QUANTIZATION OF A PARTICLE IN A BACKGROUND YANG-MILLS FIELD

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Abstract

Two classes of observables defined on the phase space of a particle are quantized, and the effects of the Yang-Mills field are discussed in the context of geometric quantization.

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Short title: Particle in background Yang-Mills field.
I. Introduction.

Let $Q$ be a Riemannian manifold considered as the configuration space of a particle, the purpose of this paper is to discuss the quantization of the observables on the phase space $T^*Q$ of this particle when it is moving under the influence of a background Yang-Mills field, the Yang-Mills potential is a connection $\alpha$ on a principal bundle $N$ over $Q$.

The free $G$–action on $N$ can be lifted to a Hamiltonian $G$–action on $T^*N$ with an equivariant moment map $J : T^*N \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$, and denote by $O_\mu$ the coadjoint orbit through $\mu$. Then $J^{-1}(O_\mu)/G$ has a canonical symplectic structure given by the Marsden-Weinstein reduction [1]. This reduced phase space is the appropriate phase space of a particle in a background Yang-Mills field $\alpha$ of charge $\mu$ [2].

We will denote by $Q(X)$ the quantization of the phase space $X$, suppressing in our notation the choices of polarizations and pre-quantization line bundles etc, via the standard procedure of geometric quantization [3, 4]. Suppose we choose the vertical polarization on $T^*N$ so that the quantization $Q(T^*N)$ of $T^*N$ gives $L^2(N)$. Moreover, suppose the co-adjoint orbit $O_\mu$ is integral, so that the quantization of this coadjoint orbit gives a irreducible representation space $\mathcal{H}_\mu$ of $G$ [5] [6]. A theorem of Guillemin-Sternberg [7] (see also [8]) then suggests that the quantization of $J^{-1}(O_\mu)/G$ is given by $\text{Hom}_G(\mathcal{H}_\mu, L^2(N))$, the space of $G$–equivariant linear maps from $\mathcal{H}_\mu$ to $L^2(N)$. And this result holds independent of whether there is a Yang-Mills field present in the background. Thus when some technical assumptions are made so that the procedure of geometric quantization can be carried out smoothly, the Yang-Mills field plays no role in the quantization of the phase space $J^{-1}(O_\mu)/G$.

We will discuss the effect of the Yang-Mills field in quantizing observables that are lifted from functions on $T^*Q$. We will show that the resulting quantum operators are expressed in terms of the covariant derivatives, which is defined by the connection $\alpha$. In particular, we will show that the quantum operators for $f$ of the form $\frac{1}{2}||p||^2 + V(q)$ are expressed in terms of the covariant Laplace operator and the Ricci curvature. This is obtained by a standard Blattner-Kostant-Sternberg (BKS) pairing approach [9]. Our results are in agreement with those of Landsman [10] who arrived at the conclusion via deformation quantization.

Outline of this paper is as follows; In section 2 we give a detailed exposition of our problem in order to standardize the notations used throughout this paper. As a prelude to our result, we note that if the gauge group is abelian, we will recover the Dirac quantization of a charge particle in the presence of an electro-magnetic field. In section 3 we introduce local coordinates to facilitate our calculation, and state some results concerning the Hamiltonian vector fields for our observables. We follow closely the treatment of [8] on the polarization chosen for the phase space in section 4. Our results when $f$ is polarization preserving are given in section 5, and the quantization of $\frac{1}{2}||p||^2 + V(q)$ using BKS pairing appears in section 6.

II. Preliminary discussions.

Let $N$ be a principal $G$–bundle over $Q$ where $G$ is compact, with Lie algebra $\mathfrak{g}$, and the group action is on the right. We define two functions $R_g : N \to N$ and
\[ \hat{n} : G \rightarrow N, \text{ where} \]
\[ R_g(n) = \hat{n}(g) = ng, \quad (1) \]
and we denote by \( F_\ast \) the Jacobian of \( F \). A connection is a linear map \( \alpha(n) : T_nN \rightarrow \mathfrak{g} \) for each \( n \in N \) satisfying

i. \( \text{Ad}_{g^{-1}}\alpha(n) = \alpha(ng)R_{g_\ast} : T_nN \rightarrow \mathfrak{g}, \)

ii. \( \alpha(n)\hat{n}_* = \text{Id} : g \rightarrow \mathfrak{g}. \)

The free \( G \)-action on \( N \) can be lifted to a Hamiltonian \( G \)-action on \( T^*N \) with moment map \( J : T^*N \rightarrow \mathfrak{g}^* \) given by \( J(\xi, n) = \hat{n}^* \cdot \xi \).

