TESTING COMPLETE POSITIVITY

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*Abstract*

We study the modified dynamical evolution of the neutral kaon system under the condition of complete positivity. The accuracy of the data from planned future experiments is expected to be sufficiently precise to test such a hypothesis.
It has been suggested that quantum gravity effects at Planck’s scale could result in loss of quantum coherence, leading to the transformation of pure into mixed states [1,2]. The neutral kaon system is a natural subnuclear laboratory to study such phenomena [3,4]. As for any decaying system, the standard quantum time-evolution for the kaon density matrix $\rho(t)$ is of the type

$$\frac{\partial}{\partial t}\rho(t) = -iH\rho(t) + i\rho(t)H^\dagger,$$

where $H$ is the effective Weisskopf-Wigner Hamiltonian. This evolution transforms pure states into pure states, although probability is not conserved: $\text{Tr}[\rho(t)] \leq \text{Tr}[\rho(0)]$, since $H \neq H^\dagger$. Loss of quantum coherence shows up when the standard evolution equation is modified as follows:

$$\frac{\partial}{\partial t}\rho(t) = -iH\rho(t) + i\rho(t)H^\dagger + T[\rho(t)].$$

The non-standard (dissipative) part $T[\rho(t)]$ is a linear transformation that, in absence of Weisskopf-Wigner terms, would generate a semigroup of maps $\tau_t$ transforming density matrices into density matrices, preserving the positivity of their spectra, the trace, and increasing their von Neumann entropy. In the presence of the Weisskopf-Wigner contribution, the trace is not preserved, but the linear maps $\gamma_t : \rho \mapsto \rho(t)$ generated by (1) are still positive and form a semigroup: $\gamma_t \circ \gamma_s = \gamma_{t+s}$, $t, s \geq 0$. When [5]-[7]

$$T[\rho(t)] = -\frac{1}{2}(R\rho(t) + \rho(t)R) + \sum_j A_j \rho(t) A_j^\dagger,$$

where $A_j$ and $R = \sum_j A_j^\dagger A_j$ are bounded $2 \times 2$ matrices, then the time-evolution maps $\gamma_t$ are completely positive (they form a so-called quantum dynamical semigroup and entropy increase is guaranteed by choosing $A_j = A_j^\dagger$ [8]).

This condition is stronger than simple positivity and has the following physical meaning [9]-[12]. Let us couple the 2-dimensional kaon system $S$ with a $n$-dimensional system $E_n$ and let us extend the evolution maps $\gamma_t$ to the global system $S + E_n$ in such a way that $E_n$ is not affected: $\tilde{\gamma}_t = \gamma_t \otimes 1_n$, $1_n$ being the $n \times n$ unit matrix. Given a state $\tilde{\rho}$ of the compound system $S + E_n$, one would like $\tilde{\gamma}_t[\tilde{\rho}]$ to be again a state, independently of $n$ and $\tilde{\rho}$. In general, this is not true, unless $\gamma_t$ is completely positive. In fact, if $\gamma_t$ is only positive, then states which are not of the separate form $\tilde{\rho} = \rho_S \otimes \rho_{E_n}$ (or convex linear combinations
of these) might, for some $n$, develop negative eigenvalues under the action of the global
time-evolutions $\tilde{\gamma}_t$ [13].

As usual, to the time-evolution $\gamma_t$ of states of $S$ there corresponds a dual time-evolution
$\gamma'_t$ of the observables $X$ (bounded operators) of $S$: $\text{Tr}(\rho \gamma'_t[X]) = \text{Tr}(\gamma_t[\rho]X)$. The latter can
be naturally extended to a linear transformation $\tilde{\gamma}'_t : [X_{ij}] \mapsto [\gamma'_t[X_{ij}]]$ of the observables of
the compound system $S+E_n$ which are $n \times n$ matrices $[X_{ij}]$ whose entries $X_{ij}$ are observables
of $S$. Now, if $\tilde{\gamma}'_t$ transforms positive $[X_{ij}]$ into positive operators, then $\gamma'_t$ is called $n$-positive.
If this property remains true for all $n$, then $\gamma'_t$ is called completely positive. In this case
$\gamma'_t$ is fixed to have the general form [12] $\gamma'_t[X] = \sum_j V_j^\dagger(t) X V_j(t)$ on the observables of $S,$
for some bounded operators $V_j(t)$ such that also $\sum_j V_j^\dagger(t)V_j(t)$ is bounded. By duality, the
time-evolution $\gamma_t$ given by (1) and (2) has the form

$$\gamma_t[\rho] = \sum_j V_j(t) \rho V_j^\dagger(t) \quad .$$

Notice that the standard quantum mechanical evolution, arising from (1) when $T[\cdot]$ is
absent, is as in (3) with $j = 1$ and $V_1(t) = \exp(-iH t)$ and, therefore, is completely positive.

