Degenerate Metric Phase Boundaries

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Abstract
The structure of boundaries between degenerate and nondegenerate solutions of Ashtekar’s canonical reformulation of Einstein’s equations is studied. Several examples are given of such “phase boundaries” in which the metric is degenerate on one side of a null hypersurface and non-degenerate on the other side. These include portions of flat space, Schwarzschild, and plane wave solutions joined to degenerate regions. In the last case, the wave collides with a planar phase boundary and continues on with the same curvature but degenerate triad, while the phase boundary continues in the opposite direction. We conjecture that degenerate phase boundaries are always null.

1 Introduction
The notion that the spacetime metric is a kind of “order parameter” $\langle g_{\mu\nu} \rangle \neq 0$ has been suggested many times in many contexts. The peculiar fact that the metric is nonvanishing in vacuum could be understood in this way. In string theory, where the metric is just a tiny part of a richer set of stringy degrees of freedom, such an interpretation appears particularly natural. In fact it has been suggested\cite{1} that above the Hagedorn temperature there is a transition to a “topological” phase with greater symmetry in which the metric vanishes. Degenerate metrics also occur in the loopy approach to canonical quantum Einstein gravity\cite{2, 3}. In a quantum state whose wavefunction is based on holonomies of the spin connection, the spatial metric has, microscopically, rank one on loops away from loop intersections.

One way to explore the consequences of admitting degenerate metrics is to adopt a polynomial formulation of Einstein gravity. In fact there are many ways to express Einstein
gravity in polynomial form, and indeed the possibility that degenerate metrics should be admitted as *bona fide* solutions has been discussed many times in the past (even by Einstein himself[4]). One reason why this possibility has attracted little notice is that one must choose whether it is the covariant metric, the contravariant metric, or some other related object that is to be allowed to become degenerate. The choice seems arbitrary, yet the resulting extensions of the theory have quite different properties.

In this paper we adopt Ashtekar’s Hamiltonian formulation[5], which is polynomial in the canonical variables and so admits a degenerate extension, and we study the structure of boundaries between nondegenerate and degenerate phases. Even within this category of degenerate extensions, there are inequivalent variations, and several investigations of these have already appeared [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The variation we adopt is the original formulation given by Ashtekar. It may turn out that this is not the “right” one. In fact, it has a negative energy problem[9] (which does not exist at least for one other degenerate extension of GR[17]), and some recent work suggests that the formulation with scalar lapse and Hamiltonian constraint of unit weight might be more relevant for the quantum theory than Ashtekar’s original formulation with a Hamiltonian constraint of weight two[18]. Nevertheless, it seems worthwhile to get a feeling for the degenerate behavior of the various formulations, and a lot of work has already been done in quantum gravity using the formulation we are studying.

In Ashtekar’s formulation one of the canonical variables is the spatial pull-back of the self-dual spin connection. Its canonical conjugate $E^i_a$ is thus a triad of contravariant spatial vector densities with tensor density weight one. (Equivalently, their Levi-Civita duals are spatial two-forms.) This triad is related to the spatial metric by

$$E^a_i E^b_i = q q^{ab} \Rightarrow \det E^a_i = \det q_{ab} \equiv q .$$

(1)

(Our notation is that $a, b, c, ...$ are spatial vector indices, while $i, j, k, ...$ are $SO(3)$ indices.)

The phase space action has an elegant and polynomial form when expressed in these variables. In this paper we will study “regular” solutions to Ashtekar’s Hamiltonian equations, that is, solutions in which the canonical variables, shift vector, and lapse density all take finite values which, except for the lapse density, are allowed to vanish. Note that Ashtekar’s lapse density $N$ is related to the usual scalar lapse $M$ by

$$N = \frac{M}{\sqrt{q}} .$$

(2)

If the spatial metric becomes degenerate, i.e. if $q \to 0$, then the requirement that $N$ should stay finite is a non-trivial requirement relating the coordinates and the covariant spacetime metric. We will use both the canonical formalism and the covariant spacetime viewpoint to study the degenerate solutions.

The particular question that we wish to focus on concerns the “phase boundary” between two regions of spacetime having a degenerate metric on one side of the boundary. This question has been considered before by one of us [7]: the conclusion then was that the exterior Schwarzschild solution can be matched to a degenerate region across the event
horizon. While this is true, we will see that it is also very misleading, since in fact the
join can take place on much more general null surfaces. For example, a lightlike plane in
Minkowski space can serve as such a phase boundary, and in the Schwarzschild solution one
can join to a degenerate metric across any spherically symmetric null surface.

