All consistent interactions for exterior form gauge fields

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We give the complete list of all first-order consistent interaction vertices for a set of exterior form gauge fields of form degree \(>1\), described in the free limit by the standard Maxwell-like action. A special attention is paid to the interactions that deform the gauge transformations. These are shown to be necessarily of the Noether form “conserved antisymmetric tensor” times “p-form potential” and exist only in particular spacetime dimensions. Conditions for consistency to all orders in the coupling constant are given. For illustrative purposes, the analysis is carried out explicitly for a system of forms with two different degrees \(p\) and \(q\) (\(1 < p < q < n\)).

It has been known for some time that electromagnetism has a generalization to \(n\)-index antisymmetric tensor potentials, \(n = 1\) being electromagnetism, and that these potentials couple naturally to \(n - 1\) dimensional extended objects. These potentials and extended objects arise throughout supergravity and string theory. There are many partial results known about their interactions, but to our knowledge, there is no systematic treatment of this subject. In this Rapid Communication, we provide the complete list of all interactions that are consistent to first order in the coupling constant and explicitly treat the case when only its gauge variation under (2) vanishes up to a surface term when the equations of motion (3) hold. This can be seen directly by expressing that the sum \(I + gV\) is gauge invariant up to (and including) order \(g\) under gauge transformations that differ from (2) by terms of at least order \(g\). Thus, the determination of all consistent interactions is equivalent, to first order in the coupling constant, to the determination of all the “observables” given by the spacetime integral of a local \(n\)-form.

Consistent interactions of a given gauge theory may be classified into three categories: (i) those that do not modify the gauge transformations; (ii) those that modify the gauge transformations without changing their algebra; and (iii) those that modify both the gauge transformations and their algebra. For the first type, the gauge variation \(\delta_A V\) of the vertex \(V\) vanishes (up to a surface term) off-shell and not just on-shell. For the second and third types, \(\delta_A V\) vanishes only on-shell,

\[ \delta_A V = \sum_k \int \frac{\delta I}{\delta A^{(k)}_{\mu_1...\mu_p \rho_k}} b^{(k)}_{\mu_1...\mu_p \rho_k} d^n x \]

with \(b^{(k)}_{\mu_1...\mu_p \rho_k} \neq 0\). The modification of the gauge transformations is given, to first order in the coupling constant \(g\), by

\[ \delta^\text{NEW}_{A^{(k)}_{\mu_1...\mu_p \rho_k}} = (dA)_{\mu_1...\mu_p \rho_k} - gb_{\mu_1...\mu_p \rho_k} \]

since then, the gauge variation \(\delta^\text{NEW}_A (I + gV)\) vanishes to order \(g^2\). If \(b\) is gauge invariant, the second variation \(\delta^\text{NEW}_A \delta^\text{NEW}_A A^{(k)}\) is of order \(g^2\) and the interaction does not modify the gauge algebra to order \(g\) [1].

Interactions of each type exist for a set of free vector fields \(A^\mu\). The interactions that do not deform the gauge transformations are given by the functions of the curvatures \(F^\mu_{\nu \rho} = \partial_\nu A^\rho - \partial_\rho A^\nu\), and their derivatives – as in the Euler-Heisenberg effective Lagrangian for electrodynamics –, as well as by the Chern-Simons terms in odd spacetime dimensions [2]. There has been no systematic

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study of the interactions of the second type but an example is given in 3 spacetime dimensions by the Freedman-Townsend vertex [3] $e^{\lambda \mu \nu} F^{\lambda}_{\mu \nu} A_0^{abc} f_{abc}$. This coupling has been studied recently in [4]. Finally, the Yang-Mills coupling, which exists in any number of spacetime dimensions, deforms both the gauge transformations and the gauge algebra, which is no longer abelian even on-shell.

As we shall see, the possible interactions of exterior forms of higher degree are by contrast much more constrained. The interaction vertices that deform the gauge transformations exist only when special conditions are met on the spacetime dimension and the degrees of the exterior forms. Furthermore, there is no analog of the Chern-Simons terms. These interactions are consistent connections for extended objects [6].

The interaction Lagrangians that do not deform the gauge transformations are described by the following theorem, which is our main result.

