Canonical Quantization of Cosmological Perturbations in the One Bubble Open Universe

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Faddeev and Jackiw’s method for constrained systems is used to derive a gauge invariant formulation of cosmological perturbations in the one bubble inflationary universe. For scalar perturbations in a flat universe, reduction of the action to the one with a single physical degree of freedom has been derived in the literature. A straightforward generalization of it to the case of an open universe is possible but it is not adequate for quantizing perturbations in the one bubble universe, because of the lack of Cauchy surfaces inside the bubble. Therefore we perform the reduction of the action outside the lightcone emanating from the center of the bubble or nucleation event, where the natural time constant hypersurfaces are no longer homogeneous and isotropic and as a result the conventional classification of perturbations in terms of scalar and tensor modes is not possible. Nevertheless, after reduction of the action we find three decoupled actions for three independent degrees of freedom, one of which corresponds to the scalar mode and the other two to the tensor modes. Implications for the one bubble open inflationary models are briefly discussed. As an application of our formalism, the spectrum of long wavelength gravity waves is simply obtained in terms of the real part of the reflection amplitude for a one dimensional scattering problem, where the potential barrier is given in terms of the bubble profile.

I. INTRODUCTION

Models which reconcile inflation with a non-critical density, \(\Omega \neq 1\), have been recently proposed in the literature [1]. Although these models are somewhat more involved than standard inflation, they may end up being favored by observations [2,3]. In this scenario, one starts with a scalar field trapped in a false vacuum that drives a de Sitter-like phase of inflation. This field undergoes a first order phase transition by forming an \(O(3,1)\) symmetric bubble. Inside the bubble, the scalar field in the new phase slowly rolls down the potential, driving a short second period of inflation. Our observable universe would be contained inside a single bubble, whose symmetry accounts for the observed large scale homogeneity and isotropy. The second period of inflation is needed to solve the entropy problem.

A complete study of cosmological perturbations in open inflation involves the quantization of fields in the presence of a bubble. So far, progress has been made by quantizing the scalar field but ignoring the selfgravity of these fluctuations. [4–8]. The quantization of tensor modes has been considered in [9–14]. Some interesting features have been found. There are some scalar modes - the so called supercurvature modes- which are not normalizable in the open hyperboloids but do contribute to the microwave background anisotropies. Some of these, which correspond to fluctuations of the bubble wall, are found to be such that they can be rewritten as tensor modes, due to their special

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Gravitational perturbations contain gauge degrees of freedom. Perhaps the most elegant way to get rid of the “unphysical” gauge modes is the gauge invariant theory of cosmological perturbations. \[18-21\]. In a universe dominated by a single scalar field, there is only one physical degree of freedom for scalar perturbations. Hence to quantize perturbations, one has to reduce the number of variables in the action to a single variable by using the constraints. This program has been carried out for scalar perturbations in a spatially flat universe \[21-23\] and in a spatially closed universe \[24\]. The final form of the action resembles the one for a scalar field with a time-dependent mass term. One might expect that the extension of this program to the case of an open universe would be straightforward. In fact, it would be so if the hypersurfaces of homogeneity and isotropy of an open universe were Cauchy surfaces on which the canonical commutation relations could be set up. However, in the case of one bubble inflationary scenario, the whole universe is contained in a single bubble and the open hypersurfaces foliate only the interior of the lightcone emanating from the center of the bubble \[25\]. In particular, one cannot deal with supercurvature modes on these hypersurfaces. Hence the quantization should be carried out outside the lightcone where Cauchy surfaces exist but no hypersurface of homogeneity and isotropy exists. The purpose of this paper is to carry out this non-trivial task, i.e., to find the reduced action for cosmological perturbations appropriate for the scenario of the one bubble inflationary universe.

Dirac’s procedure \[26\] has been for a long time the canonical way to treat constrained systems. Faddeev and Jackiw (FJ), however, have proposed \[27\] an alternative approach which leads to the same results without following all of Dirac’s steps. As they point out, two aspects of Dirac’s procedure can be avoided. First, it is not necessary to distinguish between different classes of constraints: all of them can be treated on equal footing without ambiguities. Second, it is not necessary to define conjugate momenta for those velocities which appear linearly in the Lagrangian, as is customary done in Dirac’s approach. Applied to our case, the method gives the linearized action for cosmological perturbations in terms of three gauge invariant degrees of freedom, one corresponding to the scalar mode and two to the tensor modes. This action is then ready for canonical quantization.

The paper is organized as follows. In Section II we describe the method of reduction. In Section III we apply it to cosmological perturbations in open inflation. In Section IV we derive, as an application of our formalism, the spectrum of gravity waves in open inflation. In Section V we summarize our conclusions. Some technical issues are left to the appendices.

II. REDUCTION METHOD

In the Faddeev-Jackiw approach one begins with an action first order in time derivatives,

\[ S = \int \left( a_\mu(\xi, t) \dot{\xi}_\mu + L_0(\xi, z, t) \right) dt, \]  

(2.1)

where \( \xi^\mu \) are the phase space variables of the system, and \( z \) are a subset of those which do not appear in the kinetic term. The basis of the method \[27\] is to use the Euler-Lagrange equations of motion that contain no time derivatives (the real constraints of the theory) to reduce the phase space. The equations of motion of the \( z \) coordinates belong to this category. One starts by solving this set of equations first for as many \( z \)’s as possible and then, if there are any \( z \)’s that appear linearly in the Lagrangian, for as many \( \xi \)’s as possible. After substituting these relations into the original action, it takes the form

\[ S^* = \int \left( f_i(\xi, t) \dot{\xi}_i + L_0^*(\xi, t) \right) dt, \]  

(2.2)

where now the label \( i \) spans fewer coordinates than \( \mu \). From now on, a * means that the known constraints have been substituted.

If there are further constraints in the theory, they manifest themselves as combinations of the equations of motion which contain no time derivatives. Writing the equations of motion from (2.2) in the form

\[ \frac{\delta S^*}{\delta \dot{\xi}_i} = f_{ij} \dot{\xi}_j + \frac{\partial L_0^*}{\partial \xi_j} - \partial_t f_i =: f_{ij} \dot{\xi}_j + G_i(\xi, t) = 0, \]  

(2.3)

where \( f_{ij} := \partial_i f_j - \partial_j f_i \), each zero mode \( \xi^i_0 \) of the kinetic matrix \( f_{ij} \), i.e. \( \xi^i_0 f_{ij} = 0 \), will give us the constraint equation

\[ \xi^i_0 G_i(\xi, t) = 0. \]  

(2.4)

These constraints can be used again to reduce the phase space. The process is repeated until we end up with a nonsingular \( f_{ij} \), which indicates that we have identified the reduced phase space in which the Lagrangian is unconstrained. The equivalence of this method with Dirac’s is discussed e.g. in \[28\].

For our present purpose, we do not have to follow the Faddeev-Jackiw approach step by step. In the problem of cosmological perturbations, the constraints are first class and become the generators of gauge transformations. In
their Poisson brackets are just numbers. Therefore this weak equality can be interpreted as a strong one. For any square-integrable functions $f$, where $\hat{S}$

Now, we can show the gauge invariance of $\hat{C}_\mu$ loses its canonical form but still keeps gauge invariance. This can be deduced from the well known fact that the constraints $\hat{C}_\mu$ are first class: $\{\hat{C}_\mu, \hat{C}_\nu\} = 0$. Then we find

where $\delta \hat{S} = \left. \frac{\delta}{\delta \hat{C}_\mu} \right|_{\hat{C}_\nu = 0} = 0$. \(\text{(2.8)}\)

Writing the gauge transformation of a given variable in terms of operators acting on the gauge parameters, i.e. $\delta Q^n = : \hat{Q}^\dagger_n [\hat{\lambda}]$, gauge invariance can also be written as

which implies that

where $\delta \hat{Q}^\dagger_n$ is the operator conjugate to $\delta \hat{Q}^n$ defined as

for any square-integrable functions $f$ and $g$. Thus, recalling the chain rule, $S^*$ should depend on $q^\mu$ only through the following combination of variables

**In general, this equality holds only in a weak sense. But, in our present problem, $\hat{C}_\mu$ are linear in the variables and hence their Poisson brackets are just numbers. Therefore this weak equality can be interpreted as a strong one.**
It is easily shown that the new set of reduced variables $Q^a$ is gauge invariant. In an actual calculation, all we have to do is just set $q^a = 0$ in $S^*$ and reinterpret $Q^a$ as their gauge invariant counterparts, $Q^a$. In this way, we go from the set of variables $(Q^a, q^a, p_\mu)$ to the reduced set $Q^a$.

Equation (2.10) can be also derived directly from the full Faddeev-Jackiw procedure. If instead of taking advantage of the gauge invariance we followed the method step by step, after substituting the four constraints $C_\mu$ we would find that the kinetic matrix has four zero modes. The constraint equations (2.4) for these zero modes turn out to be

$$\frac{\delta S^*}{\delta q^\mu} + \delta Q^a_\mu \frac{\delta S^*}{\delta Q^a_i} = 0,$$

but, as we have seen, due to the gauge invariance, the left-hand side vanishes identically. As explained, these identities point out that the action can be written in terms of the reduced gauge invariant set of variables (2.12).

### III. OPEN INFLATION

As mentioned in the introduction, the interior of an O(3,1) symmetric bubble is isometric to an open Friedmann-Robertson-Walker (FRW) universe. Unfortunately, the hypersurfaces of homogeneity and isotropy of this universe are not appropriate for setting canonical commutation relations or normalizing modes [25], because they are not Cauchy surfaces for the whole space time. Therefore, we shall need to quantize on spacelike hypersurfaces which cut right through the bubble, and which are therefore not homogeneous. This renders the decomposition into scalar, vector and tensor modes into a somewhat unfamiliar form. In the end, however, the three standard physical degrees of freedom will be identified.

The open FRW chart

$$ds^2 = -dt^2 + a(t)^2 d\Omega_H^3, \quad d\Omega_H^3 = dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \varphi d\varphi^2),$$

(3.1)

covers only the interior of the lightcone emanating from the center of the bubble $t = 0$, $r = 0$, which we shall call the nucleation event $N$. Following [5], we shall call the interior of the lightcone region $R$ and the outside of this lightcone region $C$. Region $C$ can be covered by analytically continuing the coordinates $t$ and $r$ in region $R$ to the complex plane. By taking $t = \tau + i$ and $r = \chi + \tau (\pi / 2)$, with $\tau$ and $\chi$ real, the line element becomes

$$ds^2 = d\tau^2 + a_E(\tau)^2 d\Omega_{ds}, \quad d\Omega_{ds} = \gamma^{ij}_{ds} dx^i dx^j = -d\chi^2 + \cosh^2 \chi (d\theta^2 + \sin^2 d\varphi^2),$$

(3.2)

where $a_E(\tau) = -i a(i \tau)$, and $d\Omega_{ds}$ is the metric of a (2+1) dimensional de Sitter space. In this chart, $\tau$ is a ‘radial’ spacelike coordinate, whereas $\chi$ is timelike. Now the spacelike hypersurface $\chi = 0$ is a Cauchy surface for the entire space-time [25]. It is convenient to introduce the conformal ‘radial’ coordinate $\eta_E$, with $d\tau = -a_E d\eta_E$. Close to the lightcone emanating from the nucleation event $N$, the scale factor behaves as $a_E(\tau) \approx \tau$, and $\eta_E \rightarrow +\infty$. As we move away from $N$ along the $\tau$ direction, the scale factor rises to a maximum and then decreases again, reaching another zero at the so called antipodal point $A$, which corresponds to $\eta_E \rightarrow -\infty$.