Completely positive maps have been used to model a large variety of physical situations,
ranging from the description of reduced statistical systems [9]- [12], [14], to the interaction of
a mycosystem with a macroscopic measuring apparatus [15,16], to a consistent description
of the wave-packet reduction in ordinary Quantum Mechanics [17]. In the following, they
will be used to describe the decay of the neutral kaon system. More in general, the time
evolution of any open quantum system can be conveniently modeled by a quantum dynamical
semigroup.

The evolution and decay of the $K-\bar{K}$ system is conventionally modeled on a 2-dimensional
Hilbert space [18]. In the basis of the $CP$-eigenstates,

$$|K_1\rangle = \frac{1}{\sqrt{2}} [|K\rangle + |\bar{K}\rangle] \quad , \quad |K_2\rangle = \frac{1}{\sqrt{2}} [|K\rangle - |\bar{K}\rangle] \quad ,$$

the generic density matrix can be expanded as $\rho = \rho^\mu \sigma_\mu$ in terms of the Pauli matrices $\sigma_i,$
$i = 1, 2, 3,$ and the identity $\sigma_0$. We shall require the modified time-evolution of the kaon
system to be completely positive and thus generated by (1), (2), with \( A_j = A_j^\dagger \) in order to ensure entropy increase. As mentioned before, this choice is based on an effective approach to the dynamics of the system; it has the advantage of being independent from the details of the microscopic mechanism responsible for the loss of quantum coherence.

On the four-vector of components \( \rho^\mu \) the dissipative term \( T[\cdot] \) acts as the \( 4 \times 4 \) real, symmetric matrix [19]

\[
[T_{\mu\nu}] = -2 \begin{pmatrix}
0 & 0 & 0 \\
0 & a & b & c \\
0 & b & \alpha & \beta \\
0 & c & \beta & \gamma
\end{pmatrix}.
\] (5)

The parameters \( a, \alpha \) and \( \gamma \) are non-negative; further, the following inequalities must be satisfied:

\[
a \leq \alpha + \gamma , \quad 4b^2 \leq \gamma^2 - (a - \alpha)^2 ,
\]
\[
\alpha \leq a + \gamma , \quad 4c^2 \leq \alpha^2 - (a - \gamma)^2 ,
\]
\[
\gamma \leq a + \alpha , \quad 4\beta^2 \leq a^2 - (\alpha - \gamma)^2 .
\] (6)

The evolution equation (1), with \( T[\cdot] \) acting as in (5), generates a semigroup of completely positive maps \( \gamma_t \) which could be, in principle, explicitly worked out. The solutions \( \rho(t) \equiv \gamma_t[\rho] \) can be used to compute certain characteristic quantities of the \( K-\bar{K} \) system that are directly accessible to experiments [3,4]. These are associated with the decay of neutral kaons into pion or semi-leptonic final states. For instance, the decay rates into two or three pions are explicitly given by

\[
R_{2\pi}(t) = \frac{\text{Tr} \left[ \rho(t)O_{2\pi} \right]}{\text{Tr} \left[ \rho(0)O_{2\pi} \right]} , \quad R_{3\pi}(t) = \frac{\text{Tr} \left[ \rho(t)O_{3\pi} \right]}{\text{Tr} \left[ \rho(0)O_{3\pi} \right]} ,
\] (7)

where \( O_f \) is the operator describing the final decay state \( f \). Other experimentally observed quantities are associated with the decay of an initial \( K \) state as compared to the corresponding decay of an initial \( \bar{K} \) state. These so-called asymmetries take the general form
where \( \rho_{\bar{K}}(t) \) and \( \rho_K(t) \) are the solutions of (1) with the initial conditions of having a pure \( \bar{K} \) and pure \( K \) at \( t = 0 \), respectively.