In section 2 of this paper we explain the general construction for obtaining a solution
with degenerate phase boundary by starting with nondegenerate Ashtekar initial data, mak-
ing a degenerate coordinate transformation, and evolving by Ashtekar’s hamiltonian. We
then apply this method to flat space data to join Minkowski space to a degenerate region,
and show how this solution can be generalized to include “connection waves” [14] in the
degenerate region.

In section 3 we adopt a covariant approach to studying degenerate phase boundaries,
imposing appropriate conditions on the covariant metric that follow from the Ashtekar the-
tory. Using this method we find solutions in which portions of Minkowski and Schwarzschild
spacetimes are joined to degenerate regions. We also study the stability of the flat space
phase boundary, through the simple device of hitting it with a plane gravitational wave
from the non-degenerate side. It turns out that the phase boundary remains null and the
plane wave continues into the degenerate region where it becomes a degenerate plane wave
solution with the same curvature.

In section 4 the question whether or not the phase boundary must always be null is
formulated and considered. We show that the field equations must play a role if this is to
be the case, and we conjecture that it is related to the fact that the characteristic surfaces
of the nondegenerate theory are null. Finally, we close in section 5 with a discussion of our
results and some open questions.

2 Canonical approach

To begin with, let us show how any nondegenerate Ashtekar initial data set \((E^0_i, A^0_i)\) can be
modified to produce a solution with a degenerate phase boundary at any initial spatial two-
surface. This general construction was first discussed by Varadarajan[9]. Let the spatial
coordinates be called \((x, y, \bar{z})\), and parametrise \(\bar{z}\) by a function \(\bar{z}(z)\) in such a way that
\(\bar{z}' := d\bar{z}/dz \to 0\) at \(z = 0\). Since the connection \(A^i_z\) is a covariant tensor, and the triad
density \(E^a_i\) is the Levi-Civita dual of a covariant tensor \(\Sigma_{bc} = \epsilon_{abc} E^0_i\), the components of
the initial data remain finite in the new coordinate system \((x, y, z)\). In fact, we have

\[
(A^i_x, A^i_y, A^i_z) = (\bar{A}^i_x, \bar{A}^i_y, \bar{z}' \bar{A}^i_z) \tag{3}
\]

and

\[
(E^x_i, E^y_i, E^z_i) = (\bar{z}' \bar{E}^x_i, \bar{z}' \bar{E}^y_i, \bar{E}^z_i) \tag{4}
\]

Thus \(A^i_z, E^x_i, E^y_i\) go to zero at the surface \(z = 0\). Now it is possible adopt this data
for positive values of \(z\), and to smoothly join it to data for negative values of \(z\) in such
a way that \(A^i_z, E^x_i, E^y_i\) remain zero for all negative \(z\). The initial value constraints
will of course be satisfied for positive $z$, since we have only made a regular coordinate transformation there. For negative $z$ the constraints require
\begin{align}
\partial_z E^z_i &= 0 \\
E^z_i \partial_z A^a_i &= 0,
\end{align}
(5)
as discussed in [14]. This new data has a “phase boundary” between a nondegenerate phase for $z > 0$ and a degenerate phase for $z < 0$. The data can now be evolved using Ashtekar’s hamiltonian, with finite lapse and shift, to yield a regular solution to Ashtekar’s degenerate extension of general relativity. Let us see what this procedure yields when performed starting with flat initial data.

