Chaos in Black Holes Surrounded by Gravitational Waves

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The occurrence of chaos for test particles moving around Schwarzschild black holes perturbed by a special class of gravitational waves is studied in the context of the Melnikov method. The explicit integration of the equations of motion for the homoclinic orbit is used to reduce the application of this method to the study of simple graphics.

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1 Introduction

There are two main lines of research of chaotic behavior in General Relativity: one deals with chaoticity associated to inhomogeneous cosmological models, as in Bianchi IX model [1]. The invariance of the relativistic theory challenges the proper concept of chaos in this approach [2]. Although, a gauge invariant method based on Maupertuis principle is proposed in [3] and applied with little changes to the Bianchi IX model [4]. The method points to the occurrence of local instability in this cosmological model near its singularity. This adds to a plethora of early analysis and numerics (see for instance the references cited in [4]), which reinforces the increasingly accepted idea that Bianchi IX is chaotic in some meaningful sense.

The other line assumes a given geometry and looks for chaotic behavior of geodesic motion in this background. In this case, the geometry can be taken as either an approximate or an exact solution of Einstein equations. Examples of chaotic geodesic motion are considered in [5] – [8] (exact geometry) and in [9, 10] (approximate).

Due to its universality and intrinsic mathematical interest, models in which unstable periodic orbits (UPOs) are subjected to small periodic perturbations has been one of the main paradigms of deterministic chaos [11]. An analytical tool to study these models is the Melnikov function that describes the transversal distance between the unstable and stable manifolds emanating from an UPO. Its isolated odd zeros indicate the crossing of these manifolds, hence the onset of chaos [12, 13]. Examples of applications of the Melnikov method in gravitation are: the gravitational collapse of cosmological models [14], the study of orbits around a black hole perturbed by either gravitational radiation [9] or an external quadrupolar shell [10]. Also for par-
articles moving in several models of attractive centers periodically perturbed this method has been applied [15].

In this work we study the occurrence of chaos for test particles moving around a Schwarzschild black hole perturbed by a special class of gravitational waves [16]. Since this perturbation is an exact solution of the Regge-Wheeler equations (see for instance [17]) we can study the range of the parameters of the model in order to have chaos. In [9] a similar situation is studied but the perturbations, albeit more general that ours, are known only in the high frequency limit.

In the next section we present a summary of the Melnikov method. In Sec. 3 we review the homoclinic orbits for Schwarzschild solution, and present an explicit solution $t = t(r)$ for this orbit. In sections 4 and 5 we consider the perturbations and the the application of the Melnikov method, respectively. In the last section we discuss some of the previous results and also compare our work with the application of the Melnikov method presented in [9].

2 The Melnikov method

Let us consider a Hamiltonian,

$$H_0 = \frac{p^2}{2m} + V(q),$$  \hspace{1cm} (1)

that admits at least one UPO with the corresponding homoclinic orbit, and also a small periodic perturbation described by the Hamiltonian function $\epsilon H_1(q, p, t)$. Then the transversal distance, in phase space, between the perturbed unstable and the perturbed stable manifolds emanating from the UPO is proportional to [13]

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\} dt,$$  \hspace{1cm} (2)
where the integral is taken along the *unperturbed* homoclinic orbit, \( t_0 \) is an arbitrary initial time, and

\[
\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p},
\]

(3)

with \((q, p)\) being canonical conjugated variables. If there is an isolated intersection for some \( t_0 \), i.e., an isolated odd zero of \( M(t_0) \), then there will be one for every \( t_0 \). This infinitely many crossings of manifolds will produce a tangle that is the signature of the homoclinic chaos [12, 13].

In particular, if \( H_1 \) has the form

\[
H_1(p, q, t) = \hat{H}_1(p, q) \cos(\omega t),
\]

(4)

we have [13]

\[
M(t_0) = \cos(\omega t_0) \int_{-\infty}^{\infty} \{H_0, \hat{H}_1\} \cos(\omega t) dt + \sin(\omega t_0) \int_{-\infty}^{\infty} \{H_0, \hat{H}_1\} \sin(\omega t) dt
\]

(5)

Then, in the generic case, we will have isolated zeros when at least one of the integrals of the previous formula is different from zero.