Let \( N^\# \) be the pullback bundle over \( T^*Q \): Explicitly,
\[
N^\# = \{(p, n) \mid p \in T^*_nQ \text{ where } q = \pi(n)\}.
\]

Define a diffeomorphism
\[
\chi : N^\# \times \mathfrak{g}^* \rightarrow T^*N, \quad (p, n, \mu) \mapsto (\xi, n) \quad \text{where} \quad \xi = \pi^*(p) + \langle \mu, \alpha \rangle \in T^*_nN.
\]

This map in turn induces an \( \alpha \)-dependent projection \( \pi_\alpha : T^*N \rightarrow T^*Q \), and their corresponding symplectic forms are related by
\[
\Omega_{T^*N} = \pi_\alpha^*\Omega_{T^*Q} + d\langle \mu, \alpha \rangle. \quad (3)
\]

One shows that the moment map is simply the projection \( N^\# \times \mathfrak{g}^* \rightarrow \mathfrak{g}^* \), i.e., \( \chi^{-1}(J^{-1}(\mu)) = N^\# \times \{\mu\} \). Thus for each \( \mu \), the \( G \)-action on \( N^\# \) induces a \( G \)-action on \( J^{-1}(\mu) \). This action is non-canonical (\( R^\#_g \Omega_{T^*N} \neq \Omega_{T^*N} \)) in general:
\[
(\xi, n) \mapsto (\xi_g, ng) \quad \text{where} \quad \xi_g = R^\#_{g^{-1}}\xi + R^\#_{g^{-1}}[(\text{Ad}_{g^{-1}} - \text{Id})\mu]\alpha(n).
\]

However, this action coincides with the canonical \( G \)-action when restricted to the isotropy subgroup
\[
H = \{g \in G \mid \text{Ad}_g^\#\mu = \mu\}
\]
of \( \mu \). (There \( \text{Ad}_{g^{-1}}^\#\mu = \mu \) and \( \xi_g = R^\#_{g^{-1}}\xi \).) The relevant phase space becomes
\[
J^{-1}(O_\mu)/G = J^{-1}(\mu)/H = N^\# / H \times \{\mu\}.
\]

Given an observable on the phase space of the particle \( \hat{f} : T^*Q \rightarrow \mathbb{R} \), by the projection \( \pi_\alpha \), the pullback map, which we continue to denote by \( f \),
\[
f = \pi_\alpha^*f : T^*N \simeq N^\# \times \mathfrak{g}^* \rightarrow \mathbb{R}, \quad (4)
\]
is invariant with respect to both the canonical \( G \)-action and the non-canonical one. In particular, \( f \) is independent on the charge variables \( \mu \in \mathfrak{g}^* \).

We assume that \( O_\mu \) is integral, so that \( Q(O_\mu) = \mathcal{H}_\mu \) is an irreducible representation space of \( G \) induced by \( \rho_\mu : H \rightarrow U(1) \). Choose an orthonormal basis \( \{\phi_i\} \) for \( \mathcal{H}_\mu \), where \( \phi_i \) is a holomorphic function on the Kähler manifold \( G/H \) [5]. Then \( \Psi \in Q(J^{-1}(\mu)/H) = \text{Hom}_G(\mathcal{H}_\mu, L^2(N)) \) is determined by \( \Psi(\phi_i) = \Psi_i \in L^2(N) \).

Using orthonormality of \( \phi_i \) and \( G \)-equivariance of \( \Psi \), we write
\[
\Psi = \sum \Psi_i\phi_i : N \times G/H \rightarrow N \times_G G/H \rightarrow \mathbb{C}
\]
which is uniquely determined by \( \psi = \Psi(\cdot, eH) : N \to \mathbb{C} \) with the condition 
\( \psi(nh) = \rho_\mu(h^{-1})\psi(n) \) for all \( h \in H \). So we see that the Yang-Mills potential plays no role in quantizing the relevant phase space, it simply picks up the multiplicity of the charge sector \( O_\mu \) in \( L^2(N) \) (cf. [7], [8]).