On physical basis, it is plausible to assume that the parameters in (5) are small, of the order of the kaon mass squared over Planck’s mass [3]. Therefore, for the explicit computation of the quantities in (7) and (8), a perturbative solution of (1) is sufficient (the detailed analysis can be found in [19]). A comparison of the analytic computations with the present experimental results gives bounds on the phenomenological parameters \( a, b, c, \alpha, \beta \) and \( \gamma \) which are compatible with zero [20]. However, more precise values for these parameters are expected when data from planned new experiments on the \( K-\bar{K} \) system become available. This will result into a stringent test of the inequalities (6), and hence of complete positivity as a viable physical condition on possible modifications of the neutral kaon dynamics. To our knowledge, this is the first time that complete positivity can be put to test to such a high accuracy in the study of a subnuclear system.

The new experiments on the \( K-\bar{K} \) system will also be able to detect physical inconsistencies that might arise when the map \( T[\cdot] \) in (1) is taken to be simply positive and not completely positive. An example is given by maps \( T[\rho(t)] \) in (1) of the form (5) with \( a = b = c = 0, \alpha \neq \gamma, \beta \neq 0 \) and \( \alpha \gamma \geq \beta^2 \). Such a modification of the standard quantum mechanical treatment of the neutral kaon dynamics has been proposed and discussed in [3,4]. In this case, (1) generates a semigroup of simply positive dynamical maps \( \gamma_t \). In fact, due to the inequalities (6), complete positivity would require \( \alpha = \gamma \) and \( \beta = 0 \), thus providing a trivial \( T[\cdot] \).

\[ A(t) = \frac{\text{Tr}[\rho_{\bar{K}}(t)\mathcal{O}_f] - [\rho_K(t)\mathcal{O}_f]}{\text{Tr}[\rho_{\bar{K}}(t)\mathcal{O}_f] + [\rho_K(t)\mathcal{O}_f]}, \quad (8) \]

\[ \]

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\[ ^1\text{Notice that the inequalities (6) put stringent bounds on possible hierarchies between the non-standard parameters. In particular, the possibility of recovering from the completely positive approach the simply positive one discussed in [1]–[4] as an effective description by making } a, b, c \]
In order to study the properties of such simply positive maps $\gamma_t$, let us first consider the time-evolution $\tau_t[\rho] = \exp(tT[\cdot])[\rho]$ generated by the non-standard part $T[\cdot]$. Its action on density matrices $\rho = \rho^\mu\sigma_\mu$ written as four-vectors of components $\rho^\mu$ can be explicitly worked out starting from (5) with $a = b = c = 0$; it is given by the $4 \times 4$ matrix

$$
[\tau_{\mu\nu}] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A(t) & B(t) \\
0 & 0 & B(t) & C(t)
\end{pmatrix},
$$

(9)

where

$$
A(t) = \frac{1}{\lambda_+ - \lambda_-} \left[ (\lambda_+ + 2\alpha) e^{\lambda_+ t} - (\lambda_- + 2\alpha) e^{\lambda_- t} \right],
$$

$$
B(t) = \frac{2\beta}{\lambda_+ - \lambda_-} \left( e^{\lambda_+ t} - e^{\lambda_- t} \right),
$$

(10)

$$
C(t) = \frac{1}{\lambda_+ - \lambda_-} \left[ (\lambda_+ + 2\alpha) e^{\lambda_+ t} - (\lambda_- + 2\alpha) e^{\lambda_- t} \right],
$$

and $\lambda_\pm = -(\alpha + \gamma) \pm \sqrt{(\alpha - \gamma)^2 + 4\beta^2}$ are both negative due to the positivity condition $\alpha\gamma \geq \beta^2$. If $\tau_t$ were completely positive, we should be able to write its action as in (3); this is possible only when $\lambda_+ = \lambda_-$ or equivalently when $\alpha = \gamma, \beta = 0$.

The map $\tau_t$ is an example of a positive linear transformation which is not 2-positive. Indeed, as previously discussed, let us extend $\tau_t$ to $\tilde{\tau}_t = \tau_t \otimes 1_2$ acting on the two-dimensional kaon system coupled to another arbitrary two-dimensional system. Then, $\tilde{\tau}_t$ transforms the (entangled) state

$$
\rho_S = \frac{1}{4} \left( \sigma_0 \otimes \sigma_0 - \sum_{i=1}^3 \sigma_i \otimes \sigma_i \right),
$$

(11)

into

$$
\tilde{\tau}_t[\rho_S] = \frac{1}{4} \left( \sigma_0 \otimes \sigma_0 - \sigma_1 \otimes \sigma_1 - A(t) \sigma_2 \otimes \sigma_2 \\
- C(t) \sigma_3 \otimes \sigma_3 - B(t) (\sigma_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_2) \right).
$$

(12)

very small with respect to $\alpha, \beta$ and $\gamma$ seems to be ruled out [21], [20].
This operator is no more positive as can be seen by computing its mean value on the states

\[ |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

(13)

\[ |v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \].

