SYMMETRIES IN TWO-DIMENSIONAL DILATON GRAVITY WITH MATTER *

Miguel Navarro†

Instituto de Matemáticas y Física Fundamental, CSIC. Serrano 113-123, 28006 Madrid, Spain.

and

Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias, Universidad de Granada. Campus de Fuentenueva, 18002, Granada, Spain.

Abstract

The symmetries of generic 2D dilaton models of gravity with (and without) matter are studied. It is shown that $\delta_2$, one of the symmetries of the matterless models, can be generalized to the case where matter fields of any kind are present. The general (classical) solution for some of these models, in particular those coupled to chiral matter, which generalizes the Vaidya solution of Einstein Gravity, is also given.

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†http://www.ugr.es/~mnavarro; mnavarro@ugr.es
1 Introduction

At present, one of the main challenges of Theoretical Physics is to devise a quantum theory which will provide a complete description of gravity. Due to the complexity of the theories involved, the Einstein-Hilbert gravity theory for instance, this task faces imposing technical difficulties. However, equally important or even more serious are the conceptual problems that arise. These are not only due to a variety of unfamiliar features which are peculiar to diffeomorphism-invariant theories but are also due to our present lack of an adequate formulation and interpretation of Quantum Mechanics. To an important extent, we ignore what a quantum theory really is – ie, we ignore what the adjective “quantum” really means – and how we should interpret these theories. Because of that, we ignore not only how to produce a quantum theory of gravity but also what would constitute a successful completion of this task.

In this context “toy theories” should have a crucial concept-clarifying role to play; no wonder a variety of them are currently under study. Prominent among them are the 2D dilaton models of gravity, which are two-dimensional general covariant models whose gravity sector involves, along with the space-time metric $g_{\mu\nu}$, a scalar field $\phi$, the dilaton (for a review, see Ref. [1]). These models are much simpler to handle that their higher-dimensional cousins but share with them not only the conceptual problems which are peculiar to diffeomorphism-invariant theories but also their most relevant physical features, such as the formation of black holes and their subsequent evaporation. Some of these models can be obtained, via dimensional reduction, from realist, higher-dimensional theories, but this is not a necessary, nor even convenient, fact to have in mind when approaching these models as they may be regarded as a sort of two-dimensional Brans-Dicke theories.

If attention is restricted to the usual category of models, that is, those with second-order Euler-Lagrange equations of motion, a very useful result to take into account is that, modulo isolated pathologies which may arise, the actions of these models can all be brought, by means of appropriate redefinitions of the dilaton field and conformal redefinitions of the metric, to the generic form [2, 3]

$$S_{G,D,G} = S_V - S_M$$

where

$$S_V = \int d^2x \sqrt{-g} \left( R\phi + V(\phi) \right)$$

and $S_M$ is a gravity-matter interaction term which may involve, the dilaton field as well as the metric.

Despite their being much simpler that their higher-dimensional cousins, few of these models have been exactly solved classically, not to mention quantum mechanically, when matter fields are present. As is well known, solvability, classical or quantum, is usually related to the presence of invariances, which is the reason that classical solv-
ability usually implies quantum solvability. Notwithstanding this fact, little attention 
had been paid to the symmetries of these models, apart from the conformal ones, 
until quite recently. A large variety of symmetries, which are in general non-conformal, 
has recently been uncovered for the matterless 2D dilaton models and they have been 
shown to explain the (classical) solvability of these theories [4].

In this paper we shall consider the generic 2D dilaton gravity with matter, eq. (1), 
and shall explore how much of what have been done in the matterless case can be 
extended to theories with matter. Nonetheless, in order to make the present paper as 
self-contained as possible and to make it clear how the symmetries actually underly 
the solvability of these models, a brief but systematic view of the matterless theories 
shall be presented in Sect. 2. Previously it is worth discussing some general features 
of these models.

The general variation of the Lagrangian in eq. (1) yields

\[
\delta \mathcal{L} = \sqrt{-g} \left\{ [R + V'(\phi) - T_\phi] \delta \phi 
+ \left[ g_{\mu\nu} \Box \phi - \frac{1}{2} g_{\mu\nu} V(\phi) - \nabla_\mu \nabla_\nu \phi - T_{\mu\nu} \right] \delta g^{\mu\nu} 
+ (E - L)_A \delta f^A 
- \nabla_\alpha s^\alpha \right\}
\]

where \( \nabla_\alpha s^\alpha \) includes all those terms which appear due to “integrations by part” which 
are required to produce the equations of motion.