If we are to quantize an observable that is a pullback of \( f \) on the phase space of the particle \( T^*Q \), the connection \( \alpha \) plays an important role. As an illustration, suppose the charge \( \mu \in \mathfrak{g}^* \) is \( G \)-invariant, then \( H = G \). This will be the case if for instance \( G \) is abelian. Under this condition, \( J^{-1}(\mu)/H \) is diffeomorphic to \( T^*Q \) via the projection \( \pi_\alpha \), and the canonical symplectic form on the reduction space pushes forward onto \( T^*Q \). So \( J^{-1}(\mu)/H \) is symplectomorphic to \( T^*Q \) if \( T^*Q \) is equipped with the “effective” symplectic form \( \Omega_{\text{eff}} = \Omega_{T^*Q} + \langle \mu, \Omega_\nabla \rangle \) where \( \Omega_\nabla \) is a two-form on \( Q \) which pulls back to the curvature form \( d\alpha \) on \( N \) [11]. It is with respect to this effective symplectic form that the quantization procedure must be carried out. Since the adjustment is a two form on \( Q \), quantization of \( T^*Q \) using the vertical polarization still gives \( L^2(Q) \). However, Hamiltonian vector fields \( \mathcal{H}_f \), associated with observables \( f \) and the Poisson bracket are defined in terms of the effective form:

\[
\Omega_{\text{eff}}(\mathcal{H}_f, -) = -df, \quad \{f_1, f_2\} = \Omega_{\text{eff}}(\mathcal{H}_{f_1}, \mathcal{H}_{f_2}).
\]

The quantization of observables must preserve Poisson bracket

\[
[\mathcal{Q}(f_1), \mathcal{Q}(f_2)] = i\hbar\mathcal{Q}(\{f_1, f_2\}).
\]

When carry out the geometric quantization with respect to \( \Omega_{\text{eff}} \), the result is the Dirac quantization with \( \alpha \) as the vector potential associated with an electro-magnetic field (cf. [3]).

### III. The Hamiltonian vector fields.

Let us first introduce local canonical coordinates \( (\xi_i, n_i) \) on \( T^*N \) so that \( \Omega_{T^*N} = d\xi_i dn_i \), similarly \( (p_a, q_a) \) on \( T^*Q \) with \( \Omega_{T^*Q} = dp_a dq_a \), and let \( \alpha = A_{si} dn_i \) where \( i = 1 \ldots \dim N \), \( a = 1 \ldots \dim Q \), \( s = 1 \ldots \dim G \), and repeated indices are summed. For each \( n \in N \), let us denote the horizontal lift \( T_q Q \to T_n N \) by the matrix \( M_{ia}(n) \), \( i = 1 \ldots \dim N \), \( \sigma = 1 \ldots \dim Q \). We have

\[
\frac{\partial q_a}{\partial n_i} M_{ib} = \delta_{ab}
\]

and the covariant derivative is the horizontal lift of \( \frac{\partial}{\partial q_a} \):

\[
\mathcal{D}_a = M_{ia} \frac{\partial}{\partial n_i}.
\]

In these coordinates, the canonical one-form and the symplectic two-form of \( T^*N \) can be calculated using (3)

\[
\chi^*\xi_idn_i = \left( p_a \frac{\partial q_a}{\partial n_i} + \mu_s A_{si} \right) dn_i
\]

\[
\chi^*\Omega_{T^*N} = \frac{\partial q_a}{\partial n_i} dp_a dn_i + \mu_s \frac{\partial A_{si}}{\partial n_j} dn_j dn_i + A_{si} d\mu_s dn_i.
\]

\[
\chi^*\Omega_{T^*Q} = \frac{\partial q_a}{\partial n_i} dp_a dn_i + \mu_s \frac{\partial A_{si}}{\partial n_j} dn_j dn_i + A_{si} d\mu_s dn_i.
\]
Let $f : N^\# \times g^* \to \mathbb{R}$ be a pullback function from $T^*Q$, and let $\mathcal{H}_f$ be its Hamiltonian vector field

$$\mathcal{H}_f = B_a \frac{\partial}{\partial p_a} + C_i \frac{\partial}{\partial n_i} + U_s \frac{\partial}{\partial \mu_s}.$$  

Using $\chi^* \Omega_{T^*N} (\mathcal{H}_f, -) = -df$ we get

$$\begin{align*}
\frac{\partial f}{\partial q_a} \frac{\partial q_a}{\partial n_i} &= - B_a \frac{\partial q_a}{\partial n_i} + \mu_s \left( \frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) C_j - A_{si} U_s, \\
\frac{\partial f}{\partial p_a} &= \frac{\partial q_a}{\partial n_i} C_i, \\
\frac{\partial f}{\partial \mu_a} &= A_{si} C_i.
\end{align*}$$

(8)

Since $f$ is invariant with respect to the canonical $G$-action, $\mathcal{H}_f$ is tangent to the subspace $J^{-1}(\mu) \simeq N^\# \times \{\mu\}$, thus $U_s = 0$. As remarked after (4), $f$ is independent of $\mu$, thus $A_{si} C_i = 0$, which implies $\mathcal{H}_f$ is horizontal. Moreover, letting $B_a = -\frac{\partial f}{\partial q_a} + E_a$, we have

$$\mathcal{H}_f = \left[ - \frac{\partial f}{\partial q_a} \frac{\partial}{\partial p_a} + C_i \frac{\partial}{\partial n_i} \right] + E_a \frac{\partial}{\partial p_a}.$$  

The terms in the bracket is the horizontal lift of the Hamiltonian vector field of $f$ with respect to the usual symplectic form $\Omega_{T^*Q}$ on $T^*Q$. $E_a$ satisfies

$$E_a \frac{\partial q_a}{\partial n_i} = \mu_s \left( \frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) C_j.$$  

(9)

So we summarize the properties of $\mathcal{H}_f$ needed for our purpose.

