Non-Perturbative Heterotic $D=6,4$, $N=1$ Orbifold Vacua

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Abstract

We consider $D = 6$, $N = 1$, $Z_M$ orbifold compactifications of heterotic strings in which the usual modular invariance constraints are violated. It is argued that in the presence of non-perturbative effects many of these vacua are nevertheless consistent. The perturbative massless sector can be computed explicitly from the perturbative mass formula subject to an extra shift in the vacuum energy. This shift is associated to a non-trivial antisymmetric B-field flux at the orbifold fixed points. The non-perturbative piece is given by five-branes either moving in the bulk or stuck at the fixed points, giving rise to Coulomb phases with tensor multiplets. The heterotic duals of some Type IIB orientifolds belong to this class of orbifold models. We also discuss how to carry out this type of construction to the $D = 4$, $N = 1$ case and specific $Z_M \times Z_M$ examples are presented in which non-perturbative transitions changing the number of chiral generations do occur.
1 Introduction

While studying $D$-dimensional perturbative heterotic vacua one of the most important ingredients is one-loop modular invariance. This property automatically guarantees the absence of gauge and gravitational anomalies. This led some people to believe that the important concept was that of modular invariance in closed string theory and that absence of anomalies was just a mere consequence of that (more important) principle.

With the recent developments of the last three years it has become clear that closed and open strings are on equal footing and that the fundamental concept is indeed anomaly cancelation and not modular invariance. If that is the case, a natural question arises: if we consider a $D$-dimensional, e.g. heterotic construction, which does not fulfill the usual criteria for one-loop modular invariance, can it still be consistent as a quantum theory? Certainly there are examples of $D = 6$ theories whose perturbative sector is anomalous and still are consistent quantum theories. The prototypes are heterotic $SO(32)$ or $E_8 \times E_8$ compactifications on $K3$ with instanton number $k < 24$ embedded in the gauge group. In other words, smooth $K3$ compactifications in which the size of $24 - k$ instantons has been set to zero. These string models have gauge and/or gravitational anomalies at the perturbative level. However, it was shown in refs. [1, 2] that there are non-perturbative effects (which occur no matter how small the string coupling is) which provide additional gauge and/or tensor and/or hypermultiplets which render the theories anomaly-free.

It would be very interesting to know, given a certain heterotic string compactification which is anomalous at the perturbative level, when it will be completed by non-perturbative effects that cancel the anomalies. In the meantime, while trying to obtain new heterotic vacua one should not really impose as a crucial ingredient additional constraints associated to modular invariance.

In this paper we apply the above philosophy to the construction of $N = 1$, $D = 6$, 4 toroidal heterotic orbifolds. In the past, perturbative Abelian orbifold vacua have been extensively studied [3, 4] in which certain modular invariance constraints are imposed on the gauge embedding of the $Z_M$ twist. We consider the modding of $SO(32)$ and $E_8 \times E_8$ $D = 6$ heterotic strings by gauge embeddings not verifying those constraints. In order to get consistent spectra certain shifts in the left-handed vacuum energy (associated to an
antisymmetric B-field flux) must be added. One obtains in this way perturbative string vacua which are naively anomalous. We find that some of them can be understood as orbifold versions of the smooth $K3$ compactifications with some instantons set to zero size [1, 2]. Hence, such orbifold models are expected to be completed by non-perturbative effects to produce a final anomaly-free theory. They correspond to the presence of five-branes with their world-volume in the uncompactified dimensions.

We find that some orbifold models of the $SO(32)$ heterotic string require the addition of five-branes stuck at the fixed points. They seem to correspond to theories that are in a Coulomb phase in which tensor multiplets appear, as in the transitions considered in [5]. In this class there are models which seem to be the heterotic duals of the $Z_M$, $M = 2, 3, 4, 6$, Type IIB orientifolds of ref. [6, 7]. For some other orbifold models in which there are no appropriate twisted moduli to repair the orbifold singularities nor transitions of the type studied in ref. [5], we lack at the moment a proper interpretation.

One can also consider the construction of $N = 1$, $D = 4$ orbifold theories with gauge embeddings not verifying the usual modular invariance constraints. However, our knowledge of the possible non-perturbative effects which could render such theories consistent is much more incomplete. Still, in some $Z_M \times Z_M$, $D = 4$, $N = 1$ orbifold constructions, some conclusions can be extracted on the basis of $D = 6$ information. In particular, $D = 6$ transitions to a Coulomb phase in which 29 hypermultiplets turn into a tensor multiplet [8, 9] have as a 4-dimensional consequence transitions in which the number of chiral ($e.g.$ $E_6$) generations varies. A similar observation has recently been made in ref. [10], using a different approach.

The organization of this paper goes as follows. In the next chapter we discuss some of the known methods to get perturbative $D = 6$, $N = 1$ heterotic vacua. This includes $Z_M$, $(0, 4)$ toroidal orbifold constructions as well as smooth $K3$ compactifications in the presence of non-Abelian or Abelian gauge bundles. As a byproduct we show that, in the same way that one can characterize $E_8 \times E_8$ compactifications by giving a pair of instanton numbers $(k_1, k_2)$, one can classify $Spin(32)/Z_2$ compactifications in terms of a pair of instanton numbers $(k_{NA}, k_A)$ corresponding to a non-Abelian and an Abelian instanton background respectively. We provide all the perturbative $Z_2$ and $Z_3$ symmetric orbifold heterotic vacua (without quantized Wilson lines) and discuss connections with
smooth $K3$ compactifications as well as with F-theory models. The theories obtained from smooth $K3$ compactifications with zero size instantons are also briefly recalled.

In Chapter 3 we consider the construction of $D = 6$ heterotic orbifolds in which the gauge embedding violates the usual modular invariance constraints. We construct a family of $Z_3$ models that provide orbifold versions of $K3$ compactifications with some zero size instantons. They require the presence of a quantized number of five-branes in order to cancel anomalies. The different behaviour of small instantons in $SO(32)$ and $E_8 \times E_8$ naturally appears. In the case of the standard embedding, there exist possible transitions in which 2 instantons per fixed point are converted into 2 five-branes. Some properties of this class of models are further clarified if one considers the relevant index theorem formulae for instantons on $Z_M$ ALE spaces. This is done in Chapter 4.

In Chapter 5 we describe non-perturbative $Z_3$ orbifold models on $SO(32)$ which seem to correspond to vacua in a Coulomb phase in which tensor multiplets appear. These are the first explicit heterotic compactifications realizing the transitions considered in ref. [5] which appear when a certain minimum number of small instantons accumulate on a $Z_M$ singularity. The simplest example of these models corresponds to the heterotic dual of the $Z_3^A$ Type IIB orientifold of ref. [6, 7]. In Chapter 6 we discuss some aspects of even $M, Z_M$ orbifolds. In particular, we show how the heterotic duals of the $Z_4^A$ and $Z_6^A$ Type IIB orientifolds of ref. [6, 7] can be understood as $SO(32)$ heterotic orbifolds with certain (non-modular invariant) gauge embeddings. The same is true for some other $Z_2$ orientifolds considered in ref. [11, 12]. All these theories are in a Coulomb phase so that extra tensor multiplets appear. We also show how the version of the Gimon-Polchinski [13] model with all five-branes at the same fixed point seems to be dual to a $Z_2$ heterotic orbifold with a non-modular invariant shift. Other examples of $Z_4$ orbifolds but with vector structure are also presented.

In Chapter 7 we discuss the possibility of extending this kind of analysis to $D = 4, N = 1$ models. As mentioned before, in certain $Z_N \times Z_M$ orbifolds one can obtain some partial information in terms of the $D = 6$ subsectors of the theory. The $Z_2 \times Z_2, D = 4$ Type IIB orientifold of ref. [14] can be understood as a certain $Z_2 \times Z_3$ orbifold of the $SO(32)$ heterotic string in which the $Z_2$ gauge embeddings do not verify the usual modular invariance constraints. We consider analogous $Z_3 \times Z_3$ heterotic orbifolds and show how
$D = 6$ transitions to Coulomb phases reflect themselves in four dimensions into transitions that can change the number of chiral generations. As an example, the standard $Z_3 \times Z_3$, $E_8 \times E_8$ orbifold with discrete torsion appears to have non-perturbative transitions to a model with just three $E_6$ generations from the untwisted sector. Some final comments are left for Chapter 8.

2 Perturbative Heterotic D=6, N=1 Vacua

Before introducing the non-perturbative heterotic orbifold models let us recall some facts about perturbative $D = 6$, $N = 1$ vacua which will be of interest for the later discussion. An interesting class of $D = 6$, $N = 1$ heterotic vacua can be obtained from symmetric toroidal orbifold compactifications on $T^4/Z_M$. The construction of these models parallels that of $T^6/Z_M$ orbifolds [3, 4] as considered in refs. [15, 16, 17]. Here we briefly review the notation and the salient points relevant to our discussion. Acting on the (complex) bosonic transverse coordinates, the $Z_M$ twist $\theta$ has eigenvalues $e^{2\pi i v_a}$, where $v_a$ are the components of $v = (0, 0, \frac{1}{M}, -\frac{1}{M})$. $M$ can take the values $M = 2, 3, 4, 6$. The embedding of $\theta$ on the gauge degrees of freedom is usually realized by a shift $V$ such that $M V$ belongs to the $E_8 \times E_8$ lattice $\Gamma_8 \times \Gamma_8$ or to the $Spin(32)/Z_2$ lattice $\Gamma_{16}$. This shift is restricted by the modular invariance constraint

$$M (V^2 - v^2) = \text{even} \quad (2.1)$$

All possible embeddings for each $M$ are easily found. In the $E_8 \times E_8$ case, we find 2 inequivalent embeddings for $Z_2$, 5 for $Z_3$, 12 for $Z_4$ and 59 for $Z_6$, leading to different patterns of $E_8 \times E_8$ breaking to rank 16 subgroups. For $Spin(32)/Z_2$ we find 3 inequivalent embeddings for $Z_2$, 5 for $Z_3$, 14 for $Z_4$ and 50 for $Z_6$. Each of these models is only the starting point of a bigger class of vacua, generated by adding Wilson lines in the form of further shifts in the gauge lattice satisfying extra modular invariance constraints, by permutations of gauge factors, etc..

The spectrum for each model is subdivided in sectors. There are $M$ sectors twisted by $\theta^j$, $j = 0, 1, \cdots, M - 1$. Each particle state is created by a product of left and right vertex operators $L \otimes R$. At a generic point in the four-torus moduli space, the massless
states follow from

\[ m_R^2 = N_R + \frac{1}{2} (r + j v)^2 + E_n - \frac{1}{2} \quad ; \quad m_L^2 = N_L + \frac{1}{2} (P + j V)^2 + E_j - 1 \quad (2.2) \]

Here \( r \) is an \( SO(8) \) weight with \( \sum_{i=1}^{4} r_i = \text{odd} \) and \( P \) a gauge lattice vector with \( \sum_{i=1}^{16} P' = \text{even} \). \( E_j \) is the twisted oscillator contribution to the zero point energy and it is given by \( E_j = j(M - j)/M^2 \). The multiplicity of states satisfying eq. (2.2) in a \( \theta^j \) sector is given by the appropriate generalized GSO projections [17]. In the untwisted sector there appear the gravity multiplet, a tensor multiplet, charged hypermultiplets and 2 neutral hypermultiplets (4 in the case of \( Z_2 \)). In the twisted sectors only charged hypermultiplets appear. The generalized GSO projections are particularly simple in the \( Z_2 \) and \( Z_3 \) case since essentially all massless states survive with the same multiplicity. The spectra for all \( Z_2 \) and \( Z_3 \) embeddings are shown in Tables 1 and 2.

It is instructive to compare these orbifold vacua with the \( D = 6, N = 1 \) models obtained upon generic heterotic compactifications on smooth \( K3 \) surfaces in the presence of instanton backgrounds [18, 2, 8]. In the \( E_8 \times E_8 \) case there are instanton numbers \((k_1, k_2)\) satisfying \( k_1 + k_2 = 24 \), as required by anomaly cancelation. It is convenient to define \( k_1 = 12 + n, k_2 = 12 - n \) and assume \( n \geq 0 \) without loss of generality. For \( n \leq 8 \), an \( SU(2) \) background on each \( E_8 \) leads to \( E_7 \times E_7 \) unbroken gauge group with hypermultiplet content

\[ \frac{1}{2} (8 + n)(56, 1) + \frac{1}{2} (8 - n)(1, 56) + 62(1, 1) \quad (2.3) \]

Due to the pseudoreal character of the \( 56 \) of \( E_7 \), odd values of \( n \) can also be considered.

Understanding the spectrum corresponding to \( n > 8 \) requires some knowledge on \( E_8 \times E_8 \) vacua in the presence of five-branes. In non-perturbative \( E_8 \times E_8 \) compactifications on \( K3 \), cancelation of gravitational anomalies requires

\[ k_1 + k_2 + n_B = 24 \quad (2.4) \]

where \( n_B \) is the number of \( E_8 \times E_8 \) five-branes. These are five-branes of M-theory, each one carrying a tensor multiplet and a hypermultiplet [2]. When \( 9 \leq n \leq 12 \), \( k_2 \) is not large enough to support an \( SU(2) \) background, the instantons become small and turn into five-branes that give rise to extra tensor multiplets. The unbroken gauge group is now
<table>
<thead>
<tr>
<th>Shift $V$</th>
<th>Group</th>
<th>Untwisted matter</th>
<th>Twisted matter</th>
<th>$(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}(1, 1, 0, \cdots, 0) \times (0, \cdots, 0)$</td>
<td>$E_7 \times SU(2) \times E_8$</td>
<td>$(56, 2) + 4(1, 1)$</td>
<td>$8(56, 1) + 32(1, 2)^*$</td>
<td>$(24, 0)$</td>
</tr>
<tr>
<td>$\frac{1}{2}(1, 0, \cdots, 0) \times (1, 1, 0, \cdots, 0)$</td>
<td>$SO(16) \times E_7 \times SU(2)$</td>
<td>$(1, 56, 2) + 4(1, 1, 1)$</td>
<td>$8(16, 1, 2)$</td>
<td>$(16, 8)$</td>
</tr>
<tr>
<td>$\frac{1}{3}(1, 1, 0, \cdots, 0) \times (0, \cdots, 0)$</td>
<td>$E_7 \times U(1) \times E_8$</td>
<td>$(56, 1) + 3(1, 1)$</td>
<td>$9(56, 1) + 18(1, 1)^*$</td>
<td>$(24, 0)$</td>
</tr>
<tr>
<td>$\frac{1}{3}(2, 0, \cdots, 0) \times \frac{1}{3}(2, 0, \cdots, 0)$</td>
<td>$SO(14) \times SO(14) \times U(1)^2$</td>
<td>$(14, 1) + (64, 1) + (1, 14) + (64, 1) + (2, 1, 1, 1) + (64, 2)$</td>
<td>$9(14, 1) + 9(1, 1, 1, 1) + 18(1, 1)^*$</td>
<td>$(12, 12)$</td>
</tr>
<tr>
<td>$\frac{1}{3}(1, 1, 1, 2, 0, 0, 0) \times (0, \cdots, 0)$</td>
<td>$SU(9) \times E_8$</td>
<td>$(84, 1) + 2(1, 1)$</td>
<td>$9(36, 1) + 18(9, 1)^*$</td>
<td>$(24, 0)$</td>
</tr>
<tr>
<td>$\frac{1}{3}(1, 1, 2, 0, \cdots, 0) \times \frac{1}{3}(1, 1, 0, \cdots, 0)$</td>
<td>$E_6 \times SU(3) \times E_7 \times U(1)$</td>
<td>$(27, 3, 1) + (1, 1, 56) + 3(1, 1, 1) + 18(1, 3, 1)^*$</td>
<td>$9(27, 1, 1) + 9(1, 3, 1)$</td>
<td>$(18, 6)$</td>
</tr>
<tr>
<td>$\frac{1}{3}(1, 1, 1, 2, 0, \cdots, 0) \times \frac{1}{3}(1, 1, 1, 2, 0, 0, 0)$</td>
<td>$SU(9) \times E_6 \times SU(3)$</td>
<td>$(1, 27, 3) + (84, 1, 1) + 2(1, 1, 1) + 18(1, 1, 1)^*$</td>
<td>$9(9, 1, 3)$</td>
<td>$(15, 9)$</td>
</tr>
</tbody>
</table>