The equations of motion can be brought to the form:

\[
\begin{align*}
R + V'(\phi) &= T_\phi \\
\Box \phi &= V + T \\
\nabla_\mu \nabla_\nu \phi &= \frac{1}{2} g_{\mu\nu} V + g_{\mu\nu} T - T_{\mu\nu} \\
(E - L)_A &= 0
\end{align*}
\]

where \( T \) is the trace of \( T_{\mu\nu} \).

Although much of our present study is minded to apply to all kind of matter, we 
shall exemplify much of it with a massless scalar fields \( \xi \), with action

\[
S_M = \frac{1}{2} \int d^2 x \sqrt{-g} \Omega(\phi)(\nabla \xi)^2
\]

and unspecified function \( \Omega \). [The particular case of dimensionalaly reduced spherically 
symmetric Einstein-Hilbert gravity minimally coupled to a massless scalar field is re-
covered with \( V = 2/\sqrt{\phi} \) and \( \Omega = G\phi \), with \( G \) the Newton constant].
For this example we have

\[ T_\phi = \frac{\Omega(\phi)}{2} (\nabla^2 \xi)^2 \]
\[ T_{\mu\nu} = \frac{\Omega}{2} \left\{ \nabla_\mu \xi \nabla_\nu \xi - \frac{1}{2} g_{\mu\nu} (\nabla^2 \xi)^2 \right\} \]  

(6)

Let us now go back to eq. (4). It is apparent that the second(-from-above) equation is redundant as it follows from the third one. However, the same is true for the first one. To show this let us first indicate that invariance under diffeomorphisms of \( S_M \) implies that, when the equations of motion of the matter fields are satisfied, the following generalized conservation law for the energy-momentum tensor \( T_{\mu\nu} \) must be fulfilled

\[ 2 \nabla^\nu T_{\mu\nu} + T_\phi \nabla_\mu \phi = 0 \]  

(7)

which implies

\[ T_\phi = -\frac{2}{(\nabla \phi)^2} \nabla^\mu \phi \nabla^\nu T_{\mu\nu} \]  

(8)

On the other hand, the affine connection \( \nabla_\mu \) and the Riemann tensor \( R_{\alpha\mu\beta\nu} \) obey

\[ \left[ \nabla_\nu, \nabla_\beta \right] \zeta_\mu = -\zeta_\rho R^\rho_{\mu\beta\nu} \]  

(9)

Particularizing this equality to \( \zeta_\mu = \nabla_\mu \phi \) and using that in two-dimensions \( R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \) yield, after a bit of algebra

\[ R \nabla_\mu \phi + V' \nabla_\mu \phi = -2 \nabla^\nu T_{\mu\nu} \]  

(10)

which, together with eq. (8) yields the desired equality.

Therefore, all the equations of motion of the theory are encapsulated in the last and last-but-one equations in eq. (4). As we shall see, this result facilitates finding the general classical trajectories.

As in the models we are dealing with it is easier to find out the conserved currents than the associated (Noether) symmetries, let us remind the reader, before proceeding, of the Noether theorem, suitably versioned for the occasion: Let \( \mathcal{L} = \mathcal{L}(\Psi^a) \) be an arbitrary Lagrangian with general variation \( \delta \mathcal{L} = (E - L)_a \delta \Psi^a - \nabla_\mu s^\mu \). Let \( j_0^{\mu} \) be a current which is conserved on shell and \( \delta_0 \Psi^a \) a transformation of the fields. Then \( j_0^{\mu} \) is the Noether current associated to \( \delta_0 \Psi^a \) iff, without using the equations of motion, the following equality holds as an identity:

\[ (E - L)_a \delta_0 \Psi^a = \nabla_\mu j_0^{\mu} \]  

(11)

In general, and due to semi-invariance, the current \( j_0^{\mu} \) will not equal \( s^\mu(\delta_0 \Psi^a) \).
2 Symmetries and general solution for the matter-less theories