**Theorem:** The only first-order consistent interactions that deform the gauge transformations are given by the generalized Noether coupling, i.e., are of the form

$$ V^{(k)} = \sum_k V^{(k)} $$

where $S^{(k)}_{\mu_1 \cdots \mu_p_k}$ are non-trivial gauge-invariant anti-symmetric tensors which are conserved modulo the equations of motion,

$$ \delta_{\mu_1} S^{\mu_1 \cdots \mu_p_k} \approx 0, \quad \delta_A S^{\mu_1 \cdots \mu_p_k} = 0. $$

In form notations,

$$ V^{(k)} = \int J^{(k)} \wedge A^{(k)}, $$

where $J^{(k)}$ is the non-trivial, gauge-invariant, conserved $(n-p_k)$-form dual to $S^{(k)}_{\mu_1 \cdots \mu_p_k}$,

$$ dJ^{(k)} \approx 0, \quad \delta_A J^{(k)} = 0. $$

**Proof:** That (8), (9) define first-order consistent interactions is rather obvious because $\delta_A (J^{(k)} \wedge A^{(k)}) = J^{(k)} \wedge \delta_A A^{(k)} = J^{(k)} \wedge d\Lambda^{(k)} \approx d(\pm J^{(k)} \wedge \Lambda^{(k)})$, where the $\pm$ sign occurs when $J^{(k)}$ is a form of even (odd) degree. That these are the only interactions that deform the gauge symmetries is harder to prove and is based on the cohomological reformulation of the observables, which are known to be in bijective correspondence with the elements of the BRST cohomology $H^0(s, d)$ at ghost number zero [7–9]. One can work out $H^0(s, d)$ by following exactly the pattern developed in [10] for analysing the mod $d$ BRST cocycles in the Yang-Mills case, as well as the results of [13] on the characteristic cohomology for exterior gauge fields described by the action (1). If one does this, one finds that the antifield-independent part of the antifield-dependent cocycles (which are the cocycles that deform the gauge symmetry [11, 12]) can always be brought to the form given in the theorem. The details will be reported elsewhere [14].

The major difference between the allowed couplings between exterior forms of degrees $\geq 2$ and the Yang-Mills cubic coupling, which may be written as in (6), but with the non gauge-invariant current $F_0^{\mu \nu} A_\mu^{abc} f_{abc}$, is that the latter deforms non-trivially the gauge algebra already at order $g$ in the coupling constant, while the former leaves it abelian at that order. This is because the vertex (6) is linear in the non-gauge invariant form $A$. Therefore, upon integration by parts, one finds that $\delta V^{(k)}$ is a combination of the Euler-Lagrange derivatives of the free Lagrangian with coefficients that are gauge invariant. Thus, according to (4) and (5), the first-order modification of the gauge transformations is gauge invariant and does not change the abelian nature of the gauge algebra. By contrast, the conserved current entering the Yang-Mills coupling is not gauge invariant. This possibility is not present here because the conserved $m$-forms ($m \leq n-2$) are necessarily gauge invariant up to trivial terms [13].

Since the non-trivial conserved $m$-forms are known to be, for $m \leq n-2$, the polynomials in the curvature forms $F^{(k)}$ and their duals $F^{(k)}$ [13], the theorem provides the most general interaction vertices that can be added to the free action consistently to first order

$$ I \rightarrow I + \sum_{(A)} g(A) V^{(A)}, $$

$$ V^{(A)} = \int F^{(k_1)} \cdots F^{(k_l)} F^{(l_1)} \cdots F^{(l_s)} A^{(t)}. $$

From now on, we shall drop the wedge symbol in exterior products. In (11), $V^{(A)}$ contains at least one dual $F^{(k)}$ since otherwise it reduces to a Chern-Simons term and does not deform the gauge symmetry.

A remarkable feature of the first-order vertices $V^{(A)}$ is that they are in finite number. Indeed, one can form only a finite number of polynomials (11) of form degree $n$ (we exclude the rather direct case of forms of degree $n-1$, for which the field strengths are $n$-forms and the theory
has no local degree of freedom, since the duals are then 0-forms of which one can take arbitrarily high powers). For a given spacetime dimension and a given set of exterior forms, all the vertices deforming the gauge transformations can be listed explicitly. There may actually even be no conserved \((n - p_k)\)-form that could match the form-degree \(p_k\) of \(A^{(k)}\) in (6) to make an \(n\)-form, in which case there would be simply no consistent vertex that would deform the gauge transformations. An example is given in [5].