Suppose the original “barred” data is just $\bar{A}^i_a = 0$ and $\bar{E}^a_i = \delta^a_i$. Then in the new coordinate system we have $A^i_a = 0$ and
\begin{equation}
E^a_i = \text{diag}(h(z), h(z), 1)
\end{equation}
(7)
where $h(z) = z'$ for $z > 0$ and $h(z) = 0$ for $z < 0$. This data satisfies all the constraints. Let us evolve it according to the equations of motion with unit lapse and vanishing shift[3]:
\begin{align}
\partial_t A^i_a &= i \epsilon_{ijk} E^b_j F^k_a \\
\partial_t E^a_i &= -i \epsilon_{ijk} E^b_j D_b E^a_k
\end{align}
(8)(9)
where $D_b$ denotes the covariant derivative with respect to the connection $A^i_a$. Given our choice of initial data, $A^i_a$ and $E^a_i$ are thus constant, equal to their initial values. The remaining equations are best written in terms of the complex linear combinations
\begin{equation}
E^a_\pm := E^a_1 \pm i E^a_2,
\end{equation}
(10)
viz.
\begin{equation}
\partial_t E^a_\pm = \pm \partial_z E^a_\pm.
\end{equation}
(11)
The general solution is thus given by $E^a_\pm = E^a_\mp(z \pm t)$. Evaluating at time $t = 0$ gives
\begin{equation}
E^a_\pm = h(z \pm t), \quad E^a_\mp = \pm i h(z \pm t).
\end{equation}
(12)
The spacetime metric with unit lapse (density) and vanishing shift is given in terms of the Ashtekar variables by
\begin{equation}
ds^2 = -E dt^2 + q_{ab} dx^a dx^b,
\end{equation}
(13)
where $E = \det E^a_i$ and $q_{ab}$ is the inverse of $q^{ab} = E^{-1} E^a_i E^b_i$. In the present case we have $E = h_+ h_-$ and $q_{ab} = \text{diag}(1, 1, h_+ h_-)$, where $h_{\pm} = h(z \pm t)$, and the metric becomes
\begin{equation}
ds^2 = h_+ h_- (d\zeta^+ d\zeta^-) + dx^2 + dy^2,
\end{equation}
(14)
where $\zeta^\pm = z \pm t$. Since one can integrate $h_{\pm} d\zeta_{\pm} = d\xi_{\pm}$, this is evidently just flat spacetime as long as $h(z)$ is nowhere vanishing.
Figure 1: Minkowski spacetime spliced onto a degenerate solution across a pair of intersecting null surfaces. The numbers \((m, n)\) indicate the rank of \(E^a_i\) and \(E^a_i E^{bi}\) respectively.

Now suppose we choose initial data of the form (7) with \(h(z) = 0\) for \(z < 0\) and \(h(z) > 0\) for \(z > 0\). Then the solution is nondegenerate only in the right hand wedge \(z > |t|\) (see figure). The boundary of this wedge is the “phase boundary”. Minkowski spacetime has been smoothly spliced onto a degenerate solution across this pair of intersecting null surfaces. The triad takes the form:

\[
E^a_i = \begin{pmatrix}
    h_+ & h_- & 0 \\
    ih_+ & -ih_- & 0 \\
    0 & 0 & 1
\end{pmatrix}
\]

where \(h_\pm := h(z \pm t)\), and the rows are \(x, y, z\) components and the columns are \(+, -, 3\) components. In the left hand wedge both \(h_+\) and \(h_-\) vanish, so both \(E^a_i\) and \(E^a_i E^{bi}\) have rank one. The triad is therefore of type \((1,1)\) in the Lewandowski-Wiśniewski notation[15]. In the forward and backward wedges \(h_-\) and \(h_+\) vanish respectively, so \(E^a_i\) has rank two but \(E^a_i E^{bi}\) still has rank one. The triad is therefore of type \((2,1)\).

In the left hand wedge the triad takes the form

\[
E^a_i = \text{diag}(0, 0, 1),
\]

which is the type of degenerate triad studied in [14]. There it was found that transverse connection wave excitations are compatible with this form of triad and propagate at the speed of light. What happens if such a connection wave is launched towards the phase boundary from the left hand wedge? Of course the wave never reaches the phase boundary, which is also moving at the speed of light. However it is interesting to ask what will happen when the wave enters the forward wedge \(t > |z|\), where the triad takes the slightly less degenerate form

\[
E^x_\pm = 0 = E^y_-, \quad E^x_\pm = h(z + t) = -iE^y_+, \quad E^z = (0, 0, 1).
\]

Let us examine the constraints and equations taking the above as an ansatz for \(E^a_i\), together with the ansatz \(A^{3}_a = 0\), \(A^i_z = 0\). It turns out that this ansatz is consistent
provided that \( E^a_+ A^+_a = 0 \), that is, \( A^+_x + i A^+_y = 0 \). In particular, if we simply set \( A^+_y = 0 \), we have solutions with connection waves propagating to the right into the forward wedge, without disturbing the "flat space" form of the triad.

In the next section we will see that one can also splice Minkowski spacetime across a single null surface.