With the scalars \( \phi \) and \( \varphi \equiv (\nabla \phi)^2 \) and the vector field \( \nabla_\mu \phi \) scalars, vector and tensor fields can be built and it can be checked whether or not they are conserved. Let \( J(\phi) \) be a primitive of \( V, \; dJ/d\phi = V \). Then, the following results hold:

**Conserved scalars.** The local energy

\[
E = \frac{1}{2} \left( (\nabla \phi)^2 - J \right),
\]

is a conserved scalar: \( \nabla_\mu E = 0 \), and consequently also is \( f(E) \) for any function \( f \). Moreover, these are the only conserved scalars which can be constructed with \( \phi \) and \( (\nabla \phi)^2 \).

The Noether symmetry associated to \( E \) is given by

\[
\delta_a \phi = 0, \quad \delta_a g_{\mu\nu} = g_{\mu\nu} a_\sigma \nabla^\sigma \phi - \frac{1}{2} (a_\mu \nabla_\nu \phi + a_\nu \nabla_\mu \phi)
\]

with arbitrary constant bivector \( a^\mu \). **Conserved currents.** The conserved current of the form \( j^\mu = A(\phi, \varphi) \nabla^\mu \phi \) can all be written as follows:

\[
j^\mu_f = f(E) \frac{\nabla^\mu \phi}{(\nabla \phi)^2}
\]

for some function \( f \). The associated symmetries are:

\[
\delta_f \phi = 0, \quad \delta_f g_{\mu\nu} = -\epsilon f'(E) \left( g_{\mu\nu} - \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^2} \right) + \epsilon f(E) \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^4} \right)
\]

In particular, for \( f = 1 \), the corresponding current and symmetry are, respectively

\[
j_1^\mu = \frac{\nabla^\mu \phi}{(\nabla \phi)^2}
\]

\[
\delta_1 \phi = 0, \quad \delta_1 g_{\mu\nu} = \epsilon \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla_\mu \phi \nabla_\nu \phi}{(\nabla \phi)^4} \right)
\]

Now, let \( j_\mu_R \) be defined by \( \nabla_\mu j_\mu_R = R \). Then, it is easy to see that the following current is conserved

\[
j_2^\mu = j_\mu_R + V \frac{\nabla^\mu \phi}{(\nabla \phi)^2}
\]

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with symmetry
\[ \delta_2 \phi = \epsilon, \quad \delta_2 g_{\mu\nu} = \epsilon V \left( \frac{g_{\mu\nu}}{(\nabla \phi)^2} - 2 \frac{\nabla \mu \phi \nabla \nu \phi}{(\nabla \phi)^4} \right) \] (19)

**Conserved tensors.** There exists a great variety of conserved 2-tensors of the form
\[ S^{\mu\nu} = A(\phi, \varphi) \nabla^\mu \phi \nabla^\nu \phi + B(\phi, \varphi) g^{\mu\nu} \]
They are given by the general solution of the equation \((A_J = dA/dJ, A_\varphi = dA/d\varphi)\)
\[ \frac{3}{2} A + A_J \varphi + A_\varphi \phi + B_J + B_\varphi = 0 \]
which implies that \(A (B)\) can be written as \(A = A(J, \varphi)\) \((B = B(J, \varphi))\). For \(A = 0\), we recover the conserved scalars described above. Another useful solution is the following traceless tensor, which is basically unique
\[ S^{\mu\nu} = \frac{\nabla^\mu \phi \nabla^\nu \phi}{(\nabla \phi)^4} - \frac{1}{2} \frac{g^{\mu\nu}}{(\nabla \phi)^2} \] (20)

2.1 General local solution

Once the symmetries are at our disposal it is easy to solve the equations of motion. It can be shown [4] that the following “metric” is invariant under \(\delta_2\)
\[ \bar{g}_{\mu\nu} = \frac{1}{(\nabla \phi)^2} \left( g_{\mu\nu} - \frac{\nabla \mu \phi \nabla \nu \phi}{(\nabla \phi)^2} \right) \]
which implies
\[ g_{\mu\nu} = (\nabla \phi)^2 \bar{g}_{\mu\nu} + \frac{\nabla \mu \phi \nabla \nu \phi}{(\nabla \phi)^2} \]