Let us now turn to the consistency of the vertices to higher orders. The most expedient way to analyse this question is to rephrase the problem in terms of the master equation and its deformations [11,15]. Then, one easily sees that the obstructions to second (and higher) order consistency lie in the local BRST cohomology \(H^1(s|d)\) at ghost number one. A first-order consistent deformation is obstructed to second order if its antibracket with itself, which is BRST-closed, is not BRST-exact [11]. In particular, if the cohomology group \(H^1(s|d)\) vanishes, no first-order consistent interaction can be obstructed at higher order. This raises the question of computing \(H^1(s|d)\). The calculation of \(H^1(s|d)\) can be performed and follows exactly that of \(H^0(s|d)\). Again, the detailed calculation will be reported elsewhere [14].

Knowing \(H^1(s|d)\), one can in principle investigate the higher order consistency of any given first-order consistent vertices for an arbitrary system of forms. The procedure can be rather tedious in practice and we shall, for illustrative purposes, consider here only the explicit case when two form degrees \(2 \leq p < q \leq n - 2\) are present. We denote the \(p\)-forms by \(A_a^p (a = 1, \ldots, m)\) and the \(q\)-forms by \(B^A (A = 1, \ldots, M)\), with respective curvatures \(F^a = dA_a^p\) and \(H^A = dB^A\). These are respectively \((m + 1)\)- and \((M + 1)\)-forms, while their duals \(F^A\) and \(H^A\) are respectively \((n - m - 1)\)- and \((n - M - 1)\)-forms. Taking into account the fact that the interactions (6) exist only if \(n - p\) or \(n - q\) can be written as \(M_1 + M_2 + M_3 + M_4\), where \(M_1\) is a multiple of \(p + 1\), \(M_2\) is a multiple of \(q + 1\), \(M_3\) is a multiple of \(n - p - 1\) and \(M_4\) is a multiple of \(n - q - 1\), with \(M_3 + M_4 \neq 0\) in order to have at least one dual, we have found that there are only three types of “basic” first-order consistent interactions that deform the gauge symmetry:

(i) Chapline-Manton couplings, which are linear in the duals [16],

\[
V_1 = \int f_{AA} F^a B^A, \quad (q = p + 1),
\]

\[
V_2 = \int f_{AA_1 \ldots A_{k+1}} H^A F^{a_1} \ldots F^{a_k} A^{a_{k+1}},
\]

\[
(k(p + 1) + p = q + 1).
\]

Here, \(f_{AA}\) and \(f_{AA_1 \ldots A_{k+1}}\) are arbitrary constants. The \(f_{AA_1 \ldots A_{k+1}}\) may be assumed to be completely symmetric (antisymmetric) in the \(a\)’s if \(p\) is odd (even). The Chapline-Manton coupling (12) exists only if \(q = p + 1\); the Chapline-Manton coupling (13) exists only if \(k(p + 1) + p = q + 1\) for some integer \(k\).

(ii) Freedman-Townsend couplings, which are quadratic in the duals [3],

\[
V_3 = \int f_{BC} H^B H^C B_A,
\]

\[
V_4 = \int c_{AB} H^A F_a A^b.
\]

Here, \(f_{BC}\) and \(c_{AB}\) are constants that are arbitrary at first order but will be restricted at second order. The Freedman-Townsend vertices (14) and (15) exist only if \(q = n - 2\).

(iii) Generalized couplings, which are at least quadratic in the duals \(H^A\),

\[
V_5 = \int k_{A_1 \ldots A_{a_1} \ldots A_{a_k+1}} H^{A_1} \ldots H^{A_l} F^{a_1} \ldots F^{a_k} A^{a_{k+1}}
\]

where \(k_{A_1 \ldots A_{a_1} \ldots A_{a_k+1}}\) are arbitrary constants with the obvious symmetries. These interactions exist only if there are integers \(k, l (l \geq 2)\) such that \(l(n - q - 1) + k(p + 1) + p = n\).

None of the above interactions may be available. This would occur, for instance, for \(n = 11, p = 2, q = 5\), for which there is thus no consistent, direct interaction of the 2-forms and the 5-forms that deforms the gauge symmetries (although these forms may of course interact through the Chern-Simons terms \(F^2 B\) or \(F H A\) which do not deform the gauge symmetries or through the exchange of another field).

The Chapline-Manton first-order coupling (12) or (13) defines a fully consistent interaction that is most easily obtained by introducing the gauge-invariant field-strengths \(F = dA - g B\) in the first case or \(H = dB - g F^k A\) in the second case. The interacting theory is simply given by the free action in which the original field strengths are replaced by the gauge-invariant ones. This automatically generates the correct \(O(g^2)\)-terms.