Table 1: Perturbative $Z_2$ and $Z_3$, $E_8 \times E_8$, orbifold models. The asterisk indicates twisted states involving left-handed oscillators. The last column shows which smooth $K3$ compactification yields a similar massless spectrum after Higgsing.
<table>
<thead>
<tr>
<th>Shift V Group</th>
<th>Untwisted matter</th>
<th>Twisted matter</th>
<th>$G_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}(1,1,0,\cdots,0)$</td>
<td>$(28,2,2)+4(1,1)$</td>
<td>$8(28,1,2)+32(1,2,1)^*$</td>
<td>$SO(8)$</td>
</tr>
<tr>
<td>$SO(28) \times SU(2) \times SU(2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}(1,1,1,1,0,\cdots,0)$</td>
<td>$(12,20)+4(1,1)$</td>
<td>$8(32,1)$</td>
<td>$SO(8)$</td>
</tr>
<tr>
<td>$SO(12) \times SO(20)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{4}(1,\cdots,1,-3)$</td>
<td>$(120)+4(120)$</td>
<td>$8(16)+8(\bar{1}6)$</td>
<td>II</td>
</tr>
<tr>
<td>$SU(16) \times U(1)$</td>
<td>$+4(1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,0,\cdots,0)$</td>
<td>$(28,2)+3(1,1)$</td>
<td>$9(28,2)+18(1,1)^*$</td>
<td>$SO(8)$</td>
</tr>
<tr>
<td>$SO(28) \times SU(2) \times U(1)$</td>
<td></td>
<td>$+45(1,1)^*$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,1,2,0,\cdots,0)$</td>
<td>$(22,5)+4(1,1)$</td>
<td>$9(22,1)+9(1,10)$</td>
<td>$SO(8)$</td>
</tr>
<tr>
<td>$SO(22) \times SU(5) \times U(1)$</td>
<td>$+2(1,1)$</td>
<td>$+18(1,5)^*$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,1,1,1,1,0,\cdots,0)$</td>
<td>$(16,8)+4(1,28)$</td>
<td>$9(1,28)+18(1,1)^*$</td>
<td>$SO(8)$</td>
</tr>
<tr>
<td>$SO(16) \times SU(8) \times U(1)$</td>
<td>$+2(1,1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,\cdots,1,2,0,0,0,0,0)$</td>
<td>$(10,11)+4(1,55)$</td>
<td>$9(1,11)+9(1,10)$</td>
<td>$II$</td>
</tr>
<tr>
<td>$SO(10) \times SU(11) \times U(1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,\cdots,1,0,0)$</td>
<td>$(14,2,2)+4(1,1)$</td>
<td>$9(1,1,1)+9(1,1,1)$</td>
<td>$II$</td>
</tr>
<tr>
<td>$SU(14) \times U(1) \times SU(2) \times SU(2)$</td>
<td>$+2(1,1,1)$</td>
<td>$+18(1,1,2)^*$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Perturbative $Z_2$ and $Z_3$, $Spin(32)/Z_2$, orbifold models. The asterisk indicates twisted states involving left-handed oscillators. The last column shows the generic terminal gauge group $G_0$ after Higgsing.
$E_7 \times E_8$ with hypermultiplets [8]

$$\frac{1}{2}(8 + n)(56, 1) + (53 + n)(1, 1)$$

(2.5)

and $(12 - n)$ extra tensor multiplets.

Models with various groups can be obtained from (2.3) and (2.5) by symmetry breaking. The group from the second $E_8$ does not possess, in general, enough charged matter to be completely broken. Higgsing stops at some terminal group, depending on the value of $n$, with minimal or no charged matter [18, 17]. For instance $E_8, E_7, E_6, SO(8), SU(3)$ terminal groups are obtained for $n = 12, 8, 6, 4, 3$ while complete breaking proceeds for $n = 2, 1, 0$. On the other hand, the first $E_7$ can be completely Higgsed away. In the last column of Table 1 we show the instanton numbers $(k_1, k_2)$ of compactifications yielding, upon Higgsing, a massless spectrum similar to the corresponding orbifold. We thus see that the five $Z_3$ orbifolds of $E_8 \times E_8$ are in the same moduli space as generic $K3$ compactifications with $n = 12, 0, 12, 6, 3$ respectively. The two $Z_2$ orbifolds correspond to $n = 12, 4$ respectively. This connection between modular invariant orbifold models and instanton backgrounds will be further clarified in Chapter 4.

In the $Spin(32)/Z_2$ case, embedding a total instanton number $k = 24$ is required to cancel gravitational anomalies. An $SU(2)$ background breaks the symmetry down to $SO(28) \times SU(2)$ with hypermultiplets in $10(28, 2) + 65(1, 1)$. Hence, upon Higgsing, the generic group is $SO(8)$. This class of models is known to be [9, 20] in the same moduli space as generic $K3$ compactifications of $E_8 \times E_8$. As shown in Table 2, the first three $Z_3$ orbifolds of $Spin(32)/Z_2$ do have $SO(8)$ as generic group but the last two models have trivial gauge group after full Higgsing.

In fact this is not completely new, it was already noticed in [19] that the fourth $Spin(32)/Z_2, Z_3$ model, could lead to complete Higgsing. Also, in ref. [20] the authors construct a heterotic $Z_2$ orbifold, ‘without vector structure’, in which the resulting $U(16)$ group can be completely broken. In our language this $Z_2$ orbifold has embedding $V = \frac{1}{3}(1, \cdots, 1, -3)$ (third example in Table 2). In general, embeddings with vector structure have shifts $V$ such that $MV = (n_1, \cdots, n_{16})$, whereas embeddings without vector structure have $MV = (n_1 + \frac{1}{2}, \cdots, n_{16} + \frac{1}{2})$. Since $MV \in \Gamma_{16}, \sum I n_I = \text{even}$ in both cases.
Beyond perturbation theory the condition \( k = 24 \) is replaced by

\[ k + n_B = 24 \tag{2.6} \]

where \( n_B \) is now the number of dynamical \( SO(32) \) five-branes, which can be understood as small instantons \[1\]. One such brane carries an \( Sp(1) \) vector multiplet, but when \( r \) of them coincide at a point on the smooth \( K3 \), the group is enhanced to \( Sp(r) \). In general, the non-perturbative group is \( \prod Sp(r_i) \) with \( \sum r_i = n_B \). The five-branes also carry non-perturbative hypermultiplets. In particular, for each \( Sp(r) \) there appear 32 half hypermultiplets in the fundamental representation, together with one hypermultiplet in the antisymmetric two-index representation (decomposable as a singlet plus the rest).

Cancelation of gauge anomalies requires that the hypermultiplets in the fundamental representation be also charged under the perturbative gauge group that arises when \( SO(32) \) is broken by the background with instanton number \( k = 24 - n_B \). The results just summarized apply to compactifications on smooth \( K3 \) surfaces. When the \( K3 \) is realized as an orbifold, the five-branes can coincide at a fixed point thereby producing other gauge groups and hypermultiplet content (see below).

We have seen that the \( E_8 \times E_8 \) compactifications can be labeled by a pair of instanton numbers \((k_1, k_2)\) with \( k_1 = 12 + n, \ k_2 = 12 - n \) and \( n = 0, \cdots, 12 \). Recently it has become clear that there are in fact different types of \( Spin(32)/Z_2 \) instantons depending on the generalized second Stieffel-Whitney class \[20\]. An analysis in terms of F-theory \[21\] has shown that in a general \( Spin(32)/Z_2 \) heterotic compactification, instantons with and without vector structure are present, their contribution to the total instanton number being respectively \( 8 + 4n \) and \( 16 - 4n \), with the integer \( n \) satisfying \( -2 \leq n \leq 4 \). A simple heterotic realization of this idea can be obtained by embedding a \( U(1) \times SU(2) \) background in \( SO(32) \supset SU(16) \times U(1) \supset SU(14) \times U(1)' \times U(1) \times SU(2) \). Then the \( Spin(32)/Z_2 \) vacua can be labeled by giving the pair of instanton numbers \((k_{NA}, k_A)\) with \( k_{NA} = 8 + 4n \) and \( k_A = 16 - 4n \). The adjoint decomposition is

\[
496 = (1,0,0,3) + (\overline{14}, \frac{1}{2}, 0, 2) + (14, -\frac{1}{2}, 0, 2) + (195, 0, 0, 1) + 2(1,0,0,1) + \\
(1,1, \frac{1}{2\sqrt{2}}, 1) + (14, \frac{1}{2}, \frac{1}{2\sqrt{2}}, 2) + (91, 0, \frac{1}{2\sqrt{2}}, 1) + \\
(1, -1, -\frac{1}{2\sqrt{2}}, 1) + (\overline{14}, -\frac{1}{2}, -\frac{1}{2\sqrt{2}}, 2) + (\overline{91}, 0, -\frac{1}{2\sqrt{2}}, 1) \tag{2.7}
\]
where the two middle entries denote the $U(1)' \times U(1)$ charges. The massless spectrum that arises upon embedding $k_A = (16 - 4n)$ instantons in $U(1)$ and $k_{NA} = (8 + 4n)$ in $SU(2)$ is found using the index theorem formulae [22, 19]. For $-1 \leq n \leq 2$ we find the following $SU(14) \times U(1)' \times U(1)$ hypermultiplets

\[
(1 - \frac{n}{2})(1, 1, \frac{1}{2\sqrt{2}}) + (1 - \frac{n}{2})(1, -1, \frac{-1}{2\sqrt{2}}) + (1 - \frac{n}{2})(91, 0, \frac{1}{2\sqrt{2}}) + \\
(1 - \frac{n}{2})(9T, 0, -\frac{1}{2\sqrt{2}}) + (6 + n)(14, -\frac{1}{2}, -\frac{1}{2\sqrt{2}}) + (6 + n)(14, \frac{1}{2}, \frac{1}{2\sqrt{2}}) + \\
(2 + 2n)(14, -\frac{1}{2}, 0) + (2 + 2n)(\bar{14}, \frac{1}{2}, 0) + (33 + 8n)(1, 0, 0)
\]  

(2.8)

For $n = 3$ there are not enough instantons to support the $U(1)$ bundle. The corresponding instantons become small and give the spectrum of a pointlike instanton without vector structure [21]. The resulting model has a gauge group $SO(28) \times SU(2) \times Sp(4)$, a hypermultiplet content

\[
8(28, 2, 1) + 56(1, 1, 1) + \frac{1}{2}(28, 1, 8) + (1, 2, 8)
\]  

(2.9)

and one additional tensor multiplet. For $n = 4$ instantons without vector structure disappear and one just has the $SU(2)$ bundle with 24 instantons mentioned above. For $n = -2$, the situation is reversed, since there is left just a $U(1)$ bundle with 24 instantons. The resulting gauge group is $U(16)$ with hypermultiplets

\[
2(120, \frac{1}{2\sqrt{2}}) + 2(\bar{120}, -\frac{1}{2\sqrt{2}}) + 20(1, 0)
\]  

(2.10)

For each value of $n$, appropriate sequential Higgsing produces chains of models that match similar $E_8 \times E_8$ heterotic chains [17], for the same value of $n$, thus providing several identifications between compactifications of both heterotic strings. This equivalence is evident in the F-theory framework, since the Calabi-Yau spaces obtained upon Higgsing (taking generic polynomials in the fibration over $\mathbb{F}_n$) are identical in both types of chains. Also, on the heterotic side perturbative T-dualities have been shown to relate $Spin(32)/Z2$ and $E_8 \times E_8$ compactifications for several values of $n$ [20]. By Higgsing it can be shown that the $Z_3$ models listed in Table 2 correspond to $n = 4, 4, 4, 1, 1$, respectively. The three $Z_2$ models correspond to $n = 4, 4, 0$. 

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To summarize, we see that standard orbifold compactifications provide a rich variety of perturbative $D = 6$ vacua, and have an interesting interplay with other techniques. Our purpose in this article is to extend this to non-perturbative constructions.

3 Non-Perturbative $D = 6$ Heterotic Orbifolds and Small Instantons

3.1 Non-Modular Invariant Heterotic Orbifolds

As reviewed in the previous section, different types of non-perturbative phenomena occur when the size of the instantons on a (smooth) $K3$ compactification goes to zero. The instanton singularity is softened by the presence of new multiplets of non-perturbative origin. This yields new classes of consistent $D = 6$, $N = 1$ non-perturbative heterotic vacua.

In this type of theories there is a perturbative part corresponding to the $E_8 \times E_8$ or $Spin(32)/Z_2$ degrees of freedom with a background of (large) instantons with total instanton number smaller than 24. There is also a non-perturbative piece which is not describable by usual perturbative string theory techniques. Although the perturbative side of these models is well understood, it has the shortcoming that generically this kind of $K3$ compactifications are not free conformal field theories (CFT). One would like to construct analogous heterotic vacua in which the perturbative part is just some free CFT like, in particular, a toroidal $Z_M$ orbifold. Furthermore, $D = 6$ orbifolds of M-theory [23, 12], F-theory [12, 24, 25, 26], and IIB orientifolds [27, 28] have been recently used [6, 13, 24, 7, 11, 29] to construct new vacua. It would be interesting to see whether there are connections between those models and orbifolds of heterotic theory.

Thus, as a first step, we would like to construct in this chapter non-perturbative vacua in which the perturbative side is describable by some sort of standard heterotic $Z_M$ orbifold. It is clear that this perturbative piece need not be modular invariant by itself, the crucial criterium would be now anomaly cancelation in the complete theory. Global
consistency of the theory requires the fulfillment of the constraint

\[ \int_X dH = 0 \]  \hspace{1cm} (3.1)

where \( H \) is the three-form heterotic field strength and \( X \) is the compact space (orbifold). In the presence of five-branes and/or orbifold fixed points in the \( X \) manifold, there are extra sources for \( dH \). Already at the perturbative level, there can be a non-vanishing flux of the antisymmetric tensor at orbifold singularities. At the non-perturbative level, there can be five-branes whose worldvolume fills the 6 uncompactified dimensions and which act as magnetic sources for the antisymmetric field. They can be understood as duals of type I D-branes in the \( SO(32) \) case and as M-theory five-branes in the \( E_8 \times E_8 \) case. In both cases the total charge associated to the antisymmetric tensor must vanish. This is

\[ \sum_f Q_f + n_B = 0 \]  \hspace{1cm} (3.2)

where \( Q_f \) is the magnetic charge associated to each fixed point and \( n_B \) is the number of five-branes present. We would like first to obtain heterotic orbifold models that would be the analogue of the smooth \( K3 \) compactifications in the presence of five-branes. In constructing this class of non-perturbative orbifold models we will be guided by the smooth \( K3 \) compactifications of \( E_8 \times E_8 \) in the presence of wandering five-branes considered in refs. [2, 8]. In order to make contact with these smooth compactifications we have to consider \( Z_M \) orbifolds in which there are massless oscillator modes appropriate to repair the orbifold singularities.

Let us start with the perturbative sector of the orbifold. We can represent the action on the \( E_8 \times E_8 \) or \( SO(32) \) degrees of freedom by a shift \( V \) in the gauge lattice. However, in searching for consistent models, we are now allowed to give up the modular invariance constraint eq.(2.1) since there are extra contributions which can help to cancel anomalies. We are interested in computing the massless spectra of such kind of models. The untwisted perturbative sector is obtained as in usual orbifold models and contains the \( N = 1, D = 6 \) supergravity sector, gauge multiplets in the \( E_8 \times E_8 \) (or \( SO(32) \)) subgroup invariant under the shift \( V \), charged hypermultiplets and the untwisted singlet moduli hypermultiplets. However the twisted sectors are novel in some respects since the fixed points are now sources for the antisymmetric field. Since the right-handed sector is supersymmetric, we
do not expect any special change for the corresponding zero modes of the mass formula. However, for the left-moving string it is reasonable to expect the presence of an appropriate shift in the vacuum level due to the antisymmetric field flux. Thus we will allow for an extra term $E_B(j)$ in the corresponding $\theta^j$ twisted sector mass formula

$$m_L^2 = N_L + \frac{1}{2} (P + j V)^2 + E_j + E_B(j) - 1$$  \hspace{1cm} (3.3)

Matching between left and right-movers will thus require

$$M (V^2 - v^2) + 2ME_B(j) = \text{even}$$  \hspace{1cm} (3.4)

for an order $M$ twisted sector. This can be understood as an orbifold version of the Bianchi identity $dH = \text{tr} F^2 - \text{tr} R^2$ as further discussed in the next chapter. If we are interested in vacua connected to smooth $K3$ compactifications, we have to impose the existence of oscillator moduli associated to the blowing-up of the singularities at the fixed points. For $M = 2$ one only finds $N_L = \frac{1}{3}$ solutions of eq.(3.3) for $V = 0$. The $M = 3$ case is richer and we discuss it in detail in the following, postponing the case of even $N$ to Chapter 6.