Writing the perturbed line element and the perturbed scalar field in the form

$$ds^2 = a_E(\eta_E)^2 \{ (1 + 2A) d\eta_E^2 - 2S_{ij} dx^i dx^j + (\gamma^{ij}_{ds} - h_{ij}) dx^i dx^j \},$$

(3.3)

the second order action for small perturbations is given by

$$\delta_2 S = -\frac{1}{2\kappa} \int d^4 x a_E^2 \sqrt{-\gamma^{ds}} \{-2(2H^2 + H')A^2 + (S_{ij}) - \frac{1}{2} h_{ij}\}^2 - \frac{1}{4} (2h^{ij}k_{ij}k_{ij} + 2h^{ij}k_{ij}k_{ij} + h^{i}j k_{i}j k_{ij}) + A^i (h_{ij} - h_{ij}) \}
+ 2\kappa (\psi^{\prime}_0 \delta \varphi A - a_E^2 V_{\varphi} \delta \varphi A)
- 4(S_{ij} - \frac{1}{2} h_{ij})(\frac{1}{2} \varphi^{\prime}_0 \delta \varphi - \mathcal{H} A)(2A^2 + 2Ah_{ij} - \frac{1}{2} h_{ij}^2)\langle \psi^{\prime}_0 \delta \varphi - \mathcal{H} A),
\quad (3.4)$$

where $\kappa = 8\pi G$, $\mathcal{H} = \dot{a}/a$ and a prime denotes a derivative with respect to the conformal ‘radial’ coordinate $\eta_E$. This can be found e.g. from Appendix A just replacing $a^2 \rightarrow a_E^2$ and $\gamma_{ij} \rightarrow -\gamma^{ds}_{ij}$ in expression (A8).

As mentioned before, the usual expansion of the metric perturbations (see e.g. B1) in scalar, vector and tensor modes with respect to the 3-hyperboloid, used inside the lightcone, cannot be used outside. The reason is that, in region $C$, the corresponding 3-hyperboloid on which these harmonics of various types are defined no longer gives a spatial section of the spacetime. Instead we shall expand in scalar and vector modes with respect to the 2-sphere.
Using a conformal time-like coordinate $\rho$ defined through the relation $\cosh \chi d\rho = d\chi$, the metric element for the de Sitter space can be written as

$$ds^2_{\text{dS}} = c_E(\rho)^2(-d\rho^2 + \omega_{AB}dx^Adx^B),$$

(3.5)

with $c_E(\rho) \equiv \csc \rho$, and where $A$, $B$, ... run over $\theta$ and $\varphi$. For convenience, we define $h_E = \dot{c}_E/c_E$, where a dot indicates derivative with respect to the time-like variable $\rho$.

We write the metric perturbations as

$$S_\rho = -S,$$
$$S_\phi = T_{||A} + V_A,$$
$$h_{\rho\rho} = 2c_E^2\zeta,$$
$$h_{\rho A} = -c_E^2(\xi_{||A} + W_A),$$
$$h_{AB} = c_E^2((w + (2)\triangle)v_{AB} - 2v_{||AB} + 2F_{(A||B)}),$$

(3.6)

where $||A$ and $(2)\triangle$ stand for the covariant derivative and scalar Laplacian associated with $\omega_{AB}$, respectively. The meaning of $(2)\triangle$ when operates on a vector or tensor quantity is explained immediately below. The fields $S$, $T$, $\zeta$, $\xi$, $V$ and $v$ are scalar modes, and $V_A$, $W_A$ and $F_A$ are divergenceless vector modes with respect to the 2-sphere metric $\omega_{AB}$. More explicitly, we can write, say, $S = \sum_l m S^{lm}(\rho)Y^{lm}(\Omega)$ for scalar modes and $V_A = \sum_l \epsilon^{lm}v^{lm}(\rho)\epsilon_{A}^{\ B}Y^{lm}(\Omega)$ for vector modes, where $Y^{lm}(\Omega)$ are the ordinary spherical harmonics which satisfy $(2)\triangle Y^{lm}(\Omega) = -l(l+1)Y^{lm}(\Omega)$, $\epsilon_{AB}$ is the unit anti-symmetric tensor on the unit 2-sphere ($\epsilon_{\theta\varphi} = \sin \theta$ etc.) and $\epsilon_{A}^{\ B} = \omega^{BC}\epsilon_{CA}$. Now we can clearly state the meaning of $(2)\triangle$. It should be understood just as $-l(l+1)$ when it is decomposed into modes. In order to avoid writing the summation over $l$ and $m$ for notational simplicity, we use $(2)\triangle$ instead of $-l(l+1)$.

The scalar and vector modes transform differently under the parity transformation. Thus, inserting the decomposition (3.6) into the action (3.4), the scalar and vector modes decouple, so they evolve independently. According to the change of signature under the parity transformation, we refer to the scalar (vector) modes as even (odd) parity modes.

We shall see that the odd parity modes contain one physical degree of freedom, which corresponds to odd parity tensor modes when analytically continued inside the lightcone [10, 11]. The even parity modes contain two degrees of freedom, one corresponding to the usual scalar and the other to even parity tensor modes.

### A. Odd parity modes

First we consider the odd parity modes. The result provided in this subsection is essentially the same as that given in [11]. However, our present approach based on the FJ method is quite different from the conventional Dirac’s method used in the previous work [11]. Furthermore, the analysis of the even parity modes discussed in the next subsection is rather complicated compared with the odd parity modes. So, also to understand our strategy, it will be convenient to present the analysis of the odd parity modes first.

The Lagrangian density for odd parity modes is

$$\nu \mathcal{L} = \frac{a_E^2\sqrt{\omega}}{4c_E\kappa}\left\{ (\dot{V}_A - 2hEV_A + c_E^2W_A)^2 + (V^A - c_E^2F^A)^{(2)\triangle} + 2)(V_A - c_E^2F_A') \right. $$
$$-c_E^2(\dot{F}^A + W^A)(^{2)\triangle} + 2)(\dot{F}_A + W_A) \right\}.$$  

(3.7)

By the definition of conjugate momenta, we find

$$\Pi^A_V := \frac{\partial \nu \mathcal{L}}{\partial \dot{V}_A} = \frac{a_E^2\sqrt{\omega}}{2c_E}\left( V^A - 2hEV^A + c_E^2W^A \right),$$
$$\Pi^A_F := \frac{\partial \nu \mathcal{L}}{\partial \dot{F}_A} = \frac{a_E^2c_E\sqrt{\omega(3)\triangle} + 2)(\dot{F}^A + W^A).}$$

(3.8)

Here we have raised indices with $\omega^{AB}$. If we cast it into first order form we obtain

$$\nu \mathcal{L} = \nu \mathcal{L}_1 - \nu \mathcal{H} - c_E^2W_A,$$
$$\nu \mathcal{L}_1 = \Pi^A_V \dot{V}_A + \Pi^A_F \dot{F}_A,$$
$$\nu \mathcal{H} = -\Pi^A_F \frac{\kappa}{a_E^2c_E\sqrt{\omega(3)\triangle} + 2)(\Pi_{FA} + \frac{\kappa c_E}{a_E^2\sqrt{\omega}}\Pi^A_F \Pi_{FA} + 2h_E\Pi^A_V \Pi_{FA}$$

(3.9)
These equations admit the following interpretation. The first tells us that
\[ Y = \sum \pi \] where \( C_W^A = -\Pi^A + c_E^2 \Pi^A' \).

To find the reduced phase space of the Lagrangian (3.9), we solve \( C_W^A = 0 \) for \( \Pi^A \), and substitute it back into the Lagrangian. Using the fact that \( C_W \) is the generator of odd parity gauge transformations (see Appendix E), the prescription given in (2.12) indicates that the gauge invariant combinations are given by \( V_A := a_E^2 \sqrt{\omega} (V_A - c_E^2 F_A') \) and \( \Pi^A := \Pi^A/(a_E^2 \sqrt{\omega}) \). The canonical first order Lagrangian for this degree of freedom is:
\[
(v) \mathcal{L} = \Pi^A V_A + \kappa c_E a_E^2 \sqrt{\omega} \left( c_E^2 \Pi^A + \frac{1}{(\triangle + 2)} \Pi^A - \Pi^A \left( \frac{\lambda^2 + 2 + 2c_E^2 \mathcal{H}'}{(\triangle + 2)^2} \right) \Pi^A \right)
- 2h_E \Pi^A V_A + \frac{V^A(\lambda^2 + 2) V_A}{4 \kappa c_E a_E^2 \sqrt{\omega}}. \tag{3.10}
\]

Solving for the velocity \( \dot{V}_A \), we find the second order reduced action for \( \Pi_A \). It is convenient to express the divergenceless vector as \( \Pi_A := \Pi|_{\epsilon_{BA}} \). With this we obtain
\[
(v) S^{(2)} = 2\kappa \int \frac{\sqrt{-g(4)}}{2} \Pi \left( \frac{\lambda^2 + 2}{\Lambda^2 + 2} \left( \Box + \frac{2\mathcal{H}'}{a_E^2} \right) \right) \Pi d^3x
= 2\kappa \int \frac{\sqrt{-dS}}{2} (a_E \Pi) \left( \frac{\lambda^2}{\Lambda^2 + 2} \left( ds \Box - 1 - \hat{R} \right) \right) (a_E \Pi) d\eta_E d^3x, \tag{3.11}
\]
where \( \Box \) stands for the four dimensional d’Alembertian, \( dS \Box \) is the d’Alembertian on the (2+1) dimensional de Sitter space, and \( \hat{R} \) is the operator defined as
\[
\hat{R} := -\frac{d^2}{d\eta_E^2} + \frac{\kappa}{2} \eta_E^2. \tag{3.12}
\]
The above action is really very simple when expanded in eigenmodes. If one absorbs the factor
\[
\frac{2\kappa \lambda^2}{\Lambda^2 + 2} = \frac{2\kappa \ell(\ell + 1)}{(\ell + 2)(\ell - 1)} = \frac{1}{(N^\ell)^2}, \tag{3.13}
\]
by redefinition of \( \Pi \), we basically obtain the action for an ordinary scalar field, \( \hat{\Pi} \), living in the curved background describing the bubble geometry, with an \( \eta_E \)-dependent mass term.

We decompose the field \( \Pi \) into modes as
\[
\Pi = \sum_{p \ell m} N^\ell \pi_{p \ell m} U^{p \ell m}_T(x), \tag{3.14}
\]
where \( \pi_{p \ell m} \) is the coefficient which represents the amplitude and \( U^{p \ell m}_T(x) \) is a suitably normalized mode function which is also an eigenfunction of the operators \( \hat{R} \) and \( \Lambda^2 \). Then the renormalized field \( \hat{\Pi} \) is defined by \( \hat{\Pi} := \sum_{p \ell m} \pi_{p \ell m} U^{p \ell m}_T(x) \). It is convenient to normalize \( U^{p \ell m}_T(x) \) by means of the Klein-Gordon norm with respect to the renormalized field \( \hat{\Pi} \). With this choice of normalization, when we go to the quantum theory by setting the canonical commutation relation between the operator counter part of \( \Pi \) and its conjugate, \( \pi_{p \ell m} \) can be recognized as the anihilation operator which satisfies \( [\pi_{p \ell m}, \pi_{p' \ell m}^\dagger] = 1 \). Writting \( U^{p \ell m}_T \) in the form,
\[
U^{p \ell m}_T(x) = a_E^1 u^p(\eta_E) \gamma^{p \ell m}(x'), \tag{3.15}
\]
the equation of motion separates into
\[
dS \Box \gamma^{p \ell m} = (p^2 + 1) \gamma^{p \ell m}, \tag{3.16}
\]
\[
\hat{R}[u^p] = p^2 u^p. \tag{3.17}
\]
These equations admit the following interpretation. The first tells us that \( \gamma^{p \ell m} \) behave as scalar fields of mass \( (p^2 + 1) \) living in a (2+1) dimensional de Sitter spacetime. The spectrum of masses is determined by (3.17), which is a one
On the other hand, the explicit expression for \( \rho_A^F \), which is independent of \( P \), where \( \epsilon_{\ell m} \) is the standard one for the Schrödinger problem. For definiteness, here we choose then the (3+1) normalization condition reduces to

\[
\int_{-\infty}^{\infty} u'^2 \tilde{u}^2 \, d\eta_E = \delta(p - p'),
\]

which is the standard one for the Schrödinger problem. For definiteness, here we choose

\[
\gamma_{\ell m} := \frac{\ell+1}{\sqrt{2}} \mathcal{P}_{\ell m}(\rho) \gamma_{\ell m}(\Omega),
\]

where \( \mathcal{P}_{\ell m} \) is defined by using the associated Legendre function of the first kind as

\[
\mathcal{P}_{\ell m}(\rho) := \frac{P_{\ell m}(i \hbar_E)}{\sqrt{\kappa_E}}.
\]

Notice that the factor \( (\ell+2) \Delta + 2 \) becomes zero when \( \ell = 1 \). In this case, we have to go back to the original Lagrangian (3.7). From Eq. (3.8) we find \( \Pi^E_{\ell m} = 0 \), which means that one extra constraint arises. Therefore there remain no physical degrees of freedom for \( \ell = 1 \) mode. In fact, this case can be quickly treated along the lines of Faddeev-Jackiw approach. Substituting \( (\ell+2) \Delta + 2 = 0 \) into the Lagrangian (3.7), and casting it into first order form, we obtain

\[
(\nu) \mathcal{L}_{\ell=1} = \Pi^{\nu}_V \dot{V}_A + \frac{\kappa_E}{a_E \sqrt{\omega}} \Pi^{\nu}_{\ell m} \Pi_{\ell m} - 2 \hbar_E V_A \Pi^{\nu}_{\ell m} - c^2_E W_A \Pi^{\nu}_{\ell m},
\]

which is independent of \( F_A \). The variation with respect to \( W_A \) gives the constraint \( \Pi^{\nu}_{\ell m} = 0 \). Then the normalizability of the mode functions requires \( \Pi^{\nu}_{\ell m} = 0 \). After substituting this constraint, the equation of motion for \( V_A \) becomes also a constraint, which enforces the Lagrangian for \( \ell = 1 \) to vanish. Modes with \( \ell = 0 \) are also absent from the action by construction. The absence of modes with \( \ell = 0, 1 \) is what we expect, because the odd parity mode represents one of the tensor degrees of freedom inside the lightcone, for which the modes \( \ell = 0, 1 \) do not exist.