### 3 The homoclinic orbits

We shall consider a relativistic particle moving in a fix spacetime described by the metric \( g_{ab} \). The world-line of a particle will be denoted by \( x^a(s) \) and its mass by \( \mu \). The motion of the particle can be obtained from the action

\[
S[x] = \frac{\mu}{2} \int g_{ab} \dot{x}^a \dot{x}^b ds.
\]

(6)

This action is not reparametrization invariant, but is the simpler for our purposes. \( s \) is the proper time along the world-line. The canonical conjugate momentum to \( x^a \) is \( p_a = \mu g_{ab} \dot{x}^b \) and satisfies the mass shell constraint
\[ g^{ab} p_a p_b = -\mu^2. \] The Hamiltonian of the system is
\[ H = \frac{1}{2\mu} g^{ab} p_a p_b. \] (7)

The background metric will be considered as the metric of a non rotating black hole, i.e., the Schwarzschild metric,
\[ ds^2 = -f dT^2 + f^{-1} dR^2 + R^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \] (8)
where \( f = 1 - 2M/R \). Since, \( T \) and \( \varphi \) are cyclic variables we have the conserved quantities:
\[ E \equiv -p_T = \mu f \frac{dt}{ds}, \]
\[ L \equiv p_\varphi = \mu R^2 \sin^2 \vartheta d\varphi. \] (9)

For planar motion \( \vartheta = \pi/2 \) (\( \dot{\vartheta} = p_\vartheta = 0 \)) we have the equivalent one dimensional problem of a particle of mass \( 2\mu^2 \) moving in the the potential \( V \) and constant energy \( E^2 - \mu^2 \),
\[ \mu^2 \left( \frac{dR}{ds} \right)^2 + V(R) = E^2 - \mu^2 \]
\[ V(R) = -\frac{2M\mu^2}{R} + \frac{L^2}{R^2} - \frac{2ML^2}{R^3}. \] (10)

It is convenient to work in dimensionless variables. Defining \( r = R/2M \) and \( U = (2M/L)^2 V \) we find
\[ U = -\frac{(1 - \beta^2)}{3r} + \frac{1}{r^2} - \frac{1}{r^3}, \] (11)
where \( \beta = \sqrt{1 - 12M^2\mu^2/L^2} < 1 \). Since \( U(\infty) \sim -O(1/r) \), in order to have a local maximum (or unstable point, \( r = r_u \)) with \( U(r_u) < 0 \), we need \( 0 \leq \beta \leq 1/2 \).
In Fig.1 we plot $U$ as a function of $r = R/2M$ for various values of $\beta$. The unstable periodic orbit (UPO) correspond in phase space to the point $(r, p_r) = (r_u, 0)$. Now by taking $U(r_u) = (2M/L)^2(E^2 - \mu^2)$ we get the motion equation for the homoclinic orbit,

$$\frac{dr}{dt} = \pm \frac{w_\beta (r - 1)(r - r_u)(r_m - r)}{r^{3/2}}, \quad (12)$$

where $r_m = 3/(1 - 2\beta)$, and $w_\beta = \frac{3}{2}\sqrt{3(1 - 2\beta)/(2 - \beta)^2}$. We have used the constant of motion $E$ to change the parameter of evolution $s$ to the dimensionless time coordinate $t = T/2M$. So the homoclinic orbit is limited by $r = r_u$ and the maximum value $r = r_m$ (turning point). This motion equation admits the primitive,

$$\pm w_\beta t = \left[r(r_m - r)\right]^{1/2} / r_m + (2 + 2r_u + r_m) \arctan \sqrt{(r_m - r)/r}$$

$$- \frac{2r_u^{5/3}}{(1 - r_u)\sqrt{r_m - r_u}} \arctanh \sqrt{r_m/r - 1} / r_m - 1$$

$$+ \frac{2}{(1 - r_u)\sqrt{r_m - 1}} \arctanh \sqrt{r_m - r} / r(r_m - 1). \quad (13)$$