### 3 Covariant approach

In this section we study the form of degenerate phase boundaries by working directly with the spacetime metric. We use basically the same simple device as was used in the canonical approach. The idea (which has occurred to others before us) is to start from a non-degenerate metric which solves Einstein’s equations, and then reparametrize one of the coordinates. This reparametrization is chosen so that it is not a diffeomorphism at some particular value of the coordinate. Adopting the new coordinate, the solution can be smoothly matched to a solution to the Ashtekar equations with a degenerate metric at the surface where the transformation misbehaves.

What are the regularity conditions we should impose at the phase boundary? If the covariant spatial metric is finite, then a finite Ashtekar triad density necessarily exists: if the metric is diagonalized, \( \mathbf{q}_{ab} = \text{diag}(q_{xx}, q_{yy}, q_{zz}) \), then \( \mathbf{E}^a_i = \text{diag}(\sqrt{q_{yy}} q_{zz} \sqrt{q_{zz}} q_{xx} \sqrt{q_{xx}} q_{yy}) \).

We shall thus look for solutions where the covariant spatial metric becomes degenerate, but remains finite, on a surface. (Note, however, that given a degenerate Ashtekar triad density \( \mathbf{E}^a_i \), the covariant spatial metric is not necessarily finite. For example, if \( \mathbf{E}^a_i = \text{diag}(1, 1, 0) \) then \( q_{zz} \) is infinite. Thus we exclude from the outset such possibilities from our study in this section.) In fact we shall restrict attention to cases where the full covariant spacetime metric is regular. In addition, we must require that the lapse with weight minus one be well defined. Because of eq. (2) this is a nontrivial condition which implies in particular that the scalar lapse must go to zero at the phase boundary.

#### 3.1 Flat spacetime

The simplest solution that we can consider is the Minkowski space metric

\[
\text{ds}^2 = -dT^2 + dX^2 + dY^2 + dZ^2 .
\]

Now perform the coordinate transformation

\[
Z = Z(z) \quad \Rightarrow \quad \text{ds}^2 = -dT^2 + dX^2 + dY^2 + Z'^2 dz^2
\]

where \( Z' = dZ/dz \). Since we do not insist that the reparametrization should be a diffeomorphism we can choose

\[
Z = z^3/3, \quad z \geq 0 \quad \quad Z = 0, \quad z < 0
\]

in which case the metric becomes degenerate at \( z = 0 \). Indeed this example can be generalized to provide metrics that are well defined but degenerate anywhere we please, and since
the spin connection also transforms covariantly it will always be left well defined by such a transformation.

That one can use this device to produce degenerate covariant metrics has been noticed before, for instance by Horowitz [19] who also quotes theorems to the effect that if one considers a dense set of smooth coordinate transformations then the generic situation is that one obtains three dimensional hypersurfaces where the metric has rank three and isolated points where it has rank two.

Now we add the requirement that Ashtekar’s lapse should be well defined as well. The situation then becomes more restrictive; in the example we just considered

\[ N = \frac{1}{Z'} . \]  

This would diverge if \( Z' \to 0 \), so that the example must be rejected. We therefore try something more general:

\[ T = T(t, z) \quad Z = Z(t, z) . \]  

To keep things simple we insist that we should have \( g_{tz} = 0 \) also after the transformation, which is ensured by the choice

\[ \dot{T} = Z' \quad T' = \dot{Z} \quad \Rightarrow \quad \dot{Z} = Z'' \]  

where a dot and a prime denote differentiation with respect to \( t \) and \( z \) respectively. In words, the simplification implies that \( Z(t, z) \) and \( T(z, t) \) obey the two-dimensional massless wave equation, or in other words that the most general coordinate transformation still at our disposal expresses \( Z \) and \( T \) as sums of “left” and “rightmovers”:

\[ Z = f(z + t) + g(z - t) \]  

\[ T = f(z + t) - g(z - t) + \text{constant} . \]  

The line element becomes

\[ ds^2 = 4f'g'(-dt^2 + dz^2) + dX^2 + dY^2 . \]  

This time Ashtekar’s lapse \( N \) is unity, and we can allow the metric to become degenerate through the choice

\[ \lim_{s \to 0^+} g'(s) = 0 . \]  

Again the coordinate transformation fails to be a diffeomorphism at the surface of degeneracy. The latter is given by \( z = t \), which when reexpressed in the original coordinates is simply the lightlike plane \( T = Z + \text{constant} \).