### 3.2 A Class of Non-Perturbative Heterotic $Z_3$ Orbifolds

For $Z_3$ orbifolds we only need consider the $\theta$ sector with extra energy shift $E_B$. The two moduli associated to each of the nine fixed points of the orbifold are given by $\alpha^i_{-\frac{1}{3}} |0\rangle$, where $i = 1, 2$, label the two compact complex dimensions and $|0\rangle$ is a twisted ground state, singlet under the non-Abelian gauge group of the model. Such $N_L = \frac{1}{3}$ oscillator modes will only be massless for

$$V^2 = \frac{8}{9} - 2 E_B$$  \hspace{1cm} (3.5)

Thus the maximum shift in the vacuum energy will correspond to $E_B = \frac{4}{9}$ (obtained for $V = 0$). The other extreme case is $V^2 = \frac{8}{9}$, in which we have $E_B = 0$ corresponding to some of the modular invariant (perturbative) models displayed in Tables 1 and 2. Models with $V^2 > \frac{8}{9}$ will not have the required oscillator states in their twisted spectra. (We will see in the next chapters that these longer shifts are of interest in certain $SO(32)$ heterotic transitions involving tensor multiplets, but we disregard them for the time being.)
Let us consider first the $SO(32)$ heterotic string. Consider the class of shifts $V$ with $3V \in \Gamma_{16}$ of the form

$$V = \frac{1}{3} (1, \cdots, 1, 0, \cdots, 0)$$

(3.6)

with an even number $m$ of $\frac{1}{3}$ entries and $m \leq 8$. The unbroken group is $U(m) \times SO(32 - 2m)$ and the untwisted sector contains hypermultiplets transforming as $(m, 32 - 2m) + (\frac{m(m - 1)}{2}, 1) + 2(1, 1)$. The twisted sectors have an extra vacuum shift $E_B = (\frac{8-m}{18})$ and the mass formula gives massless hypermultiplets in each twisted sector transforming as

$$\left( \frac{m(m - 1)}{2}, 1 \right) + 2(1, 1)$$

(3.7)

for $m = 0, 2, 4, 6, 8$.

There are two other $Z_3$ models with singlet moduli in the twisted sector. One of them, with shift $V = (\frac{2}{3}, 0, \cdots, 0)$, has gauge group $SO(30) \times U(1)$ and its spectrum is given in Table 3. The other model has shift $V = \frac{1}{6} (1, \cdots, 1)$, $3V$ being a spinorial weight. It is thus a $SO(32)$ embedding without vector structure, a $Z_3$ analogue of the $Z_2$ orientifolds constructed in [13]. The gauge group is $U(16)$ and the hypermultiplets are also displayed in Table 3, along with the spectra of the rest of these models.

Except for the $m = 8$ case, the rest of these models, as they stand, have gauge and gravitational anomalies and the corresponding shifts do not fulfill the perturbative modular invariance constraints. However, it turns out that the addition of an appropriate number of five-branes renders them consistent, very much in the same way that a smooth $K3$ compactification with instanton number $k < 24$ becomes consistent upon including $24 - k$ small instantons (five-branes). Indeed, one can check that adding $3(8 - m)$ five-branes to the vacua in eq.(3.6) (12 five-branes in the other two cases) leads to anomaly-free results. If we consider all of the five-branes coinciding at the same point (and away from singularities) a non-perturbative gauge group $Sp(n_B)$ is expected to appear, along with hypermultiplets transforming as

$$\frac{1}{2} (m, 1, 2n_B) + \frac{1}{2} (m, 1, 2n_B) + \frac{1}{2} (1, 32 - 2m, 2n_B) + (1, 1, \frac{2n_B(2n_B - 1)}{2} - 1) + (1, 1, 1)$$

(3.8)

with respect to $U(m) \times SO(32 - 2m) \times Sp(n_B)$. It is straightforward to check that all non-Abelian gauge and gravitational anomalies do cancel. Thus, our construction provides a
<table>
<thead>
<tr>
<th>Shift V Group</th>
<th>Untwisted matter</th>
<th>Twisted matter</th>
<th>$E_B$</th>
<th>$n_B$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, ···, 0)</td>
<td>2(1,1)</td>
<td>18(1,1)*</td>
<td>$\frac{4}{9}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$SO(32)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,0,···,0)$</td>
<td>(28,2)+3(1,1)</td>
<td>9(1,1)+18(1,1)*</td>
<td>$\frac{3}{5}$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$SO(28) \times SU(2) \times U(1)$</td>
<td></td>
<td></td>
<td></td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(2,···,0)$</td>
<td>(30)</td>
<td>9(30)+18(1)*</td>
<td>$\frac{2}{5}$</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$SO(30) \times U(1)$</td>
<td>+ 2(1,1)</td>
<td></td>
<td></td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,1,0,···,0)$</td>
<td>(24,4) + (1,6)</td>
<td>9(1,6)+18(1,1)*</td>
<td>$\frac{2}{5}$</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$SO(24) \times SU(4) \times U(1)$</td>
<td>+ 2(1,1)</td>
<td></td>
<td></td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,1,1,1,0,···,0)$</td>
<td>(20,6) + (1,15)</td>
<td>9(1,15) + 18(1,1)*</td>
<td>$\frac{1}{5}$</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>$SO(20) \times SU(6) \times U(1)$</td>
<td>+ 2(1,1)</td>
<td></td>
<td></td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,···,1)$</td>
<td>(120)+2(1)</td>
<td>18(1)*</td>
<td>$\frac{2}{5}$</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$SU(16) \times U(1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,1,1,1,1,1,0,···,0)$</td>
<td>(16,8)+(1,28)</td>
<td>9(1,28)+18(1,1)*</td>
<td>0</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>$SO(16) \times SU(8) \times U(1)$</td>
<td>+ 2(1,1)</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}(1,1,1,1,1,1,1,1,0,···,0)$</td>
<td>(16,8)+(1,28)</td>
<td>9(1,1)</td>
<td>$\frac{3}{5}$</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>$SO(16) \times SU(8) \times U(1)$</td>
<td>+ 2(1,1)</td>
<td></td>
<td></td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Non-Perturbative $Z_3$, Spin(32)/$Z_2$, orbifold models. $n_B$ gives the number of five-branes needed to obtain cancelation of anomalies. The last example, discussed in Chapter 5, requires instead just nine tensor multiplets and no enhanced gauge group nor extra hypermultiplets. It is connected with the next to last model which is in fact perturbative.
new class of consistent non-perturbative orbifold heterotic vacua.

Notice that the models obtained require the addition of 6s, s = 4, 3, 2, 1, 0, five-branes. They contribute one unit of magnetic charge each. Thus, in order to have an overall vanishing magnetic charge, each of the fixed points (which in these particular models are identical) must carry magnetic charge \( Q_f = -\frac{n_f}{n_f} \), \( n_f \) being the number of fixed points (9 in the case at hand). Thus the fixed points carry charges \( Q_f = -\frac{8}{3}, -2, -\frac{4}{3}, -\frac{2}{3}, -\frac{4}{3} \) respectively for each of the first six models in Table 3. Notice also that the shift in the left-handed vacuum energy is in each case given by \( E_B = -\frac{Q_f}{6} \).

The \( E_8 \times E_8 \) case is to some extent similar but has some peculiarities. Consider the class of models generated by gauge shifts of the form

\[
V = \frac{1}{3}(1, \cdots, 1, 0, \cdots, 0) \times \frac{1}{3}(1, \cdots, 1, 0, \cdots, 0)
\]

with an even number \( m_1 \) \( (m_2) \) of \( \frac{1}{3} \) entries in the first (second) \( E_8 \) and with \( m_1 + m_2 \leq 8 \). Models with appropriate oscillator moduli in the twisted sector have \( (m_1, m_2) = (0,0), (2,0),(4,0), (2,2), (2,4) \) and \( (4,4) \). The corresponding gauge groups and hypermultiplet spectra are given in Table 4 (recall that some shifts shown in the table can be written in the form (3.9) through \( E_8 \) lattice automorphisms). Again, all of these models (except for \( (m_1, m_2) = (4,4) \)) do not fulfill the perturbative modular invariance constraints and are also anomalous. However, unlike the \( SO(32) \) case, they do not have non-Abelian gauge anomalies. We can check that they miss an equivalent of \( 3(8 - m_1 - m_2) \times 30 \) hypermultiplets in order to cancel gravitational anomalies. But this is precisely the contribution corresponding to the presence in the spectrum of \( 3(8 - m_1 - m_2) \) tensor multiplets and the same number of hypermultiplets. These missing modes match the non-perturbative spectrum corresponding to setting this same number of instantons to zero size in \( E_8 \times E_8 \).

This is a nice check of our procedure since the simple addition of a shift in the vacuum energy automatically takes into account the difference between the \( SO(32) \) and \( E_8 \times E_8 \) heterotic strings, yielding no gauge anomalies in the second case. The \( Z_3 \) models under consideration are orbifold analogues of the \( E_8 \times E_8 \) vacua in the presence of wandering branes considered in refs. [2, 8]. In fact, if we use the hypermultiplets to Higgs the theory as much as possible we find that the final massless spectra corresponds to that found in smooth (fully Higgsed) \( K3 \) compactifications with instanton numbers \((k_1, k_2)\), as given in
the final column of Table 4, and \((24 - k_1 - k_2)\) M-theory five-branes.

A few comments on anomaly cancelation are in order. In the \(E_8 \times E_8\) case, although the pure quartic gauge anomaly vanishes, there is no full factorization of the total anomaly, as expected. In the anomaly polynomial there is a factorized piece which is canceled by the exchange of the usual perturbative tensor multiplet and an extra piece of the form \(n_B(F_1^2 - F_2^2)^2\) which is canceled \([8, 30]\) by the contribution of the \(n_B\) tensor multiplets coming from the five-branes. Concerning \(U(1)\) anomalies, they do not factorize, neither in \(SO(32)\) nor in \(E_8 \times E_8\) models. In fact they are spontaneously broken much in the same way as in the models with anomalous \(U(1)\)'s of refs. \([22, 19, 20]\). In contrast, one can check that the \(U(1)\)'s appearing in modular invariant perturbative orbifolds do always have standard factorization.

To end this section we will describe smooth \(K3\) compactifications with the same kind of spectra as the non-perturbative orbifolds with embedding \((3.6)\). Consider then instantons in the \(U(1)\) background in \(SO(32) \supset SU(m) \times SO(32 - 2m) \times U(1)\). This \(U(1)\) has generator \(Q = \frac{1}{\sqrt{2m}}(1, \cdots, 1, 0, \cdots, 0)\), with \(m\) non-zero entries. The massless spectrum resulting from embedding \(k\) instantons in this \(U(1)\) follows from the index theorem \([22, 19]\).

We find

\[
\begin{align*}
(2k - 1)[(m(m - 1), 2, \sqrt{2m}) & + (m(m - 1), 1, -\sqrt{2m})] \\
+ (k - 1)[(m, 32 - 2m, \sqrt{2m}) & + (m, 32 - 2m, -\sqrt{2m})] + 20(1, 1, 0)
\end{align*}
\] (3.10)

Neglecting \(U(1)\) charges, we then see that for \(k = 3m\) eq. \((3.10)\) reduces to

\[
\begin{align*}
[(m, 32 - 2m) + (m(m - 1), 1)] & + 2(1, 1) \\
+ 9[(m(m - 1), 1) + 2(1, 1)]
\end{align*}
\] (3.11)

This is precisely the perturbative untwisted plus twisted content of the non-modular invariant orbifolds that we have considered. Notice also that the number of required five-branes is \(n_B = 3(8 - m)\) as we found before. A similar exercise can be carried out for the two remaining embeddings.
<table>
<thead>
<tr>
<th>Shift $V$</th>
<th>Group</th>
<th>Untwisted matter</th>
<th>Twisted matter</th>
<th>$E_B$</th>
<th>$n_B$</th>
<th>$(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, \cdots, 0) \times (0, \cdots, 0)$</td>
<td>$E_8 \times E_8$</td>
<td>2(1,1)</td>
<td>18(1,1)*</td>
<td>4/5</td>
<td>0</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$1/3 (1,1,0, \cdots, 0) \times (0, \cdots, 0)$</td>
<td>$E_7 \times U(1) \times E_8$</td>
<td>(56,1)+3(1,1)</td>
<td>9(1,1)+18(1,1)*</td>
<td>3/5</td>
<td>6</td>
<td>(12,0)</td>
</tr>
<tr>
<td>$1/3 (2, \cdots, 0) \times (0, \cdots, 0)$</td>
<td>$SO(14) \times U(1) \times E_8$</td>
<td>(14,1)+(64,1)</td>
<td>9(14,1)+18(1,1)*</td>
<td>2/5</td>
<td>6</td>
<td>(12,0)</td>
</tr>
<tr>
<td>$1/3 (1,1,0, \cdots, 0) \times 1/3 (1,1,0, \cdots, 0)$</td>
<td>$E_7 \times U(1) \times E_7 \times U(1)$</td>
<td>56(1)+1(56)</td>
<td>18(1,1)+18(1,1)*</td>
<td>2/5</td>
<td>6</td>
<td>(12,0)</td>
</tr>
<tr>
<td>$1/3 (2, \cdots, 0) \times 1/3 (2, \cdots, 0)$</td>
<td>$SO(14) \times SO(14) \times U(1)^2$</td>
<td>(14,1)+(64,1) + (1,14) + (1,64)</td>
<td>9(14,1)+9(1,14) + 18(1,1)*</td>
<td>1/5</td>
<td>6</td>
<td>(12,12)</td>
</tr>
<tr>
<td>$1/3 (2, \cdots, 0) \times 1/3 (2, \cdots, 0)$</td>
<td>$SO(14) \times SO(14) \times U(1)^2$</td>
<td>(14,1)+(64,1)</td>
<td>9(1,1)</td>
<td>3/5</td>
<td>6</td>
<td>Sp(1)^9</td>
</tr>
</tbody>
</table>

Table 4: Non-Perturbative $Z_3$, $E_8 \times E_8$, orbifold models. $E_B$ denotes the extra vacuum energy shift. $n_B$ refers to the number of M-theory five-branes needed to cancel anomalies. The last example, discussed in Chapter 5, requires different dynamics involving the presence of extra gauge group and charged hypermultiplets. The next to last example is modular invariant, but has been included for comparison with the last one, since they are connected by a transition.
3.2.1 Orbifolds with Wilson Lines

A larger class of similar models, both for $E_8 \times E_8$ and $SO(32)$, can be obtained if there are additional quantized Wilson lines [4]. The corresponding action on the gauge degrees of freedom can be represented by shifts $a_i, i = 1, 2$, where $3a_i$ belongs to the $E_8 \times E_8$ or $Spin(32)/Z_2$ lattice and $i$ labels the two $T^2$ tori in $T^4 = T^2 \times T^2$. Now, each of the fixed points has associated to it one of the 9 possible shifts $V' = V + n_1a_1 + n_2a_2$ with $n_1, n_2 = 1, 0, -1$. In order to get a model with massless oscillators in all twisted sectors all shifts $V'$ have to obey similar constraints as before, although now the shift $E_B$ in each twisted sector will be different. Two $E_8 \times E_8$ examples that require respectively the addition of 8 and 4 five-branes are described in Tables 5 and 6. In the first example three of the fixed points have magnetic charge $Q_f = -\frac{4}{3}$ and the other six have $Q_f = -\frac{2}{3}$. In the second model three of the fixed points have $Q_f = -\frac{4}{3}$ and the other six $Q_f = 0$. Thus, in this second example only three of the fixed points cause the non-perturbative phenomena. Notice that the total number of five-branes required in this class of models is given by

$$n_B = -\sum_f Q_f = 6 \sum_f E_B(f)$$

(3.12)

where the sum goes over all fixed points. In this class of models one can get theories with any even number of five-branes from 0 to 24.