As \(\nabla^\mu \phi \bar{g}_{\mu\nu} = 0\) we must have
\[ \bar{g}_{\mu\nu} = A k_\mu k_\nu \]
with \(k^\mu\) the vector density
\[ k^\mu = \frac{\epsilon^{\mu\nu} \nabla^\nu \phi}{\sqrt{-g}} \] (21)

Now, as \(\nabla_\mu k_\nu = \nabla_\nu k_\mu, k_\mu\) is, at least locally, a total derivative: \(k_\mu = \nabla_\mu t\) for some function \(t\). It is then natural to choose as local coordinates \(r \equiv \phi\) and \(t\). Now, it is easy to see that \(k^\mu\) is a Killing vector [3], which implies \(A = A(r)\). The equations of motion (4) imply \(A = 1\). Therefore, we finally arrive at the general solution
\[ \phi = r \]
\[ g_{\mu\nu} = (2M - J(r))\nabla_\mu t\nabla_\nu t - \frac{\nabla_\mu r\nabla_\nu r}{2M - J(r)} \]  \hspace{1cm} (22)

where \( M = -E \) is an arbitrary constant.

Another elegant way of arriving at the general solutions is by using free-field methods. Here we choose the conformal gauge

\[ ds^2 = 2e^\rho dx^+ dx^- \]  \hspace{1cm} (23)

for which we have \( R = -2e^{-\rho}\partial_+ \partial_- \rho \) and \( \Box = 2e^{-\rho}\partial_+ \partial_- \).

In the conformal gauge, the conservation law for the traceless tensor in eq. (20) takes the form

\[ 0 = \partial_+ T_{--} = \partial_+ \left( \frac{\partial_- \phi \partial_+ \phi}{(\nabla \phi)^4} \right), \quad 0 = \partial_- T_{++} = \partial_- \left( \frac{\partial_+ \phi \partial_+ \phi}{(\nabla \phi)^4} \right) \]

which imply \( \frac{\partial_+ \phi}{(\nabla \phi)^2} = \partial_+ p, \quad \frac{\partial_- \phi}{(\nabla \phi)^2} = \partial_- m \), for some \( p = p(x^+) \) and \( m = m(x^-) \).

A bit of algebra leads us to

\[ e^\rho = 2(\nabla \phi)^2 \partial_+ p \partial_- m = 2(2E + J)\partial_+ p \partial_- m \]

By fixing the residual gauge as \( p = \frac{1}{2}x^+, \ m = -\frac{1}{2}x^- \), we finally arrive at:

\[ \int_\phi^{\phi'} \frac{d\tau}{2M - J(\tau)} = -\frac{1}{2}(x^+ - x^-) \]

\[ e^\rho = \frac{1}{2}(2M - J) \]  \hspace{1cm} (24)

### 3 Symmetries in 2D dilaton gravity with matter

Let us now introduce matter fields. Our most general result in this case is that the symmetry (current) \( \delta_2 (j_2^\mu) \) above can be generalized for whatever kind of matter is present. More precisely: if \( S_M \) is invariant under diffeomorphisms and the equations of motion (4) are obeyed, the following current is conserved

\[ j_2^\mu = j_R^\mu + V \frac{\nabla^\mu \phi}{(\nabla \phi)^2} + 2 \frac{\nabla_\nu \phi}{(\nabla \phi)^2} T^{\mu\nu} \]  \hspace{1cm} (25)

To show this it suffices to make use of the equations of motion (4), eq. (8) and of the fact that in two dimension, and for any tensor \( N_{\mu\nu} \), the following quantity vanishes identically (\( N = N^\rho_{\rho} \))
\[
(N^\mu - \frac{1}{2} \delta^\mu \alpha N)(N^\alpha - \frac{1}{2} \delta^\alpha \nu N) - \frac{1}{2} \delta^\mu \nu (N^\beta - \frac{1}{2} \delta^\beta \alpha N)(N^\alpha - \frac{1}{2} \delta^\alpha \beta N)
\]

or, equivalently,

\[
N_{\mu \alpha} N^{\alpha \nu} = \frac{1}{2} \delta_{\mu \nu} - \frac{1}{2} \delta_{\nu \mu} N^2
\]  \hspace{1cm} (26)