**Proposition 1.** If $f : T^*N \to \mathbb{R}$ is a pullback of a function on $T^*Q$, then for all $\mu \in g$ the Hamiltonian vector field $\mathcal{H}_f$ is a vector field on the subspace $J^{-1}(\mu)$, and as the total space of a principal bundle over $T^*Q$ with connection $\alpha$, $\mathcal{H}_f$ is horizontal. If differs from the horizontal lift of the standard Hamiltonian vector field on $T^*Q$ by a field in the vertical direction. With respect to the local coordinates chosen, we have explicitly:

$$\mathcal{H}_f = \left[ - \frac{\partial f}{\partial q_a} \frac{\partial}{\partial p_a} + M_{ia} \frac{\partial f}{\partial p_a} \frac{\partial}{\partial n_i} \right] + \mu_s M_{ia} M_{jb} \left( \frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) \frac{\partial f}{\partial p_i} \frac{\partial}{\partial p_a}.$$  

(9)

Furthermore, we have

$$\langle \chi^* \xi_i dn_i, \mathcal{H}_f \rangle = p_a \frac{\partial f}{\partial p_a}.$$  

(10)

**IV. Polarization.**

We first state some well known results concerning the quantization of integral coadjoint orbits $O_\mu$. Let $\mathfrak{h}$ be the Lie algebra of $H$, we say that $O_\mu$ is integral if the map

$$v \in \mathfrak{h} \to 2\pi i \langle v, \mu \rangle$$

(11)
is the derivative of a global character, i.e., there is a group homomorphism $\rho_\mu : H \to U(1)$ such that $\rho_{\mu^*}$ is the map given in (11). A version of the Borel-Weil theorem, due to Kirillov [5] and Kostant [6] asserts that there is a one-to-one correspondence between the integral orbits of $G$ and its unitary irreducible representations, and these representations can be constructed by the method of geometric quantization applied to the coadjoint orbit $O_\mu$, which we will briefly explain.

Let $L$ be the prequantization line bundle $G \times_{\rho_\mu} \mathbb{C}$ over $O_\mu \simeq G/H$ with connection induced by the map $\rho_\mu$. It is known that $O_\mu$ is a Kähler manifold with complex coordinates with respect to which the $G$–action is holomorphic. There is a standard $G$–equivariant polarization quantizing with respect to the line bundle and this polarization gives $\mathcal{H}_\mu$ whose elements are holomorphic functions on $O_\mu$.

The polarization, known as the positive Kähler polarization, is given by left translation of a set of $v_k \in g \otimes \mathbb{C}$, the complexification of $g$, so that the polarization is generated by

$$V_k(\text{Ad}_g\mu) = g_*v_k \in T_{\text{Ad}_g\mu}O_\mu.$$  

As a polarization, $v_k$ thought of as vectors in $T_\mu O_\mu$ satisfies

$$\Omega_{O_\mu}(v_h, v_k) = 0$$

where $\Omega_{O_\mu}$ is the canonical symplectic form on $O_\mu$. The specifics of the choices of $v_k$ will not be important in what follows. It is worth mentioning that $V_k$ is contained in the vertical polarization on $T^*N$.

Consider the complex distribution on $N^# \times O_\mu$ generated by $\{\partial_{p_n}, V_k\}$, it is $G$–equivariant thus projects onto $N^# \times_G O_\mu \simeq N^#/H \times \{\mu\}$. One checks that the image is a polarization which we denote by $\mathcal{P}$ [8].

It is easy to represent $\mathcal{P}$ in local coordinates on $N^#$; $\partial_{p_n}$ are vector fields on $N^#$, and $V_k$ corresponds to $\hat{n}_*v_k$ as complex vector fields on $N$, $\hat{n}_*$ as in (1). This is so since the assignment $(p, n, \text{Ad}_g\mu) \mapsto (p, ng, \mu)$ defines the projection $N^# \times O_\mu \to N^# \times_G O_\mu \simeq N^#/H \times \{\mu\}$. Thus vector field generated by $G$–action on $O_\mu$ translates to vector field generated by $G$–action on $N$.