The Freedman-Townsend vertices define a consistent theory to higher orders if and only if two conditions are met: (i) the \(f_{BC}\) fulfill the Jacobi identity and thus define a Lie algebra; (ii) the \(c_{AB}\) define a representation of that Lie algebra. These restrictions arise because \(H^1(s|d)\) does not vanish and the antibracket of the cocycles defining the first order interaction is not zero in cohomology unless (i) and (ii) are fulfilled. When these conditions are met, one finds that the fully interacting theory is given, in first order form for the \(B\)-fields, by

\[
I = \int -\frac{a}{2} (2B_A \Phi^A + \tau^A \Phi A) - \frac{1}{2b} F_a^a F_a^a
\]

with \(a = (n - 1)!(-1)^n/(q + 1)!\) and \(b = (p + 1)!\). In (17), \(\beta^A\) is an independent 1-form that coincides on-shell with
the dual of $H^A$ in the free limit, while $\Phi^A$ is its curvature, $\Phi^A = d\beta^A + (g/2) f_{BC}^A \beta^B \beta^C$, and $F^a_A$ is the covariant exterior derivative of $A^a$, $F^a_A = DA^a \equiv dA^a + g f^a_{Bc} \beta^B \wedge A^c$. The action (17) is invariant under the abelian gauge transformations

$$\delta B_A = DA_A - \frac{g}{ab} t^a_{bc} F^{a}_{cb} \delta \beta^A, \quad \delta \beta^A = 0, \quad \delta A^a = DA^a, \quad (18)$$

which are still reducible on-shell because $\Phi^A \approx 0$. If one eliminates $\delta \beta^A$ by means of its equations of motion, one gets the second-order form of the Freedman-Townsend model, which is not polynomial. If one eliminates instead $B^A$, one gets the non-linear sigma-model with a minimal coupling of the exterior form $A^a$ to the flat connection $g^{-1}dg$. Note that the metrics $\delta_{AB}$, $\delta_{ab}$ with which we have lowered and raised the internal indices need not be invariant [5].

Lastly, we turn to the generalized couplings (16). These define first-order terms of a fully consistent interaction no matter how the coefficients $k_{A_1...A_{q+1}}$ are chosen. The corresponding full theory is (in first order form for $B^4$)

$$I = \int -\frac{a}{2} (2B_A \Phi^A + T^A(\beta_A)) - \frac{1}{2b} T^a A_a$$

$$+ g \int k_{A_1...A_{q+1}} \beta_{A_1} \ldots \beta_{A_{q+1}} A_{q+2} F_{q+2} \ldots F_{q+k+1} \quad (19)$$

with $a = -(-1)^{(n-q-1)(q+1)}(n-q-1)!$ and $b = (p+1)!$ and (reducible) gauge transformations

$$\delta A^a = dA_{a_1}, \quad \delta \beta^A = 0, \quad \delta B_{A_1} = dA_{A_1}$$

$$- \alpha k_{A_1...A_{q+1}} \beta_{A_2} \ldots \beta_{A_{q+1}} A^1 F_{q+2} \ldots F_{q+k+1}. \quad (20)$$

Here, $\Phi^A = d\beta^A$ and $\alpha = l(-1)^{(n-q)(n-q-1)-1}/a$. Again, upon elimination of the auxiliary field $\beta^A$, one generates the first-order vertex (16) and the corresponding higher-order terms. This theory does not appear to have been described explicitly in the previous literature. It is dual to a theory with pure Chern-Simons couplings, as it can be easily seen by eliminating $B^A$ instead of $\beta^A$.

Note that the algebra of the gauge transformations remains abelian on-shell to all orders in the coupling models for the above three models [17].

To conclude, we have shown that the gauge symmetries of exterior form gauge fields have a high degree of rigidity. Interactions that deform them do exist, but only in special dimensions. Furthermore, they never modify the gauge algebra to first order in the coupling constant. Couplings to 1-forms can be treated along similar lines. One finds additional interactions besides those given by the above theorem, which are also of the Noether form $j^\mu A_\mu$. However, the conserved current $j^\mu$ which couples to the 1-form need not be gauge-invariant. There is actually only one non gauge-invariant current that is available and it leads to the Yang-Mills cubic vertex, which deforms the gauge algebra to order $g$. All other currents $j^\mu$ may be assumed to be gauge-invariant and thus do not lead to algebra-deforming interactions. There is in particular no vertex of the form $\bar{I}BA$ where $A$ are 1-forms and $B$ are p-forms ($p > 1$) with curvature $H$, which excludes charged p-forms (i.e., p-forms transforming in some representation of a Lie algebra minimally coupled to a Yang-Mills potential). An important difference with the above case is that the gauge-invariant conserved currents $j^\mu$ are in infinite number because the starting theory is free and possesses an infinite number of conserved charges. There is accordingly an infinite number of first-order consistent interaction vertices but most of them are of course inconsistent at higher order.

