A NEW CLASS OF INHOMOGENEOUS COSMOLOGICAL MODELS WITH YANG-MILLS FIELDS

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Exact solutions corresponding to spherically symmetric inhomogeneous nonstationary Tolman metrics are obtained for the self-consistent system of Einstein-Yang-Mills equations for the gauge group $SO_3$.

The static configurations of the Einstein-Yang-Mills (EYM) system have been studied in a large number of works (see, for example, [1, 2]). Investigations of the EYM system in application to cosmological models of the Universe have been much less successful, and all the basic results in this direction have been obtained for Friedmann-Robertson-Walker (FRW) spaces. For example, in [3, 4] exact solutions of the Yang-Mills (YM) equations were obtained on a background of homogeneous and isotropic Friedmann metrics. In [5], exact self-consistent solutions of the EYM equations were found with the aid of a conformally flat representation of a line element and the conformal invariance of massless YM fields. However, analysis of these results and additional investigations show that narrowing the class of the desired spherically symmetric solutions of the EYM system to homogeneous Friedmann models severely limits the possibility of understanding the role of YM fields in cosmological processes, especially during the epoch of the very early Universe [6]. It is obvious that any point source of YM fields should destroy the homogeneity of space. The question of a possible substantial role of YM fields in cosmological inflation as a result of their strong nonlinearity was discussed in [7] from a general standpoint (irrespective of the class of metrics). Accordingly, to clarify all basic aspects of the effect of YM fields on the evolution of the Universe, the admissible forms of the metrics must be extended as much as possible. On these grounds, in the present paper we propose to abandon the requirement of homogeneity of space (retaining its isotropy) in order to investigate how self-gravitating YM fields can influence the regimes of inflation of the Universe and whether or not in this case the structure of space can asymptotically in time reach the structure of homogeneous isotropic Friedmann spaces, as is observed in the present epoch.
We shall study the EYM system on the basis of the $SO_3$ non-Abelian gauge theory, described by the action.

$$S = -\frac{1}{8\pi}\int_M \sqrt{-g} d^4x \left\{ \frac{\mathcal{R}}{2\kappa} + \frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu} \right\}$$  

(1)

Here $M$ is the pseudo-Riemannian space-time with signature $(+----)$, $g = \det(g_{\alpha\beta})$ is the determinant of the metric tensor, $\mathcal{R}$ is the scalar curvature,

$$F_{\mu\nu} = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + i e f_{bc}^a W_\mu^b W_\nu^c$$  

(2)

$W_\mu^a$ is the isorhithetical YM fields in the $SO_3$ model, $e$ is a characteristic constant, $f_{bc}^a$ are the structure constants of the $SO_3$ group, and $\kappa = 8\pi G$. The Euler-Lagrange equations for the action (1) have the form

$$G^\alpha_\beta = \mathcal{R}^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta \mathcal{R} = \kappa T^\alpha_\beta,$$  

$$D_\beta(\sqrt{-g} F^{\alpha\beta}) = 0$$  

(3) (4)

where $\mathcal{R}^\alpha_\beta$ is the Ricci tensor, $G^\alpha_\beta$ is the Einstein tensor, $D_\beta$ is the covariant derivative, and the energy-momentum tensor of the YM fields equals

$$T^\alpha_\beta = \frac{1}{4\pi} \left( F^{a\alpha\mu} F^a_{\beta\mu} - \frac{1}{4} \delta^a_\beta F^{a\mu\nu} F^a_{\mu\nu} \right)$$  

(5)

Spherically symmetric cosmological models are described by metrics for which the space-time interval has the following general form:

$$ds^2 = dt^2 - U(r,t)dr^2 - V(r,t)d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$  

(6)

written in a comoving synchronous coordinate system. For YM fields, the general spherically symmetric ansatz can be written in the form

$$W_i^a = \varepsilon_{iab} x^b \frac{K(r,t) - 1}{er^2} + \delta_i^a \frac{S(r,t)}{er} + x^a x_i \frac{T(r,t)}{er},$$  

$$W_0^a = x^a \frac{W(r,t)}{er}, \quad \Phi^a = x^a \frac{H(r,t)}{er}, \quad T(r,t) = -\frac{S(r,t)}{r^2}$$  

(7)

Here and below $\mu, \nu = 0, \ldots, 3, \ i, j = 1, 2, 3, \ a, b = 1, 2, 3,$ the functions $K, S, T, W, H$ are unknown.