The Hilbert space structure on $L^2(N)$ is given by integration with respect to the measure $\text{d}n = \text{d}z\sqrt{\det(g)}\text{d}q$, where $\text{d}z$ is a Haar measure on $G$ which we transfer to a measure on the fiber in the projection $N \to Q$ and $g$ is the metric on $Q$. Using the half-form bundle formalism the wavefunctions are of the form $\psi(n)\sqrt{\det n}$. It is clear the the Haar measure will play no role in our consideration as the polarization and all Hamiltonians in question are $G$–invariant. To keep the half-form bundle formalism to a minimum, we may identify the wavefunctions as $\psi(n)\det g^{1/4}$. We will determine explicitly the differential operators corresponding to $f$ so that

$$\psi(n)\det g^{1/4} \mapsto [\mathcal{Q}(f)\psi(n)]\det g^{1/4}.$$  

In quantizing $f$ that is linear in the momentum variables, the $\det g^{1/4}$ term will give rise to the covariant divergence, and for $f = \frac{1}{2m}\vert p \vert^2 + V(q)$, it results in the Ricci curvature. The appearance of the Ricci curvature is also reported in [4].

V. Polarization preserving case.

If $f : T^*Q \to \mathbb{R}$ is linear in $p$, $f = K(a\mu)p_a$, then one easily checks, using (9), that $\exp t\mathfrak{h}_f^*\mathcal{P} = \mathcal{P}$. In fact, we have
Proposition 2.

\[
\left[ \mathcal{H}_f, \frac{\partial}{\partial p_a} \right] = \frac{\partial K_a}{\partial q_b} \frac{\partial}{\partial p_b}, \tag{15}
\]

\[
[\mathcal{H}_f, V_k] \in \text{span} \left\{ \frac{\partial}{\partial p_a} \right\}, \tag{16}
\]

where the brackets refer to the Lie algebra bracket on vector fields.

Proof. Equation (15) is by direct computation. We have

\[
\mathcal{H}_f = \left( E_b - p_c \frac{\partial K_c}{\partial q_b} \right) \frac{\partial}{\partial p_b} + K_b M_{ib} \frac{\partial}{\partial n_i} \tag{17}
\]

where \( E_b \) is independent of \( p_a \), and (15) results.

To show (16), we first realize from Proposition 1 that \( \mathcal{H}_f = W_1 + W_2 \) where \( W_1 \) is the horizontal lift of a vector field on \( T^* Q \), thus \([W_1, V_k] = 0\) as \( V_k \) is generated by the group action on \( N \). \( W_2 \) is of the form \( F_a \frac{\partial}{\partial p_a} \). Since \( V_k \) is independent of \( p_a \), \([W_2, V_k] = V_k(F_a) \frac{\partial}{\partial p_a} \), where \( V_k(F_a) \) refers to applying the vector field as a differential operator to the coefficient function \( F_a \). \( \square \)

The importance of (16) is that \([\mathcal{H}_f, V_k] \) is a combination of vectors fields in \( \mathcal{P} \) which does not involve the \( V_h \) vector fields. According to (7.12) of [3], the quantization of \( f \) is then given by:

\[
Q(f)\psi = \left\{ \mathcal{H}_f(f(n) \det g^{1/4}) + \frac{1}{2} \sum_{a=1}^{\dim Q} \left( \frac{\partial K_a}{\partial q_a} (f(n) \det g^{1/4}) \right) \right\}
\]

\[
= -i\hbar \left( \mathcal{H}_f(f) + \frac{1}{2} \left( \frac{1}{\sqrt{\det g}} \mathcal{H}_f(\sqrt{\det g}) + \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a} \right) \right) \tag{18}
\]

The Hamiltonian vector field \( \mathcal{H}_f \) projects to a vector field \( V^\# \) on \( N \), which is the horizontal lift to the projection of \( \mathcal{H}_f \) onto \( Q \) where \( V = K_b \frac{\partial}{\partial q_b} \). Note that \( \mathcal{H}_f(\sqrt{\det g}) = V(\sqrt{\det g}) \). The divergence of the vector field \( V \) on \( Q \) is defined [12] through the relation

\[
d \ast V = \text{div} V \sqrt{\det g} dq
\]

The covariant divergence on \( N \) is defined as the divergence of the horizontal lift \( V^\# \). We have

\[
\text{div} V = \frac{1}{\sqrt{\det g}} V(\sqrt{\det g}) + \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a}.
\]

Using (17), (18) and the fact that \( \psi \) is independent of \( p_a \), we have

Proposition 3. \( Q(f)\psi = -i\hbar (K_a D_a + \frac{1}{2} \text{div} V) \psi \) where \( D \) is the covariant derivative with respect to the connection \( \alpha \).