<table>
<thead>
<tr>
<th>$V = \frac{1}{3}(1,1,0\cdots,0) \times \frac{1}{3}(1,1,0,\cdots,0)$</th>
<th>$a_1 = \frac{1}{3}(0,\cdots,0,1,1) \times (0,\cdots,0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(12) \times U(1)^2 \times E_7 \times U(1)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$SO(12) \times U(1)^2 \times E_7 \times U(1)$</td>
</tr>
<tr>
<td>Sector</td>
<td>Matter</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Untwisted</td>
<td>$(32,1)+ (1,56)+4(1,1)</td>
</tr>
<tr>
<td>$V' = V$</td>
<td>6(1,1)+6(1,1)*</td>
</tr>
<tr>
<td>$V' = V + a_1$</td>
<td>3(12,1)+9(1,1)+6(1,1)*</td>
</tr>
<tr>
<td>$V' = V - a_1$</td>
<td>3(12,1)+9(1,1)+6(1,1)*</td>
</tr>
</tbody>
</table>

Table 5: A $Z_3$ non-perturbative $E_8 \times E_8$ orbifold model with one Wilson line. Anomaly cancelation requires the presence of 8 five-branes.
\[ V = \frac{1}{3}(1,1,0,\cdots,0) \times \frac{1}{3}(1,1,0,\cdots,0) \quad a_1 = \frac{1}{3}(0,\cdots,0,2) \times (0,\cdots,0) \]

\[ \text{SO}(10) \times \text{SU}(2) \times U(1)^2 \times E_7 \times U(1) \]

<table>
<thead>
<tr>
<th>Sector</th>
<th>Matter</th>
<th>(Q_f)</th>
<th>(E_B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Untwisted</td>
<td>(10,2,1)+(1,1,56)+4(1,1,1)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>(V' = V)</td>
<td>6(1,1)+6(1,1)*</td>
<td>-4/3</td>
<td>2/9</td>
</tr>
<tr>
<td>(V' = V + a_1)</td>
<td>3(10,1,1)+3(16,1,1)+6(1,1,1) +3(1,2,1) +6(1,2,1)* +6(1,1,1)*</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(V' = V - a_1)</td>
<td>3(10,1,1)+3(16,1,1)+6(1,1,1) +3(1,2,1) +6(1,2,1)* +6(1,1,1)*</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6: A \(Z_3\) non-perturbative \(E_8 \times E_8\) orbifold model with one Wilson line. Anomaly cancelation requires the presence of 4 five-branes.

### 3.3 Transitions between Perturbative and Non-Perturbative Orbifold Vacua

An interesting question is whether there is any shift \(V\) in \(E_8 \times E_8\) or \(\text{Spin}(32)/\mathbb{Z}_2\) which admits both spectra with and without five-branes. This is interesting because it can indicate possible transitions between perturbative and non-perturbative vacua which proceed through the emission of five-branes to the \(K3\) bulk. Comparing Tables 1 to 4 we see that indeed, there is a unique case corresponding to the 'standard embedding', \(V = \frac{1}{3}(1,1,0,\cdots,0) \times (0,\cdots,0)\) \((V = \frac{1}{3}(1,1,0,\cdots,0)\) for \(\text{Spin}(32)/\mathbb{Z}_2\)\) in which there are both a model without five-branes and a model with 18 five-branes. Both models have identical untwisted perturbative spectrum but differ in that the twisted spectrum of the perturbative model has extra hypermultiplets, transforming (in the \(E_8 \times E_8\) case) as \((56,1)+7(1,1)\), while the non-perturbative one contains just three singlets per fixed point. Also, the fixed points in the non-perturbative model have magnetic charge \(Q_f = -2\). This suggests that there can be transitions by which, around a fixed point in the perturbative model, a set of hypermultiplets \((56,1)+4(1,1)\) is converted into two five-branes which are emitted to the bulk. The magnetic charge is conserved in the process since each fixed
point has charge $Q_f = -2$ and each of the five-branes has charge $+1$. This can happen fixed point by fixed point so that there should exist similar models to the second model in Table 4 with any even number of five-branes in between 2 and 18. Thus, in this standard embedding models there is a discrete degree of freedom which corresponds to having pairs of zero size instantons.

These are the unique heterotic $Z_3$ models (of the type being considered) that admit both perturbative and non-perturbative realizations. Notice that this is related to the fact that, unlike what happens with the other $Z_3$ models, one can locally cancel the magnetic charge in each fixed point by moving two five-branes into each fixed point. This is not possible in the other $Z_3$ examples. We will see other types of transitions involving a Coulomb phase in $SO(32)$ in chapter 5.

We have discussed in this chapter heterotic orbifold models rendered consistent by the addition of five-branes which move in the bulk. In all these models there are 20 massless singlets corresponding to the underlying $K3$ moduli and then the known physics of small instantons in smooth $K3$ compactifications is expected to apply. We have not considered shifts of the form $V = \frac{1}{3}(2,1,1,0,\cdots,0)$. This case is special since, unlike the other models, the twisted oscillators are not singlets under the non-Abelian gauge group and their interpretation as $K3$ moduli is not obvious. In fact, models obtained by setting $E_B = \frac{1}{9}$ lack some hypermultiplets to cancel gravitational anomalies.

A natural question is what happens with orbifold models in which some moduli are absent. This happens in $Z_3$ if the gauge shifts have $V^2 > \frac{8}{9}$, as we discussed above. That also happens in the above orbifold models if, for instance, we increase the vacuum energy $E_B \rightarrow E_B + \frac{1}{3}$. In this case the singular orbifold cannot be smoothed out and some five-branes can be stuck at the orbifold singularities. We then need to know the physics of small instantons at $Z_M$ singularities. The present understanding of this topic is still incomplete (see refs. [31, 21, 5, 32, 33, 34]). Some interesting results are known about the existence of new Coulomb phases in $SO(32)$, $D = 6$ theories when a sufficiently large number of small instantons sit at an orbifold singularity [5]. Concerning the equivalent situation in $E_8 \times E_8$, results about the non-perturbative enhanced gauge groups are known in terms of F-theory [34], but we lack as yet sufficient information about the hypermultiplets. We will show in Chapter 5 how indeed the general class of orbifold models discussed in
this chapter might correspond in some cases to $SO(32)$ heterotic vacua in one of these Coulomb phases with extra tensor multiplets.

## 4 Index Theorems and Orbifold Singularities

We would like now to interpret some of the results found in the previous section in terms of index theorems for instantons on $Z_M$ ALE spaces. This will give us a better understanding of the class of orbifold models that we are constructing. In particular, we will see that the number of hypermultiplets found in the twisted sectors of the orbifolds coincides with the dimension of the moduli space of instantons at $Z_M$ singularities, both for the $Spin(32)/Z_2$ and $E_8 \times E_8$ cases. Another interesting and important question is how to relate the modular invariance condition (2.1), or more generally the shift $V$, with the anomaly conditions (2.6) and (2.4).

To answer these questions, we consider first the $SO(32)$ heterotic string on $T^4/Z_M$. We then need to study $SO(32)$ instantons on orbifold singularities. These instantons are characterized by an integer $\ell$ and also by a rotation $\Theta$ in $SO(32)$ that takes into account the orbifold action. $\Theta$ is related to the shift $V$ by $\Theta = \text{diag}(e^{\pm 2\pi i V_1}, \ldots, e^{\pm 2\pi i V_{16}})$. The condition $MV \in \Gamma_{16}$ implies $\Theta^M = \pm 1$. Indeed, when $MV$ is a root weight of $\Gamma_{16}$, $\Theta^M = 1$ and the instantons have vector structure, according to the terminology of ref. [20]. When $MV$ is instead a spinor weight of $\Gamma_{16}$, $\Theta^M = -1$ and the instantons do not have vector structure.

For instantons with vector structure the eigenvalues of $\Theta$ are of the form $e^{2\pi i \mu/M}$, with $\mu = 0, 1, \ldots, M - 1$. We define $\bar{w}_\mu$ as the number of such eigenvalues. Notice that $\bar{w}_\mu = \bar{w}_{M-\mu}$ and $\sum_\mu \bar{w}_\mu = 32$. Then, the instanton number at a $Z_M$ fixed point turns out to be [5]

$$I_f = \ell + \sum_{\mu=0}^{M-1} \frac{\mu(M - \mu)}{4M} \bar{w}_\mu \quad \text{(4.1)}$$

We will also find convenient to define $w_\mu$ as the number of entries equal to $\frac{\mu}{M}$ in $V$. Clearly, $\bar{w}_0 = 2w_0$, $\bar{w}_\mu = w_\mu$ for $\mu < P$ and $\bar{w}_P = 2w_P$ ($\bar{w}_P = w_P$) for $M = 2P$ ($M = 2P + 1$).

The result in eq. (4.1) can also be written as

$$I_f = \ell + ME_\Theta \quad \text{(4.2)}$$
where
\[ E_\Theta = \sum_{I=1}^{16} \frac{1}{2} V_I (1 - V_I) \] (4.3)

This \( E_\Theta \) is the twisted vacuum energy associated to \( \Theta \). On the other hand, the Bianchi identity \( dH = \text{tr} F^2 - \text{tr} R^2 \), integrated around a fixed point yields

\[ Q_f = \int dH = I_f - C_2(\varepsilon_M) \] (4.4)

where \( C_2(\varepsilon_M) = (M^2 - 1)/M \) is the Euler number associated to the ALE space \( \varepsilon_M \). The latter can be written in terms of the vacuum energy \( E_\theta = (M - 1)/M^2 \), corresponding to the four compactified bosonic coordinates, as \( C_2(\varepsilon_M) = M(M + 1)E_\theta \). Then one can finally write

\[ Q_f = \ell' + M(E_\Theta - E_\theta) \] (4.5)

with \( \ell' = \ell - M + 1 \). For \( Q_f = 0 \) one recovers the usual modular invariance constraints of perturbative orbifolds, whereas for a non-vanishing \( H \)-flux at the fixed points one gets a modified level matching constraint, as we advanced in the previous chapter.

Given \( I_f \), we can also compute the total instanton number as

\[ k = \sum_f I_f \] (4.6)

It is straightforward to verify that the modular invariance constraint (2.1) on \( V \) implies the condition that the total instanton number adds to 24, for some integer \( \ell \). To this end, recall that \( T^4/Z_2 \) has 16 \( Z_2 \) fixed points; \( T^4/Z_3 \) has 9 \( Z_3 \) fixed points; \( T^4/Z_4 \) has 4 \( Z_4 \) fixed points plus 6 \( Z_2 \) fixed points; and \( T^4/Z_6 \) has 1 \( Z_6 \) fixed point, 4 \( Z_3 \) fixed points and 5 \( Z_2 \) fixed points.

In a general model (modular invariant or not), the total instanton number (4.6) satisfies (2.6). For given \( \ell \) and \( V \) we can then determine the number of allowed five-branes. To illustrate the foregoing discussion we will focus on the \( Z_3 \) orbifold. In this case, an embedding with vector structure has \( m \) eigenvalues \( e^{\pm 2\pi i/3} \) so that

\[ I_f = \ell + \frac{m}{3} \] (4.7)

The total instanton number is then

\[ k = 9\ell + 3m \] (4.8)
The condition \( k = 24 \) is satisfied for \( m = 2, 5, 8, 11, 14 \) with \( \ell = 2, 1, 0, -1, -2 \), respectively. The corresponding shifts are precisely those given in Table 2.

We can also consider the models studied in Section 3 having shift (3.6) and \( n_B = 24 - 3m \). From (4.8) and (2.6) it follows that these models have \( \ell = 0 \) and necessarily \( m \leq 8 \), since otherwise they would have \( k > 24 \) (for \( \ell = 0 \)). Notice that this bound also appeared in the orbifold construction, but for different reasons.

It is also possible to give general results for the number of massless hypermultiplets at a \( Z_M \) fixed point with instanton number \( I_f \). This number is related to the dimension of the moduli space of instantons, \( M_{\text{inst}}(M) \), and follows from the index theorem in [37] for manifolds with boundary (the case of \( Z_M \) ALE spaces, for which the boundary is at infinity is studied in [38]). For embeddings with vector structure it is found that [5]

\[
\dim M_{\text{inst}}(M) = 30I_f + \frac{1}{2} \left[ \sum_{\mu,\nu=0}^{M-1} \bar{w}_\mu \bar{w}_\nu X_{\mu\nu} - \sum_{\mu=0}^{M-1} \bar{w}_\mu X_{\mu,M-\mu} \right] \tag{4.9}
\]

where \( X_{\mu\nu} \) is defined by

\[
X_{\mu\nu} = -\frac{1}{4M} |\mu - \nu|(M - |\mu - \nu|) \tag{4.10}
\]

At a \( Z_M \) ALE space there are also \((M-1)\) blowing-up modes that must be taken into account.

Let us again illustrate these results for the \( Z_3 \) orbifold. For \( \Theta \) with \( m \) eigenvalues \( e^{\pm 2\pi i/3} \), the total number of states at a fixed point, denoted \( N_f(3) \), is given by

\[
N_f(3) = 30\ell + \frac{1}{2} m(m-1) + 2 \tag{4.11}
\]

Notice that for the modular invariant embeddings with \( m = 2, 5, 8, 11, 14 \) and \( \ell = 2, 1, 0, -1, -2 \), \( N_f(3) \) agrees with the number of states per fixed point given in the third column of Table 2. Moreover, notice that for \( \ell = 0 \), eq. (4.11) agrees precisely with the number of massless states, cf. eq. (3.7), derived from the mass formula with an extra term \( E_B = \frac{(8-m)}{18} \).

We now give a derivation of the result for \( \dim M_{\text{inst}} \), for a particular \( Z_3 \) embedding, starting directly from the index theorem on a \( Z_3 \) ALE space \( \varepsilon_3 \). This exercise will show how to generalize to the \( E_8 \times E_8 \) case. Consider then, as in Section 3.1, instantons in the \( \mathbf{U}(1) \) background in \( \mathbf{SO}(32) \supset \mathbf{SO}(32-2m) \times \mathbf{SU}(m) \times \mathbf{U}(1) \). As we have seen, this \( \mathbf{U}(1) \)
has generator $Q = \frac{1}{\sqrt{2m}} (1, \cdots, 1, 0, \cdots, 0)$, with $m$ non-zero entries. In the decomposition of the adjoint we find $m(32 - 2m)$ states of charge $q = \pm \frac{1}{\sqrt{2m}}$ and $\frac{m(m-1)}{2}$ states of charge $q = \pm \frac{2}{\sqrt{2m}}$. These states give rise to charged hypermultiplets whose number can be computed from the index theorem. In general, the number of hypermultiplets of charge $q$, denoted $D(q)$, is given by

$$D(q) = -\frac{1}{24} C_2(\epsilon_3) + q^2 I_f + \frac{1}{2} \xi_{\frac{1}{2}}(q)$$

Here $C_2(\epsilon_3) = \frac{8}{3}$ is the Euler number of $\epsilon_3$ and $I_f$ is given in eq. (4.7). The boundary correction $\xi_{\frac{1}{2}}(q)$ is given by

$$\xi_{\frac{1}{2}}(q) = \frac{1}{3} \sum_{j=1}^{2} \frac{e^{2\pi i \sqrt{2m} q j/3}}{2(1 - \cos \frac{2\pi q j}{3})}$$

where the phase in the numerator is such that a state with charge $q = \frac{1}{\sqrt{2m}}$ picks up a phase $e^{2\pi i/3}$ while going once around the fixed point. We readily obtain $\xi_{\frac{1}{2}}(\pm \frac{1}{\sqrt{2m}}) = -\frac{1}{9}$, $\xi_{\frac{1}{2}}(\pm \frac{2}{\sqrt{2m}}) = -\frac{1}{9}$. Therefore,

$$D(\frac{1}{\sqrt{2m}}) = D(-\frac{1}{\sqrt{2m}}) = \frac{\ell}{2m}$$
$$D(\frac{2}{\sqrt{2m}}) = D(-\frac{2}{\sqrt{2m}}) = \frac{2\ell}{m} + \frac{1}{2}$$

The total number of states is then

$$\frac{\ell}{m} [m(32 - 2m)] + \left(\frac{4\ell}{m} + 1\right) \frac{m(m-1)}{2} = 30\ell + \frac{m(m-1)}{2}$$

in agreement with eq. (4.11), once the two blowing-up modes are included.