Introducing the orthonormalized isoframe

$$\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),$$  

$$\mathbf{l} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta),$$  

$$\mathbf{m} = (-\sin \varphi, \cos \varphi, 0)$$  

(8)
and transforming to spherical coordinates, we find that the ansatz (7) becomes

\[
\begin{align*}
W_1 &= 0 \\
W_2 &= e^{-1} \{(K - 1)m + S l\}, \\
W_3 &= e^{-1} \{-(K - 1)l + S m\}, \\
W_0 &= e^{-1} W_n.
\end{align*}
\]

As a result, we have

\[
\begin{align*}
F_{01} &= -F_{10} = -e^{-1} W' n, \\
F_{02} &= -F_{20} = e^{-1} \left( (K + WS) m + (\dot{S} - WK) l \right), \\
F_{03} &= -F_{30} = e^{-1} \left( -(K + WS) l + (\dot{S} - WK) m \right) \sin \theta \\
F_{12} &= -F_{21} = e^{-1} \left( K' m + S' l \right), \\
F_{23} &= -F_{32} = e^{-1} \sin \theta \left( K^2 - 1 + S^2 \right) n, \\
F_{13} &= -F_{31} = e^{-1} \sin \theta \left( -K' l + S' m \right)
\end{align*}
\]

Here and below an overdot denotes the derivative \(\partial/\partial t\) and the prime denotes the derivative \(\partial/\partial r\) To analyze the system of EYM equations, it is also convenient to introduce the new fields \(A(r, t), B(r, t)\) and \(\phi(r, t)\) with the aid of the formulas

\[
K(r, t) = A(r, t) \sin \phi(r, t), \quad S(r, t) = A(r, t) \cos \phi(r, t), \quad B(r, t) = \dot{\phi}(r, t) + W(r, t)
\]

After the corresponding transformations, the EYM system acquires the following form

\[
\begin{align*}
G_0^0 &= \kappa T_0^0 = \kappa \frac{\sqrt{U}}{4\pi e^2} \left[ \frac{W'^2}{2U} + \frac{A'^2 + A^2 B^2}{V} + \frac{A'^2 + A^2 \phi'^2}{UV} + \frac{(A^2 - 1)^2}{2V^2} \right]; \\
G_1^1 &= \kappa T_1^1 = \kappa \frac{\sqrt{U}}{4\pi e^2} \left[ \frac{W'^2}{2U} - \frac{A'^2 + A^2 B^2}{V} - \frac{A'^2 + A^2 \phi'^2}{UV} + \frac{(A^2 - 1)^2}{2V^2} \right]; \\
G_2^2 &= G_3^3 = \kappa T_2^2 = \kappa T_3^3 = \kappa \frac{\sqrt{U}}{4\pi e^2} \left[ -\frac{W'^2}{2U} - \frac{(A^2 - 1)^2}{2V^2} \right]; \\
G_0^1 &= \kappa T_0^1 = \kappa \frac{\sqrt{U}}{4\pi e^2} \left[ 2 \frac{A' A + A^2 B \phi'}{V} \right]; \\
\frac{\partial}{\partial t} \left( \sqrt{U} \dot{A} \right) - \frac{\partial}{\partial r} \left( \frac{1}{\sqrt{U}} A' \right) + \sqrt{U} \left( \frac{(\phi')^2}{U} - B^2 + \frac{A^2 - 1}{V} \right) A &= 0 \\
\frac{\partial}{\partial t} \left( \sqrt{U} AB \right) - \frac{\partial}{\partial r} \left( \frac{1}{\sqrt{U}} A \phi' \right) + \sqrt{U} \dot{A} B - \frac{1}{\sqrt{U}} A' \phi' &= 0 \\
\frac{\partial}{\partial r} \left( \frac{V}{\sqrt{U}} W' \right) - 2\sqrt{U} A^2 B &= 0
\end{align*}
\]
where $G^\alpha_\beta$ are known expressions obtained for the metric (6).