Since \( \det g \) is a function of \( n \) through \( q \), the divergence and the covariant divergence are the same.
VI. BKS pairing case.

Let \( \mathcal{P} \) and \( \mathcal{P}' \) be transversal polarizations on \( T^*N \), denote their associated quantum spaces by \( \mathcal{Q} \) and \( \mathcal{Q}' \). The BKS pairing gives rise to a map \( B : \mathcal{Q}' \to \mathcal{Q} \) such that

\[
\langle B(\psi), \phi \rangle = \int_{T^*N} \psi \bar{\phi} (\det \omega)^{1/2} d\ell
\]

where \( d\ell \) is the Liouville form \( d\xi_1 \ldots d\xi_n dn_1 \ldots dn_n \). Since the volume form on \( N \) is \( dn = \sqrt{\det g} dn_1 \ldots dn_n \), we have

\[
B \psi(n) = \frac{1}{\sqrt{\det g}} \int_{T^*N} \psi (\det \omega)^{1/2} d\xi_1 \ldots d\xi_n \tag{19}
\]

Let \( f = \frac{1}{2} g^{ab} p_a p_b + V(q) \), where \( g^{ab} \) is the inverse of the metric \( g_{ab} \). From (9) we have

\[
\mathcal{H}_f = - \left( \frac{\partial V}{\partial q_a} + \frac{1}{2} \frac{\partial g^{bc}}{\partial q_a} p_b p_c \right) \frac{\partial}{\partial p_a} + M_{ia} g^{ab} p_b \frac{\partial}{\partial n_i}
\]

\[
+ \mu_s M_{ia} M_{jb} \left( \frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) g^{bc} p_c \frac{\partial}{\partial p_a} ,
\]

and note the linear dependence of the coefficients on the \( p \) variables. We denote by \( P_t = e^{t H_f} P \), here the two polarizations \( P \) and \( P_t \) do not intersect transversely. We claim

**Proposition 4.** Vector fields generated by the group action are in \( \mathcal{P} \cap P_t \).

**Proof.** It suffices to show that

\[
\Omega(\exp t \mathcal{H}_f V_k, \frac{\partial}{\partial p_a}) = 0 \tag{20}
\]

and

\[
\Omega(\exp t \mathcal{H}_f V_k, V_h) = 0 \tag{21}
\]

with \( \Omega \) as in (7). Let \( \bar{p}(p,n,t) \) and \( \bar{n}(p,n,t) \) denote the flow generated by \( \exp t \mathcal{H}_f \) at \((p,n) \in N^\#\) with \( \mu \) fixed, \((\bar{p}, \bar{n}, \mu) = \exp t \mathcal{H}_f (p,n, \mu)\). Then \( \exp t \mathcal{H}_f V_k = V_k(\bar{p}_a) \frac{\partial}{\partial p_a} + V_k(\bar{n}_i) \frac{\partial}{\partial n_i} \). So \( \Omega(\exp t \mathcal{H}_f V_k, \frac{\partial}{\partial p_a}) = \frac{\partial q_a}{\partial n_i} V_k(\bar{n}_i) \). Recall \( V_k = \hat{n}_s v_k \) is vertical and \( \frac{\partial q_a}{\partial n_i} \) is the Jacobian of the projection \( \pi : N \to Q \). Then \( \pi_s \hat{n}_s = 0 \) implies (20) holds.

Equation (21) follows from general principle; Since \( f \) is \( G \)-invariant (with respect to the non-canonical \( G \)-action), \( V_h \) is equivariant with respect to the flow: \( \exp t \mathcal{H}_f V_h(p,n) = V_h(\exp t \mathcal{H}_f (p,n)) \). Since the flow of a Hamiltonian vector field preserves the symplectic form, we have

\[
\Omega(\exp t \mathcal{H}_f V_k(p,n), V_h(\exp t \mathcal{H}_f (p,n))) = \Omega(V_k, V_h) = \Omega_{\mathcal{O}_\mu}(v_h, v_k) = 0,
\]

where \( v_h \) and \( v_k \) belongs to a polarization on \( \mathcal{O}_\mu \) to begin with (13). \( \Box \)