The equivalent formulae for the $E_8 \times E_8$ case are not available in the literature so we will repeat the same analysis for $U(1)$ instantons in $E_8$. Consider for instance the $U(1)$ in $E_8 \supset E_7 \times U(1)$ with generator $Q_1 = \frac{1}{2}(1, 1, 0, \cdots, 0)$. In the adjoint decomposition we find 56 states of charge $q = \pm \frac{1}{2}$ and one state of charge $q = \pm 1$. Since $I_f = \ell_1 + \frac{2}{3}$ in this case, from eq. (4.12) we then find $D(\pm \frac{1}{2}) = \frac{\ell_1}{4}$ and $D(\pm 1) = \ell_1 + \frac{1}{2}$. Hence

$$\dim \mathcal{M}_{inst} = \frac{\ell_1}{2} \times 56 + (\ell_1 + \frac{1}{2}) \times 2 = 30\ell_1 + 1$$

We must also include the two blowing-up modes of the $Z_3$ fixed point. Taking $\ell_1 = 2$ we then reproduce the number of twisted states in the modular invariant standard embedding.
On the other hand, taking $\ell_1 = 0$ we recover the number of states in the non-perturbative $Z_3$ model with the same embedding, $E_B = \frac{1}{3}$, and $n_B = 18$, since $9I_f = 6$. Moreover, if we embed the same $U(1)$ in both $E_8$’s, we have $I_f = \ell_1 + \ell_2 + \frac{4}{3}$ and from the index theorem we conclude $\dim \mathcal{M}_{\text{inst}} = 30\ell_1 + 30\ell_2 + 2$, since there are no states in the $E_8 \times E_8$ adjoint with mixed charges. Setting $\ell_1 = \ell_2 = 0$ gives the same number of states found from the mass formula with $E_B = \frac{2}{3}$. Also, in this case $9I_f = 12$ so that 12 five-branes are needed.

It is a simple exercise to work out the number of states for other embeddings. For example, the $U(1)$ with generator $Q_1 = \frac{1}{\sqrt{2}}(1, 1, 1, 0, \cdots, 0)$ breaks $E_8$ to $SO(14) \times U(1)$ and gives 14 states with charge $q = \pm \frac{1}{\sqrt{2}}$ plus 64 states with charge $q = \pm \frac{1}{2\sqrt{2}}$. Using $I_f = \ell_1 + \frac{4}{3}$ and the index formula (4.12) we find

$$\dim \mathcal{M}_{\text{inst}} = \frac{\ell_1}{8} \times 128 + \frac{\ell_1 + 1}{2} \times 28 = 30\ell_1 + 14 \quad (4.17)$$

The embedding $Q_1 = \frac{1}{2\sqrt{2}}(2, 0, \cdots, 0)$ leads to the same results, as it should be due to equivalences in the $E_8$ lattice. Remembering to include the two blowing-up modes of the $Z_3$ fixed point, we readily recover the results for the third model in Table 4 that has $\ell_1 = 0$ and $9I_f = 12$ so that $n_B = 12$. The perturbative model with the same embedding in both $E_8$’s is also reproduced taking $\ell_1 = \ell_2 = 0$.

The results reported in this section give the hypermultiplet number in the twisted sectors of all models in Tables 1 to 4. Notice, however, that the orbifold analysis gives not only the number of hypermultiplets but their quantum numbers with respect to the perturbative gauge group. In addition, the orbifold construction gives the spectrum of the complete model including untwisted hypermultiplets, vector, tensor and gravity multiplets.

5 Branes at Fixed Points and Tensor Multiplets

Up to this point we have constructed orbifold models that contain enough blowing-up modes in their perturbative spectrum to smooth out the singular points completely. This feature served us as a guide to obtain the relevant non-perturbative effects rendering the theory consistent. However, as mentioned at the end of Section 3.3, one frequently encounters models not containing these blowing-up modes, and for which the addition
of tensors or $Sp(n_B)$ enhanced symmetries in the prescribed way does not cancel the anomalies completely.

A natural possibility is that in these models the positions of some or all of the five-branes are trapped at the (non-removable) singular points of the variety, so that the brane dynamics differs from that found at smooth points, and their low energy excitations are in this sense exotic. Our purpose in this section is to apply some known results about branes at singularities to the understanding of other families of non-perturbative orbifolds.

The behaviour of the $SO(32)$ heterotic five-branes near orbifold points can be extracted from the recent studies of type I D-five-branes on ALE spaces [31, 5, 32, 33], and some ideas borrowed from the $Z_2$ case, extensively analyzed from the F-theory point of view in [21]. In Table 7 we show the spectrum of some worldvolume theories of $SO(32)$ five-branes at $Z_M$ singularities which we will need in the rest of the article.

Specifically, in [5] it was argued (based on the analysis of [31] and anomaly considerations, further confirmed by a detailed determination of world-sheet consistency conditions [32]) that when a large enough number $\ell$ of five-branes sit on a $Z_3$ singular point (with possible vector structure), a $Sp(\ell) \times U(2\ell + m - 8)$ gauge symmetry develops with hypermultiplets transforming as

$$
(16 - m)(2\ell, 1) + (2\ell, 2\ell + m - 8) + m(1, 2\ell + m - 8) \\
+(1, \ell + \frac{m}{2} - 4)(2\ell + m - 9)
$$

The multiplicity can be understood as gauge quantum numbers under the perturbative symmetry group. Furthermore, there also appears an extra tensor degree of freedom. It is also stressed that on the Coulomb branch parametrized by the scalar in this tensor multiplet, one of the two perturbative blowing-up modes is absent, since the singular point cannot be completely smoothed out while preserving the tensor multiplet in the spectrum. Thus the possibility of understanding perturbative spectra with missing blowing-up modes opens up.

Note that this world-volume theory makes sense as long as $\ell \geq \frac{8-m}{2}$, thus a minimum value on $\ell$ is required for having the singular point on the Coulomb phase. The $m = 8$

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1As noticed in [21], the notion of vector structure is ill defined along the tensor Coulomb branch. We follow ref. [5] in relating the existence of a vector structure to the gauge shift, as explained in section 2.
<table>
<thead>
<tr>
<th>$Z_M$</th>
<th>Gauge Group</th>
<th>Hypermultiplets</th>
<th>$n_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Sp(\ell)$</td>
<td>$\frac{32}{2}(2\ell) + (\ell(2\ell - 1))$</td>
<td>0</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$Sp(\ell) \times Sp(\ell + \frac{w_1}{2} - 4)$</td>
<td>$w_0(2\ell, 1) + w_1(1, 2\ell + w_1 - 8) + (2\ell, 2\ell + w_1 - 8)$</td>
<td>1</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$Sp(\ell) \times SO(2\ell + 8)$ [(w_1 = 0)]</td>
<td>$(2\ell, 2\ell + 8)$</td>
<td>1</td>
</tr>
<tr>
<td>$Z_3$</td>
<td>$Sp(\ell) \times U(2\ell + w_1 - 8)$</td>
<td>$w_0(2\ell, 1) + w_1(1, 2\ell + w_1 - 8)$ [+ (2\ell, 2\ell + w_1 - 8) + (1, (\ell + \frac{\mu_2}{2} - 4)(2\ell + w_1 - 9))]</td>
<td>1</td>
</tr>
<tr>
<td>$Z_4$</td>
<td>$Sp(\ell) \times U(2\ell + w_1 + w_2 - 8)$ [\times Sp(\ell + \frac{w_1}{2} + w_2 - 8)]</td>
<td>$w_0(2\ell, 1, 1) + w_1(1, 2\ell + w_1 + w_2 - 8, 1)$ [+ w_2(1, 1, 2\ell + w_1 + 2w_2 - 16) + (2\ell, 2\ell + w_1 + w_2 - 8, 1)] [+ (1, 2\ell + w_1 + w_2 - 8, 2\ell + w_1 + 2w_2 - 16)]</td>
<td>2</td>
</tr>
</tbody>
</table>

| $Z_2$ | $U(2\ell)$ | $\frac{32}{2}(2\ell) + 2(\ell(2\ell - 1))$ | 0 |
| $Z_4$ | $U(2\ell) \times U(2\ell + u_2 - 8)$ | $u_1(2\ell, 1) + u_2(1, 2\ell + u_2 - 8) + (2\ell, 2\ell + u_2 - 8)$ \[+ (1, (\ell + \frac{\mu_2}{2} - 4)(2\ell + u_2 - 9)) + (\ell(2\ell - 1), 1)\] | 1 |
| $Z_6$ | $U(2\ell) \times U(2\ell + u_2 + u_3 - 8)$ \[\times U(2\ell + u_2 + 2u_3 - 16)\] | $u_1(2\ell, 1, 1) + u_2(1, 2\ell + u_2 + u_3 - 8, 1) + (\ell(2\ell - 1), 1, 1)$ \[+ u_3(1, 1, 2\ell + u_2 + 2u_3 - 16) + (2\ell, 2\ell + u_2 + u_3 - 8, 1)\] \[+ (1, 2\ell + u_2 + u_3 - 8, 2\ell + u_2 + 2u_3 - 16)\] \[+ (1, 1, (\ell + \frac{\mu_2}{2} + u_3 - 8)(2\ell + u_2 + 2u_3 - 17))\] | 2 |

Table 7: Some world-volume theories of $SO(32)$ five-branes at $Z_M$ singularities. Here $w_\mu$ is the number of entries equal to $\frac{\mu}{M}$ in $V$ with vector structure. Similarly, $u_\mu$ is the number of entries equal to $\frac{2\mu - 1}{2M}$ in $V$ without vector structure.
case is special, since the transition to the Coulomb branch is possible even for \( \ell = 0 \). The described spectrum reduces to just one tensor multiplet, without any gauge enhancement. Remarkably enough, this is precisely the non-perturbative contribution which is required to complete the \( Z_3 \) orbifold with shift \( V = \frac{1}{3}(1, 1, 1, 1, 1, 1, 1, 1, 0, \cdots, 0) \) and \( E_B = \frac{1}{3} \). Its spectrum is shown at the bottom of Table 3, and we see that each twisted sector contributes with a singlet. Once the extra tensors are added, the model is anomaly-free. Observe that it also reproduces the \( Z_3^A \) orientifold of ref. [6, 7]. It was noted in [5] that the origin of the tensors in this orientifold could be understood in terms of such a Coulomb phase. Our orbifold construction, on the other hand, yields a global matching with the orientifold spectrum, including the untwisted and twisted sectors, thus providing the complete heterotic dual.

Notice that there exists a modular invariant perturbative orbifold with this same shift \( V \), as shown in Table 3. Each fixed point contributes with one 28 of \( U(8) \) and two blowing-up modes, in agreement with the number of states obtained from the index theorem (4.11). This model corresponds to having the singular points in the Higgs phase. The transition to the Coulomb branch is dominated by tensionless strings in analogy with the familiar zero size \( E_8 \) instanton [39, 8], and in the process the spectrum at a fixed point changes as

\[
28 + 1 \rightarrow \text{tensor},
\]  

which is consistent with anomaly cancelation conditions, because the 28 does not contribute to the pure quartic gauge anomalies. The transition can occur locally at each fixed point, so that there is a whole family of models with the number of tensors \( n_T \) varying from zero to nine.

One can easily construct a further class of \( Z_3 \) models with \( m \leq 8 \) and singular points in the Coulomb phase. Consider a \( Z_3 \) twisted sector feeling a shift \( V = \frac{1}{3}(1, \cdots, 1, 0, \cdots, 0) \) with an even number \( m \) of \( \frac{1}{3} \) entries. Including an extra energy shift \( E_B = \frac{14-m}{18} \) leads to the required perturbative spectrum, namely one singlet per fixed point, playing the role of the surviving blowing-up mode. However, note that in order to reach the Coulomb branch at least \( \frac{(8-m)}{2} \) five-branes should be located at each fixed point, and this cannot be done for all of them, due to the bound of 24 for the total instanton number. Hence, one is forced to consider models that contain two kinds of fixed points. All share the same
gauge shift, but a number $9 - r$ of them have $E_B = \frac{8 - m}{18}$ (Higgs phase points) and the remaining $r$ have $E_B = \frac{14 - m}{18}$ (Coulomb branch points). A total of $3m$ large instantons is located at the fixed points, $\ell_i$ small instantons are sitting at the $i$-th singularity, and $(24 - 3m - \sum_{i=1}^{r} \ell_i)$ five-branes wander around the $K3$ bulk. The final spectrum is easily determined by adjoining to the orbifold perturbative states (with the corresponding $E_B$ at each twisted sector) the massless modes associated to the adequate number of trapped and wandering five-branes. The result when the $\ell_i$ are set to their critical value $\ell_i = \ell_c = \frac{8 - m}{2}$ (at which the non-perturbative unitary group is absent) and the $n_B = (6 - r)\ell_c$ wandering branes are coincident, is as follows

$$U(m) \times SO(32 - 2m) \times \prod_{i=1}^{r} Sp(\ell_c) \times Sp(n_B)$$

$$+ (m, 32 - 2m; 1, \cdots, 1; 1) + \left(\frac{m(m-1)}{2}, 1; 1, \cdots, 1; 1\right)$$

$$+ 2(1, 1; 1, \cdots, 1; 1)$$

$$+(9-r) \left[\left(\frac{m(m-1)}{2}, 1; 1, \cdots, 1; 1\right) + 2(1, 1; 1, \cdots, 1; 1)\right]$$

$$+ r(1, 1; 1, \cdots, 1; 1) + \frac{1}{2}(1, 32 - 2m; 1, 2\ell_c, \cdots, 1; 1) + r \text{ tensors}$$

$$+ (m, 1; 1, \cdots, 1; 2n_B) + \frac{1}{2}(1, 32 - 2m; 1, \cdots, 1; 2n_B)$$

$$+ (1, 1; 1, \cdots, 1; n_B(2n_B - 1) - 1) + (1, 1; 1, \cdots, 1; 1)$$

where underlining means permutation. One can check that all gauge and gravitational anomalies cancel. Thus, these two possible $E_B$ in this class of models reproduce the two phases of the dynamics of branes at the singular point. In analogy with the $m = 8$ case, one can easily follow the change of states involved in the transition to the Coulomb branch ($r \rightarrow r + 1$).

With the knowledge acquired we can now interpret yet another class of models, whose fixed points have shifts $V = \frac{1}{3}(1, \cdots, 1, 0, \cdots, 0)$, with a number of nonzero entries $8 < m \leq 14$, and $E_B = \frac{14 - m}{18}$. The formula for the vacuum energy shift as function of the monodromy twist, and the resulting perturbative spectrum, just a singlet per fixed point, suggest that this kind of fixed points are frozen at the Coulomb branch and should be completed using the non-perturbative content described at the beginning of this section. As happened before, we must face a global subtlety due to the constraint on the total instanton number on $K3$. Following eq. (4.7), we see that, even if we set $\ell = 0$ for all the nine fixed points, more than 24 instantons are required. The solution consists again in
not treating all singular points symmetrically, though in this case they will also differ in the shifts they feel, by means of the introduction of quantized Wilson line backgrounds.