We now consider a class of exact solutions of the EYM equations that corresponds to the reduction $A(r, t) = 0$, which is equivalent to the condition

$$K(r, t) = S(r, t) = 0$$

This requirement does not mean that there is no YM field. As is clear from (9), the nonzero components of the YM fields are as follows:

$$W_0 = e^{-1} W_n, \quad W_2 = -e^{-1} m, \quad W_3 = e^{-1} l$$

then the nonzero components of the stress tensor are

$$F_{01} = -F_{10} = -e^{-1} W'_n, \quad F_{23} = -F_{32} = e^{-1} \sin \theta \left(K^2 - 1 + S^2\right) n,$$

The EYM system (12),(13) with $A \equiv 0$ is highly simplified. We call attention to the fact that in this case

$$G^1_0 \equiv \frac{\dot{V}}{V} - \frac{V'}{2V^2} - \frac{V'\dot{U}}{2UV} = 0$$

This equation can be solved and yields a relation between the functions $U, V$: $\sqrt{U} = (\sqrt{V})'/f(r)$, where $f(r)$ is an arbitrary function of $r$. This class of metrics comprises the well-known Tolman metrics [11] with the interval

$$ds^2 = dt^2 - \frac{(R')^2}{f^2} dr^2 - R^2 d\Omega^2,$$  

where $R = R(r, t) > 0$ is a function to be determined. For example, for $R(r, t) = a(t)g(r), f(r) = g'(r), g(r) = \{\sin r, r, \text{shr}\}$ the Tolman metric is identical to the Friedmann metric. In the general case, the Tolman metrics correspond to inhomogeneous cosmological models.

The following expressions for the Einstein tensor can be derived for the interval (15):

$$G^0_0 = \frac{F''}{2R'R^2}, \quad G^1_1 = \frac{\ddot{F}}{2RR^2}, \quad G^2_2 = G^3_3 = \frac{1}{4R'R} \left(\frac{\dddot{F}}{R}\right)'$$

where

$$F(r, t) = 2RR^2 + 2R(1 - f^2)$$

The following equations follow from the conservation law $T^\beta_\alpha ;\beta = 0$ written in the Tolman metric (15):

$$T^2_2 = T^1_1 + \frac{R}{2R'} \left(T^1_1\right)' ,$$  

4
\[ R^2 \{ (\dot{T}_0^0)R' - (T_1^1)'R \} + (T_0^0 - T_1^1) \frac{\partial (R^2)}{\partial t'} R' = 0 \]  \hspace{1cm} (19)

With allowance for relations (15)-(19), Eqs. (13) reduce to one equation, from which it follows that

\[ W' = eq(t) \frac{R'}{fR^2} \]  \hspace{1cm} (20)

where \( q(t) \) is an arbitrary function of \( t \). We now find from Eqs. (12)

\[ T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = - (8\pi)^{-1} Q^2 R^{-4} , \]  \hspace{1cm} (21)

where \( Q^2 = q^2 + g^2, \ g = e^{-1} \). Substituting the components of the energy-momentum tensor into Eq. (18) transforms this equation into an identity for the arbitrary function \( R(r, t) \), and Eq. (19) leads only to the requirement \( Q = \text{const} \), which is equivalent to the requirement \( q = \text{const} \).