We have chosen the constant of integration to have $t(r = r_m) = 0$. In Fig.2 we show the positive branch of (13) for different values of $\beta$. The particle takes an infinite amount of time to depart (arrive) from (to) the homoclinic point where the UPO is located, this is a universal behaviour for these type of orbits. The explicit expression (13) for the homoclinic orbit will play an important role in our study of the Melnikov function and its use will be a significant departure from the treatment given in [9] wherein other interesting graphics related with the homoclinic orbit are presented.
4 The perturbations

The perturbations of the black hole that we shall consider are of the particular class \( g_{ab} + \epsilon h_{ab} \) with \( g_{ab} \) given by (8) and

\[
\begin{align*}
    h_{TT} &= -f X P_l \cos(\sigma T) \\
    h_{RR} &= f^{-1} Y P_l \cos(\sigma T) \\
    h_{\varphi\varphi} &= R^2 \sin^2 \vartheta (Z P_l + WP_l \cot \vartheta) \cos(\sigma T), \\
    h_{\varphi\theta} &= R^2 (Z P_l + WP_l \cot \vartheta) \cos(\sigma T),
\end{align*}
\]

where \( X, Y, Z \) and \( W \) are functions of \( R \) to be determined by the Einstein equations \( R_{ab}(g_{cd} + \epsilon h_{cd}) = 0 \). \( P_l = P_l(\cos \vartheta) \) are the usual Legendre polynomials. When the above mentioned Einstein equations are expanded up to the first order in \( \epsilon \) we find a linear system of first order differential equations for the functions \( X, Y, Z \) and \( W \) [17]. The particular separation of the angular part of the perturbation (14) was found by Friedman [18]. Combinations of this variables reduce these equations to the Zerilli equation, i.e., to a Schrödinger type of equation with a nontrivial potential [17]. A particular solution of the first order differential equations is

\[
X = pq, \quad Y = 3Mq, \quad Z = (R - 3M)q, \quad W = Rq,
\]

where

\[
q = f^{1/2}/R^2, \quad p = M - \frac{M^2 + \sigma^2 R^4}{R^2 - 2M}.
\]

This solution was found by Xanthopoulos [16] using an ad hoc method. We note that it can be easily found in the Zerilli formulation and corresponds to zero Zerilli function \( Z^{(+)} \),

\[
Z^{(+)} = \frac{R^2}{n R + 3M} \left( \frac{3M}{R} W - Y \right),
\]
where \( n = (l - 1)(l + 2)/2 \).

From Eq. (7) we have for the perturbed system

\[
(g^{ab} - \epsilon h^{ab})p_ap_b = -\mu^2.
\]

For an even \( l \) we have that the perturbations (14) are even functions in the variable \( \cos \vartheta \), i.e., we have perturbations with reflection symmetry with respect the plane \( \vartheta = \pi/2 \). Then, for a particle moving in the plane \( \vartheta = \pi/2 \) these type of perturbations do not change the plane of the orbit. In other words, the perturbation introduces a “perpendicular force” that is an odd function of \( \cos \vartheta \). We find for particles moving in the plane \( \vartheta = \pi/2 \),

\[
-p_t = H_0 + \epsilon H_1,
\]

with

\[
H_1 = -\left[\frac{E}{2f} h_{TT} - \frac{1}{2E} \left( f^3 p_R^2 h_{RR} + \frac{L^2 f}{R^4} h_{\phi\phi} \right) \right],
\]

and

\[
\begin{align*}
h_{TT} &= -fpq N_l \cos(\sigma T), \\
h_{RR} &= \frac{3Mq}{f} N_l \cos(\sigma T), \\
h_{\phi\phi} &= R^2 (R - 3M)q N_l \cos(\sigma T),
\end{align*}
\]

where \( N_l \) is zero for odd \( l \) and

\[
N_l = (-1)^n \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n}, \quad (l = 2n).
\]