Now we relate the quantities in the outside of the lightcone with those in the inside of it. Inside the lightcone we can use the tensor harmonics to decompose the tensor part of the metric perturbation into modes. Thus the mode function \( U_{\nu \ell m} \) defined in Appendix B [11] will be the most convenient choice of the variable to specify the tensor perturbation there. In order to relate the amplitude \( \pi_{\nu \ell m} \) to \( U_{\nu \ell m} \), we compare the \( (\rho \Lambda) \)-component of the metric perturbation in the synchronous gauge \( (V_p = 0) \). Following the notation in Appendix D, we associate a subscript (or superscript) \( N \) to indicate the quantity evaluated in this gauge. From Eq. (3.8), \( h^N_{\nu \rho \Lambda} \) is evaluated as

\[
h^N_{\nu \rho \Lambda} = -2\kappa_E \Pi^{\nu}_{\ell m} = \sum_{\rho \ell m} \frac{2\kappa_E N^{\nu}_{\rho \ell m}}{a_E} \epsilon_A B U_{\nu \ell m}^B u^B.
\]

On the other hand, the explicit expression for \( \rho \Lambda \)-component of the tensor harmonics is given in [29,11]. After the analytic continuation to region C, for the odd parity gravitational wave perturbation, we obtain

\[
h^N_{\rho \Lambda} = \sum_{\rho \ell m} \sqrt{\frac{(\ell-1)(\ell+2)\Gamma(ip + \ell + 1)\Gamma(-ip + \ell + 1)}{2p^2(p^2 + 1)\Gamma(ip)\Gamma(-ip)}} c_E \mathcal{P}_{\rho \ell m} \epsilon_A B Y_{\rho \ell m}^B U_{\rho \ell m}^B.
\]

Hence we find that the amplitude \( \pi_{\rho \ell m} \) is related to the variable \( U_{\rho \ell m} \) by
\[
\pi_{p\ell m} u^p = \frac{1}{\sqrt{2\kappa} \ell + 1} \sqrt{\frac{\Gamma(-ip + \ell + 1)}{(p^2 + 1) \Gamma(ip + \ell + 1) \Gamma(ip) \Gamma(-ip)}} \frac{dU^{-}_p}{d\eta^E}.
\] (3.25)

Conversely, by using the equation satisfied by \(U^{-}\) of ref [11],
\[
\left( \frac{1}{a_E^2} \frac{d}{d\eta^E} + (p^2 + 1) \right) U^{-} = 0,
\] (3.26)

\(U^{-}\) is expressed in terms of \(\pi\) as
\[
U^{-}_{p\ell m} = -\sqrt{2\kappa} \ell + 1 \sqrt{\frac{\Gamma(ip + \ell + 1) \Gamma(-ip)}{(p^2 + 1) \Gamma(-ip + \ell + 1) \Gamma(ip)}} \frac{d(a_E u^p)}{d\eta^E}.
\] (3.27)

B. Even parity modes

The even parity modes contain two dynamical degrees of freedom. One of them is the usual scalar mode, and the other is the even parity tensor mode discussed in [11]. After lengthy algebra, complicated by the fact that the spatial sections are not homogeneous outside the lightcone from the nucleation event, the Lagrangian can be cast into second order form as the sum of a Lagrangian for the scalar mode plus a Lagrangian for the even parity tensor mode. The details of the reduction of the action are given in Appendix C. Here we only discuss the meaning of the final results.

For the scalar part, we have
\[
S^{(2)}_q = \frac{1}{2} \int \sqrt{-g} d^3 x \left\{ \frac{dS^2}{d\eta^E} + 3 \left( \hat{\mathcal{O}} q \right)^2 \right\} q d\eta^E d^3 x,
\] (3.28)

where we have introduced the Schrödinger-like operator
\[
\hat{\mathcal{O}} := -\frac{d^2}{d\eta^E^2} + \frac{\kappa}{2} \varphi'^2 + \varphi' \left( \frac{1}{\varphi'} \right)''.
\] (3.29)

where \(d^3 x\) stands as before for the d’Alembertian on the (2+1) dimensional de Sitter space of unit radius. The variable \(q\) is related to the gauge invariant potential \(\Phi_H\) of Bardeen [18] when evolved to the outside of the lightcone (see Appendix D).††

\[
q = \frac{2 a_E}{\kappa \varphi_0} \Phi^E.
\] (3.30)

Putting \(q = a_E \sum \mathcal{N}^p q_{p\ell m} U^p_{S\ell m}(x)\) with the mode function of the form \(U^p_{S\ell m} = a_E^{\frac{1}{2}} q^p(\eta^E) \mathcal{Y}^{p\ell m}(x^i)\), the equation of motion separates into (3.16) and
\[
\hat{\mathcal{O}}[q^p] = (p^2 + 4) q^p.
\] (3.31)

Just as in subsection III A, the masses \((p^2 + 1)\) of the (2+1) dimensional fields \(\mathcal{Y}^{p\ell m}\) are determined as the eigenvalues of the Schrödinger equation (3.31). Now, if we absorb the factor
\[
\hat{\mathcal{O}} = p^2 + 4 = \frac{1}{(\mathcal{N}^p)^2},
\] (3.32)

by defining \(\hat{\mathcal{O}} := \sum q_{p\ell m} U^p_{S\ell m}\), we obtain the action for an ordinary scalar field. As before, we require that \(U^p_{S\ell m}\) is normalized with respect to the Klein-Gordon norm for the renormalized field \(\hat{\mathcal{O}}\). This normalization condition reduces to

††The potential \(\Phi_H\) is given by \(\Phi_H := -h + (B - E')\) (see Appendix B). We also recall that \(\Phi_H\) is equal to \(\Phi\) of Kodama and Sasaki [19] and to \(-\Psi^{(g)}\) of Mukhanov, Feldmann and Brandenberger [21].
Since the potential of the operator $\hat{O}$ is not positive definite, we cannot determine the spectrum of $p^2$ unless we solve Eq. (3.31). If the spectrum obtained previously by ignoring degrees of freedom of the metric perturbations [5] does not change (except for the wall fluctuation mode at $p^2 = -4$; see below), the spectrum will be continuous for $p^2 > 0$ and there may be one discrete mode at $-1 < p^2 < 0$. We will discuss this issue in a forthcoming paper [30].

Note that for the modes with $p^2 = -4$ we have $\hat{O}_0[q] = 0$. If one ignores the metric perturbations, these correspond to the wall fluctuation modes [5, 15–17]. Once the metric perturbations are taken into account, however, the wall fluctuation modes are found to be contained in the continuous spectrum of the gravitational wave perturbations [13]. Hence we expect the modes with $p^2 = -4$ cease to contribute to physical fluctuations. In fact, there is a strong evidence that this is true by examining the regularity of the metric perturbations due to these modes [11]. Unfortunately, however, we have no rigorous proof for it. One possible (and probably most reasonable) standpoint is to require the square integrability of the mode functions, otherwise integration by parts cannot be performed. Then we find that the discrete modes at $p^2 = -4$ should be excluded from the spectrum. This can be seen as follows.

For a while, we neglect a positive definite term $\kappa \varphi_0'^2 / 2$ in the operator $\hat{O}$. Then Eq. (3.31) for $p^2 = -4$ becomes

$$\hat{O}_0[q] = 0, \quad \hat{O}_0 := -\frac{d^2}{d\eta_E^2} + \varphi_0' \left( \frac{1}{\varphi_0'} \right)''.$$

(3.34)

The two independent solutions are easily found as

$$q_1 = \frac{N_1}{\varphi_0'}, \quad q_2 = \frac{N_2}{\varphi_0'} \int^{\eta_E} \varphi_0'^2 d\eta_E,$$

(3.35)

where $N_1$ and $N_2$ are constants. One readily sees that $q_1$ is badly divergent at $\eta_E = \pm \infty$ since $\varphi_0'$ vanishes there (this is necessary for regularity of the instanton). As for $q_2$, it is regular at $\eta_E = -\infty$ by construction but diverges at $\eta_E = \infty$. Thus there is no square-integrable solution in the limit of weak gravitational coupling. Now, since the solution $q_2$ has no node, the lowest eigenvalue $p_0^2 + 4$ of the operator $\hat{O}_0$ will be greater than zero. Then since the term we have neglected from $\hat{O}$ is positive definite, the lowest eigenvalue of Eq. (3.31) will be greater than $p_0^2 + 4$, which is positive definite. Hence we conclude that no square-integrable solution of Eq. (3.31) exists at $p^2 = -4$.

If the operator $\hat{O}$ in front of $q$ were absent from the action (3.28), the above fact would be sufficient to exclude the modes with $p^2 = -4$. However, the action for these modes seems to be an indeterminacy of the form zero times infinity. On one hand, $\hat{O}$ vanishes, but on the other, $q$ are not integrable on the spacelike surface $\chi = 0$. As we have mentioned, however, there are evidences that they do not contribute to physical fluctuations. Hence it seems reasonable to accept the square integrability as the guiding principle and exclude these special modes from the spectrum.

For the tensor part, we have

$$S^{(2)}_{\omega} = 2\kappa \int \frac{\sqrt{-\gamma} dS}{2} \left( \hat{K} \omega \right) \left\{ \frac{ds}{\ell^2} - 1 - \hat{K} \right\} w d\eta_E d^3 x$$

(3.36)

where $\hat{K}$ is the operator defined in Eq. (3.12). If we expand $\omega$ by means of the eigenfunction of $\hat{K}$, the operator $\hat{K}$ can be replaced with the corresponding eigenvalue. Then we can absorb the factor

$$\frac{2\kappa \hat{K}}{\ell^2} \left( \left( \triangle \right)^{2} + 1 \right) = \frac{2\kappa p^2}{(\ell - 1)(\ell + 1)(\ell + 2)} =: \frac{1}{(N^{\mu}_{+})^2},$$

(3.37)

by redefinition of variable and we obtain the action for an ordinary scalar field. As before $\omega$ is decomposed as $\omega = a_E \sum N^{\mu}_{+} w_{\mu \nu} U_{\nu \mu}$. As in the case of odd parity, the spectrum is purely continuous, with $p^2 > 0$.

Now we relate $w_{\mu \nu}$ with the mode function $U_{\mu \nu}^{(+)}$ [11] defined in the inside of the lightcone (See Appendix B). In order to relate the amplitude $w_{\mu \nu}$ to $U_{\mu \nu}^{(+)}$, we focus on the traceless part of the $(AB)$-component of the metric perturbation, $v$, in the synchronous gauge $(A = S = T = 0)$. As before, we associate a subscript (or superscript) $N$ to indicate the quantity evaluated in this gauge.

