Canonical Formulation of Generalized Coupled Dispersionless System

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Abstract

We give a canonical formulation of generalized coupled dispersionless system which is recently proposed by the present authors. Symmetries of this system and their generators are given. A geometrical interpretation of this system is discussed.

Key Words: dispersionless equation, group theory, Lie algebra, Lagrangian, Hamiltonian, canonical formulation, Noether theorem

1 Introduction

The inverse scattering schemes for soliton equations are powerful tool to obtain N -soliton solutions and an infinite number of conserved quantities. The most famous one is the ZS-AKNS scheme[1, 2]. Many inverse scattering schemes such as the ZS-AKNS and its varieties have 2×2 matrix forms. There are, however, fewer generalizations of them to 3×3 or higher dimensional matrix form. It is interesting to hunt for soliton equations with general $n \times n$ inverse scattering schemes. Recently, the present authors have proposed a generalized coupled dispersionless system given by

$$S_{xt} + [S_x, [S, G]] = 0, \quad (1)$$

where the matrix field $S = S(x, t)$ and the constant matrix G are elements of arbitrary non-Abelian Lie algebra[3]. This equation has the $n \times n$ ZS-AKNS type inverse scattering scheme and nonlinearity comes from the non-Abelian character. Eq.(1) is a generalization of the coupled integrable, dispersionless equations[6, 7]:

$$\begin{aligned} q_{xt} + (rs)_x &= 0, \\ r_{xt} - 2q_x r &= 0, \\ s_{xt} - 2q_x s &= 0 \end{aligned} \quad (2)$$

based on group theoretical point of view. For $SU(1, 1) \sim O(2, 1) \sim SL(2, R)$, eq.(1) reproduces eq.(2). And for $SU(2) \sim O(3)$, we can obtain

$$\begin{aligned} q_{xt} + (rr^*)_x &= 0, \\ r_{xt} - 2q_x r &= 0, \\ r^*_{xt} - 2q_x r^* &= 0, \end{aligned} \quad (3)$$

which is equivalent to the Pohlmeyer-Lund-Regge system[4]. Eqs.(2) and (3) have been solved by the inverse scattering method under the appropriate boundary conditions and shown to be integrable[5, 6, 7]. They have important conserved quantities

$$q_x^2 + r_x s_x = q_0^2 \quad (4)$$

and

$$q_x^2 + r_x r_x^* = q_0^2, \quad (5)$$

which are obtained from the inverse scattering schemes. Here $q_0 = q_x(\pm\infty)$ is constant.

Canonical formulation of the system (2) has been given in [8] with the Dirac bracket[9]. In this paper, we present a canonical formulation of generalized coupled dispersionless system

(1). In the next section we shall review our general system. Canonical formulation is given in § 3 and in § 4 we shall consider symmetries of the Lagrangian. The last section is devoted to discussions. We also give in the last section a brief geometrical interpretation of generalized dispersionless system and an analogy between this system and a particle in external magnetic field.

2 Generalized Coupled Dispersionless System

Let $\mathcal{G}$ be a simple Lie group, $\dim \mathcal{G} = N$, and $\mathfrak{g}$ be its Lie algebra. The generators T^a of $\mathfrak{g}$ satisfy the commutation relation with the structure constant f_c^{ab} :

$$[T^a, T^b] = if_c^{ab} T^c, \quad (6)$$

and the Cartan metric η^{ab} is defined by

$$\eta^{ab} = \text{Tr}(T^a T^b). \quad (7)$$

Without loss of generality we can take η^{ab} as diagonal and f_c^{ab} as totally anti-symmetric. With T^a 's we define S by

$$S = \phi_a T^a = \eta_{ab} \phi^a T^b, \quad (8)$$

where $\phi^a = \phi^a(x, t)$ is a vector field with components $(\phi^1, \phi^2, \dots, \phi^N)$ and η_{ab} is the inverse matrix of η^{ab} . We also define a constant matrix G as

$$G = \kappa_a T^a, \quad (9)$$

with a constant vector $\kappa^a = (\kappa^1, \kappa^2, \dots, \kappa^N)$. These quantities are rotated by global gauge transformation:

$$S' = \Omega^{-1} S \Omega, \quad (10)$$

$$G' = \Omega^{-1} G \Omega, \quad (11)$$

where $\Omega \in \mathcal{G}$.