This being the case, quantization of \( f \) via BKS pairing involves integrating only over the \( p \) variables, i.e., the fiber coordinates of the projection \( \Pi : N^\# \to N \).
According to (7.20) of [3], together with the similarity transform (14) adjustment and the adjustment in the BKS pairing described in (19),

\[
Q(f)\psi(n) = \frac{1}{\det g^{1/2}} \frac{1}{\det g^{1/4}} i\hbar \frac{d}{dt} \bigg|_{t=0} \Psi_t(n) \tag{22}
\]

where \( \Psi_t(n) = (i\hbar)^{-\dim Q/2} \int_{\Pi(n)} [\det \omega_{ab}]^{1/2} \exp(i\hbar^{-1} L) \Psi(p, n, t) dp, \)

\[
\Psi(p, n, t) = \psi(\bar{n}(p, n, t)) \times [\det g(\bar{n}(p, n, t))]^{1/4} \tag{23}
\]

\[
\omega_{ab} = \Omega \left( \frac{\partial}{\partial p_a}, \exp \mathcal{H}_{f, \ast} \frac{\partial}{\partial p_b} \right), \tag{25}
\]

\[
L = t \left( \frac{1}{2} ||p||^2 + V(q) \right) - 2 \int_0^t V(\bar{n}(p, n, s)) ds. \tag{26}
\]

The manipulation follows closely that of Sniatycki [3]. Making the substitution \( x_a = tp_a, \) we have results analogous to (7.26) and (7.27) of [3]:

**Proposition 5.**

\[
\lim_{t \to 0^+} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right) = (2\pi\hbar)^{\dim Q/2} e^{\pi i \text{sgn}(g)/4} \sqrt{\det g} \delta(x). \tag{27}
\]

\[
\frac{\partial}{\partial t} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right) = \frac{i\hbar}{2} g_{ab} \frac{\partial^2}{\partial x_a \partial x_b} t^{-\dim Q/2} \exp \left( \frac{i}{\hbar} \frac{||x||^2}{2t} \right). \tag{28}
\]

**Proof.** The first equation follows from the method of stationary phase (cf. [13]), which for \( n \)-dimensional space reads

\[
\int_{\mathbb{R}^n} a(y)e^{ik\phi(y)} dy = \left( \frac{2\pi}{k} \right)^{n/2} \sum_{y:|\phi(y)|=0} e^{\pi i \text{sgn} H(y)/4} \frac{e^{ik\phi(y)} a(y)}{\sqrt{|\det H(y)|}} + O(k^{-n/2-1}).
\]

The \( \frac{1}{2} \) factor in (27) plays the role of the large parameter \( k. \) \( H \) is the Hessian of \( \phi \) which in our case is \( g^{\mu\nu}, \) thus \( \det H = \det g^{-1} \) and \( \text{sgn} \) is the signature of the metric. The only stationary point in (27) is \( x = 0, \) thus the right hand side of (27) has a (\( \dim Q \)-dimensional) delta function at \( x = 0. \)

The second equation is a straight forward computation. We need the fact that \( g_{\mu\nu}g^{\mu\nu} = \delta_{ab}, \) and \( \sum_a \sum_b g_{ab}g^{ab} = \sum_a \delta_{aa} = \dim Q \) in the course of the computation. \( \square \)

One checks that \( \omega_{ab} \) in (29) is

\[
\frac{\partial q_a}{\partial n_i} \frac{\partial \bar{n}_i}{\partial p_b} = tg^{ab} + \text{higher order terms in } t
\]

using (30) below. Then \( [\det \omega_{ab}]^{1/2} dp \sim t^{-\dim Q/2} dx, \) providing us with the needed factor to apply the results of Proposition 5. Thus in determining \( \left. \frac{d}{dt} \right|_{t=0} \Psi_t(n), \) we
need only to consider terms involving \( t, tp_a \) and \( t^2p_ap_b \) while ignoring terms of the form \( t^2p_a \) and all higher order terms. Using (9), the expansion of \( \bar{n} \), expressed in the \( t \) and \( x_a \) variables, up to the relevant terms are

\[
\tilde{n}_i(n, x) = n_i + M_{ia}g^{\mu a}x_\mu + \frac{1}{2} \left[ M_{jb} \left( \frac{\partial M_{ia}}{\partial n_j} g^{\nu b} g^{\mu a} + M_{ia} \frac{\partial g^{\nu b}}{\partial q_b} g^{ba} \right) \right] x_\mu x_\nu
\]

which is independent of \( t \). And \( \Psi_t(n) \) in (23) is reduced to

\[
\Psi_t(n) = (i\hbar)^{-\dim Q/2} \int_{\Pi^{-1}(n)} t^{-\dim Q/2} \exp \left( \frac{i ||x||^2}{\hbar 2t} \right) \Phi(n, x, t) \, dx,
\]

where

\[
\Phi(n, x, t) = \exp(-i\hbar^{-1}tV(q)) \times \psi(\tilde{n}(n, x)) \det g(\tilde{n}(n, x))^{1/4}.
\]

By applying Proposition 5, integration by parts yields

\[
Q(f)\psi(n) = \frac{(-2\pi i)^{\dim Q/2} e^{\pi i sgn/4}}{\det g^{1/4}} \left[ -\frac{\hbar^2}{2} g_{ab} \left. \frac{\partial^2 \Phi}{\partial x_a \partial x_b} \right|_{x, t=0} + i \hbar \left. \frac{d}{dt} \Phi \right|_{x=t=0} \right]
\]

The \( \sqrt{\det H} \) term that appears in the method of stationary phase formula results in a \( g^{1/2} \) factor (27) that cancels with the \( \det g^{1/2} \) on the right hand side of (22).