From Eqs. (C17), (D8) and (D9) and with the aid of the equation

$$\omega = \frac{(2) \triangle \left( \left( \triangle \right) + 2 \right) \Pi \omega}{2\kappa E c T \sqrt{\omega}},$$

(3.38)
which follows from the reduced Lagrangian (C18), \( v_N' \), is evaluated as

\[
v_N' = -\frac{\kappa N^p}{a_E c_E^2 (2 \Delta)^2 (2 \Delta + 2)} \left( 2h_E \partial_\rho + \left( \frac{\partial}{\partial \rho} \right)^2 + 2h^2 - 2c_E^2 R \right) Y^{p \ell m} w_{p \ell m} u^p
\]  

(3.39)

On the other hand, using the expression for the tensor harmonics given in [29,11], after the analytic continuation to region C, we have

\[
v_N = -\sum_{p \ell m} c_E^{-2} \sqrt{\frac{2 \Gamma(ip + \ell + 1) \Gamma(-ip + \ell + 1)}{p^2 (p^2 + 1) \Gamma(ip) \Gamma(-ip)}} \times \left( 2h_E \partial_\rho + (-\ell(\ell + 1) + 2h^2 - 2p^2 c_E^2) \right) P_{p \ell} Y^{\ell m} u^{(+) \ell m}.
\]  

(3.40)

Hence, we find that the variable \( w \) is related to the variable \( U^{(+) \ell m} \) through the relation

\[
w_{p \ell m} u^p = \frac{1}{\sqrt{2 \kappa \ell^2 + 1}} \sqrt{\frac{\Gamma(-ip + \ell + 1)}{(p^2 + 1) \Gamma(ip) \Gamma(-ip)}} \frac{a_E}{a} \frac{dU^{(+) \ell m}}{d\eta_E}.
\]  

(3.41)

As in the odd parity case, using the equation for \( U^{(+) \ell m} \) which is the same as for \( U^{(-) \ell m} \), Eq. (3.26), the inverse relation is given by

\[
U^{(+) \ell m} = -\sqrt{2 \kappa \ell^2 + 1} \sqrt{\frac{\Gamma(ip + \ell + 1) \Gamma(ip) \Gamma(-ip)}{(p^2 + 1) \Gamma(-ip + \ell + 1)}} \frac{a_E^{-2}}{a} w_{p \ell m} \frac{d(a_E u^{(+) \ell m})}{d\eta_E}.
\]  

(3.42)

IV. SPECTRUM OF GRAVITY WAVES

As an application of our formalism, let us find the spectrum of long wavelength tensor modes predicted in open inflationary models. This reduces to solving the scattering problem for the Schrödinger equation (3.17). The potential for this problem vanishes at both \( \eta_E \rightarrow -\infty \) and \( \eta_E \rightarrow \infty \), so the asymptotic behavior at \( \pm \infty \) of the two orthogonal solutions \( u_{(\pm)}^p \) for the energy \( p^2 \) can be taken just as incident plane waves from \( \pm \infty \) with momentum \( p \) which interact with the potential and produce reflected waves which return to \( \pm \infty \) with reflection amplitude \( \sigma_\pm \), and transmitted waves moving to \( \mp \infty \) with transmission amplitude \( \varrho_\pm \). That is, in the limit \( \eta_E \rightarrow \pm \infty \), the \( u_{(\pm)}^p \) are given by

\[
u_{(\pm)}^p = \begin{cases} \frac{1}{\sqrt{2 \pi}} \varrho_+ e^{ip\eta_E} + e^{-ip\eta_E} & (\eta_E \rightarrow +\infty), \\ \frac{1}{\sqrt{2 \pi}} \sigma_+ e^{-ip\eta_E} & (\eta_E \rightarrow -\infty), \end{cases}
\]  

(4.1)

and

\[
u_{(-)}^p = \begin{cases} \frac{1}{\sqrt{2 \pi}} \sigma_- e^{ip\eta_E} & (\eta_E \rightarrow +\infty), \\ \frac{1}{\sqrt{2 \pi}} \varrho_- e^{-ip\eta_E} + e^{ip\eta_E} & (\eta_E \rightarrow -\infty). \end{cases}
\]  

(4.2)

Here we note that \( p \) is non-negative. Using the Wronskian relations we obtain

\[
|\sigma_+|^2 = 1 - |\varrho_+|^2, \quad |\sigma_-|^2 = 1 - |\varrho_-|^2,
\]  

(4.3)

\[
\sigma_- = \sigma_+, \quad \sigma_+ + \sigma_- + \rho_+ = 0.
\]  

(4.4)

Using Eq. (4.4), we can show that

\[
\int d\eta_E u_{(\pm)}^p u_{(-)}^p = 0,
\]  

(4.5)

and hence \( u_{(\pm)}^p \) and \( u_{(-)}^p \) are orthogonal. Note that normalization condition (3.19) is satisfied because
Analytically continuing the solution inside the lightcone by means of \( \eta_E = -\eta_R - i\pi/2 \), where \( \eta_R \) is the conformal time in region \( \mathcal{R} \), the amplitude of perturbations well after the modes have crossed the horizon (\( \eta_R \to 0 \)) is given by

\[
|u_{(+)}^p|^2 + |u_{(-)}^p|^2 = |e^{\pi p/2}\sigma_+|^2 + |e^{\pi p/2}\varrho_+ + e^{-\pi p/2}|^2
= \frac{1}{\pi} (\cosh \pi p + R \varrho_+). \tag{4.7}
\]

As we can see from the equation above, the bubble manifests itself in the spectrum through the real part of the reflection amplitude of the Schrödinger problem.

In the thin-wall approximation, we can take the interior of the bubble as a pure de Sitter space, with scale factor given by \( a_E = 1/(H \cosh \eta_E) \) outside the lightcone from \( \mathcal{N} \). In this limit, we can integrate out the potential in (3.17) and express it as a delta function with strength \( \Delta s = \kappa \mu R_{U}/2 \), where \( \mu = \int_{0}^{\infty} a_{E}^{-1}\varphi_0^2 \, d\eta_E \) and \( R_{U} = a_{E}(\eta_w) \) are the surface tension and the radius of the wall, respectively. The reflection amplitude for this delta function potential is \( \varrho_+ = -ie^{-2ip\eta_w} \Delta s/(2p + i \Delta s) \). Using Eq. (3.42) with the fact that the scale factor inside the lightcone is given by \( a = 1/(H \sinh \eta_R) \), we recover the spectrum for \( U_{pt}^{(+)} \) given in reference [13]:

\[
(U_{pt}^{(+)}|^2 = \frac{2\pi \kappa H^2}{p(p^2 + 1) \sinh \pi p} \left( |u_{(+)}^p|^2 + |u_{(-)}^p|^2 \right)
= \frac{2\kappa H^2 \coth \pi p}{p(p^2 + 1)} \left( 1 - \frac{R}{\cosh \pi p} \left( \frac{\Delta s \cos 2p\eta_w + 2p \sin 2p\eta_w}{\Delta s} \right) \right), \tag{4.8}
\]

where \( R \) is the reflection coefficient, given by

\[
R = \frac{(\Delta s)^2}{4p^2 + (\Delta s)^2}. \tag{4.9}
\]

Of course, the validity of Eq.(4.7) is not restricted to the thin wall regime, and a more complete analysis in the general case will be presented elsewhere [30]. Notice that this equation has been derived neglecting the term \( \kappa \varphi_0^2/2 \) in the equation for the evolution of the modes inside the lightcone. This is justified for long wavelength modes, which are frozen in soon after bubble nucleation. For modes that enter the horizon at times \( t \gg H^{-1} \), which corresponds to \( p \gg 1 \), the generation of perturbations occurs during the second stage of inflation and has little to do with the bubble profile. Instead, the details of the slow roll potential will be important.

V. CONCLUSIONS AND DISCUSSION

In this paper we have applied Fadeev and Jackiw’s formalism for constrained systems to the problem of cosmological perturbations in the one bubble open inflationary universe (The cases of flat and closed universes have been also considered in Appendix B).

We have found the reduced action for a gauge invariant variable describing scalar degrees of freedom and for two gauge invariant variables describing tensor degrees of freedom. This tensor part coincides with the one found previously in [11].

The nucleation of a bubble breaks the \( O(4,1) \) symmetry of de Sitter space down to \( O(3,1) \). It is known that, neglecting the self-gravity of the bubble, there is a special scalar mode with eigenvalue \( p^2 = -4 \) which corresponds to fluctuations of the bubble wall. This mode can be seen as the Goldstone mode associated with the breaking of symmetry \( O(4,1) \) down to \( O(3,1) \). We have seen that the wall fluctuation mode disappears from the spectrum of scalar perturbations once gravity is included. This is somewhat reminiscent of what happens in gauge theories: the Goldstone mode disappears when the gauge fields (gravitational degrees of freedom) are included (the tensor modes acquire a mass term on the wall, where the Goldstone used to live, which cuts off the infrared divergence encountered in [9]). As pointed out in [13] (see also [31]), the disappearance of the wall fluctuation mode is perhaps not too surprising. Even in the absence of self-gravity, this mode can be written as a tensor mode, which contributes to microwave background anisotropies just like any gravitational wave would do [15–17]. The study of tensor modes in [11,13] showed that in the weak gravity limit, the ‘infrared’ contribution of gravity waves to microwave anisotropies reproduces the effect of bubble wall fluctuations.

As an application of our formalism, we have derived the spectrum of long wavelength tensor modes in open inflation. This is given in terms of the real part of the reflection coefficient in a one dimensional scattering problem, where the potential barrier is a function of the bubble profile. In the thin wall regime, we recover the results of [11]. A more complete study of this spectrum and that of scalar perturbations will be presented elsewhere [30].
In this appendix we derive the action for the fluctuations of a scalar field coupled to gravity on a FRW background, up to second order in the perturbation variables.

The metric is written in the ADM [32] form
\[ ds^2 = -(N^2 - N_i N^i) d\eta^2 + 2N_i dx^i d\eta + g_{ij} dx^i dx^j, \] (A1)
where \( N \) is the lapse, \( N_i \) is the shift function and \( g_{ij} \) is the metric on the constant \( \eta \) space-like hypersurfaces in which we have foliated spacetime. Up to total derivative terms, the purely gravitational part of the action can be written as
\[ S_{gr} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( -\frac{(A1)}{2} \right) \]
\[ + \frac{1}{2} \partial_i (\sqrt{-g} N^i g^{ij}) \partial_j \ln (g) + \partial_i N \partial_j (\sqrt{-g} g^{ij}) \]
where
\[ K_{ij} = \frac{1}{2N} (N_{ij} + N_{ji} - g_{ij}). \] (A3)

With a prime we denote a derivative with respect to time \( \eta \), and the vertical bar stands for the covariant derivative with respect to the spatial metric \( g_{ij} \). The action for the scalar matter field is
\[ S_m = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - V(\varphi) \right]. \] (A4)

Now we expand the metric and the scalar field over an FRW-like background solution. The perturbed metric and the perturbed scalar field read
\[ ds^2 = a(\eta) \left\{ -(1 + 2A) d\eta^2 + 2S_i dx^i d\eta + (\gamma_{ij} + h_{ij}) dx^i dx^j \right\}, \]
where \( \gamma_{ij} \) is the metric on the constant curvature space sections. The background fields \( a \) and \( \varphi_0 \) satisfy the equations
\[ \mathcal{H}^2 - \mathcal{H}' + K = \frac{\kappa}{2} \varphi_0^2, \]
\[ 2 \mathcal{H}' + \mathcal{H}^2 + K = \frac{\kappa}{2} (\varphi_0'^2 + 2a^2 V(\varphi_0)), \]
\[ \varphi_0'' + 2 \mathcal{H} \varphi_0' + a^2 V_\varphi(\varphi_0) = 0, \] (A6)
where \( \mathcal{H} := a'/a \), and \( K \) is the curvature parameter, which has the values 1, 0, -1 for closed, flat and open universes respectively.