Let us write the action of generalized coupled dispersionless system as

$$I = \int dt dx \mathcal{L}(S, S_x, S_t), \quad (12)$$

where $\mathcal{L}$ is the Lagrangian density defined by

$$\mathcal{L} = \text{Tr} \left(\frac{1}{2} S_x S_t - \frac{1}{3} G[S, [S_x, S]] \right). \quad (13)$$

This Lagrangian density is manifestly invariant under the global gauge transformation (10) and (11). The Euler-Lagrange equation can be derived from the action (12) by

$$S_{xt} - [[S, G], S_x] = 0. \quad (14)$$

From eq.(14) it is easy to show that the following quantity

$$\text{Tr}(S_x^n), \quad (15)$$

is conserved for integer n ($n \geq 1$) and also invariant under the gauge transformation.

This system has the following inverse scattering scheme

$$\begin{aligned} V_x &= UV, \\ V_t &= WV, \end{aligned} \quad (16)$$

where matrices U and W are elements of g and defined with an eigenvalue λ as

$$\begin{aligned} U &= \lambda S_x, \\ W &= W_0 + \frac{1}{\lambda} W_1 = [S, G] + \frac{1}{\lambda} G. \end{aligned} \quad (17)$$

This choice is an extension of the Kotlyarov's definition[4] for the $SU(2)$ to the general Lie algebra. The compatibility condition

$$U_t - W_x + [U, W] = 0, \quad (18)$$

then, yields the equation of motion

$$S_{xt} - [W_0, S_x] = 0, \quad (19)$$

that is, eq.(14).

To obtain conservation law from the inverse scattering scheme, let the Jost function be

$$V_i = \exp(\Sigma_i). \quad (20)$$

Then, from the compatibility condition $V_{xt} = V_{tx}$, we get

$$(\Sigma_i)_{xt} = (\Sigma_i)_{tx}. \quad (21)$$

By using eq.(16), eq.(21) reduced to the form of the conservation law

$$\frac{\partial}{\partial t} \left(\sum_j U_{ij} \frac{V_j}{V_i} \right) = \frac{\partial}{\partial x} \left(\sum_j W_{ij} \frac{V_j}{V_i} \right). \quad (22)$$

Expanding $V_j/V_i (j \neq i)$ in the power series of $1/\lambda$ and equating the terms of the same power of $1/\lambda$, we can formally obtain an infinite number of conserved quantities[10].

Here some examples are shown. For the case of $SU(1,1)$ we can classify according to $\kappa_a \kappa^a > 0, = 0$ and < 0 . For $\kappa_a \kappa^a > 0$ eq.(2) is obtained with $\phi^a = (q, \frac{1}{2}(r-s), \frac{1}{2}(r+s))$ and $\kappa^a = (1, 0, 0)$ and its inverse scattering scheme is given by

$$U = \lambda \begin{pmatrix} q_x & r_x \\ s_x & -q_x \end{pmatrix}, \quad (23)$$

$$W = \begin{pmatrix} 0 & -r \\ s & 0 \end{pmatrix} - \frac{1}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The generators of $SU(1,1)$ are

$$T^1 = \frac{\tau^3}{\sqrt{2}}, \quad T^2 = \frac{\tau^1}{\sqrt{2}}, \quad T^3 = \frac{i\tau^2}{\sqrt{2}}, \quad (24)$$

where the Pauli matrices τ^i :

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (25)$$

The other choices of κ^a , say, $\kappa^a = (0, 0, 1)$ for $\kappa_a \kappa^a < 0$ and $\kappa^a = (1, 0, 1)$ for $\kappa_a \kappa^a = 0$ yield different equations.

For $SU(2)$, without loss of generality by virtue of the gauge invariance, we can choose $\phi^a = (\frac{1}{2}(r+r^*), \frac{1}{2i}(r-r^*), q)$ and $\kappa^a = (0, 0, 1)$ and the generators

$$T^1 = \frac{\tau^1}{\sqrt{2}}, \quad T^2 = \frac{\tau^2}{\sqrt{2}}, \quad T^3 = \frac{\tau^3}{\sqrt{2}}, \quad (26)$$

and obtain eqs.(3). The inverse scheme is the same form as eq.(25) with $s = r^*$.

When we choose other Lie algebra g with dimension N and appropriate κ^a , we obtain dispersionless system with N -components having the inverse scattering scheme with general $n \times n$ matrix.