This solution can now be joined across the surface of degeneracy to a regular solution of Ashtekar’s equations which has an everywhere degenerate metric. How do we know this is possible? One argument goes as follows: Extend any timeslice from the nondegenerate
region across the phase boundary $T = Z + \text{constant}$. On this slice use the Minkowski initial data in the nondegenerate region and smoothly match to degenerate data on the other side. When this initial data is evolved, it will simply yield the Minkowski spacetime everywhere outside, since the null phase boundary coincides with the Cauchy horizon for the exterior data.

Another method is to argue that, under a degenerate "coordinate transformation", the Ashtekar variables—which can be obtained in a polynomial manner from the covariant tetrad and connection—remain finite. If the lapse and shift also remain finite then it seems reasonable to suppose that the transformed variables continue to solve the Ashtekar equations because the transformation has the same form as a diffeomorphism (although it is not a diffeomorphism). We have not yet managed to show that this is true in general, but it can be explicitly checked for the coordinate transformation (24, 25). In this case the connection remains zero, the lapse unity, and the shift zero, and the triad takes the form

$$E_i^x = \begin{pmatrix} f' + g' \\ i(f' - g') \\ 0 \end{pmatrix}, \quad E_i^y = \begin{pmatrix} -i(f' - g') \\ f' + g' \\ 0 \end{pmatrix}, \quad E_i^z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (28)$$

It is easily checked that for any $f(z + t)$ and $g(z - t)$ this is a solution to the Ashtekar equations (9). Choosing $f' \neq 0$ and $g'(s) = 0$ for $s < 0$ yields a solution with a single null degeneracy boundary. To recover the solution of the previous section with a pair of intersecting phase boundaries one must choose $f' = g'$.

### 3.2 Spherically symmetric spacetime

The general structure of the result just obtained depends neither on the plane symmetry nor on the flatness of Minkowski space. For example, it is possible to perform a coordinate transformation such that the metric becomes degenerate on the forward light cone of a point in Minkowski space. In this case the surface of degeneracy is singular at the tip of the cone, so we can not really match the solution smoothly to an everywhere degenerate solution on the other side.

This problem is avoided if instead we consider the more general spherically symmetric line element

$$ds^2 = -F(R)dT^2 + \frac{dR^2}{F(R)} + R^2d\Omega^2, \quad (29)$$

where

$$F(R) = 1 - \frac{2m}{R} \quad (30)$$

for the Schwarzschild solution, although the explicit form of $F(R)$ will not matter for the argument which follows. If we let $R$ and $T$ depend on $t$ and $r$, and then simplify things by the requirements

$$\dot{T} = \frac{R'}{F(R)}, \quad T' = \frac{\dot{R}}{F(R)} \quad (31)$$

It is easily checked that for any $f(z + t)$ and $g(z - t)$ this is a solution to the Ashtekar equations (9). Choosing $f' \neq 0$ and $g'(s) = 0$ for $s < 0$ yields a solution with a single null degeneracy boundary. To recover the solution of the previous section with a pair of intersecting phase boundaries one must choose $f' = g'$. 

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(with dot and prime denoting differentiation with respect to $t$ and $r$ respectively), we find that the shift vector vanishes and that the line element becomes

$$
\mathbf{ds}^2 = \frac{R^2 - \dot{R}^2}{F(R)} \left( -dt^2 + dr^2 \right) + R^2 d\Omega^2.
$$

(32)

Ashtekar’s lapse is

$$
N = \frac{1}{R^2 \sin \theta}.
$$

(33)

(The coordinate singularities at the poles are benign and do not concern us here.) The simplifying requirements (31) imply that

$$
\ddot{R} - R'' = F^{-1} F_R (\dot{R}^2 - R^2),
$$

(34)

where $F_R$ denotes $dF/dR$.

Now the spatial metric becomes degenerate only if

$$
R^2 - \dot{R}^2 = 0.
$$

(35)

As such a surface of degeneracy is approached, the right hand side of (34) approaches zero, so the function $R(t, r)$ approaches a solution to the two-dimensional massless wave equation. Near the surface $R(r, t)$ can thus be written in the approximate form

$$
R \approx f(r + t) + g(r - t).
$$

(36)

Now (35) implies $f'g' \approx 0$, so we have either $f' = 0$ or $g' = 0$ at the surface of degeneracy. This means that the latter is given by $r = \pm t + c$ for some constant $c$. This is a null hypersurface, as seen from the form of the line element (32). For the Schwarzschild solution this hypersurface remains regular all the way back to the white hole singularity (or to the naked singularity in the negative mass case).