The associated Noether symmetry is

\[
\delta_2 \phi = \epsilon \hspace{1cm} \delta_2 g_{\mu \nu} = \epsilon \left\{ V \left( \frac{g_{\mu \nu}}{(\nabla \phi)^2} - 2 \frac{\nabla \mu \phi \nabla \nu \phi}{(\nabla \phi)^4} \right) + 2 \frac{T_{\mu \nu}}{(\nabla \phi)^2} - 2 T \frac{g_{\mu \nu}}{(\nabla \phi)^2} \right\} \hspace{1cm} (27)
\]

The variation \( \delta f^A \) of the matter fields is such that off-shell

\[
(E - L)_A \delta f^A = 2 \frac{\nabla \nu \phi}{(\nabla \phi)^2} \nabla \mu T_{\mu \nu}
\]  \hspace{1cm} (29)

For our exemplifying scalar matter field, the transformation is

\[
\delta \xi = \frac{\nabla^\mu \phi}{(\nabla \phi)^2} \nabla \mu \xi = \frac{\nabla \phi \nabla \xi}{(\nabla \phi)^2}
\]  \hspace{1cm} (30)

It is easy to see that, in spite of what happens in the matterless case, this symmetry does not correspond to the diffeomorphism generated on-shell by the vector field \( s^\mu = \frac{\nabla^\mu \phi}{(\nabla \phi)^2} \).

Let us now consider currents of the form

\[
S^\mu = A(\phi, \varphi) \nabla^\mu \phi + B(\phi, \varphi) \nabla_\nu T_{\mu \nu}
\]  \hspace{1cm} (31)

Conservation implies

\[
0 = A_\phi (\nabla \phi)^2 + A_\varphi V(\nabla \phi)^2 + AV - \frac{1}{2} B(\nabla \phi)^2 T_\phi
\]

\[+ \left[ 2A_\varphi (\nabla \phi)^2 + A + \frac{1}{2} BV \right] T \]

\[+ \left[ B + B_\varphi (\nabla \phi)^2 \right] \left( T^2 - T_{\alpha \beta} T^{\alpha \beta} \right) \]

\[+ \left[ -2A_\varphi + B_\phi + B_\varphi V \right] \nabla_\mu \phi \nabla_\nu \phi T_{\mu \nu}
\]  \hspace{1cm} (32)

where eq. (26) has been used. It follows that
1) If $T_\phi = 0$, $T = 0$ and $V' = \beta V$ (with $\beta$ = constant) the following current is conserved

$$S^{\mu}_{\beta} = -\beta \nabla^{\mu} \phi + V \frac{\nabla^{\mu} \phi}{(\nabla \phi)^2} + 2 \frac{\nabla_{\nu} \phi}{(\nabla \phi)^2} T^{\mu\nu}$$ (33)

The Noether symmetry associated to $j_{\beta}^{\mu} = j_{2}^{\mu} - S^{\mu}_{\beta}$ is conformal and is given by [4]

$$\delta \phi = \epsilon, \quad \delta g_{\mu\nu} = -\epsilon \beta g_{\mu\nu}, \quad \delta f^{A} = 0$$ (34)

2) If $T_\phi = 0$, $T = 0$ and $T_{\alpha\beta} = 0$ (chiral matter)

$$S^{\mu}_{f} = f(E) \nabla_{\nu} \phi T^{\mu\nu}$$ (35)

is a conserved current for any function $f$. The associated symmetry transformation of the gravity sector is

$$\delta \phi = 0, \quad \delta g_{\mu\nu} = T_{\mu\nu}$$ (36)

**Conserved 2-tensors.** Consider now (symmetric) tensor fields of the form

$$S^{\mu\nu} = A(\phi, \varphi) \nabla^{\mu} \phi \nabla^{\nu} \phi + B(\phi, \varphi) g^{\mu\nu} + C(\phi, \varphi) T^{\mu\nu}$$ (37)