The order of Legendre polynomial \( l \) enters in a trivial way in this class of planar perturbations, as a multiplication by the constant \( N_l \). The frequency \( \sigma \) not only appears in the oscillating function \( \cos(\sigma T) \) but also in \( p \).
5 The Melnikov function

Now we shall use the fact that we know explicitly the function $t = t(r)$ for the homoclinic orbit to evaluate the Melnikov function. We find

$$M(t_0) = -\sin(\omega t_0)K,$$

with

$$K = \frac{E}{M} N_t J,$$  \hspace{1cm} (24)

$$J = \int_{r_u}^{r_m} F(r, \beta) \sin(\omega t(r)) dr,$$  \hspace{1cm} (25)

where $\omega = 2M\sigma$ is a dimensionless frequency and

$$F(r, \beta, \omega) = \left[ \frac{1}{2r^2} \hat{h}_{tt} - \frac{f}{2r^5} h_{tt, r} + \frac{2\gamma f^2}{r^5} (f - \frac{1}{4r}) \hat{h}_{\varphi \varphi} - \frac{\gamma f^5}{2r^4 (2M)^2} \hat{h}_{\varphi \varphi, r} + \frac{f^2}{2r^2} (1 - 2\gamma f^2) \hat{h}_{rr} - \left( \frac{dr}{dt} \right)^2 \left( \frac{1}{r^2} \hat{h}_{rr} + \frac{f}{2} \hat{h}_{rr, r} \right) \right].$$  \hspace{1cm} (26)

The constant $\gamma$ is defined in terms of $\beta$ as $\gamma = \frac{2}{2r^2}(1 + \beta)(2 - \beta)^2$, and

$$\hat{h}_{tt} = -\frac{f^{3/2}}{2r^2} \left( 1 - \frac{1 + 4\omega r^4}{2rf} \right),$$

$$\hat{h}_{rr} = \frac{3}{2r^2 f^{1/2}},$$

$$\frac{\hat{h}_{\varphi \varphi}}{(2M)^2} = f^{1/2}(2r - 3)/2.$$  \hspace{1cm} (27)

By $t = t(r)$ in (25) we mean the positive branch of (13). We have used the homoclinic orbit to map the infinite interval of the integration to a finite one. Also, all the quantities appearing in the definition of $J$ are dimensionless.

Therefore, to have the homoclinic tangle of orbits we need $J \neq 0$. The exact computation of $J$ even in the simple case of the perturbations
under consideration is hopeless. Nevertheless, the integrand of $J$ is explicitly known in terms of the variable of integration. This fact can be used to study the zeros of $J = J(\beta, \omega)$, or better the range of $\beta$ and $\omega$ in which $J \neq 0$, in a simple graphic way.

Firstly, it is instructive to plot the function $S = \sin(\omega t(r))$ for a fix $\beta$ and different values of $\omega$. We find (see Fig.3) that, depending on the value of $\omega$, $S$ does not change sign in a large portion of the interval $r_u(\beta) \leq r \leq r_m(\beta)$, e.g., the two bottom curves (from right to left). We also have a very rapid oscillation near the position of the UPO, that is not shown in the graphic. The values of the parameters are $\omega = 0.15$ (top, from right to left), 0.1, and 0.05 (bottom) and $\beta = 1/4$. Also it is illuminating to make a graphic of $F(r, \beta, \omega)$ for different values of $\beta$ and $\omega$ in the range of interest $r_u(\beta) \leq r \leq r_m(\beta)$. Fig. 4 shows that this function is very well behaved and does not change of sign in the interval of interest. The integrand of $J$, i.e., $F(r, \beta, \omega) \sin(\omega t(r))$ with $\beta = 1/4$ and $\omega = 0.1$ is presented in Fig.5. The area under the curve is clearly different from zero, then $J \neq 0$. In this case the perturbation will originate a chaotic motion of the test particle. The value $\omega = 0.1$ was chosen from the relation

$$\omega_k \sim \pi/t_k,$$

(28)

where $t_k$ is the value of the homoclinic orbit that correspond to the maximum value of its curvature $k = t''/(1 + (t')^2)^{3/2}$. For $\beta = 1/4$ we have $t_k = 29.3691$ and $\omega_k = 0.1069$. Therefore, we can say that the perturbations with $\omega < \omega_k$ will originate chaotic behaviour.
6 Discussion