Since \( n \) is fixed, we can choose a normal coordinate system \([12]\) around \( q = \pi(n) \) so that \( \frac{\partial q^\mu}{\partial q_a} = 0 \) for all \( \mu, \nu \) and \( q_a \) when evaluated at \( n \), \( i.e., at x = t = 0 \). A direct computation shows

\[
\frac{d}{dt} \left. \Phi(n, x, t) = -i\hbar^{-1}V(q)\psi(n) \det g^{1/4} \right|_{x=t=0}
\]

\[
g_{ab} \left. \frac{\partial^2 \Phi}{\partial x_a \partial x_b} \right|_{x=t=0} = \det g^{1/4} \left[ g^{\mu \nu} M_{ij} \frac{\partial M_{i\mu}}{\partial n_j} \frac{\partial \psi}{\partial n_i} + g^{\mu \nu} M_{ij} \frac{\partial^2 \psi}{\partial n_i \partial n_j} + \frac{1}{4} g^{\mu \nu} g_{ab} \frac{\partial^2 g_{\mu \nu}}{\partial q_a \partial q_b} \right]
\]

Here we have made repeated use of the identity \([12\ p.302]\),

\[
\frac{1}{\det g} \frac{\partial \det g}{\partial q_b} = \frac{1}{\det g} \frac{\partial g_{\mu \nu}}{\partial q_b} = g^{\mu \nu} \frac{\partial g_{\mu \nu}}{\partial q_b}.
\]

In normal coordinates, the covariant Laplace operator reduces to

\[
\Delta_\alpha \psi = \frac{1}{\det g^{1/2}} D_\mu (\det g^{1/2} g^{\mu \nu} D_\nu \psi)
\]

\[
= g^{\mu \nu} M_{i\mu} M_{j\nu} \frac{\partial^2 \psi}{\partial n_i \partial n_j} + g^{\mu \nu} M_{j\nu} \frac{\partial M_{i\mu}}{\partial n_j},
\]

\[
(35)
\]
and the Ricci curvature becomes
\[
R = g^{ik}(\partial_i \Gamma^j_{ji} - \partial_j \Gamma^j_{ki} + \Gamma^j_{km} \Gamma^m_{ji} - \Gamma^j_{jm} \Gamma^m_{ki}) \\
= (g^{ik} g^{\mu\nu} - g^{iv} g^{k\mu}) \partial_i \partial_k g_{\mu\nu} \\
= \frac{3}{2} g^{ik} g^{\mu\nu} \partial_i \partial_k g_{\mu\nu} \tag{36}
\]
Here \( \Gamma^k_{ij} \) are the Christoffel symbols, and the identity
\[
\Gamma^m_{ij,k} + \Gamma^m_{jk,i} + \Gamma^m_{ki,j} = 0
\]
is used to show \( g^{iv} g^{k\mu} \partial_i \partial_k g_{\mu\nu} = -\frac{1}{2} g^{ik} g^{\mu\nu} \partial_i \partial_k g_{\mu\nu} \). We must caution the readers that these expressions only hold in normal coordinates. However, by combining (32–36) we can express our final result in an coordinate invariant form:

**Proposition 6.** Quantization of \( f = \frac{1}{2} ||p||^2 + V(q) \) gives
\[
Q(f)\psi(n) = (-2\pi i)^{\text{dim} Q/2} e^{\pi \text{sign}(g)/4} \left( -\frac{\hbar^2}{2} \Delta_\alpha + \frac{1}{6} R \right) + V(q) \right) \psi(n).
\]

We conclude with a final remark. The Yang-Mills field is defined [14] as the curvature \( D\alpha \) of the Yang-Mills potential \( \alpha \), whereas the contribution of this connection in the local expression of the symplectic form is \( d\alpha \). They are related by
\[
D\alpha(v,w) = d\alpha(\text{hor} v, \text{hor} w)
\]
where \( \text{hor} \) denotes the horizontal projection. Since the vector fields of concern are all horizontal, the effect of \( d\alpha \) is equivalent to the curvature.

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References