3 Canonical Formulation

In this section we present a canonical formulation of generalized coupled dispersionless system (14). It is convenient to construct the canonical formulation in terms of ϕ^a . The Lagrangian density (13) becomes

$$\mathcal{L} = \frac{1}{2} \phi_x \cdot \phi_t + \frac{1}{3} \kappa \cdot [\phi \times (\phi_x \times \phi)], \quad (27)$$

where the symbol “ $\cdot$ ” denotes inner product as $\varphi \cdot \psi = \varphi^a \psi_a$ and “ $\times$ ” exterior product defined by $(\varphi \times \psi)_c = f^{ab}_c \varphi_a \psi_b$. The Euler-Lagrange equation is given by

$$\phi_{tx} + (\phi \times \kappa) \times \phi_x = 0 \quad (28)$$

through the variational principle. Linear independence of the basis T^a can also give eq.(28) from eq.(14). The canonical conjugate momentum is defined as

$$\pi_a \equiv \frac{\partial \mathcal{L}}{\partial \phi_t^a} = \frac{1}{2} \phi_{ax}. \quad (29)$$

It is obvious that the canonical conjugate momentum is not independent of ϕ^a . This is represented by constraint equation

$$\chi^a \equiv \pi^a - \frac{1}{2} \phi_x^a = 0. \quad (30)$$

Since the usual Poisson bracket between the constraints does not vanish, this type of constraints is called constraint of the second class. Then it is convenient to introduce the Dirac bracket[9] defined by

$$\begin{aligned} \{F(x), G(y)\}_{\text{D.B.}} &\equiv \{F(x), G(y)\}_{\text{P.B.}} \\ &- \int dz dz' \{F(x), \chi^a(z)\}_{\text{P.B.}} \Delta_{ab}^{-1}(z - z') \{\chi^b(z'), G(y)\}_{\text{P.B.}}, \end{aligned} \quad (31)$$

where the symbol $\{\cdot, \cdot\}_{\text{P.B.}}$ stands for the Poisson bracket with a convention

$$\{\pi_a(x), \phi^b(y)\}_{\text{P.B.}} = \delta_a^b \delta(x - y), \quad (32)$$

and Δ_{ab}^{-1} is the inverse of the matrix made from the Poisson bracket between constraints:

$$\Delta_{ab}(x - y) = \{\chi_a(x), \chi_b(y)\}_{\text{P.B.}}. \quad (33)$$

The Dirac bracket (31) ensures that $\{\chi_a, \chi_b\}_{\text{D.B.}} = 0$ and $\{R, \chi_b\}_{\text{D.B.}} = 0$ where R is an arbitrary function of ϕ^a . Then, the fundamental Dirac bracket between the dynamical variables ϕ^a 's is

$$\{\phi^a(x), \phi^b(y)\}_{\text{D.B.}} = \frac{1}{2} \eta^{ab} \text{sgn}(x - y), \quad (34)$$

where $\text{sgn}(x - y)$ is the sign function. This bracket relation is useful to calculate the relations between physical quantities.

According to the usual procedure, the Hamiltonian is given by

$$H = \int dx \mathcal{H}, \quad (35)$$

where $\mathcal{H}$ is the Hamiltonian density defined by,

$$\begin{aligned}\mathcal{H} &= \phi_t \cdot \pi - \mathcal{L} \\ &= -\frac{1}{3}\kappa \cdot [\phi \times (\phi_x \times \phi)].\end{aligned}\tag{36}$$

Since the Hamiltonian is the generator of time translation, time evolution of the dynamical variables is given by the Hamilton equations in terms of the Dirac bracket as

$$\phi_t^a = \{H, \phi^a\}_{\text{D.B.}},\tag{37}$$

$$\phi_{xt}^a = \{H, \phi_x^a\}_{\text{D.B.}},\tag{38}$$

which coincide with the Euler-Lagrange equation (28).

The conserved quantity (15) for $n = 2$ becomes

$$\phi_x \cdot \phi_x,\tag{39}$$

which reduces to (4) and (5) for the cases of $SU(1, 1)$ and $SU(2)$, respectively.

4 Symmetries

The matrix formed Lagrangian density (13) is invariant under the global gauge transformation (10) and (11). Hence, it is obvious that the Lagrangian density (27) is also invariant under the infinitesimal version of the gauge transformation:

$$\delta\phi = \phi \times \theta,\tag{40}$$

$$\delta\kappa = \kappa \times \theta,\tag{41}$$

where θ^a is an infinitesimal constant parameter of the gauge transformation. Since κ^a is not a dynamical variable, there are no generators of the gauge transformation for it. For a certain subgroup of $\mathcal{G}$, however, there exists a special gauge transformation where $\delta\kappa = \kappa \times \theta = 0$. The Lagrangian density (27) is really invariant under such a transformation and the generators of the special gauge transformation are conserved quantities. The Noether theorem is applicable. The following generator of the gauge transformation of ϕ^a is the case as

$$Q_G^a = \frac{1}{2} \int dx (\phi_x \times \phi)^a,\tag{42}$$

which satisfies the Lie algebra:

$$\{Q_G^a, Q_G^b\}_{\text{D.B.}} = f^{ab}_c Q_G^c,\tag{43}$$

and generates the gauge transformation:

$$\{Q_G^a, \phi^b(x)\}_{\text{D.B.}} = f^{ab}_c \phi^c. \quad (44)$$

The action is invariant under the ‘reparametrization’ of x :

$$dx' = dx + f_x(x)dx, \quad (45)$$

where $f(x)$ is an arbitrary infinitesimal function of x . As special cases, translation and scale transformation of x are included. The Noether theorem gives us the generator of the reparametrization

$$Q_R = \frac{1}{2} \int dx (\phi_x \cdot \phi_x). \quad (46)$$

Although Q_R diverges because of $\phi_x \cdot \phi_x = \text{const.}$, Q_R formally generates the correct transformation

$$\{Q_R, \phi^a(x)\}_{\text{D.B.}} = \phi^a_x(x). \quad (47)$$

Q_R is the Casimir invariant of the Lie algebra g . In fact, the Dirac bracket satisfies the relation

$$\{Q_G^a, Q_R\}_{\text{D.B.}} = 0, \quad (48)$$

i.e. Q_R is gauge invariant. The x -translational invariance means that $\frac{1}{2}\phi_x \cdot \phi_x$ is the momentum density of the field. The conservation law

$$(\phi_x \cdot \phi_x)_t = 0 \quad (49)$$

represents the momentum density conservation. Namely, the conserved quantity (15) has a physical meaning for $n = 2$. The action is also invariant under time translation, $t' = t + \epsilon$ where ϵ is a constant infinitesimal parameter and, of course, the Noether theorem yields the Hamiltonian (35).

5 Discussions

We have provided the canonical formulation of generalized dispersionless system (28), that is, eq(1). We have defined the Dirac bracket between dynamical variables and presented the Hamiltonian and the Hamilton equations for the system. We have also given the generators of the symmetries of the action including the global gauge symmetry and the reparametrization invariance of x . We have obtained the generators of the gauge transformation and found that the Lie algebra is realized by the Dirac bracket between the generators. From the reparametrization invariance, we have shown total momentum Q_R of the field as a conserved quantity.

Although dispersionless system (1) has the inverse scattering scheme and has formally, not explicitly, an infinite number of conserved quantities. Integrability of generalized dispersionless system is not clear. This is still open problem. We hope that our formulation is helpful to solve this problem.

Finally, we would like to give some comments on generalized coupled dispersionless systems. From the geometrical point of view it can be shown that the conserved quantity (15) or (39) is natural consequence of simple structure of the system (14) or (28). At time t fixed, a solution S_x^0 of eq.(14) forms a certain curve in the space S_x . The equation of motion (14) describes time evolution of this curve as

$$D_t S_x^0 = 0, \quad (50)$$

where D_t is covariant derivative defined as

$$D_t A = A_t - [W_0, A]. \quad (51)$$

Eq.(50) may be interpreted as a parallel transport of each point x of the curve along the direction of t and W_0 plays as a connection. $\text{Tr}((S_x^0)^2)$ is a conserved quantity because of eq.(50). This situation is analogous to the square of momentum of a charged particle interacting with external magnetic field. Indeed, since the Lorentz force in the magnetic field is orthogonal to the momentum of the charged particle, the absolute value of the momentum is unchanged. So, assigning W_0 as a magnetic field, we can say that coupled system (3) describes a current-conducting string with infinite length in a magnetic field for $O(3)$. Details will be discussed in a separated paper.

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References

- [1] P. G. Drazin and R. S. Johnson, *Solitons: An Introduction*, Cambridge University Press, Cambridge, 1989.
- [2] M. J. Ablowitz and H. Suger, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1985.
- [3] H. Kakuata and K. Konno, J. Phys. Soc. Japan **65** (1996) 340.
- [4] V. P. Kotlyarov, J. Phys. Soc. Japan **63** (1994) 3535.
- [5] K. Konno, Applicable Analysis, **57** (1995) 209.
- [6] K. Konno and H. Kakuata, J. Phys. Soc. Japan **64** (1995) 2707.
- [7] K. Konno and H. Kakuata, J. Phys. Soc. Japan **65** (1996) 713.
- [8] H. Kakuata and K. Konno, J. Phys. Soc. Japan **65** (1996) 1.
- [9] P. A. M. Dirac, *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, Yeshiva University, New York 1964.
- [10] M.Wadati, H.Sanuki and K.Konno, Prog. Theor. Phys. **53** (1975) 419.