Let us stress that, previous claims [7] notwithstanding, the matching to a degenerate metric can take place at any initial value of the radial coordinate $R$.

### 3.3 Collision with a plane wave

In order to learn something about the stability properties of a degenerate region we will verify that a degeneracy surface which is a null plane will remain null also if it collides with a plane gravitational wave coming from the opposite direction. The actual calculation that we will perform is a trivial reparametrization of the well known plane wave solution, so we begin with a short review of the properties of that solution[20, 21]. The line element has the form

$$
\mathbf{ds}^2 = -dUdV + F^2 dX^2 + G^2 dY^2,
$$

(37)

where $F$ and $G$ are functions of $V$ only. This metric solves Einstein’s equations if and only if

$$
FG'' + GF'' = 0.
$$

(38)
This is a plane wave moving “leftwards”. Since the functions $F$ and $G$ can be chosen at will, subject only to the condition (38), we can consider a “sandwich wave” which has non-vanishing curvature only in some finite interval of the coordinate $V$. Then spacetime is flat in front of the wave, and it reverts to being flat after the passing of the wave (see Fig. 2).

We would now like to study a collision between such a plane wave and a surface of degeneracy moving “rightwards”. This is simple enough, since we can introduce such a surface at $U = 0$ (say) through a reparametrization of the coordinate $U = U(u)$. For the Hamiltonian decomposition we use the coordinates $T = (U + V)/2$ and $Z = (V - U)/2$, with $T$ the time function and $\partial/\partial T$ the time flow vector field. One finds by means of a calculation similar to the one we gave in detail for flat spacetime that it is possible to perform the reparametrization in such a way that Ashtekar’s triad $E^a_i$ becomes degenerate at $U = 0$, while the lapse, the shift, and the connection remain unaffected. In a convenient gauge, the self-dual spin connection has the non-vanishing components

$$A_{X1} = iA_{X2} = F' \quad A_{Y2} = -iA_{Y1} = G'.$$

The only non-zero components of the self-dual curvature are

$$F'_{VX1} = iF'_{VX2} = F'' \quad F'_{VY2} = -iF'_{VY1} = G''.$$

These are untouched by the reparametrization $U = U(u)$.

In the degenerate region we must have a degenerate solution to the Ashtekar equations. The form this solution takes is the following. Define the new coordinates $(t, z)$ via

$$U = T - Z = -2g(z - t) \quad V = T + Z = t + z,$$

and introduce the SO(3) triad

$$U_i = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \quad V_i = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad W_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Figure 2: Sandwich plane wave moving leftward collides with a degenerate phase boundary moving rightward.
\begin{align*}
N &= \frac{1}{FG} & N^a &= \Lambda_i = 0 \\
E_i^a &= \frac{1}{2} GV_i + g' GU_i & E_i^b &= -\frac{i}{2} FV_i + ig' FU_i & E_i^c &= FG W_i \\
A_{xi} &= F' U_i & A_{yi} &= iG' U_i \\
F_{yz} &= -iG'' U_i & F_{xx} &= F'' U_i.
\end{align*}

All these quantities are well defined when \( g' = 0 \), even if \( g'(u) = 0 \) for all \( u \) greater than some \( u_0 \). As discussed in section 3.1, it seems likely that in general a degenerate Ashtekar configuration obtained in this manner will automatically be a solution if the original non-degenerate configuration was a solution and the lapse is finite. In this particular case it is straightforward to check explicitly that all the constraints and equations of motion are satisfied. Thus, after colliding with the degenerate phase boundary, the plane wave is converted into a degenerate plane wave with the same connection and curvature. The triad on the other hand has rank two, while \( E_i^a E^b_i \) has rank one, as can be seen from Eqn (44). This is therefore a degenerate solution of type (2,1). It is not one of the connection waves found in [14], since in that paper only type (1,1) solutions were studied.

There are two salient features here: The connection and curvature of the wave continue unaffected into the degenerate region (they are unaffected by the reparametrization), and the degeneracy surface remains null and nonsingular as viewed from the nondegenerate side. It should also be said that there is a global difficulty with this degenerate plane wave solution. The surfaces of constant \( U \) are focused by the wave and eventually develop caustic singularities at some value of \( V \). This is a coordinate singularity related to the fact that the plane wave spacetimes do not admit Cauchy surfaces [21]. These caustics present a problem for us since it is not at the moment clear how to make sense of a surface of degeneracy which is not smooth.