The simplest such example has

\[
V = (0, \cdots, 0) \\
A = \frac{1}{3}(1, \cdots, 1, 0, \cdots, 0)
\]

(5.4)

with the Wilson line \( A \) having \( m \) nonzero entries. We have six points feeling the shift \( A \) under study and three with trivial monodromy. Let us briefly discuss its spectrum. The untwisted sector contains the gauge group \( U(m) \times SO(32 - 2m) \) and two singlets. In the twisted sector, the three points with shift \( V = 0 \) contribute with two blowing-up singlets each, as expected from index theory, while the remaining six points give one perturbative singlet each. The total instanton number used up to now is \( 2m \), so the non-perturbative spectrum to be added corresponds to the massless modes of \( 24 - 2m \) wandering five-branes and six fixed points on the Coulomb phase. The final spectrum (the wandering branes are taken coincident, for notational convenience) is

\[
U(m) \times SO(32 - 2m) \times Sp(24 - 2m) \times U(m - 8)^6 \\
\begin{array}{c}
14(1, 1; 1, 1, \cdots, 1) \\
+ (m, 1; 48 - 4m; 1, \cdots, 1) + \frac{1}{2}(1, 32 - 2m; 48 - 4m; 1, \cdots, 1) + \\
(1, 1; 48 - 4m; 47 - 4m; 1, \cdots, 1) + (1, 1; 1, 1, 1, \cdots, 1) \\
+ (m, 1; m - 8, \cdots, 1) + (1, 1; \frac{m - 8(m - 9)}{2}, \cdots, 1) + 6 \text{ tensors}
\end{array}
\]

(5.5)

All gauge and gravitational anomalies cancel.

Another, more complicated, family of models can be constructed with the choice

\[
V = \frac{1}{3}(1, \cdots, 1, 0, \cdots, 0) \\
A = \frac{1}{3}\left(-\frac{1}{2}, \cdots, -\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right)
\]

(5.6)

with \( 8 \leq m \leq 14 \). It contains three sets of twisted subsectors (with three fixed points
each) feeling the gauge shifts

\begin{align*}
V &= \frac{1}{3}(1, \ldots, 1, 0, \ldots, 0), \quad E_B = \frac{m-8}{15} \\
V + A &= \frac{1}{3}(1, 2, \ldots, 1) \quad , \quad E_B = \frac{2}{9} \\
V + 2A &= \frac{1}{3}(0, \ldots, 0, 1, \ldots, 1) \quad , \quad E_B = \frac{14-m}{15}
\end{align*}

Thus, three fixed points do not have vector structure (their perturbative spectrum is shown in Table 3, and has been described in Section 3), three have a shift with \(16 - m \leq 8\) nonzero entries (and are in the Higgs branch) and three have a shift with \(m \geq 8\) nonzero entries (so are frozen at the Coulomb phase). In that situation the model must be completed non-perturbatively with four wandering branes and some Coulomb branch content, the final result being completely free of gauge and gravitational anomalies. The wandering branes can be used to put some of the Higgs phase points in the Coulomb phase (a process nicely accounted for through a change in the corresponding \(E_B\), as described above), always leading to consistent results.

We stress that the models we have described are the first global constructions in which one can follow the transitions to the Coulomb branch due to the piling up of small instantons at singular points. It is remarkable that the simple recipe of introducing \(E_B\) allows us to compute easily not only the number of states being swallowed in that transition, but also their quantum numbers under the perturbative and non-perturbative gauge symmetries.

Finally, let us point out that these results concern exclusively the \(SO(32)\) heterotic orbifolds, since only for them the dynamics of five-branes at singular points of \(K3\) has been determined with sufficient detail. Unfortunately such a detailed knowledge is still lacking for the \(E_8 \times E_8\) heterotic non-perturbative effects. However, some information can be extracted from the F-theory analysis of [34] for the case of unbroken \(E_8 \times E_8\). It can be shown that when four small \(E_8\) instantons coalesce on top of a \(Z_3\) singular point, a non-perturbative \(SU(2)\) appears, as well as four tensor multiplets, and hypermultiplets transforming as \(4(2)^\prime\)’s. As in the \(SO(32)\) case, one of the two blowing-up modes of the singular points disappears.

It is easy to construct a non-perturbative orbifold in which this proposal is realized.
Consider for example the $Z_3$ orbifold with $V=(0,\cdots,0)$ and $E_B = \frac{4}{9}$ in eight fixed points and $E_B = \frac{7}{9}$ in the remaining. The perturbative gauge group is $E_8 \times E_8$ and there are two untwisted moduli. The fixed points with $E_B = \frac{4}{9}$ generate two blowing-up modes each, while the fixed point with $E_B = \frac{7}{9}$ gives just one singlet. The missing modes signal the existence of instantons frozen at the singular point. Consequently we must add the spectrum just mentioned, associated to four $E_8$ instantons at an $A_2$ singularity. The model is finally rendered consistent by adding twenty wandering five-branes. We stress that the appearance of $E_8$ instantons stuck at singular points is due to the shift $E_B \rightarrow E_B + \frac{1}{3}$ in close analogy with the $SO(32)$ case (actually, this is consistent with the equivalence of both theories upon compactification to $D=5$ on a further $S^1$ [34]). We also notice that the F-theory version of this model is provided by the mirror of the last Calabi-Yau on page 32 of ref. [35], and its interpretation is consistent with the conjecture in [36].

It would be interesting to extend this constructions to the case of fixed points with non-trivial gauge twists, in order to check that the orbifold perturbative spectrum is completed by the appropriate non-perturbative contribution. As already mentioned, these spectra have not been determined in the literature. But reversing the viewpoint we can try to extract this information precisely by imposing the consistency of our non-perturbative orbifolds with missing blowing-up modes, which do not obey the usual ‘tensor + hypermultiplet’ rule.

Consider for example the $E_8 \times E_8$ heterotic $Z_3$ orbifold defined by the shift $V = \frac{1}{3}(1,1,1,1,0,0,0,0,0) \times (1,1,1,1,0,0,0,0,0)$ and $E_B = \frac{1}{3}$, whose perturbative spectrum is described in Table 4. The gauge group is $SO(14)^2 \times U(1)^2$, the untwisted hypermultiplet content is

$$(\mathbf{14}, \mathbf{1}) + (\mathbf{64}, \mathbf{1}) + (\mathbf{1}, \mathbf{14}) + (\mathbf{1}, \mathbf{64}) + 2(\mathbf{1}, \mathbf{1})$$

(5.8)

Also, taking into account the shift $E_B$, each twisted sector contributes with just one singlet hypermultiplet. Contrary to the usual $E_8 \times E_8$ case, besides gravitational anomalies, this spectrum presents severe $SO(14)$ gauge anomalies. There are nine missing $\mathbf{14}$’s of each $SO(14)$ that should arise non-perturbatively. A curious solution to the gravitational and gauge anomaly cancelation conditions is provided by adding a non-perturbative $Sp(1)^9$ gauge group and hypermultiplets transforming as $\frac{1}{2}(\mathbf{14}, \mathbf{1}; \mathbf{2}, \cdots, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{14}; \mathbf{2}, \cdots, \mathbf{1}) + 2(\mathbf{1}, \mathbf{1}; \mathbf{2}, \cdots, \mathbf{1})$. 
The situation is reminiscent of what happens in the $SO(32)$ case. This is perfectly sensible, since this orbifold model is associated to a heterotic compactification on a smooth $K3$ with $(12,12)$ instantons embedded in $E_8 \times E_8$, thus $n = 0$ in the notation of Section 2. This model is known to develop non-perturbative gauge symmetries at special loci in its moduli space, as required by heterotic/heterotic duality [2], and moreover these effects have been interpreted as the shrinking of instantons in the $SO(32)$ T-dual version [20]. Thus our choice of additional contributions should correspond to this kind of effects.

Note that the analogy with the Witten content is not complete, since the proposed spectrum does not contain the two-index antisymmetric representation of the $Sp(1)$’s. These singlets would parametrize the positions of the (T-dual) five-branes on $K3$, so its absence shows that the non-perturbative dynamics is frozen at the fixed points in the orbifold. In particular, one cannot make the dual five-branes coalesce, since no gauge enhancement is possible without those singlets.

Moreover, this model has a modular invariant relative with the same untwisted sector, but twisted matter containing the required $14$’s (see Table 1), and both are connected by Higgsing of the non-perturbative symmetry. Note that in this process the $64$’s are not touched (since they live in the untwisted sector) and so the dynamics involved does not correspond to the shrinking of an $E_8$ instanton $^2$.

In this sense, one encounters again that the procedure of shifting the energy by $E_B$ naturally determines the non-perturbative effects relevant to each model. This last model presents evidence in favour of new dynamics in $E_8 \times E_8$ heterotic compactification on singular varieties. As was mentioned above, some results on this topic have been obtained in [34] for the case of unbroken $E_8 \times E_8$, but a similar analysis as symmetry breaking takes place would be required to check the proposed spectrum.

$^2$A $U(1)$ bundle construction immediately shows that an $E_8$ instanton yields an anomaly-free combination of $14$’s and $64$’s.
6 Even $M \ Z_M$ Models and the Heterotic Duals of Type IIB Orientifolds

It is possible to generalize the non-perturbative $Z_3$ orbifold construction in order to treat $Z_2$, $Z_4$ and $Z_6$ orbifolds. However, there are some peculiarities. To start with, most of the $Z_2$ gauge shifts lead to orbifolds which have no singlets adequate to play the role of $K3$ moduli and the models obtained often do not correspond to smooth compactifications with some zero size instantons (as happened in the $Z_3$ models described in Chapter 3). Also, since $Z_4$ and $Z_6$ orbifolds have $Z_2$ subsectors, they share the same property. Take as an example the $Z_2$ orbifold in $Spin(32)/Z_2$ with standard embedding and $E_B = \frac{1}{2}$. There are no oscillator modes left that could be identified with $K3$ twisted moduli and the model has hypermultiplets in four $28$’s of $SO(28)$. To cancel gauge anomalies, sixteen $28$’s are needed that could come from sixteen small instantons. However, since there are no $K3$ moduli available, there could be one small instanton (with vector structure) stuck at each $Z_2$ fixed point. The non-perturbative spectrum corresponding to this case has not been determined in the literature. Notice that in this case the results in Table 7 are of no use since with $w_1 = 2$ we need a minimum of $l = 3$ at each fixed point and that would lead to an inconsistent vacum with total instanton number bigger than 24.

There are however some cases where this difficulty is absent, and we show below how the heterotic dual of the Gimon-Polchinski (GP) $Z_2$ orientifold with eight dynamical five-branes at the same fixed point can be understood as a particular heterotic $Z_2$ orbifold with small instantons. Also, we showed in the previous section how certain orbifold heterotic models without $K3$ moduli can be understood as models in a Coulomb phase with extra tensor multiplets. We will show that this is also the case in some $Z_4$ and $Z_6$ heterotic orbifolds. In the last subsection we construct a $Z_2$ orbifold with the same spectrum as orientifold models constructed by Dabholkar and Park [11] and Gopakhumar and Mukhi [12].
### 6.1 Heterotic Duals of Type IIB Orientifolds

To start with, consider the models presented in Table 8. These are the four $Z^A_M$ (in the notation of ref. [6]) Type IIB orientifolds [6, 7]. We will construct the heterotic duals of all these ($M$ even) models as orbifolds, with gauge embedding not verifying the modular invariance constraints, in the presence of small instantons.

<table>
<thead>
<tr>
<th>$Z^A_M$</th>
<th>$n_T$</th>
<th>(99)-Gauge Group</th>
<th>(99)-Hyper.</th>
<th>$Spin(32)/Z_2$ Shift</th>
<th>$E_B(j)$</th>
<th>$n_H(j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_2^A$</td>
<td>0</td>
<td>$U(16)$</td>
<td>2(120)</td>
<td>$V = \frac{1}{4}(1,\ldots,1)$</td>
<td>$\frac{1}{4}$</td>
<td>16</td>
</tr>
<tr>
<td>$Z_3^A$</td>
<td>9</td>
<td>$SO(16) \times U(8)$</td>
<td>$(1,28) + (16,8)$</td>
<td>$V = \frac{1}{5}(1,1,1,1,1,1,1,0,\ldots,0)$</td>
<td>$\frac{1}{5}$</td>
<td>9</td>
</tr>
<tr>
<td>$Z_4^A$</td>
<td>4</td>
<td>$U(8) \times U(8)$</td>
<td>$(28,1) + (1,28)$</td>
<td>$V = \frac{1}{8}(1,1,1,1,1,1,1,3,\ldots,3)$</td>
<td>$\frac{3}{16}$</td>
<td>4</td>
</tr>
<tr>
<td>$Z_6^A$</td>
<td>6</td>
<td>$U(4) \times U(4) \times U(8)$</td>
<td>$(6,1,1) + (1,6,1)$</td>
<td>$V = \frac{1}{12}(1,1,1,5,5,5,5,3,\ldots,3)$</td>
<td>$\frac{1}{3}$</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 8: Type IIB $Z^A_M$ orientifolds and their heterotic duals.

The second column gives the number of tensor multiplets, whereas the third and fourth show the multiplet content in the (99) sectors in the orientifolds. The (99) vector multiplets are precisely those expected to be reproduced from a perturbative heterotic model. It is easy to find shifts $V$ with $NV$ belonging to the $Spin(32)/Z_2$ lattice such that the untwisted sector of an heterotic orbifold contains exactly these (99) vector and hypermultiplets. Such shifts for the different $Z^A_M$ are shown in the fifth column of Table 8. The $Z_3^A$ model was discussed in Chapter 5, it can be understood in terms of a transition to a Coulomb phase involving nine tensor multiplets.

The $Z_2^A$ model is the GP orientifold [13], shown to be related to certain $Z_2$ orbifolds of $Spin(32)/Z_2$ and $E_8 \times E_8$ [20]. The closed string spectrum produces twenty $K3$ moduli. The complete open string spectrum has a $U(16)_9 \times U(16)_5$ gauge group with hypermultiplets in $2(120, 1) + 2(1, 120) + (16, 16)$. This is the case if all eight dynamical D-five-branes coincide at the same fixed point. If half a five-brane is located at each of the 16 fixed points, the gauge group is $U(16)_9 \times U(1)_5^{16}$ with hypermultiplets transforming
as $2(120) + 16(16) + 20(1)$. In fact the $U(1)^{16}_5$ is broken and swallows sixteen of the singlet hypermultiplets in a variation of the Green-Schwarz mechanism [20]. This is the particular GP model that admits a $SO(32)$ heterotic dual which is a free field theory [20]. Indeed, a standard perturbative (modular invariant) $Z_2$ orbifold of $Spin(32)/Z_2$ with shift $V = \frac{1}{4}(1, \cdots, 1, -3)$ has exactly the same spectrum (see Table 2). While the 16(16) hypermultiplets of this model originate in a standard (perturbative) twisted sector, in the orientifold they originate in the (59) open string sector. It is also interesting to consider the instanton number and the number of states at the $Z_2$ fixed points. Since the shift is modular invariant, we expect $I_f = \frac{3}{2}$ so that $16I_f = 24$. The dimension of $\mathcal{M}_{inst}$ can be worked out taking into account that $V$ breaks $SO(32)$ to $U(16)$ [20]. It is found that $\dim \mathcal{M}_{inst} = 30(I_f - 1) = 15$. Including the blowing-up mode gives 16 states per fixed point as expected.

We would like now to obtain the heterotic dual of the particular GP model with gauge group $U(16)_9 \times U(16)_5$. Consider a $Z_2$ orbifold with $Spin(32)/Z_2$ embedding $V = \frac{1}{4}(1, \cdots, 1)$. This shift, unlike that discussed in the previous paragraph, is not modular invariant by itself since $2(V^2 - \frac{1}{2})$ is not even. The perturbative gauge group is $U(16)$ with untwisted hypermultiplets $2(120) + 4(1)$. Level matching can be achieved by adding an extra vacuum energy $E_B = \frac{1}{4}$ that then leads to 16(1) extra hypermultiplets.

Anomaly cancelation requires the presence of 16(16) hypermultiplets of $U(16)$. In fact notice that this specific $V$ corresponds to a $Z_2$ without vector structure. As discussed in [21], in such a situation, whenever 8 small instantons coalesce at an orbifold singularity, a $U(16)$ non-perturbative gauge group is generated with non-perturbative matter content $16(16) + 2(120)$. Therefore, considering both perturbative and non-perturbative contributions, the full GP model is reproduced. Notice that all twenty $K3$ moduli are present here. Moving the small instantons away from the singularity corresponds to Higgsing down to $Sp(8)$ by giving a vev to a $120$, leaving a standard small instanton content. It is consistent to require eight five-branes since in this case $I_f = 1$ as shown in [20]. Also, notice that $\dim \mathcal{M}_{inst} = 30(I_f - 1) = 0$, meaning that there is only one (blowing-up) state per fixed point.