Expanding the total action, keeping terms of second order in perturbations, and using the background equations, we find
\[ S = S_{gr} + S_m = S_0 + \delta_2 S, \] (A7)
where \( S_0 \) is the action for the background solution and \( \delta_2 S \) is quadratic in perturbations:
\[ \delta_2 S = \frac{1}{2\kappa} \int d^4x a^2 \sqrt{\gamma} \left\{ -2(2\mathcal{H}^2 + \mathcal{H}') A^2 + (S_{ij}) - \frac{h_{ij}}{2} \right\}^2 \]
\[ + \frac{1}{2} (2h_{ij} h_{kj}) - h_{ij} h_{kj} - 2h_{ij} h_{kj} + h_{ij} h_{kj} + h_{ij} h_{kj}) + A^{ij} (h_{ij} - h_{ij}) \]
\[ + 2\kappa (\delta \varphi^2 - \delta \varphi^i \delta \varphi^i - a^2 V_\varphi \delta \varphi^2) - 2\kappa (\varphi_0' \delta \varphi^i A + a^2 V_\varphi \delta \varphi A) \]
\[ + 4(S_{ij}) - \frac{h_{ij}}{2}) (\frac{1}{2} \varphi_0' \delta \varphi - \mathcal{H} A) + K (2A^2 - 2Ah_i^i - \frac{1}{2} h_i^i h_j^j + h_{ij} h_{ij}). \] (A8)
We have raised and lowered spatial indices with $\gamma_{ij}$.

In the open inflation case, comparing (A5) with (3.3), we see that the action for small perturbations outside the lightcone can be found just replacing $a^2 \to -a_E^2$ and $\gamma_{ij} \to -\gamma_{ij}^{dS}$ in expression (A8). The result is equation (3.4) in the text.

**APPENDIX B: REDUCTION INSIDE THE LIGHTCONE**

Although in the case of open inflation the $t = \text{const.}$ surfaces are not Cauchy surfaces for the whole spacetime, it is known [11] that they can be used to normalize the subcurvature modes, i.e. those modes for which the eigenvalue of the Laplacian on the hyperboloids of homogeneity and isotropy is smaller than -1. Furthermore, the resulting reduced action can be, in some heuristic sense to be discussed later, analytically continued to the outside of the lightcone. Then it is found that we obtain the correct result even for supercurvature modes, i.e. modes other than the subcurvature modes. Compared with the reduction in the outside of the lightcone, the analysis in the inside of the lightcone is very simple. Therefore, for an alternative less rigorous but rapid derivation, in this appendix we consider the reduction of the Lagrangian directly inside the lightcone. We shall simultaneously consider also the case of flat and closed spatial sections.

Inside the lightcone the metric perturbations can be decomposed into scalar, vector and tensor modes [18,19,21], regarding the way in which the modes transform under spatial coordinate transformations. On a homogeneous background, the modes are decoupled in the action, and evolve independently. Thus we expand the metric perturbations as

$$h_{ij} = -2\psi\gamma_{ij} + 2E_{ij} + 2F_{ij} + t_{ij},$$

$$S_i = B_i + V_i,$$  \hspace{1cm} (B1)

where $\psi$, $B$ and $E$ are scalar modes, $F_i$ and $V_i$ are vector modes,$^\dagger$ and $t_{ij}$ is a tensor mode. $F_i$ and $V_i$ are divergenceless, and $t_{ij}$ is transverse traceless (TT), i.e.

$$F_{ij} = 0 = V_{ij} = 0.$$  \hspace{1cm} (B2)

Substituting the decomposition (B1) in (A8), the action is decoupled into three pieces

$$\delta_2 S = (s) \delta_2 S + (v) \delta_2 S + (t) \delta_2 S.$$  \hspace{1cm} (B3)

### 1. Scalar Perturbations

The action for scalar perturbations reads

$$(s) \delta_2 S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left\{ -6\psi^2 - 12\mathcal{H}A\psi' + 2\Delta \psi (2A - \psi) - 2(\mathcal{H}' + 2\mathcal{H}^2)A^2 
\right.
\left. + \kappa (\Delta \psi^2 + \psi \Delta \psi - a^2 V_{,\varphi\varphi} \delta \varphi^2) + 2\kappa (3\varphi_0' \psi \delta \varphi - \varphi_0' \delta \varphi A - a^2 V_{,\varphi} A \delta \varphi) + \kappa (-6\psi^2 + 2A^2 + 12\psi A + 2(B - E') \Delta (B - E')) \right\} + 4\Delta (B - E') \left( \frac{\kappa}{2} \varphi_0' \delta \varphi - \psi' - \mathcal{H}A \right),$$  \hspace{1cm} (B4)

where $\Delta$ is the laplacian associated with $\gamma_{ij}$.

To apply FJ formalism [27], we first have to cast the Lagrangian in first order form, defining momenta for the variables whose time derivative appears quadratically in the Lagrangian. As usual, the conjugate momenta are defined as

$$\Pi_\psi := \frac{\delta}{\delta \psi'} (s) \delta_2 S = \frac{2a^2 \sqrt{\gamma}}{\kappa} \left( -3\psi' + \Delta E' + 3 \frac{\kappa}{2} \varphi_0' \delta \varphi - \Delta B - 3 \mathcal{H} A \right),$$

$$\Pi_\varphi := \frac{\delta}{\delta \varphi'} (s) \delta_2 S = a^2 \sqrt{\gamma} \left( 3 \varphi_0' \delta \varphi - \varphi_0 A \right),$$

$$\Pi_E := \frac{\delta}{\delta E} (s) \delta_2 S = \frac{2a^2 \sqrt{\gamma} \Delta}{\kappa} \left( \kappa E' + \psi' - \frac{\kappa}{2} \varphi_0' \delta \varphi - \kappa B + \mathcal{H} A \right).$$  \hspace{1cm} (B5)

$^\dagger$It should be noted that $F_i$ and $V_i$ defined here are different from $F_A$ and $V_A$ defined in Eq. (3.6).
The first order Lagrangian for scalar perturbations turns out to be

\[ (s) \mathcal{L} = (s) \mathcal{L}_1 + (s) \mathcal{L}_0 = (s) \mathcal{L}_1 - (s) \mathcal{H} - B \mathcal{C}_B - A \mathcal{C}_A, \]  
\[ (s) \mathcal{L}_1 = \Pi \psi \psi' + \Pi \delta \varphi' + \Pi E' E', \]  
\[ (s) \mathcal{H} = \frac{\kappa}{4 a^2 \sqrt{\gamma}} \left( -\kappa \Pi^2 + 2 \Pi \psi \Pi E + \frac{3 \Pi E}{\triangle} + 2(\triangle + 3 \kappa) \Pi \delta \varphi' \right) + \frac{\kappa}{2} \varphi_0 \psi \Pi \delta \varphi \]
\[ + \frac{a^2 \sqrt{\gamma}}{\kappa} \left( (\Delta + 3 \kappa) \psi^2 - \frac{\kappa}{2} \left( \Delta + 3 \kappa - \mathcal{H}^2 - \mathcal{H}' + \frac{\varphi''}{\varphi_0} \right) \delta \varphi^2 \right), \]
\[ \mathcal{C}_B = \Pi E, \]
\[ \mathcal{C}_A = -\mathcal{H} \Pi \psi + \varphi_0' \Pi \delta \varphi + \frac{2 a^2 \sqrt{\gamma}}{\kappa} \left( -\frac{\kappa}{2} \Delta + 3 \kappa \right) \psi + \frac{\kappa}{2} \left( \mathcal{H} \varphi_0' - \varphi_0'' \right) \delta \varphi. \]

We observe that neither \( A \) nor \( B \) enters into \( \mathcal{L}_1 \), so there is no dynamical evolution for these fields. They correspond to \( \delta N^\mu \) in the notation of the introduction. These fields appear linearly in the lagrangian, and their equations of motion, \( \mathcal{C}_{A/B} = 0 \) contain no time derivatives. They allow us to evaluate two of the momenta in terms of the other fields. Moreover, the constraints \( \mathcal{C}_A \) and \( \mathcal{C}_B \) are the generators of the infinitesimal gauge transformations associated with diffeomorphisms. Under a scalar diffeomorphism generated by the vector \( \lambda^\mu = (\lambda^0, \lambda^i) \), the metric transforms into \( g_{\mu
u} + \delta g_{\mu
u} \), from which we can read the variation of all the scalar components of the metric perturbation. By commutation with \( \lambda^0 \mathcal{C}_A + \lambda \mathcal{C}_B \) we recover the transformation law for the canonical fields:

\[
\delta \varphi = -\mathcal{H} \lambda^0, \quad \delta \Pi_\psi = \frac{2 a^2 \sqrt{\gamma}}{\kappa} (\Delta + 3 \kappa) \lambda^0, \quad \delta \Pi_\varphi' = \frac{a^2 \sqrt{\gamma}}{\kappa} (\varphi_0'' - \varphi_0' \mathcal{H}) \lambda^0, \quad \delta \Pi_E = 0.
\]

The constraints and the scalar Hamiltonian \( (s) \mathcal{H} \) satisfy the following algebra:

\[
\{ (s) \mathcal{H}, \mathcal{C}_A \} = \partial_\eta \mathcal{H} C_A - \mathcal{H} \partial_\eta \mathcal{C}_A + \mathcal{C}_B, \quad \{ (s) \mathcal{H}, \mathcal{C}_B \} = 0, \quad \{ \mathcal{C}_A, \mathcal{C}_B \} = 0.
\]

The time derivative of the constraints appears due to its explicit time dependence. This derivative acts only on background quantities but not on canonical coordinates. Under a gauge transformation, \( (s) \mathcal{L}_1 \) transforms as

\[
\delta \varphi \mathcal{L}_1 = -\mathcal{H} \lambda^0 \left( \partial_\eta - \partial_\eta \right) C_A - \lambda \left( \partial_\eta - \partial_\eta \right) C_B.
\]

Using these results and the fact that the action is invariant under a gauge transformation, we can recover the transformation law for the lagrange multipliers [33]:

\[
\delta \varphi A = \lambda^{0r} + \mathcal{H} \lambda^0, \quad \delta \varphi B = \lambda' - \lambda^0.
\]

Now we proceed with the phase space reduction. We start by solving the constraints \( \mathcal{C}_B = 0 \) and \( \mathcal{C}_A = 0 \) for \( \Pi E \) and \( \Pi \delta \varphi \), respectively. After substitution of the constraint, there is no \( E \) dependence in the Lagrangian, so the Lagrangian becomes a functional of only \( \Pi_\psi, \psi \) and \( \delta \varphi \). \( (s) \mathcal{L}^* = (s) \mathcal{L}^* \Pi_\psi, \psi, \delta \varphi \). The dissappearance of \( E \) is related to the fact that there is no variable other than \( E \) itself whose gauge transformation depends on \( \lambda \) besides the Lagrange multipliers. Hence it is not possible to construct a gauge invariant combination which contains \( E \). Therefore \( E \) necessarily vanishes from \( (s) \mathcal{L}^* \).

Applying the formula given in (2.12) or equivalently looking at the gauge transformation law given in Eqs (B7), the gauge invariant combinations are found to be constructed from the remaining variables as

\[
\Psi = \psi + \frac{\mathcal{H}}{\varphi_0} \delta \varphi, \quad \Pi_\Psi = \Pi_\psi - \frac{2 a^2 \sqrt{\gamma}}{\kappa \varphi_0} (\Delta + 3 \kappa) \delta \varphi.
\]

The action expressed as a functional of \( \Pi_\Psi \) and \( \Psi \) can be obtained, up to total derivative terms, by simply putting \( \delta \varphi = 0 \) in \( (s) \mathcal{L}^* \Pi_\psi, \psi, \delta \varphi \) and replacing \( \Pi_\psi \) with \( \Pi_\Psi \) and \( \psi \) with \( \Psi \). We finally obtain
(s) $\mathcal{L}^* = \Pi_{\Psi} \Psi' - \frac{a^2\sqrt{\gamma}}{\kappa^2 \varphi'_0} ((\triangle + 3 \mathcal{K}) \Psi + \frac{\kappa \mathcal{H}}{2a^2 \sqrt{\gamma}} \Pi_{\Psi})^2$

$$- \frac{a^2 \sqrt{\gamma}}{\kappa} \Psi (\triangle + 3 \mathcal{K}) \Psi + \frac{1}{4a^2 \sqrt{\gamma}} \Pi_{\Psi} \frac{\kappa \mathcal{K}}{(\triangle + 3 \mathcal{K})} \Pi_{\Psi}. \quad (B12)$$

Notice that the procedure is equivalent to fixing a gauge where $E$ and $\delta \varphi$ (the variables conjugate to the momenta we have solved the constraints for) are set to zero.