4 Is the phase boundary always null?

The examples studied above suggest that the phase boundary is always null. We have not so far been able to prove this, however we offer in this section some remarks that may be useful in addressing this problem. In particular, we argue that the equations of motion are at least in part necessary, and we conjecture why they might be sufficient.

First of all, what exactly do we mean by the phase boundary being null? Since the metric on the nondegenerate side is perfectly regular all the way up to the phase boundary, it is meaningful to say the boundary is null “as viewed from the nondegenerate side”. More precisely, suppose the surface on which the metric becomes degenerate is given by \( f = 0 \), where \( f \) is a smooth function with \( \nabla_a f |_{f=0} \neq 0 \) with respect to local charts in which the metric does not degenerate as \( f \to 0 \). We consider the surface to be null if \( g^{ab} \nabla_a f \nabla_b f \to 0 \) as the surface is approached.
The regularity of the Ashtekar variables at the surface seems to be necessary for the null conjecture to have any chance of being true. Counterexamples serve to illustrate this. First, consider the reparametrization $Z = z^3/3$ of Minkowski space discussed in section 3.1. There the degeneracy surface $Z = 0$ is clearly timelike. However, when the coordinate $z$ is adopted, the Ashtekar lapse density diverges, so this configuration does not qualify as a regular Ashtekar solution. As a second example consider the FRW line element $ds^2 = -dt^2 + t(dx^2 + dy^2 + dz^2)$. For this line element, the spatial metric becomes degenerate on the spacelike surface $t \to 0^+$. Here also the lapse density diverges so, again, one does not have a regular Ashtekar solution at the boundary.

The regularity of the lapse density, although necessary, is not sufficient to ensure nullness of the phase boundary. It seems that, at least to some degree, the equations of motion are required. In the spherically symmetric case, analyzed in section 3.2, it was found that the phase boundary is always null. No use was made of the particular form of the function $F(R)$ in the line element (29), and of course for arbitrary $F(R)$ one does not have a solution to the Einstein equation. On the other hand, one might argue that part of the equations of motion has been used in restricting to the case $g_{tt}g_{rr} = -1$, and perhaps this part of the equations of motion is enough to imply nullness of the phase boundary in this very symmetrical case. To further explore this question here we will generalize the previous analysis to allow for independent coefficients of $dt^2$ and $dr^2$ in the spherically symmetric line element. We find that the boundary is not null in the general (non-solution) case.

Consider a line element of the form

$$ds^2 = -F(R)dt^2 + G(R)dR^2 + R^2d\Omega^2,$$

where

$$H := FG$$

is not necessarily constant. The case $H(R) = 1$ corresponds to the previously considered situation. Now let $R$ and $T$ depend on $t$ and $r$, and make the simplifying requirements

$$\dot{T} = R'/F \quad \quad T' = GR$$

so that the metric will remain diagonal. The line element then becomes

$$ds^2 = F^{-1}(R^2 - H\dot{R}^2)(-dt^2 + Hdr^2) + R^2d\Omega^2,$$

and Ashtekar’s lapse density is

$$N = \frac{1}{H^{1/2}R^2\sin\theta}.$$

The simplifying requirements (49) imply that

$$H\dot{R} - R'' = F^{-1}F_R(H\dot{R}^2 - R'^2) - H_R\dot{R}^2,$$

where the subscript $R$ denotes differentiation with respect to $R$. 

Now the spatial metric becomes degenerate at the surface

\[ f := R^2 - H \dot{R}^2 = 0. \]  

To see whether this surface is null we examine the normal vector

\[ g^{ab} \nabla_a f \nabla_b f = F(H f)^{-1} [f'^2 - H f^2] \]  

in the limit that the surface \( f = 0 \) is approached. Factorizing the last term, we are led to evaluate

\[ f' \pm H^{1/2} \dot{f} = (2R'' \pm H^{1/2} \dot{R} - H R \dot{R}^2)(R' \mp H^{1/2} \dot{R}) \pm 2H^{1/2} \dot{R} (F^{-1} F_R f + H R \dot{R}^2), \]  

where (52) has been used to eliminate the \( \ddot{R} \) term. If \( H \) is constant, then the last term of (55) vanishes as the \( f = 0 \) surface is approached, and we have

\[ (f' + H^{1/2} \dot{f})(f' - H^{1/2} \dot{f}) \propto (R' + H^{1/2} \dot{R})(R' - H^{1/2} \dot{R}) = f = 0, \]  

so the right hand side of (54) vanishes, so the \( f = 0 \) surface is null. If \( H \) is not constant, however, the surface appears not to be null.