To better understand the particular perturbation (14) we have computed the Kretschmann scalar for the perturbed metric for the particular values of $l = 2$ and $l = 3$, we find

$$R_{abcd}R^{abcd} = \frac{48M^2}{R^6} + \epsilon k_l \frac{M}{R^8}(3M - R)\sqrt{1 - \frac{2M}{R}} P_l(\cos \vartheta),$$

(29)

the numerical constant $k_l$ is $k_2 = 58$ and $k_3 = -3/2$. Thus, we have a Kretschmann scalar decaying as $R^{-7}$ for $R \gg 2M$, indicating that part of the wave has origin in the black hole.

In this work we have shown that for a particular class of gravitational perturbations we have that the particles moving along timelike geodesics will present a chaotic motion when the frequency of the perturbation is $\omega < \omega_k$. In the complementary case, $\omega > \omega_k$ our simple graphic method does not apply. But, anyway one can compute numerically the integral $J$ with a great precision (the integrand is known explicitly). We found that for $\omega > \omega_k$, $J$ has positive and negative values depending on the values of $\beta$. Then for certain specific values of $\beta$ we will have $J(\beta, \omega) = 0$ and the Melnikov method does not apply to these cases. From the previous analysis it does look like that the generic situation is chaotic, i.e., only for a numerable set of frequencies $\omega$, given a value of $\beta$, we should not have chaos. The number of elements of this set increases with $\omega$, but is still numerable. To be more precise, the Melnikov method fails to predict chaos only for a a numerable set of values of the parameters. We note that seldom one can find so easily the parameter range of a given problem that will produce a chaotic situation. Moreover, for parameters outside this range it is reasonable to guess that only for a numerable set of frequencies the integrability is preserved. Therefore,
based in our previous analysis, we can conjecture that chaos is generic for the class of perturbed systems studied. In our case, the limit \( \lim_{\omega \to \infty} J(\beta, \omega) \) is not defined. Note that in \( F(\beta, \omega) \) appears \( \omega^4 \) and in the integrand we also have \( \cos \omega t \). In the case analyzed in [9] the perturbation are quite general, but they are known explicitly only in the limit \( \omega \to \infty \). Their integral equivalent to our \( J \) has \( \pm \infty \) integration limits and in the integrand they have, essentially, an oscillating function with frequency \( \omega \) multiplied by another function proportional to \( \omega \). Since, in [9], every thing is known only in the limit of infinite frequency, in order to predict chaotic behaviour, it is necessary to show that their integral is different from zero in the same limit. This point is unclear in the quoted paper.

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References


FIGURE CAPTIONS

Fig.1. The effective $U$ potential is plotted for the dimensionless parameter $eta = 1/3$ (top curve), 1/4, and 1/5 (bottom curve); the maximum value is located at $r_{un} = 3/(1 + \beta)$.

Fig.2. A graphic of the positive branch of Eq. (13) is pictured for values of the parameter $eta = 1/3$ (top curve), 1/4, and 1/5 (bottom curve). We have that the particle takes a finite time to travel from $r_m$ to the vicinity of $r_{un}$ and then takes an infinite time to arrive to $r_{un}$ itself, wherein is located the UPO.

Fig.3. A graphic of $S = \sin[\omega\tau(r)]$ for $\beta = 1/4$, and $\omega = 0.05$ (bottom curve, from right to left), 0.1, and 0.15 (top curve).

Fig.4. A plot of $F(r, \beta, \omega)$ for $\beta = 1/4$.

Fig.5. A plot of $F(r, \beta, \omega) \sin(\omega t(r))$ for $\beta = 1/4$, and $\omega = 0.1$. We have a rapid oscillation near $r_{un}$ ($=12/5$) (not shown in the figure), that produce very fine spines in the range [-0.03, 0.03].