**Notes added**

1. As stated in the conclusion, the interaction vertices of this paper are still available in the presence of 1-forms. They just fail to exhaust then all the possible vertices. In particular, the vertices (i), (ii) and (iii) given above for a mixed system of $p$ and $q$-forms are still available when $p = 1$; the analysis of their higher-order consistency is also equally valid.

In a recent preprint [18], Brandt and Dragon have described an interaction between two 1-forms $A^a$ ($a = 1, 2$) and one 2-form $B^1$ in four dimensions that actually fits into the general Freedman-Townsend structure (14), (15) by taking $f_{BC}^A = 0$ ($A = 1$, abelian one-dimensional algebra) and $t_{11}^1 = 1$, other components of $t_{ab}^1 = 0$. [In their notations, $A^1 = W$ and $A^2 = A$]. It is clear that $[t_1, t_1] = 0$, so this 2 by 2 matrix defines a representation of the abelian one-dimensional Lie algebra with $f_{BC}^A = 0$. Therefore, the higher-order consistency condition are fulfilled and the full interaction is given by Eq. (17). Elimination of the auxiliary field $\beta$ is easily checked to reproduce the action of [18]. Thus, this action precisely falls in what we have called the Freedman-Townsend category. Note that due to the triangular form of the matrix $t_1$, one can dualize not only the 2-form $B^1$ (to get a scalar), but also the potential $A^2$ (to get another one-form). If one does so, one obtains two 1-forms and one scalar coupled through a standard Chern-Simons term. The vertex (15) specialized to that peculiar set of fields and to a similar choice of coefficients was considered previously to first order in $g$ by Brandt in the supergravity context [19].

Our approach, which covers the general cases of both abelian and non-abelian Lie algebras as well as arbitrary representations, and which also covers higher-degree forms in higher spacetime dimensions, shows explicitly that the familiar concepts on which Yang-Mills theory is based (curvature, covariant derivatives, representations) are shared by the model (17). It is precisely the recognition of this similarity that enabled us to construct the interaction vertex (17) to all orders.

2. The basic interaction vertices described above can of
course be combined, or can be combined with Chern-Simons terms. This leads, in general, to additional constraints on their coefficients (which may actually have no non trivial solutions in some cases). One example of a non-trivial combination is the description of massive vector fields worked out in [3,20], which combines the Freedman-Townsend vertex with a Chern-Simons term [21]. Another example is given in [22], where both the Freedman-Townsend vertex and the Yang-Mills vertex are introduced simultaneously.

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[1] We include in the first category the interaction vertices that do not fulfill  \( \delta \Lambda = 0 \) off-shell, but can be made to do so by a field redefinition. Similarly, we include in the second category the interaction vertices that deform non trivially the gauge transformations but deform trivially their algebra, in the sense that the gauge algebra can be made abelian by appropriate redefinition. This will always be understood in the sequel.


[17] With three (or more) different degrees, however, one may modify the algebra at order \( g^2 \). For instance, if \( A, B \) and \( C \) are respectively 3-, 4- and 7-forms, the Lagrangian \( \sim F \wedge F + \overline{H} \wedge H + \overline{G} \wedge G \) is invariant under the gauge transformations \( \delta A = d\epsilon + g\Lambda, \delta B = d\Lambda \) and \( \delta C = d\mu + g\epsilon dB - g^2 \Lambda B \), where \( \epsilon, \Lambda \) and \( \mu \) are respectively 2-, 3- and 6-forms. Here, \( F = dA - gB, H = dB \) and \( G = dC - gAdB + (1/2)g^2 B^2 \). The commutator of two \( \Lambda \)-transformations is a \( \mu \)-transformation with \( \mu = g^2 \Lambda_1 \Lambda_2 \). The fact that the gauge algebra may become non-abelian at order \( g^2 \) shows that the connection interpretation, which excludes this possibility, is not always appropriate.