It can be show that

1) For $T_\phi = 0$ and $T = 0$ the tensor fields with

$$C = C(\phi), \quad C \text{ an arbitrary function}$$
$$B = (C_\phi + \alpha) \varphi + 2\alpha J + \frac{1}{2} \int \varphi V C_\phi$$
$$A = -C_\phi - 2\alpha$$ (38)

are conserved. If $T \neq 0$, these tensors are still conserved if $\alpha = 0$. The tensor with $C = \phi$, $\alpha = -\frac{1}{2}$,

$$J^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \left((\nabla \phi)^2 - J(\phi)\right) + \phi T^{\mu\nu}$$ (39)

can be regarded as the generalization of the local energy of the massless models. The conserved tensor with $C = 0$ is purely kinematical – it does not depend on the matter fields:

$$S^{\mu\nu} = \nabla^{\mu} \phi \nabla^{\nu} \phi - \frac{1}{2} g^{\mu\nu} (\nabla \phi)^2 - g^{\mu\nu} J$$ (40)
3.1 Chiral matter

We define chiral matter as that whose energy-momentum tensor obeys

\[ T_\phi = 0, \quad T^\mu_\mu = 0 \quad \text{and} \quad T_{\mu\nu}T^{\mu\nu} = 0 \quad (41) \]

In the conformal gauge we have

\[ 0 = T_{\mu\nu}T^{\mu\nu} = 2e^{-2\rho}T_{++}T_{--} \quad (42) \]

Therefore the set of all solution splits in two sectors of left-moving and right-moving fields. Let us choose \( T_{--} = 0 \). The conservation law for the tensor eq. (39) implies:

\[
\begin{align*}
\partial_- E &= 0 \quad \Rightarrow \quad E = P(x^+) \\
\partial_+ E + \nabla^+ \phi T_{++} &= 0 \quad \Rightarrow \quad \nabla^+ \phi = p(x^+) 
\end{align*}
\]

Thus, we are led to the equations

\[
\begin{align*}
e^{-\rho} \partial_- \phi &= p(x^+) \\
\partial_+ \phi &= \frac{P}{p} + \frac{1}{2p} J \\
\frac{\partial_+ P}{p} &= -T_{++}
\end{align*}
\]

However, it does not appear that, in the present coordinates, these equations can be solved in general for arbitrary \( J \) and \( T_{++} \). To go further, let us make a change of coordinates

\[ x^+ = u, \quad x^- = x^-(u, r) \quad (45) \]

so that \( \phi \equiv r \). The metric and the energy-momentum tensor in this coordinates takes the form

\[
\begin{align*}
ds^2 &= e^{\lambda}(2d\phi du + A(du)^2) \\
T &= T_{uu}(du)^2 \quad (46)
\end{align*}
\]

with \( T_{uu} = T_{uu}(u) \). Now, by writting down in this gauge the equations of motion (4), it is easy to show that \( e^\lambda \) is pure gauge, and finally arrive at the following general solution

\[ ds^2 = 2dud + (2M(u) - J)(du)^2 \quad (47) \]
with
\[ M(u) = \int^u d\tilde{u} T_{uu}(\tilde{u}) \] (48)

This metric generalizes, for an arbitrary potential \( V \), the Vaidya solution of Einstein gravity.

4 Conclusions

We have analysed in a rather systematic way the conserved currents and the symmetries of the 2D dilaton models of gravity with and without matter. In particular we have shown that \( \delta_2 \) can be extended to models coupled to any kind of matter. We have also shown how the existence of invariances – and hence of conserved currents – is directly related with analytic solvability. In fact, almost all the models which are known to be solvable fall in one of the categories of symmetric models we have described in the present paper. The Jackiw-Teitelboim model, \( V = 4\lambda^2 \phi \), coupled to conformal matter may be regarded as an exception to this rule, as it is solvable \[5\] even though, apart from \( \delta_2 \), it is not known to have any additional symmetry. This model is solvable because, in the conformal gauge, the first equation in eq. (4) is a Liouville equation, whose solvability does not appear to be related with any invariance. In fact, the whole problem of solving these models (with conformal matter) can be regarded as a generalization of Liouville theory. It would be interesting to see if the machinery of integrable systems is useful here.

The massless 2D dilaton models have been shown to be related to Poisson-\( \sigma \)-models, which seems to explains their highly symmetric nature (see, for instance, Ref. \[6\]). The underlying reason why \( \delta_2 \) is so general is a mistery to us as of the writting of this paper. It seems to indicate that these models have a degree of unity previously unexpected. This unity may have important consequences in relation to their solvability.

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References