For flat universes we can easily recover the results of [22] or [21,23]. If $\mathcal{K} = 0$, using the equation of motion for $\Pi_{\Psi}$ we can eliminate the momenta $\Pi_{\Psi}$ in favor of the velocity $\Psi'$, and we will find the following second order Lagrangian:

$$\mathcal{L}_{\Psi} = \frac{1}{2} \frac{a^2 \varphi'_0}{\mathcal{H}^2} (\Psi^2 + \Psi \triangle \Psi), \quad (B13)$$

which after the rescaling

$$\vartheta = z \Psi, \quad z = \frac{a\varphi'_0}{\mathcal{H}}, \quad (B14)$$

becomes the Lagrangian of a scalar field in flat spacetime with time dependent mass,

$$\mathcal{L}^{(2)}_{\vartheta} = \frac{1}{2} \left( \vartheta'^2 - \vartheta_{,i} \vartheta^{,i} + \frac{z''}{z} \vartheta'^2 \right). \quad (B15)$$

The reduced gauge invariant variable $\vartheta$ coincides with the one found in ref [23]:

$$\vartheta = a(\delta \varphi + \varphi'_0 \psi). \quad (B16)$$

In the general case we can perform the canonical transformation

$$\Psi = \frac{\kappa \varphi'_0}{4} \tilde{q} - \frac{2 \kappa \mathcal{H}}{a^2 \sqrt{\gamma}} \varphi'_0 \frac{1}{\triangle + 3 \mathcal{K}} \tilde{p},$$

$$\Pi_{\Psi} = \frac{a^2 \sqrt{\gamma}}{2 \mathcal{H}} \varphi'_0 (\triangle + 3 \mathcal{K}) \tilde{q} + \frac{2}{\kappa \varphi'_0} \tilde{p}, \quad (B17)$$

and solve for the momenta $\tilde{p}$ to obtain the second order Lagrangian

$$\mathcal{L}^{(2)}_{\tilde{q}} = - \frac{\sqrt{-4}}{2} \tilde{q} (\triangle + 3 \mathcal{K}) \tilde{q} \left( \Box - m^2[a, \varphi_0] \right) \tilde{q},$$

$$m^2[a, \varphi_0] = - \frac{1}{a^2} \left( 4 \mathcal{K} - 2 \mathcal{H}' + \varphi'_0 \left( \frac{1}{\varphi'_0} \right)'' \right). \quad (B18)$$

Here $\Box$ stands for the four dimensional scalar d’Alembertian. The action for this lagrangian can also be written as

$$S^{(2)}_{\text{lin}} = \frac{1}{2} \int \sqrt{\gamma} (\triangle + 3 \mathcal{K}) q_{\text{lin}} \left( \hat{\mathcal{O}} + \triangle + 3 \mathcal{K} \right) q_{\text{lin}} d\eta d^3x,$$

$$\hat{\mathcal{O}} = - \frac{d^2}{d\eta^2} + \frac{\kappa}{2 \varphi'_0} \psi^2 + \varphi'_0 \left( \frac{1}{\varphi'_0} \right)'' \quad (B19)$$

where $q_{\text{lin}} = a \tilde{q}$, which is formally analogous to (3.28). Note, however, that $(\hat{\mathcal{O}} q)$ is changed by $(\triangle - 3) q_{\text{lin}}$. These operators are not the analytic continuation of one another, but both have the same eigenvalues on solutions of the equations of motion.

We note that the reduced gauge invariant variable $\tilde{q}$ is proportional to $\Phi_{\mathcal{H}}$ of Bardeen [18],

$$\tilde{q} = \frac{2}{\kappa \varphi'_0} \left( \Psi - \frac{\mathcal{H} \kappa}{2a^2 \sqrt{\gamma} (\triangle + 3 \mathcal{K})} \Pi_{\Psi} \right) = \frac{2}{\kappa \varphi'_0} (\psi - \mathcal{H}(B - E')) = - \frac{2}{\kappa \varphi'_0} \Phi_{\mathcal{H}}. \quad (B20)$$

For flat universes ($\mathcal{K} = 0$), $\tilde{q}$ is proportional to $Q$ of ref [22],

$$\tilde{q} = \frac{Q}{\sqrt{-\triangle}}, \quad (B21)$$

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The Lagrangian is already in a canonical form and has no constraints. We can decompose the gravitational waves. Then, it becomes manifest that there are two decomposed degrees of freedom for each \( p, \ell, m \) and the action (B18) reduces to the one found in ref. [22] if we replace \( \tilde{q} \) with \( Q \). If we take an ansatz of the form \( \tilde{q} = a^{-1} q^p(\eta) Y^{p\ell m}(x^i) \), where \( Y^{p\ell m} \) are the scalar harmonics on the homogenous spatial section, the equation of motion separates into

\[
\triangle Y^{p\ell m} = (-p^2 + \mathcal{K}) Y^{p\ell m},
\]

\[
\mathcal{O}[q^p] = (p^2 - 4\mathcal{K}) q^p.
\]

2. Vector and Tensor Perturbations

As we have said, neither tensor nor vector modes couple to the scalar perturbations. The vector part of the action carries, as we will see, no dynamics. We find for it

\[
^{(v)} \delta_2 S = -\frac{1}{4\kappa} \int d^4x \, a^2 \sqrt{\gamma} \left( \dot{V}_m - \dot{F}_m \right)^2, \tag{B23}
\]

where

\[
\dot{V}_m = \sqrt{-\triangle - 2\mathcal{K}} V_m, \quad \dot{F}_m = \sqrt{-\triangle - 2\mathcal{K}} F_m. \tag{B24}
\]

We can compute the corresponding first order Lagrangian,

\[
^{(v)} \mathcal{L} = \tilde{\pi}^m \dot{F}_m - \frac{\kappa}{a^2 \sqrt{\gamma}} \tilde{\pi}^m \tilde{\pi}_m + \tilde{\pi}^m \dot{V}_m. \tag{B25}
\]

The field \( \dot{V}_m \) has no conjugate momenta, and enters as a Lagrange multiplier which enforces \( \tilde{\pi}^m \) to vanish. After substituting \( \tilde{\pi}^m = 0 \), we will end with a vanishing Lagrangian. In this model vector modes are pure gauge.

For the tensor modes we find

\[
^{(t)} \delta_2 S = \frac{1}{8\kappa} \int d^4x \, a^2 \sqrt{\gamma} \left\{ t^{ij} \dot{t}'_{ij} - t^{ij} \dot{t}^{jk}_i \dot{t}_{jk} - 2\mathcal{K} t^{ij} t_{ij} \right\}. \tag{B26}
\]

The action can be cast easily in a first order form,

\[
^{(t)} \delta_2 S = \int d^4x \left\{ \tilde{\pi}^{ij} \dot{t}'_{ij} - \frac{2\kappa}{a^2 \sqrt{\gamma}} \tilde{\pi}^{ij} \pi_{ij} + \frac{a^2 \sqrt{\gamma}}{8\kappa} \{ (\triangle - 2\mathcal{K}) t_{ij} \} \right\}. \tag{B27}
\]

The Lagrangian is already in a canonical form and has no constraints. We can decompose \( t_{ij} \) by using the normalized transverse-traceless tensor harmonics \([29,11], Y_{ij}^{(+)}p\ell m \) and \( Y_{ij}^{(-)}p\ell m \), as

\[
t_{ij} = \sum_{p\ell m} U_{ij}^{(+)p\ell m}(\eta) Y_{ij}^{(+)p\ell m} + \sum_{p\ell m} U_{ij}^{(-)p\ell m}(\eta) Y_{ij}^{(-)p\ell m}. \tag{B28}
\]

Then, it becomes manifest that there are two decomposed degrees of freedom for each \( p, \ell, m \), which correspond to gravitational waves.

APPENDIX C: EVEN PARITY PHASE SPACE REDUCTION OUTSIDE THE LIGHTCONES

Inserting the mode decomposition (3.6) in the action (3.4), the even parity modes decouple from the odd parity ones. The even parity part reads

\[
^{(e)} \mathcal{L} = \frac{a^2 c_E \sqrt{\omega}}{2\kappa} \left\{ S \left( \triangle^{(2)} \xi' + 2h_E \left( w' + 2\xi' \right) - 2\kappa \varphi_0^2 \delta \phi + 4\mathcal{H} A - 4\mathcal{H} w - \frac{\triangle^{(2)} \dot{T}}{c_E^2} \right) + \left( -2 - \frac{\triangle^{(2)} \delta \phi}{c_E^2} \right) S^2 - 2S' w + \frac{(\triangle^{(2)} \delta \phi)}{c_E^2} T^2 - \frac{\triangle^{(2)} \dot{T}^2}{2c_E^2} + T \left( -4^{(2)} \delta \phi \dot{\phi} \varphi_0 - (2^{(2)} \triangle \dot{\phi} \phi - 2^{(2)} \triangle \varphi_0) w' + 2^{(2)} \kappa \triangle \omega' + 2^{(2)} \triangle h_E \xi' \right) \right\}. \]

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where
\[
\begin{align*}
\Pi_A &= \frac{a_E^2\sqrt{\omega}}{2c_E\kappa} \left( \sqrt{\omega} + h_E (w + (2) \Delta v) + 2 H S + (2) \Delta \xi + 2 h_E \xi \right), \\
\Pi_T &= -\frac{a_E^2\sqrt{\omega}}{2c_E\kappa} \left( T + 2 h_E T + S + c_E^2 \xi \right), \\
\Pi_w &= -\frac{a_E^2\sqrt{\omega}}{2c_E\kappa} \left( \sqrt{\omega} - 2 \dot{A} + 2 S' + 4 H S + 2 h_E \xi + (2) \Delta \xi \right), \\
\Pi_v &= \frac{a_E^2\sqrt{\omega}}{2c_E\kappa} \left( (2) \Delta + 2) \dot{v} + 2 h_E A - (2) \Delta \xi \right), \\
\Pi_{\delta \phi} &= \frac{a_E^2\sqrt{\omega}}{2c_E\kappa} (\delta \dot{\phi} - \phi'_0 S).
\end{align*}
\]