Explicitly, one finds

\[ \langle u | \tilde{\tau}_t [\rho_S] | u \rangle = -\langle v | \tilde{\tau}_t [\rho_S] | v \rangle = \frac{1}{2} (A(t) - C(t)) . \]

(14)

This quantity never vanishes for \( t > 0 \) and has a definite sign depending on the relative magnitude of \( \alpha \) and \( \gamma \); therefore, if \( A(t) - C(t) \) is positive in one case, it is negative in the other. This means that \( \tilde{\tau}_t [\rho_S] \) develops negative eigenvalues. If one adds to the generator \( T[\rho] \) the Weisskopf-Wigner part as in (1), this result is not altered as we shall see later.

At first sight, this conclusion might seem of little physical relevance. Indeed, the coupling of the subsystem of interest to an abstract \( n \)-level system is regarded as too artificial by those who consider the condition of complete positivity of the reduced dynamics as a mere technical request [22]. The point is that in our case, the additional two-dimensional system can be taken to be another kaon system. Precisely this physical situation is commonly encountered in the so-called \( \phi \)-factories. In these experimental setups, a \( \phi \) meson decays into an entangled antisymmetric state of two neutral kaons that in our formalism can be written as [4]

\[ |\Psi_S\rangle = \frac{1}{\sqrt{2}} (|K_1\rangle \otimes |K_2\rangle - |K_2\rangle \otimes |K_1\rangle) . \]

(15)

As a density matrix \( |\Psi_S\rangle \langle \Psi_S| \) this state exactly corresponds to the “singlet” \( \rho_S \) in (11).

The global time-evolution of states like (15) is obtained by the tensor product \( \Omega_t = \gamma_t \otimes \gamma_t \) of the time-evolutions of the single kaons. In the standard approach, with purely Weisskopf-Wigner dynamics, positive operators remain positive and negative eigenvalues are never
generated. Indeed, this is guaranteed by any completely positive time-evolution $\gamma_t$ as comes out from (3):

$$\Omega_t[\rho] = \sum_{ij} [V_i(t) \otimes V_j(t)] \rho [V_i^\dagger(t) \otimes V_j^\dagger(t)]. \quad (16)$$

On the contrary, the positivity of the evolving states need not be preserved by modified dynamics $\gamma_t$ that are only simply positive. Again, the single kaon time-evolution given by (9) provides us with an example of this fact. The action of $\mathcal{T}_t = \tau_t \otimes \tau_t$ on $|\Psi_S\rangle\langle\Psi_S| = \rho_S$ gives

$$\mathcal{T}_t[\rho_S] = \frac{1}{4} [\sigma_0 \otimes \sigma_0 - \sigma_1 \otimes \sigma_1$$

$$- (A^2(t) + B^2(t))\sigma_2 \otimes \sigma_2 - (B^2(t) + C^2(t))\sigma_3 \otimes \sigma_3$$

$$- B(t)(A(t) + C(t)) (\sigma_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_2)], \quad (17)$$

and, therefore, the presence of negative eigenvalues can be ascertained as before by computing the mean values on the vectors (13):

$$\langle u | \mathcal{T}_t[\rho_S] | u \rangle = -\langle v | \mathcal{T}_t[\rho_S] | v \rangle$$

$$= \frac{1}{2} (A^2(t) - C^2(t)) \neq 0. \quad (18)$$

We now show that this pathology cannot be cured by considering also the Weisskopf-Wigner contribution to the time-evolution. Let us call $\mathcal{W}_t$ the full time-evolution $\Omega_t$ when the non-standard dissipative part is absent. It corresponds to the ordinary quantum mechanical evolution on the compound two-kaon system. Then, $\Omega_t$ can be obtained from $\mathcal{W}_t$ and $\mathcal{T}_t$ through the Trotter product formula

$$\Omega_t[\rho] = \lim_{n \to \infty} \left( \mathcal{W}_{t/n} \circ \mathcal{T}_{t/n} \right)^n [\rho]. \quad (19)$$