After the calculations are performed and Eq. (16) is taken into account, the Einstein equations (12) reduce to three equations:

\[ F' = \frac{\kappa}{4\pi} Q^2 \frac{R'}{R^2}, \quad \dot{F} = \frac{\kappa}{4\pi} Q^2 \frac{\dot{R}}{R^2}, \quad \left( \frac{\dot{F}}{\dot{R}} \right)' = -2 \frac{\kappa}{4\pi} Q^2 \frac{\dot{R}}{R^3}, \]  \hspace{1cm} (22)

the last of which follows from the first two. The first two equations, however, lead to one equation for the function \( R(r, t) \):

\[ (R\dot{R})^2 = (f^2 - 1)R^2 + \delta R - GG^2 \]  \hspace{1cm} (23)

Just as in the standard Tolman model [12], the solutions can be divided into three separate classes in accordance with the conditions: \( f^2 = 1, \ f^2 > 1, \ f^2 < 1 \).

1. **Parabolic model \( (f^2 = 1) \).** In this case, the solutions for \( R(r, t) \) can be represented in the form

\[ R(r, t) = R^p(r, t) = \delta^{-1}Q^2 + \delta^{-1} \left[ X_+^{1/3}(r, t) + X_-^{1/3}(r, t) \right]^2, \]  \hspace{1cm} (24)

where \( \delta = \text{const} > 0 \)

\[ X_\pm(r, t) = Ht - \beta(r) \pm \sqrt{G^3 Q^6 + (Ht - \beta(r))^2}, \ H = \pm 3\delta^2/4 \]

Here and below, \( \beta(r) \) is an arbitrary differentiable function of \( r \).

2. **Hyperbolic model \( (f^2 > 1) \).** The general integral of Eq. (23) for \( R = R^h(r, t) \) can be written in the form

\[ t - \beta(r) = (f^2 - 1)^{-1} \left[ (f^2 - 1)R^2 + \delta R - GG^2 \right]^2 - \frac{\delta}{2} (f^2 - 1)^{-3/2} \ln \left\{ 2(f^2 - 1)^{1/2} \left[ (f^2 - 1)R^2 + \delta R - GG^2 \right]^{1/2} + 2(f^2 - 1)R + \delta \right\}, \]  \hspace{1cm} (25)
In the particular case $\delta = 0$ the solution for $R$ can be found in explicit form. Specifically,

$$
R(r, t) = R_0^b(r, t) = (f^2 - 1)^{1/2} \left[ (f^2 - 1) (t - \beta(r))^2 + GQ^2 \right]^{1/2}.
$$

(26)

3. Elliptical model ($f^2 < 1$). A real solution for $R(r, t) = R^e(r, t)$ exists, if

$$
\delta > 0, \; \delta^2 \leq 4GQ^2, \; 1 - \delta^2(4GQ^2)^{-1} < f^2 < 1.
$$

In this case the general integral can be written in the form

$$
t - \beta(r) = (1 - f^2)^{-1} \left[ -(1 - f^2) R^2 + \delta R - GQ^2 \right]^{1/2} +
$$

$$
+ \frac{\delta}{2} (1 - f^2)^{-3/2} \arcsin \left\{ \frac{\delta - 2(1 - f^2)R}{[(1 - f^2)R^2 + \delta R - GQ^2]^{1/2}} \right\}
$$

(27)

The solutions obtained correspond to a space-time filled with a YM field possessing only a radial electric component $\mathbf{E}_r$ and only a radial magnetic component $\mathbf{B}_r$ which have the form

$$
\mathbf{E}_r = \mathbf{F}_{r0} = q \frac{R'}{f R^2} \mathbf{n}, \; \mathbf{B}_r = -\frac{1}{2} \sqrt{q^*} \varepsilon_{rjk} \mathbf{F}^{jk} = g \frac{R'}{f R^2} \mathbf{n},
$$

where $g^* = \det(g_{ij}) = R^4 R^2 \sin^2 \theta / f^2(r)$. Hence one can see that the constants $q$ and $g$ have the meaning of electric and magnetic charges, respectively.

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