We conclude from this analysis that the phase boundary generated by the degenerate coordinate transformation is null only if \( H := FG \) is constant in the line element (47). Thus at least part of the Einstein equation is required. We suspect that the nullness of the phase boundary is related to the fact that the characteristic surfaces of the (nondegenerate) Einstein equation are null. Since the Einstein equation locally preserves the rank of the metric, it seems likely that the rank can only change across a characteristic surface. We conjecture that this is the underlying reason for the nullness of the phase boundary in the cases examined so far.

5 Discussion

We have obtained in this paper examples of solutions of Ashtekar’s equations in which degenerate and non-degenerate metrics coexist, separated by a “phase boundary”. These “geometries” differ from the kind of degenerate solutions that occur in other degenerate extensions of general relativity.

We think that our examples provide a reasonable amount of support for our conjecture that the boundary of the degenerate region is always null, as viewed from the non-degenerate side. This appears to make some sense, since the characteristic surfaces on the nondegenerate side are null. It also ties in well with Matschull’s observation [13] that the specific way in which Ashtekar’s variables allow the metric to become degenerate is such that a local causal structure of spacetime is preserved in the form of a partly collapsed lightcone. In the

\[ \text{It is important here that } \nabla_a f \text{ is a non-zero vector where } f = 0, \text{ i.e. that } \dot{f} \text{ and } f' \text{ are nonzero there. Although we have not proven that they are nonzero, there appears to be no reason why they should vanish.} \]
examples we have studied, the phase boundary is also null, in Matschull’s sense, as viewed from the degenerate side, so the phase boundary propagates causally in that sense too.

All of our examples have the feature that the phase boundary is a smooth null hypersurface (except at the focal surface in the plane wave spacetime). It is not at all clear whether degenerate and nondegenerate solutions can be matched across null surfaces that are not smooth. Since caustics on null surfaces are generic, the scope of the investigation should be expanded to determine what happens at caustics. This could even be investigated in the flat space example, taking as the initial phase boundary a dimpled surface whose null normal congruence develops a caustic.

In this paper we have restricted attention to the vacuum case only. For a general matter stress-energy tensor the static, spherically symmetric solution does not in general have constant $H$, so the results of section 4 seem to suggest that even if the Einstein equation is satisfied the phase boundary may fail to be null. However, if matter is included, one must also require that the canonical matter variables in the Ashtekar formulation are regular as the surface of degeneracy is approached. If they are not, then the phase boundary is not allowed in a regular solution. It would be interesting to check this in some examples. The Reissner-Nordstrom solution has constant $H$, so one needs to look elsewhere. For instance, a self-interacting, self-gravitating scalar field configuration could provide an example, or a stellar interior with perfect fluid matter. Another interesting example would be an electrovac plane wave. Scalar and electromagnetic matter fields[22] and perfect fluids[8] have all been incorporated into the Ashtekar formulation already, so the groundwork has been laid.

A question we have ignored, but which should be answered if the degenerate extension of the theory is to be taken seriously, is whether the initial value problem is in fact well-posed. The answer probably depends on the way in which the metric is allowed to become degenerate. For example, when it is the covariant metric that is allowed to be degenerate, it was shown by Horowitz[19] that the topology of spacetime is not determined by initial data since the topology can change unpredictably. In the degenerate Plebanski formulation, Reisenberger[12] showed that initial data does not determine a solution for the fields even in fixed topology. The status of the initial value problem in the degenerate Ashtekar theory is not known (as far as we know), but the existence of the (degenerate) causal structure[13] suggests that the dynamics may indeed be determined by data on a spacelike surface. In particular, the causal structure implies the existence of a nowhere vanishing “timelike” vector field which, together with a causality condition ruling out closed timelike curves, may be enough to yield the result (proved by Geroch[23] for nondegenerate Lorentzian metrics) that topology change in a spatially compact universe is impossible.

Lest we get carried away extolling the virtues of the degenerate Ashtekar theory, it is worth restating the fact, mentioned in the introduction, that this theory seems to admit negative energy configurations[9], whereas other degenerate extensions of general relativity do not[17]. It remains very much an open question which, if any, degenerate extension is physically reasonable, not to mention which, if any, is actually correct.
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