The $Z^4_4$ orientifold [6, 7] has a similar structure. Consider the non-modular invariant
shift

\[ V = \frac{1}{8}(1, 1, 1, 1, 1, 1, 1, 3, \ldots, 3) \]  \hspace{1cm} (6.1)

It is easy to check that the untwisted sector of the orbifold corresponding to this embedding reproduces the (99)-sector of the orientifold model. By including an energy shift \( E_B = \frac{3}{16} \) it achieves level matching in the twisted \( \theta \) sector and produces four singlets. Another 10 singlets are obtained from the \( \theta^2 \) sector with \( E_B = \frac{1}{4} \) (notice that \( 2V \) corresponds to the GP shift discussed above). Again, this model is a \( Z_4 \) without vector structure and after adding \( E_B(j) \) only one singlet per twisted sector survives. The absence of some blowing-up modes suggests, in analogy with our experience with \( Z_3 \), that there is a transition to a Coulomb phase in which tensor multiplets appear. In this case the non-perturbative content can be read from the results of refs. [5, 32]. In order to use the analysis of instantons at \( Z_M \) ALE spaces we must take into account that the \( Z_4 \) orbifold has four \( Z_4 \) fixed points and six \( Z_2 \) fixed points. Hence, there are four \( Z_4 \) ALE spaces \( \varepsilon_4 \) and six \( Z_2 \) ALE spaces \( \varepsilon_2 \).

The embedding in (6.1) corresponds to instantons without vector structure. From refs. [5, 32] one learns that in the case of small instantons on a \( Z_4 \) singularity there is an enhanced gauge group

\[ U(2\ell_4) \times U(2\ell_4 + u_3 - 8) \] \hspace{1cm} (6.2)

where \( \ell_4 \) is an integer large enough to guarantee \( 2\ell_4 + u_3 - 8 \geq 0 \), and \( u_3 \) is the number of \( \frac{3}{8} \) entries in \( V \). Likewise, \( u_4 = (16 - u_3) \) is the number of \( \frac{1}{8} \) entries in \( V \). In the case at hand \( u_3 = 8 \). There is also one tensor multiplet and hypermultiplets transforming as shown in Table 7. The \( Z_4^4 \) orientifold model is reproduced by setting \( \ell_4 = 4 \) at one of the \( Z_4 \) fixed points and \( \ell_4 = 0 \) at the other three fixed points. From the four small instantons at the first fixed point one gets an \( U(8) \times U(8) \) non-perturbative gauge group with \( 8(8, 1) + 8(1, 8) + (8, 8) + (28, 1) + (1, 28) \) hypermultiplets and one tensor multiplet. From the other three we get one tensor multiplet from each and no enhanced gauge symmetry. One can check that the magnetic charge at the six \( \varepsilon_2 \) is \(-\frac{1}{2}\) whereas that at the four \( \varepsilon_4 \) is \(-\frac{1}{4}\), so that the total charge coming from the singularities cancels that coming from the four small instantons. The given multiplicity of the hypermultiplets can now be understood as the gauge quantum numbers under the perturbative groups. Putting
together the perturbative spectrum from the heterotic orbifold plus these non-perturbative contributions one matches the spectrum of the $Z_4^A$ model. The connection with orientifold models was already noticed in ref. [5]. By putting together all contributions we are then able to provide a realization of the complete perturbative plus non-perturbative $U(8)^4$ model. Notice that we are left with sixteen moduli. The four missing blowing-up modes signal the presence of tensor multiplets in a Coulomb phase.

The $Z_6^A$ model is also realized in a similar, but more intricate, way. Consider the non-modular invariant shift without vector structure given by

$$V = \frac{1}{12}(1,1,1,1,5,5,5,5,3,\ldots,3)$$

This embedding breaks $SO(32)$ to $U(4) \times U(4) \times U(8)$ and implies perturbative untwisted hypermultiplets transforming as $(6,1,1) + (1,6,1) + (4,1,8) + (1,4,8) + 2(1,1,1)$. This matches the $(99)$ sector of $Z_6^A$ orientifold. To fulfill level-matching in the $\theta^j$ sectors we add an extra vacuum energy $E_B(j) = \frac{1}{4}, \frac{1}{3}, \frac{1}{4}$, for $j = 1, 2, 3$. The mass formula then leads to massless twisted hypermultiplets transforming as singlets. Specifically, the number of singlet hypermultiplets in the $\theta^j$ sector is $n_H(j) = 2, 5, 5$, for $j = 1, 2, 3$.

We look next for the non-perturbative piece of the spectrum. The $(55)$ and $(59)$ content will follow from non-perturbative transitions in $\varepsilon_6$ and $\varepsilon_3$ ALE spaces. Recall that in the $Z_6$ orbifold there is one $\varepsilon_6$, four $\varepsilon_3$ and five $\varepsilon_2$ ALE spaces. The non-perturbative gauge group comes from small instantons at the single $\varepsilon_6$. Small instantons without vector structure at a $Z_6$ singularity generate an enhanced gauge group [5, 32]

$$U(2\ell_6) \times U(2\ell_6 + u_2 + u_3 - 8) \times U(2\ell_6 + u_2 + 2u_3 - 16)$$

where $\ell_6$ is an integer large enough to guarantee positive arguments. Here $u_2$ ($u_3$) is the number of $\frac{3}{12}$ ($\frac{5}{12}$) entries in $V$. Also, $u_1 = (16 - u_2 - u_3)$ is the number of $\frac{1}{12}$ entries in $V$. In this example $u_2 = 8, u_3 = 4$. In addition, there are two tensor multiplets and hypermultiplets transforming as given in Table 7. The $Z_6^A$ orientifold model is reproduced by setting $\ell_6 = 2$ at the $Z_6$ fixed point and $\ell = 0$ at the other fixed points. One then gets a non-perturbative gauge group $U(4) \times U(8) \times U(4)$ with matter in $4(4,1,1) + 8(1,8,1) + 4(1,1,4) + (4,8,1) + (1,8,4) + (6,1,1) + (1,1,6)$. The given multiplicity can now be understood as the gauge quantum numbers under the perturbative groups. We
also get two tensors from this \( \varepsilon_6 \). The rest of the tensors, as mentioned before, are produced after a non-perturbative transition in all four \( \varepsilon_3 \) ALE’s. Indeed, these are the \( \varepsilon_3 \) spaces appearing in the \( Z_3^A \) model, which as explained in Section 5, lead to one tensor each.

Notice that the present \( Z_6 \) orbifold model has only fourteen moduli. The six missing moduli signal the presence of six tensor multiplets in a Coulomb phase.

We thus see that the heterotic duals of the \( Z_M^A \) Type-IIB orientifolds can be understood as \( Z_M \) orbifolds with non-modular invariant embeddings without vector structure in the presence of small instantons.

### 6.2 \( Z_4 \) Heterotic Orbifolds with Vector Structure

Consistent \( Z_4 \) models with vector structure can also be constructed. Some examples are listed in Table 9. They correspond to a generic shift

\[
V = \frac{1}{4}(1, \cdots, 1, 2, \cdots, 2, 0, \cdots, 0)
\]

with \( w_1 \frac{1}{4} \) entries and \( w_2 \frac{2}{4} \) entries. The gauge group and corresponding untwisted matter sector content are

\[
G = U(w_1) \times SO(2w_2) \times SO(2w_0)
\]

\[
U = (\mathbf{w}_1, 1, 2\mathbf{w}_0) + (\mathbf{w}_1, 2\mathbf{w}_2, 1) + 2(1, 1, 1)
\]

where \( w_0 = 16 - w_1 - w_2 \).

Modular invariant models satisfy the constraint

\[
4w_2 + w_1 = 2 \mod 8
\]

This also follows from the instanton numbers \( I_f(4) \) and \( I_f(2) \) at the \( Z_4 \) and \( Z_2 \) fixed points. Indeed, from eq. (4.1) we find

\[
I_f(4) = \ell_4 + \frac{3w_1}{8} + \frac{w_2}{2}
\]

\[
I_f(2) = \ell_2 + \frac{w_1}{4}
\]

The total instanton number is \( k = 4I_f(4) + 6I_f(2) \).
It is also interesting to determine the number of states at the $Z_M$ fixed points, denoted $\mathcal{N}_f(M)$. From eq. (4.9) we find

\begin{align*}
\mathcal{N}_f(4) &= \dim \mathcal{M}_{\text{inst}}(4) + 3 = 30\ell_4 + \frac{w_1}{2}(w_1 + w_2 - 1) + w_2(\frac{w_1}{2} + w_2 - 1) + 3 \\
\mathcal{N}_f(2) &= \dim \mathcal{M}_{\text{inst}}(2) + 1 = 30\ell_2 + \frac{w_1}{2}(w_1 - 1) + 1
\end{align*}

(6.9)

where we have included the contribution from the blowing-up modes. It is instructive to check how these formulae correctly give the number of hypermultiplets in the twisted sectors of the modular invariant $Z_4$ orbifolds in Table 9 (first and third examples). To this purpose let us now study the twisted states in more detail. If $R_1$ denotes the whole set of massless hypermultiplets in the $\theta$ sector, the total contribution is $4R_1$. The factor 4 takes into account the four $\theta$ fixed points. In the $\theta^2$ sector there are twelve additional $Z_2$ fixed points, completing six $\theta$-invariant pairs. The total contribution from the $\theta^2$ sector is $5R_2 + 3R_3$, where $R_2$ and $R_3$ are subsets of the massless states. Notice that we have divided by two so that only particles or antiparticles are counted. Schematically we have the distribution

\begin{align*}
\theta : & \quad 4R_1 \\
\theta^2 : & \quad 5R_2 + 3R_3
\end{align*}

(6.10)

In order to compare with the index theorem results (6.9), the structure (6.10) can be rewritten to show the explicit contribution from the four $\epsilon_4$ and six $\epsilon_2$ ALE spaces as

\begin{align*}
\theta + \theta^2 : & \quad 4[R_1 + \frac{1}{2}R_2] + 6[\frac{1}{2}R_2 + \frac{1}{2}R_3]
\end{align*}

(6.11)

Therefore, the $\mathcal{N}_f(M)$ are given by

\begin{align*}
\mathcal{N}_f(4) &= R_1 + \frac{1}{2}R_2 \\
\mathcal{N}_f(2) &= \frac{1}{2}(R_2 + R_3)
\end{align*}

(6.12)

The $R_i$ found in the orbifold analysis match exactly the index theorem results when the appropriate values for $\ell_4$ and $\ell_2$ are taken. These are shown on the last column of Table 9. These choices also lead, upon substitution in equation (6.8), to a total instanton number of twenty four.
This exercise can also be performed for the non-modular invariant orbifolds that we
are to describe in the following. In each case, we find agreement between the spectrum
predicted by the index theorem and that found in the orbifold. Recall that whenever
there are fixed points in the Coulomb branch, the number of states (actually \( \dim \mathcal{M}_H \))
is given by \( \mathcal{N}_f(M) = \dim \mathcal{M}_{\text{inst}} + M - 1 - 29P \), where \( M = 2P \) (or \( M = 2P + 1 \)) [5].

A similar analysis is possible for the models without vector structure of Section 6.1.
Just notice that by computing the number of hypermultiplets minus the number of vector
multiplets from the results in Table 7, we obtain

\[
\mathcal{N}_f(4) = \dim \mathcal{M}_{\text{inst}} + 3 - 29 \tag{6.13}
\]

where

\[
\dim \mathcal{M}_{\text{inst}} = 30\ell_4 + \frac{1}{2}u_2(u_2 - 1) \tag{6.14}
\]

and similarly [5]

\[
\mathcal{N}_f(2) = 30\ell_2 + 1 \tag{6.15}
\]

We also stress that the contribution of the total instanton number found from (6.8) plus
the number of five-branes required always adds to twenty four.

In some cases it is possible to add an \( E_B \) energy shift in twisted sectors and produce
a transition to a non-perturbative model in the same way it happened for \( Z_3 \). Non-
modular invariant shifts are also possible. Examples of the first situation are the first
(standard embedding) case as well as the \( U(14) \times U(1)^2 \) model corresponding to the shift
\( V = \frac{1}{4}(2, 1, \cdots, 1, 0) \). In both cases adding \( E_B(1) = \frac{1}{4} \) in the \( \theta \) sector kills some of the
perturbative matter but singlets that can be identified with \( K3 \) moduli survive. Then,
non-perturbative contributions from standard small instantons are generated with the
adequate content to render the model consistent. The non-perturbative groups are \( Sp(8) \)
and \( Sp(4) \) respectively.

The last two models in Table 9 are examples in which transitions to Coulomb phases
appear. For instantons with vector structure on a \( Z_4 \) singularity one expects an enhanced
gauge symmetry [5]

\[
Sp(\ell_4) \times U(2\ell_4 + w_1 + w_2 - 8) \times Sp(\ell_4 + \frac{w_1}{2} + w_2 - 8) \tag{6.16}
\]
There are also two tensor multiplets and the hypermultiplets shown in Table 7. The models have also \( Z_2 \) singularities and instantons with vector structure on them give an enhanced gauge group

\[
Sp(\ell_2) \times Sp(\ell_2 + \frac{w_1}{2} - 4)
\]

(6.17)

In addition, there is one tensor multiplet and the extra hypermultiplets displayed in Table 7.

The \( U(8) \times SO(16) \) \((w_1 = 8, w_2 = 0)\) model in the table is somewhat similar to the \( Z_3 \) example considered in Chapter 5. The shift in this case is, however, non-modular invariant. Inclusion of \( E_B(1) = \frac{1}{16} \) and \( E_B(2) = \frac{1}{4} \) leads to a consistent model if six tensor multiplets are included. These tensors can be interpreted as originating from small instantons sitting at each of the six \( \varepsilon_2 \) ALE spaces. In fact, as eq.(6.17) shows, \( w_1 = 8, \ell_2 = 0 \) are critical values. Notice that it is also possible to include a higher energy shift \( E_B(1) = \frac{5}{16} \) in the \( \theta \) sector. The model becomes consistent with four extra tensor multiplets. It would signal a transition \( 28 + 1 \rightarrow \text{tensor at each } \varepsilon_4 \text{ ALE space} \) as we have encountered before. Nevertheless, notice that this would not correspond to the situation treated in [5] since, as we remarked before, two tensor multiplets are expected there at each \( \varepsilon_4 \), furthermore a critical \( \ell_2 \) value with \( w_1 = 8 \) is not possible.

Notice that all the examples discussed in this section correspond to orbifolds of the \( Spin(32)/Z_2 \) heterotic. It would be interesting to explore the equivalent type of models in the \( E_8 \times E_8 \) case, for which less is known about the behaviour of instantons at singularities.