The corresponding first order Lagrangian reads:

\[
\begin{align*}
(\mathcal{L}) &= (\mathcal{L}_1) - (\mathcal{H} - C_S S - C_\xi \xi - C_\xi'), \\
(\mathcal{L}_1) &= \Pi_A \dot{A} + \Pi_T \dot{T} + \Pi_w \dot{w} + \Pi_v \dot{v} + \Pi_{\delta \phi} \dot{\phi}, \\
(\mathcal{H}) &= -\frac{a_E^2\sqrt{\omega}}{2c_E\kappa} \left\{ (2) \Delta T^2 + T \left( -4 \Delta \Delta H A - \frac{4 h_E}{a_E^2 \sqrt{\omega}} \frac{\Pi_T}{c_E^2 + 2 \kappa (2) \Delta \delta \phi \phi'_0} \right) \\
&\quad - \frac{h_E}{2} (2) \Delta v - \frac{h_E}{2} (2) \Delta \Delta + (2) \Delta v' + w \left( (2) \Delta + 4 c_E^2 \right) A + \frac{\kappa h_E}{a_E^2 \sqrt{\omega}} \frac{(\Pi_A + 2 \Pi_w)}{(2) \Delta + 2) \Delta c_E^2 \omega} \\
&\quad - \frac{h_E}{2} (2) \Delta v - \frac{h_E}{2} (2) \Delta \Delta + (2) \Delta v' + w \left( (2) \Delta + 4 c_E^2 \right) A + \frac{\kappa h_E}{a_E^2 \sqrt{\omega}} \frac{(\Pi_A + 2 \Pi_w)}{(2) \Delta + 2) \Delta c_E^2 \omega} \\
&\quad + \frac{2 A^2}{2} \left( (2) \Delta + 4 c_E^2 \right) A + \frac{\kappa h_E}{a_E^2 \sqrt{\omega}} \frac{(\Pi_A + 2 \Pi_w)}{(2) \Delta + 2) \Delta c_E^2 \omega} \\
&\quad + \frac{c_E^2}{2} \delta \phi^2 \left( (2) \Delta - a_E^2 \delta \phi A' - 4 \kappa a_E^2 \delta \phi A \right) \right\}, \\
&\quad \frac{\kappa}{a_E^2 \sqrt{\omega}} \left\{ c_E^2 \Pi_T^2 - \Pi_A \Pi_w - \frac{\Pi_v^2}{(2) \Delta + 2) \Delta c_E^2 \omega} \right\},
\end{align*}
\]

where
\[
\begin{align*}
C_S &= \phi'_0 \Pi_{\delta \phi} - 2 H \Pi_w - \Pi_T - a_E \left( \frac{\Pi_A}{a_E} \right)' + \frac{a_E^2 h_E c_E \sqrt{\omega}}{\kappa} \left( \mathcal{H} (w + (2) \Delta v) + (2) \Delta v' + \frac{(2) \Delta T}{c_E^2} \right), \\
C_\xi &= \Pi_v - (2) \Delta \Pi_w + c_E^2 \Pi'_T.
\end{align*}
\]
\[ C_\zeta = -2h E \Pi_w - \frac{a^2_E c_E \sqrt{\omega}}{\kappa} \left( \frac{1}{2} (2) \triangle + 2 \right) (w + (2) \triangle v) + 2c_E^2 \mathcal{H} w' + c_E^2 w'' + 2c_E^2 A' - (2) \triangle A + 2c_E^2 (2\mathcal{H}^2 + \mathcal{H}' - 1)A - 2\mathcal{H} (2) \triangle T - (2) \triangle T' - \kappa a^2_E c_E^2 V_{\phi\phi}, \delta \phi - \kappa c_E^2 \phi_0 \delta \phi' \right). \]

Notice that when \( \ell = 0,1 \), equation (C2) is meaningless because of the factors involving the laplacian. For the moment, we assume that \( \ell \neq 0,1 \), and postpone the discussion of the fate of these two modes until the end of this section.

We note that the fields \( \xi, \zeta \) and \( S \) do not appear in the canonical form \( \mathcal{L}_1 \), and only appear linearly in the lagrangian, i.e., they are \( \delta N^\mu \)-like variables. The constraints are \( \mathcal{C}_S = 0, \mathcal{C}_\xi = 0 \) and \( \mathcal{C}_\zeta = 0 \), which generate even parity gauge transformations (see Appendix E).

To reduce the phase space we proceed following the way discussed in section II. By taking linear combinations of the constraints \( \mathcal{C}_S = 0, \mathcal{C}_\xi = 0 \) and \( \mathcal{C}_\zeta = 0 \) we can construct \( \hat{C}_\mu \) which takes the form \( \hat{C}_\mu = p_\mu - \hat{p}_\mu q \), where \( p_\mu = \{ \Pi_{\delta\phi}, \Pi_\nu, \Pi_w \} \) and \( q = \{ A, \delta \phi, T, v, w, \Pi_{\delta\phi}, \Pi_T \} \). Now we can apply the formula (2.12) to obtain the gauge invariant combinations of variables as

\[ \Phi = A - \frac{1}{a_E} \left( \frac{a \delta \phi}{\sqrt{\omega}} \right)' , \]
\[ T = T + \frac{\delta \phi}{\sqrt{\omega}} + c_E^2 v', \]
\[ \Pi_\Phi = \Pi_A - \frac{a^2_E c_E \sqrt{\omega}}{\kappa h E} \left( h c_E^2 \omega' + c_E^2 \omega + \frac{(2) \triangle \omega}{2} \right), \]
\[ \Pi_T = \Pi_T + \frac{a^2_E c_E \sqrt{\omega}}{\kappa h E} \left( h c_E^2 \omega' + \omega' \right), \]

where

\[ \hat{\omega} = w + 2\mathcal{H} \frac{\delta \phi}{\sqrt{\omega}} + (2) \triangle v. \]  

Substituting the constraints into the original first order Lagrangian to eliminate \( p_\mu = \{ \Pi_{\delta\phi}, \Pi_\nu, \Pi_w \} \), we obtain the Lagrangian that depends only on \( q \). Then, simply taking \( \delta \phi = v = w = 0 \) and replacing \( A, T, \Pi_A \) and \( \Pi_T \) by \( \Phi, T, \Pi_\Phi \) and \( \Pi_T \), respectively, we finally get an action which depends only on the two pair of canonically conjugate gauge invariant fields:

\[ \mathcal{L}^* = \Pi_\Phi \hat{\Phi} + \Pi_T T - \mathcal{H}^*, \]
\[ \mathcal{H}^* = \frac{\Pi_{\delta\phi}^2}{2 a^2 E^2 c_E \sqrt{\omega}} - \frac{2 h E \Pi_{\delta\phi} \Phi}{\kappa (2) \Delta (2) \triangle + 2} + \frac{(2) \triangle (2) \triangle + 2}{\kappa a^2 E^2 c_E \sqrt{\omega}} \frac{\Pi_{\delta\phi}^2}{2} + \frac{\kappa \Pi_{\delta\phi}^2}{a^2 E^2 c_E \sqrt{\omega}} \frac{\Pi_\Phi}{2} \frac{\Pi_T}{2}, \]

where \( \Pi_{\delta\phi}, \Pi_\nu, \Pi_w \) are functions of \( T, \Pi_T, \Phi \) and \( \Pi_\Phi \) determined from the constraints and given by

\[ \Pi_{\delta\phi} = \Pi_T + \Pi_\Phi - \mathcal{H} \Pi_\Phi - \frac{a^2_E h E \sqrt{\omega}(2) \triangle T}{\kappa c E \phi_0'} + \frac{a^2_E \mathcal{H} c_E \sqrt{\omega}}{\kappa h E \phi_0'}, \]
\[ \Pi_\nu = -\frac{a^2_E}{2h E} \mathcal{H} \Pi_T, \]
\[ \Pi_w = \frac{a^2 E c_E \sqrt{\omega}}{2 h E}, \]
\[ \mathcal{F} = (2) \Delta \Phi + 2c_E^2 (1 - 2\mathcal{H}^2 - \mathcal{H}') \Phi - 2c_E^2 \mathcal{H} \Phi' + (2) \triangle (2) \mathcal{H} T + T'. \]

To disentangle the two degrees of freedom, we proceed in the following way. This Hamiltonian carries the counterparts of the scalar and even parity tensor modes inside the lightcone. Since they are decoupled there, we can expect that they are also decoupled outside the lightcone. Hence we choose new coordinates \( \{ s, v \} \) such that when they are analytically continued inside the lightcone, they reduce to pure scalar and pure tensor variables in a particular gauge. A convenient choice is the longitudinal gauge for the scalar modes and the synchronous gauge for the tensor modes, in which \( S_i = 0 \) and the traceless part of the spatial metric perturbation inside the lightcone becomes a purely tensor

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quantity. Let us call this gauge the Newton gauge for convenience. Then it is easy to see that \( \hat{\Phi} \) is a purely scalar type quantity. For tensor modes, we use the fact that \( S_i = 0 \) hence \( T = 0 \) and \( v \) is a purely tensor type variable in the Newton gauge. Then with the help of the equations of motion, we can show that the following pair of fields \( s, v \) become purely scalar and purely tensor type variables (see Appendix D for a brief discussion about this subject):

\[
\begin{align*}
\dot{s} &= a_E v_0 \phi, \\
\dot{v} &= \hat{K}[a_E T] - \frac{d}{d\eta_E} (a_E \phi).
\end{align*}
\]  

(C7)

Then we can find new canonically conjugate momenta \( \Pi_s, \Pi_v \) such that the Hamiltonian \( (s)\mathcal{H}^* \) decouples into two pieces. Just for reference, we recall the definition of the operators \( \hat{\Pi} \) and \( \hat{K} \),

\[
\hat{\Pi} = -\frac{d^2}{d\eta_E^2} + \frac{\kappa}{2} \phi'^2 + \phi' \left( \frac{1}{\phi_0} \right)'' , \quad \hat{K} = -\frac{d^2}{d\eta_E^2} + \frac{\kappa}{2} \phi'^2 . \quad \text{(C8)}
\]

It is useful to keep in mind the following relation between \( \hat{\Pi} \) and \( \hat{K} \):

\[
\frac{d}{d\eta_E} \hat{\Pi} \phi_0' = \hat{K} \frac{d}{d\eta_E} \phi_0' . \quad \text{(C9)}
\]

To find the appropriate momenta basis in which the Hamiltonian decouples, we propose an ansatz for it, namely

\[
\begin{align*}
\Pi_{\phi} &= a_E \phi_0' \Pi_s + a_E \Pi_v + \hat{\Pi}[\phi] + \hat{\Pi}[T], \\
\Pi_T &= a_E \hat{K}[\Pi_v] + \hat{T}[T] + \hat{\Pi}[\phi],
\end{align*}
\]

(C10)

where \( \hat{\Pi} \) and \( \hat{T} \) are in principle arbitrary differential operators, but \( \hat{\Pi}_{\phi} \) and \( \hat{\Pi}_T \) must be related in order to keep the transformation canonical. The momenta dependence of the transformation is found by requiring the transformation (C7)-(C10) to be canonical. Now we compute the canonical equations of motion for \( s, v \) using the old basis, and express the result in terms of the new basis. We find that \( \dot{s} \) is independent of \( \Pi_v \), and that \( \dot{v} \) is independent of \( \Pi_s \).

Then we define the operators involved in the definition of the new momenta basis in order to completely decouple these two equations. It can be shown that if we choose the operators as

\[
\begin{align*}
\hat{\Pi}[\phi] &= -a_E^2 c_E \sqrt{\omega} \left( \phi_0'^2 - 2(\mathcal{H}^2 - 1)c_E \right) \phi, \\
\hat{\Pi}[T] &= -a_E c_E \sqrt{\omega} \left( \phi_0'^2 + 2c_E \right) \frac{d}{d\eta_E} (a_E T) - \frac{a_E^2 \mathcal{H} c_E \sqrt{\omega}}{\kappa} T, \quad \text{(C11)}
\end{align*}
\]

\[
\begin{align*}
\hat{T}[T] &= -a_E c_E \sqrt{\omega} \left( \phi_0'^2 + 2c_E \right) \frac{d}{d\eta_E} (a_E T) - \frac{a_E^2 \mathcal{H} c_E \sqrt{\omega}}{\kappa} T, \\
\hat{\Pi}[\phi] &= a_E \frac{a_E^2 \mathcal{H} c_E \sqrt{\omega}}{\kappa} \phi + \frac{2a_E^2 \mathcal{H} c_E \sqrt{\omega}}{\kappa} \phi,
\end{align*}
\]

(C12)

\( \dot{s} \) and \( \dot{v} \) turn out to be

\[
\begin{align*}
\dot{s} &= \frac{1}{c_E \sqrt{\omega}} \hat{\Pi}[\Pi_s], \\
\dot{v} &= \frac{\kappa}{2 c_E \sqrt{\omega}} \left( 1 - 4 \hat{K} \frac{\phi_0'^2}{(2 \Delta + (2) \Delta + 2) \phi_0'^2} \right) \hat{\Pi}[\Pi_v] \\
&+ \frac{c_E^2 \phi_0'^2 + 2c_E^2 (2 \Delta + (2) \Delta + 2) (1 - \hat{K}) c_E^2 - 4(1 + \hat{K}) c_E^2}{(2 \Delta + (2) \Delta + 2) \phi_0'^2} \hat{\Pi}[\phi].
\end{align*}
\]

