psi} (f, \Phi \psi \sigma^j \lambda + f, \Phi \bar{\psi} \sigma^j \bar{\lambda})$$
$$+ \dot{\Phi} (f_i \bar{\psi} \sigma^i \lambda + i f, \Phi \psi \bar{\psi} + i f, \Phi \lambda \bar{\lambda}) + \dot{\Phi} (f_i \psi \sigma^i \lambda - i f, \Phi \psi \bar{\psi} - i f, \Phi \lambda \bar{\lambda})$$
$$+ i f, \Phi (\lambda \sigma^i \bar{\psi} \bar{\lambda} - \psi \sigma^i \lambda \bar{\psi} \bar{\lambda}) + i f, i \Phi (\psi \sigma^i \lambda \lambda - \sigma^i \bar{\psi} \bar{\psi})$$

$$\tag{B.2}$$

\[-f D^2 + D (f_i (\psi \sigma^i \psi - \lambda \sigma^i \bar{\lambda}) - 2 f, \Phi \psi \lambda - 2 f, \Phi \bar{\psi} \bar{\lambda}) - f F F^* + F (f_i \psi \lambda - f, \Phi \bar{\psi} \bar{\psi} + i f, i \lambda \sigma^i \bar{\lambda}) - F^* (f, \Phi \bar{\lambda} \bar{\lambda} - f, \Phi \psi \psi + i f, i \psi \sigma^i \bar{\lambda})\]

• Auxiliary fields – These are given $Spin(5)$ language using the results of appendix A.

$$D = \frac{1}{2} f^{-1} f, i \eta^\alpha, \gamma^i_{\alpha \beta} \bar{\eta}^\beta$$
$$F = \frac{i}{2} f^{-1} f, i \bar{\eta}^\alpha, \gamma^i_{\alpha \beta} \bar{\eta}^\beta$$
$$F^* = \frac{i}{2} f^{-1} f, i \eta^\alpha, \gamma^i_{\alpha \beta} \eta^\beta$$

• Manifest SUSY

$$\delta x^i = -i (\epsilon \sigma^i \lambda + \bar{\epsilon} \sigma^i \bar{\lambda})$$
$$\delta \lambda_\alpha = \dot{x}^i (\epsilon \sigma^i)_\alpha + i D \epsilon_\alpha$$
$$\delta \bar{\lambda}_\alpha = \dot{x}^i (\bar{\epsilon} \sigma^i)_\alpha - i D \bar{\epsilon}_\alpha$$
$$\delta D = \dot{\epsilon} \bar{\lambda} - \dot{\bar{\epsilon}} \lambda$$
$$\delta \Phi = 2 \epsilon \psi$$
$$\delta \bar{\Phi} = -2 \bar{\epsilon} \bar{\psi}$$
$$\delta \psi_\alpha = -i \Phi \bar{\epsilon}_\alpha + F \epsilon_\alpha$$
$$\delta \bar{\psi}_\alpha = -i \Phi \epsilon_\alpha + F^* \bar{\epsilon}_\alpha$$
$$\delta F = -2 i \epsilon \bar{\psi}$$
$$\delta F^* = -2 i \bar{\epsilon} \psi$$

• Non-manifest SUSY

$$\delta x^i = -i (\epsilon \sigma^i \psi - \bar{\epsilon} \sigma^i \bar{\psi})$$
$$\delta \lambda_\alpha = i \dot{\Phi} \epsilon_\alpha + F^* \bar{\epsilon}_\alpha$$
$$\delta \bar{\lambda}_\alpha = -i \dot{\Phi} \bar{\epsilon}_\alpha - F \epsilon_\alpha$$
$$\delta D = \dot{\epsilon} \bar{\psi} - \dot{\bar{\epsilon}} \bar{\psi}$$
$$\delta \Phi = -2 \bar{\epsilon} \lambda$$
$$\delta \bar{\Phi} = 2 \epsilon \bar{\lambda}$$
$$\delta \psi_\alpha = \dot{x}^i (\epsilon \sigma^i)_\alpha + i D \epsilon_\alpha$$
$$\delta \bar{\psi}_\alpha = -\dot{x}^i (\bar{\epsilon} \sigma^i)_\alpha + i D \bar{\epsilon}_\alpha$$
$$\delta F = -2 i \epsilon \bar{\lambda}$$
$$\delta F^* = -2 i \bar{\epsilon} \psi$$

* There is a sign ambiguity in expressions of the form $\psi \sigma^i \lambda$ or $\bar{\psi} \sigma^i \bar{\lambda}$, corresponding to lower or upper indices on the spinors. We universally take spinors with lower indices. No such ambiguity arises if one of the spinors is barred.
• Spin(5) SUSY – These follow from (B.4) and (B.5) using (B.3) and the results of appendix A.

\[ \delta x^i = i \left( \epsilon^\alpha \gamma^i_{\alpha \beta} \bar{\eta}^\beta - \bar{\epsilon}^\alpha \gamma^i_{\alpha \beta} \eta^\beta \right) \]

\[ \delta \eta_\alpha = \dot{x}^i \gamma^i_{\alpha \beta} \epsilon^\beta + \frac{i}{2} f^{-1} f^{\beta} \left( \eta^\gamma \gamma^i_{\gamma \delta} \bar{\eta}^\delta \epsilon_\alpha + \eta^\gamma \gamma^i_{\gamma \delta} \eta^\delta \bar{\epsilon}_\alpha \right) \]  \hspace{1cm} (B.6)

\[ \delta \bar{\eta}_\alpha = -\dot{x}^i \gamma^i_{\alpha \beta} \bar{\epsilon}^\beta - \frac{i}{2} f^{-1} f^{\beta} \left( \eta^\gamma \gamma^i_{\gamma \delta} \bar{\eta}^\delta \bar{\epsilon}_\alpha + \eta^\gamma \gamma^i_{\gamma \delta} \eta^\delta \epsilon_\alpha \right). \]

References


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