As already noticed the Weisskopf-Wigner evolution $\mathcal{W}_t$ preserves the positivity of operators. From (9) it can be checked that $\mathcal{T}_t$ is trace and hermiticity preserving. Further, as proved above, the operator $\mathcal{T}_{t/n}[\rho_S]$ has negative eigenvalues and therefore can be written as the
sum of non-trivial positive and negative parts which are not altered by the action of \( \mathcal{W}_{t/n} \). Therefore, we can write:

\[
(\mathcal{W}_{t/n} \circ \mathcal{T}_{t/n})[\rho_S] = \rho_+ + \rho_- ,
\]

where \( \rho_+ \) is a positive and \( \rho_- \) a negative operator. The iteration of this action cannot destroy the presence of a negative part. In fact, since \( \mathcal{T}_t \) is trace preserving one has

\[
\text{Tr}(\rho_-) = \text{Tr}(\mathcal{T}_{t/n}[\rho_-]) = \text{Tr}(\mathcal{T}_{t/n}[\rho_-])_+ + \text{Tr}(\mathcal{T}_{t/n}[\rho_-])_- .
\]

Therefore, since \( \text{Tr}(\mathcal{T}_{t/n}[\rho_-])_+ \geq 0 \), while the trace of any negative operator is obviously \( \leq 0 \), it turns out that

\[
\left| \text{Tr}(\mathcal{T}_{t/n}[\rho_-])_- \right| \geq |\text{Tr}(\rho_-)| .
\]

Further, notice that

\[
\frac{\partial}{\partial s} \text{Tr}(\mathcal{W}_s \left[\mathcal{T}_{t/n}[\rho_-]\right]_-) = \frac{\partial}{\partial s} \left( (\Gamma \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \Gamma) \mathcal{W}_s \left[\mathcal{T}_{t/n}[\rho_-]\right]_- \right) \leq -2\lambda \text{Tr}(\mathcal{W}_s \left[\mathcal{T}_{t/n}[\rho_-]\right]_-) ,
\]

where \( \Gamma \) is the width matrix of the Weisskopf-Wigner hamiltonian and \( \lambda \) its largest eigenvalue. Integrating this inequality with respect to \( s \) from 0 to \( t/n \), one obtains

\[
\text{Tr}(\mathcal{W}_{t/n}\left[\mathcal{T}_{t/n}[\rho_-]\right]_-) \leq e^{-2\lambda t/n} \text{Tr}(\mathcal{T}_{t/n}[\rho_-])_- .
\]

Putting together (22) and (23), one gets

\[
\left| \text{Tr} \left( \left( \mathcal{W}_{t/n} \circ \mathcal{T}_{t/n} \right)[\rho_-] \right)_- \right| \geq e^{-2\lambda t/n} |\text{Tr}(\rho_-)| .
\]

In deriving this formula we have used the fact that \( \mathcal{W}_t[\rho]_- = \mathcal{W}_t[\rho_-] \), for any \( \rho \); this follows from the invertibility and the positivity preserving property of \( \mathcal{W} \) and from the uniqueness of the decomposition of self-adjoint operators into positive and negative parts.
Iterating $n - 1$ times the above estimates, one finally obtains

$$\left| \text{Tr} \left( \left[ (\mathcal{W}_{t/n} \circ \mathcal{T}_{t/n})^n [\rho_S] \right]_\sim \right) \right| \geq e^{-2\lambda t} |\text{Tr}(\rho_-)| > 0.$$  \hfill (26)

Therefore, the negative part in $(\mathcal{W}_{t/n} \circ \mathcal{T}_{t/n})^n [\rho_S]$ will survive the limit $n \to \infty$; hence $\Omega_t[\rho_S]$ will always have negative eigenvalues. In other words, the full time-evolution $\Omega_t$ on the correlated “singlet” state of the two kaons will always generate negative probabilities.

In conclusion, we have discussed how the condition of complete positivity on the time-evolution might be tested in actual experiments involving neutral kaons. On one hand, this can be done by verifying certain inequalities that must be fulfilled by the parameters of any completely positive modification of the standard phenomenological Weisskopf-Wigner description of a single kaon system. On the other hand, by studying time-correlations of entangled two kaon systems one can distinguish between completely positive vs simply positive modified dynamics due to the emergence, in the latter case, of negative probabilities.
REFERENCES


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