(C13)

which are already decoupled. It can be verified that the \( \Pi_s \) and \( \Pi_v \) defined by (C10) with the help of (C11)-(C12) are canonical conjugates of \( s, v \), so the two equations we have computed are two canonical equation of motion of the system. Therefore, to find the Hamiltonian in the new basis, we only need to know the two remaining canonical equations of motion. Computing \( \hat{\Pi}_s \) and \( \hat{\Pi}_v \) we obtain:
\[ \hat{\mathcal{O}}[\Pi_s] = c_E^3 \sqrt{\omega} \left[ (3 + \frac{(2)\triangle}{c_E^2}) s - \hat{\mathcal{O}}[s] \right] \]

\[ \hat{\mathcal{K}}[\Pi_v] = \frac{2 c_E^5 \sqrt{\omega} \hat{K}}{\kappa} \left( \frac{(2)\triangle(4 + (2)\triangle(1 - \hat{K})) - 4 c_E^2 (\hat{K} (2)\triangle - 2 + (1 + \hat{K}) c_E^2)}{(2)\triangle((2)\triangle + 2) h_E^2} + \frac{(2)\triangle + 2}{c_E^2 \hat{K}} \right) v - \frac{\hat{K}^2 c_E^2}{(2)\triangle((2)\triangle + 2) h_E} \left( 2 (4 + (2)\triangle(1 - \hat{K})) c_E^2 \right) + \frac{(2)\triangle((2)\triangle + 2) - 4(1 + \hat{K}) c_E^4}{\Pi_v} \]  

(C14)

Note that all the \( \eta_E \) dependence has been absorbed in the differential operators \( \hat{\mathcal{O}} \) and \( \hat{\mathcal{K}} \). Expanding (C13) and (C14) in terms of eigenfunctions of these operators, we can read directly from them the coefficients of the Hamiltonian in the new basis. The corresponding first order lagrangian is

\[ ^{(s)} \mathcal{L}^s = \mathcal{L}_q[s, \Pi_s] + \mathcal{L}_w[v, \Pi_v], \]  

(C15)

\[ \mathcal{L}_q = \Pi_s \dot{s} - \Pi_s \frac{\hat{\mathcal{O}}}{2 c_E \sqrt{\omega}} \Pi_s + \frac{c_E^3 \sqrt{\omega}}{2} \frac{1}{\hat{\mathcal{O}}} \left( 3 + \frac{(2)\triangle}{c_E^2} \right) s, \]

\[ \mathcal{L}_w = \Pi_v \dot{v} - \frac{1}{2} \Pi_v \frac{\kappa \hat{K}}{2 c_E \sqrt{\omega}} \left( 1 - \frac{4 \hat{K} c_E^2 (2)\triangle - 2 + (1 + \hat{K}) c_E^2}{(2)\triangle((2)\triangle + 2)} \right) \Pi_v \]

\[- \frac{\Pi_v \hat{K} c_E^2 (2)\triangle((2)\triangle + 2) + 2 (4 + (2)\triangle(1 - \hat{K})) c_E^2 - 4(1 + \hat{K}) c_E^4}{(2)\triangle((2)\triangle + 2) h_E^2} v \]

\[ + \frac{1}{2} v \frac{2 c_E^5 \sqrt{\omega}}{\kappa} \left( \frac{(2)\triangle(4 + (2)\triangle(1 - \hat{K})) - 4 c_E^2 (2)\triangle - 2 + (1 + \hat{K}) c_E^2}{(2)\triangle((2)\triangle + 2) h_E^2} \right) v \]

\[ + \frac{(2)\triangle + 2 c_E^2}{\hat{K}^2 c_E^4} v. \]

The lagrangian \( \mathcal{L}_w \) can be put easily in second order form. Solving for the momenta \( \Pi_s \), and defining \( q \) through

\[ s = \hat{\mathcal{O}}[q], \]  

(C16)

we find equation (3.28) of the text.

The lagrangian \( \mathcal{L}_w \) needs a little extra work. Performing the following canonical transformation

\[ w = \frac{h_E}{c_E \sqrt{\omega}} \Pi_v - \frac{(2)\triangle + 2 c_E^2}{\hat{K}^2} \Pi_v, \]

\[ \Pi_w = \frac{\kappa \hat{K} (2 \hat{K} c_E^2 - h_E^2) - (2)\triangle}{(2)\triangle((2)\triangle + 2)} \Pi_v + 2 c_E \sqrt{\omega} \frac{2 h_E (2)\triangle + c_E^2 - \hat{K} ((2)\triangle + 2 c_E^2) c_E^2}{(2)\triangle((2)\triangle + 2) h_E^2} \left( 2 (4 + (2)\triangle(1 - \hat{K})) c_E^2 \right) + \frac{(2)\triangle((2)\triangle + 2) - 4(1 + \hat{K}) c_E^4}{\Pi_v} \]  

(C17)

we find for \( \mathcal{L}_w \)

\[ \mathcal{L}_w = \Pi_w \dot{w} + \kappa w \hat{K} c_E \sqrt{\omega} \frac{(2)\triangle - (1 + \hat{K}) c_E^2}{(2)\triangle((2)\triangle + 2)} w - \Pi_w \frac{(2)\triangle((2)\triangle + 2)}{4 \kappa \hat{K} c_E^4 \sqrt{\omega}} \Pi_w. \]  

(C18)

Solving for the momenta \( \Pi_w \), we find equation (3.36) in the text.

When \( \ell = 0, 1 \), simply looking at the definition of momenta (C1), we can see that new constraints arise, due to the fact that some of them vanish. For \( \ell = 0 \), \( \Pi_T \) and \( \Pi_v \) become zero. In fact, the second order lagrangian \( ^{(s)} \mathcal{L} \) for \( \ell = 0 \) is independent of \( \xi, T \) and \( v \). In this case we are left with a Lagrangian that depends only on three fields plus two lagrangian multipliers, therefore \( ^{(s)} \mathcal{L}_{\ell=0} \) only contains one degree of freedom. Applying the Faddev-Jackiw formalism, the action for this degree of freedom turns out to be the one for the scalar degree of freedom, \( S_q^{(2)} \), with \( \ell = 0 \). Similarly, if \( \ell = 1 \) the lagrangian \( ^{(s)} \mathcal{L}_{\ell=1} \) is independent of \( v \), so we have four fields and three lagrangian multipliers. As before, \( ^{(s)} \mathcal{L}_{\ell=1} \) only contains one degree of freedom. As expected, in this case we recover the action \( S_q^{(2)} \) for \( \ell = 1 \). This is consistent with the fact that \( w \) represents one of the tensor degrees of freedom inside the lightcone, so it must be absent for \( \ell = 0, 1 \). On the other hand, \( q \) represents the scalar degree of freedom, so it must exists for all \( \ell \).
As we have said, the guess for the variables which disentangle the lagrangian (C5) is motivated by their expression when evolved inside the lightcone in a particular gauge (Newton gauge, defined in Appendix C), where we know they are purely scalar-type or purely tensor-type variables. To show this, we need to derive some useful and well known relations between the scalar potentials $\Phi_H$ and $\Phi_A$. By a subscript (or superscript) $N$ we indicate that the quantity is evaluated in the Newton gauge.

First we consider the variables inside the lightcone. In the Newton gauge $B_N$ and $E_N$ vanish, so the constraint $C_B = 0$ reduces to

$$\psi'_N - \frac{\kappa}{2} \varphi'_0 \delta \varphi_N + H A_N = 0.$$  (D1)

Substituting it into the definition of $\Pi_\psi$, Eq. (B5), we find that this momentum also vanishes,

$$\Pi^N_\psi = 0.$$  (D2)

Expressing the equations of motion for $\Pi_\Psi$ and $\Psi$, which follow from Eq. (B12), in terms of $\psi_N$ and $\delta \varphi_N$, and eliminating $\delta \varphi'_N$ between them, we find

$$\frac{1}{a} (a \psi'_N) - \frac{\kappa}{2} \varphi'_0 \delta \varphi_N = 0.$$  (D3)

Comparing with (D1), we recover the well known relation

$$(\Phi_A =) A_N = \psi_N (= -\Phi_H).$$  (D4)

The analytic continuation of these relations (D3) and (D4) to the outside of the lightcone does not change their form.

Now, returning to the outside of the lightcone, we justify our choice of variables. In the Newton gauge, evaluating $\Phi$ defined in (C3) with the aid of (D3) and (D4), we find

$$\Phi = \frac{2}{\kappa a E \varphi'_0} \left[ \frac{2a E}{\kappa \varphi'_0} \psi_N \right].$$  (D5)

so $\Phi$, and therefore $s$ and $q$, are already a purely scalar-type variable. Recalling the definition of $q$ given by Eqs. (C7) and (C16): $\Phi = \hat{O}[q]/(a E \varphi'_0)$, we find

$$q = \frac{2a E \psi_N}{\kappa \varphi'_0}.$$  (D6)

By means of the fact $\Phi_H = -\psi_N$ inside the lightcone, we find Eq. (3.30) in the text.

To find the tensor-type variable, we use the fact that in the Newton gauge $v_N$ is a purely tensor-type variable. The strategy is to construct a combination of $\Phi$ and $T$ proportional to $v_N$. First, recalling that $T_N = 0$, we evaluate $T$ defined in (C3) in the Newton gauge as

$$T = \frac{2}{\kappa a E \varphi'_0} (a E \psi'_N) + c^2_E v'_N.$$  (D7)

Then if we define

$$\tilde{T} := a E T - \frac{2}{\kappa \varphi'_0} (a E \psi'_N) = a E c^2_E v'_N,$$  (D8)

we find this quantity becomes a purely tensor-type variable. Acting with $\tilde{K}$, we finally find the desired variable

$$v := \tilde{K}[\tilde{T}] = \tilde{K}[a E T] - \frac{d}{d\eta} \frac{1}{a E \varphi'_0} \hat{O}[\frac{2a E}{\kappa \varphi'_0} \psi_N]$$

$$= \tilde{K}[a E T] - \frac{d}{d\eta} (a E \Phi),$$  (D9)

where we have used the relation (C9).

\[\text{---}\]

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\[\text{---}\]
We can verify that under an even parity gauge transformation generated by \( \lambda^0 C_S + \lambda^\rho C_\zeta + \lambda C_\xi \), i.e. a diffeomorphism \( x^\mu \rightarrow x^\mu + \lambda^\mu \) with \( \lambda^\mu = (\lambda^0, \lambda^\rho, \lambda^\parallel A) \), the canonical scalar fields transform according to:

\[
\begin{align*}
\delta_\rho A &= \lambda^0 + H\lambda^0, \\
\delta_\rho w &= -2H\lambda^0 - 2h_E\lambda^\rho - (2)\Delta\lambda, \\
\delta_\rho T &= -\lambda^0 - c_E^2\lambda', \\
\delta_\rho \delta \varphi &= \varphi_0 \lambda^0.
\end{align*}
\] (E1)

The scalar constraints and the scalar Hamiltonian \( ^{(s)}H \) satisfy the following algebra:

\[
\begin{align*}
\{ ^{(s)}H, C_S \} &= \delta_\rho C_S - H C_\zeta, \\
\{ ^{(s)}H, C_\xi \} &= \delta_\rho C_\xi, \\
\{ ^{(s)}H, C_\zeta \} &= \delta_\rho C_\zeta - c_E^2 C_S - h C_\zeta + C_\xi, \\
\{ C_\alpha, C_\beta \} &= 0.
\end{align*}
\] (E2)

Notice that the partial derivative with respect to \( \rho \) acting on the constraints only affects background quantities. Using Eqs. (E1) and (E2), and the fact that the action (C2) is invariant under a gauge transformation, we find the transformation law for the lagrangian multipliers as

\[
\delta_\rho S = \lambda^0 - c_E^2 \lambda^\rho', \quad \delta_\rho \xi = \dot{\lambda} - \lambda^\rho, \quad \delta_\rho \zeta = \dot{\lambda} + h_E \lambda^\rho + H \lambda^0.
\] (E3)

For the odd parity modes, under a diffeomorphism generated by \( \lambda^\mu = (0, 0, \lambda^A) \), where \( \lambda^A \) is divergenceless; \( \lambda^A|_{\parallel A} = 0 \), the fields transform according to

\[
\delta_\rho V_A = -c_E^2 \lambda_A', \quad \delta_\rho F_A = -\lambda_A.
\] (E4)

The algebra satisfied by the Hamiltonian \( ^{(c)}H \) and the constraint is:

\[
\{ ^{(c)}H, C^A_W \} = \delta_\rho C^A_W.
\] (E5)

The action is invariant if we transform \( W \) as

\[
\delta_\rho W_A = \dot{\lambda}_A.
\] (E6)