### 6.3 Heterotic Dual of DPGM \( Z_2 \) Orientifolds

We now wish to consider a \( Z_2 \) orbifold of heterotic \( SO(32) \) which yields the same spectrum as the \( Z_2 \) orientifold constructed by Dabholkar and Park [11] and model C of Gopakumar and Mukhi [12]. This is a \( D = 6, N = 1 \) model with gauge group \( SO(8)^8 \), seventeen tensors and four hypermultiplets. It can be obtained in terms of F-theory compactified on the standard \( Z_2 \times Z_2 \) orbifold, as a compactification of M-theory on \( T^5/Z_2 \times Z_2 \) and as a type IIB orientifold. Here we will obtain it as a heterotic \( SO(32) \) \( Z_2 \) orbifold (with a non-modular invariant shift). We will embed the \( Z_2 \) twist in terms of a shift \( V \) in the
<table>
<thead>
<tr>
<th>Shift/ Sectors</th>
<th>(w₂, w₁)</th>
<th>Group</th>
<th>(ℓ₄, ℓ₂, n_B) / E_B</th>
</tr>
</thead>
<tbody>
<tr>
<td>V = ½(1,1,0,...,0)</td>
<td>(0,2)</td>
<td>U(2) × SO(28)</td>
<td>(3,1,0)</td>
</tr>
<tr>
<td>U : (2,28) + 2(1,1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ : 4[(1,1) + (14)] + 2(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ² : 5(2,28) + 16((1,1) + (1,1))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V = ½(1,1,0,...,0)</td>
<td>(0,2)</td>
<td>U(2) × SO(28)</td>
<td>(1,1,8)</td>
</tr>
<tr>
<td>U : (2,28) + 2(1,1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ : 4[(1,1) + (14)]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ² : 5(2,28) + 16((1,1) + (1,1))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V = ½(1,1,...,1,0)</td>
<td>(1,14)</td>
<td>U(1) × U(14) × U(1)</td>
<td>(−2,−2,0)</td>
</tr>
<tr>
<td>U : 2(14) + 2(14) + 2(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ : 4[(14) + (14)] + 2(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ² : (5 + 3)((14) + (14)) + 2(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V = ½(1,1,...,1,0)</td>
<td>(1,14)</td>
<td>U(1) × U(14) × U(1)</td>
<td>(−3,−2,4)</td>
</tr>
<tr>
<td>U : 2(14) + 2(14) + 2(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ : 4[(1) + (1)]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ² : (5 + 3)((14) + (14)) + 2(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V = ½(1,1,1,1,1,1,1,1,0,...,0)</td>
<td>(0,8)</td>
<td>U(8) × SO(16)</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>U : (8,16) + 2(1,1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ : 4[(28,1) + 3(1,1)]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ² : 6 Tensors</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V = ½(1,1,1,1,1,1,1,1,0,...,0)</td>
<td>(0,8)</td>
<td>U(8) × SO(16)</td>
<td>(0,0,0)</td>
</tr>
<tr>
<td>U : (8,16) + 2(1,1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ : 4[(1,1) + 4 Tensors]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ² : 6 Tensors</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9: Examples of consistent $Z_4$ SO(32) orbifold models with $E_B \neq 0$. The asterisk indicates twisted states involving left-handed oscillators. Only the perturbative matter and tensors are shown. $n_B$ gives the number of five-branes in the bulk.
The two Wilson lines break the symmetry down to $SO(8)^4$ whereas the $V$ shift projects out all charged multiplets from the untwisted sector. Only the four untwisted moduli hypermultiplets remain in that sector. Now, the sixteen twisted sectors split into four sets of four fixed points each which are subject to shifts $V, V + a_2, V + a_1 + a_2$ and $V + a_1$ respectively. The first three sets of four fixed points are all similar, the corresponding shift has $w = 8, \frac{1}{2}$ entries. Thus $V^2 = 2$ and there are no massless hypermultiplets at any of those twelve fixed points. However, we already mentioned that the value $w = 8$ for embeddings with vector structure is critical for five-branes sitting at a $Z_2$ singularity [5]. Indeed, one tensor and a gauge group $Sp(\ell) \times Sp(\ell + w - 4)$ appear (see Table 7). Since in our case $w = 8$, we get one tensor for each of the twelve fixed points and no enhanced gauge group for $\ell = 0$.

The other four fixed points with shift $V + a_1$ have a different behavior. Indeed, this shift is trivial and hence we have $w = 0$ for those fixed points. As remarked in ref. [33], five-branes at a $Z_2$ singularity with $w = 0$ give transitions to a Coulomb phase with one tensor multiplet and a gauge group $Sp(\ell) \times SO(2\ell + 8)$ (see Table 7). Thus, in our case, with $\ell = 0$ at each of the fixed points we have altogether a non-perturbative group $SO(8)^4$ and four tensor multiplets. Putting all the contributions together we get the total content $SO(8)^8$, seventeen tensor multiplets and four singlet hypermultiplets. Notice how the 16 twisted sectors are in a Coulomb phase, twelve of them with $w = 8$ yielding only tensors and the other four have $w = 0$ yielding in addition the required non-perturbative $SO(8)^4$.

A similar construction can be carried out in the $E_8 \times E_8$ heterotic, as a $Z_2$ orbifold with the same Wilson line structure embedded in $E_8 \times E_8$ in a symmetrical way. One obtains the same untwisted sector, and the remaining tensors and vector multiplets are expected to arise from non-perturbative effects due to small $E_8$ instantons (possibly at fixed points). Their dynamics, however, has been only partially determined [34], so that a complete check of how the spectra match is not available at the moment. Notice that the presence of a Wilson line breaking either group to $SO(16) \times SO(16)$ shows that
both heterotic constructions are related by T-duality. An interesting issue in this respect would be to understand how the different non-perturbative effects are mapped under this transformation.

One can also construct a non-perturbative $Z_2$ orbifold yielding the spectrum of the model of ref. [40] (model B in ref. [12]). Consider the following shift structure

$$V = \frac{1}{2} (1,1,1,1,1,1,1,0,\ldots,0)$$
$$A = \frac{1}{4} (1,\ldots\ldots,1)$$

The untwisted sector contributes the gauge group $U(8)^2$ and hypermultiplets transforming as $\mathbf{4}(\mathbf{1}, \mathbf{1}) + (\mathbf{8}, \overline{\mathbf{8}}) + (\overline{\mathbf{8}}, \mathbf{8})$. The fixed points feeling the critical shift $V$ generate 8 tensors without gauge group, and those feeling the shift $V + A$ give 8 neutral hypermultiplets. Observe that the total instanton number adds up to 24, and that all anomalies cancel. The spectrum matches exactly that in [40], and upon Higgssing reproduces model B in [12].

7 $D = 4$, $N = 1$ Non-Perturbative Orbifolds and Chirality Changing Transitions

7.1 $D = 4$, $N = 1$ Non-Perturbative Orbifolds

In principle, the idea explored in previous chapters for the $D = 6$ case could be extended to $D = 4$, $N = 1$. One would construct heterotic orbifold vacua with perturbative and non-perturbative sectors in which the perturbative (but non-modular invariant) sector could be understood in terms of simple standard orbifold techniques. One must also add a non-perturbative piece, but we face the problem that non-perturbative phenomena in $N = 1$, $D = 4$ theories are poorly understood at the moment. In $D = 6$ we were guided by the known results of $D = 6$ small instanton dynamics, but in $D = 4$ there are no clear guidelines.

However, we can concentrate on certain restricted classes of $D = 4$ orbifolds in which much of the structure is expected to be inherited from $D = 6$. In particular, one can
consider $Z_N \times Z_M$ orbifolds in $D = 4$ with unbroken $N = 1$ supersymmetry. Such type of orbifolds have two general classes of twisted sectors, those that leave a 2-torus fixed and those that only leave fixed points. The first type of twisted sectors is essentially 6-dimensional in nature, the twist by itself would lead to an $N = 2$, $D = 4$ theory which would correspond to $N = 1$, $D = 6$ upon decompactification of the fixed torus. For this type of twisted sectors we can use our knowledge of non-perturbative $D = 6$, $N = 1$ dynamics and the results of the previous chapters. Twisted sectors of the second type are purely 4-dimensional in nature and we would need extra information about 4-dimensional non-perturbative dynamics. To circumvent this lack of knowledge, one can restrict to a particular class of $Z_N \times Z_M$ orbifolds with gauge embeddings such that these purely 4-dimensional twisted sectors are either absent or else are not expected to modify substantially the structure of the model. A first step could be to try to reproduce known $D = 4$, $N = 1$ orientifolds. A simple example is the $Z_3$, $D = 4$ orientifold of ref. [29] (see also [41]), but this example is known to have a perturbative heterotic dual. The next simplest $D = 4$ Type IIB orientifold is the $Z_2 \times Z_2$ example of Berkooz and Leigh (BL). Below we will describe the heterotic dual of the $Z_2 \times Z_2$ BL orientifold. We then consider a class of $Z_3 \times Z_3$, $D = 4$ non-perturbative orbifolds in which certain interesting non-perturbative chirality-changing transitions occur.

7.2 The Heterotic Dual of the $Z_2 \times Z_2$ Berkooz-Leigh $D = 4$ Orientifold

In the BL model the Type IIB string is twisted with respect to world-sheet parity combined with the standard $Z_2 \times Z_2$ action on six compact (toroidal) dimensions. This model has three sets of five-branes $5_i$, $i = 1, 2, 3$, whose world-volume fills the four uncompactified dimensions plus the $i$-th complex plane. The largest gauge symmetry arises when thirty two D-five-branes (eight dynamical five-branes) of type $5_i$ coincide at the same fixed point (fixed with respect to the $Z_2$ action not touching the $i$-th complex plane). Each of these sets induces an $Sp(8)$ maximal gauge symmetry. The open 99-brane sector also gives rise to an $Sp(8)$ and the total gauge group is $Sp(8)_9 \times Sp(8)_{5_1} \times Sp(8)_{5_2} \times Sp(8)_{5_3}$. There are also chiral multiplets transforming as $3(\underline{120}, 1, 1) + (\underline{16}, 16, 1, 1)$, where the underlining
means permutation. In addition to the gravitational sector, the closed string spectrum includes $16 \times 3 + 6$ extra singlet chiral multiplets (moduli) plus the axi-dilaton chiral field.

It is easy to find a $Z_2 \times Z_2$ orbifold of the $SO(32)$ heterotic string whose invariant (untwisted) sector corresponds to the (99)-open string sector of the BL model. In this case both $Z_2$ actions cannot be realized simultaneously by shifts in the lattice. One can realize one of the $Z_2$ actions through the shift

$$V = \frac{1}{4}(1, \cdots, 1)$$

(7.1)

This same shift realizes the twist in the heterotic version of a non-perturbative (from the heterotic point of view) GP model, as we described in the previous chapter. The other $Z_2$ embedding can be realized in the gauge degrees of freedom by a permutation $\Pi$ whose action on the sixteen bosonic coordinates $F_I$ is

$$\Pi : (F_1, F_2, \cdots, F_8, F_9, \cdots, F_{16}) \rightarrow (F_9, F_{10}, \cdots, F_{16}, F_1, F_2, \cdots, F_8)$$

(7.2)

Notice that $\Pi V = V$ so that both actions commute as they should. Each of these actions separately break the gauge symmetry down to $U(16)$. But only a common $Sp(8)$ subgroup is invariant under both. One can also check that, associated to the three complex planes there are three untwisted chiral multiplets transforming as the antisymmetric $120$ of $Sp(8)$. Thus, this $Z_2 \times Z_2$ indeed reproduces the (99) open string spectrum of the $Z_2 \times Z_2$ BL orientifold. Neither $V$ nor $\Pi$ verify the usual modular invariance conditions of perturbative orbifolds, in particular $2(V^2 - \frac{1}{2}) \neq$ even. We are then led to add an appropriate shift $E_B = \frac{1}{4}$ in each of the twisted sectors to recover level matching, as we discussed in the $D = 6$ examples. With such $E_B$, sixteen singlets per twisted sector would survive. They will correspond to the twisted moduli. Non-perturbative effects should give rise to the rest of the particles in order to match the orientifold spectra. Each of the twisted sectors is indeed a $D = 6$ orbifold, so one can apply what is known in six dimensions and then project. In particular, for all eight five-branes of each $i$-th sector coinciding at a fixed point one expects a non-perturbative group $U(16)_i, i = 1, 2, 3$. The projection with respect to the other $Z_2$ symmetry should break each of these groups to $Sp(8)_i$. One also expects the non-perturbative generation of chiral multiplets transforming as $16$ under the perturbative $Sp(8)$ giving rise to the charged hypermultiplets appearing in
the orientifold construction. Thus, one concludes that the heterotic dual of the BL model is the $Z_2 \times Z_2$ heterotic orbifold just described, in which the perturbative part is computed in the standard way (modulo a shift in the vacuum energy) and the non-perturbative piece can be understood in terms of known $D = 6$ small instanton physics.

### 7.3 A Class of Non-Perturbative $Z_3 \times Z_3$ Orbifold vacua

We have seen in previous chapters that the simplest non-perturbative heterotic orbifolds in $D = 6$ are obtained for $Z_3$ twists. Thus, it is natural to consider $D = 4$ $Z_3 \times Z_3$ orbifolds with gauge embeddings of the restricted form described in Chapter 3, with length-squared sufficiently small. Let us first review a few points about generic $Z_3 \times Z_3$ orbifolds (see ref. [43] for more details). The point group is generated by twists $\theta$ and $\omega$ with twist vectors given by $a = \frac{1}{3}(0, 1, 0, -1)$ and $b = \frac{1}{3}(0, 0, 1, -1)$ respectively. The gauge embedding shifts are given by 16-dimensional vectors $A, B$ satisfying $3A, 3B \in \Gamma_{16}$. Apart from the untwisted sector there are eight twisted sectors. Those with gauge embeddings $A, B, (A - B)$ and their inverses are 6-dimensional in nature. The remaining two twisted sectors have twist vector $(a + b) = \frac{1}{3}(0, 1, 1, -2)$ and gauge shift $(A + B)$ and their inverses. They comprise a standard $Z$ orbifold with twenty seven fixed points. These sectors are purely 4-dimensional in nature, and we want to circumvent them. In some particular cases, as we will show below, they have no massless fields and hence we can proceed by using essentially only $D = 6$ information. In most of the cases, however, there are massless fields. In this case we should restrict to models with shifts $A, B$ verifying the usual $Z$-orbifold modular invariance constraint for the $(A + B)$ shift, namely In this way we will avoid purely 4-dimensional non-modular invariant shifts. On the other hand, the $A, B$ and $(A - B)$ shifts can be allowed to violate modular invariance constraints. We now present several specific examples with a variety of interesting properties.

**Example 1**

This model is interesting because it shows the existence of non-perturbative $D = 4$ transitions between perturbative and non-perturbative vacua in which the particle spectrum changes. It also seems to have a Type IIB orientifold dual. Consider the shifts

$$A = \frac{1}{3}(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$$


\[ B = \frac{1}{3}(0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0) \]  

in \( \Gamma_{16} \). Notice that each of these shifts by itself would give rise in \( D = 6 \) to the \( U(8) \times SO(16) \) models discussed in the Section 5 which are in turn related to the \( Z_3^A \) GJ orientifold. Thus, this example is expected to be dual to a \( D = 4 \) \( Z_3 \times Z_3 \) Type IIB orientifold generalization of the \( Z_3^A \), very much like the \( Z_2 \times Z_2 \) BL orientifold is a \( D = 4 \) generalization of the GP model (a similar observation concerning this \( Z_3 \times Z_3 \) model has been recently made in ref.[42]). One can easily check that for these specific \( A \) and \( B \) the usual modular invariance constraints are verified in all twisted sectors. The gauge group is \( U(4)^3 \times SO(16) \) and the complete chiral multiplet spectrum (except for the dilaton \( S \) and untwisted moduli) is displayed in Table 10.

<table>
<thead>
<tr>
<th>Sector</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_1 )</td>
<td>((4,1,1,8_v)+(6,1,1,1)+(1,\bar{4},\bar{4},1))</td>
</tr>
<tr>
<td>( U_2 )</td>
<td>((1,4,1,8_v)+(1,6,1,1)+(\bar{4}, 1, \bar{4},1))</td>
</tr>
<tr>
<td>( U_3 )</td>
<td>((1,1,4,8_v)+(1,1,6,1)+(\bar{4}, \bar{4}, 1,1))</td>
</tr>
<tr>
<td>( A, \bar{A} )</td>
<td>(9(6,1,1,1)+9(1,6,1,1)+18(1,1,1,1))</td>
</tr>
<tr>
<td>( B, \bar{B} )</td>
<td>(9(1,6,1,1)+9(1,1,6,1)+18(1,1,1,1))</td>
</tr>
<tr>
<td>( A - B, \bar{A} - \bar{B} )</td>
<td>(9(6,1,1,1)+9(1,6,1,1)+18(1,1,1,1))</td>
</tr>
<tr>
<td>( A + B )</td>
<td>(27(1,1,1,1) + 27(1,1,1,8_s))</td>
</tr>
</tbody>
</table>

Table 10: Spectrum of the \( Z_3 \times Z_3, U(4) \times U(4) \times U(4) \times SO(8) \) perturbative model.

But we can equally consider a non-perturbative orbifold in which the \( D = 6 \) subsectors \( A, B \) and \( A - B \) have a left-handed vacuum energy shifted by \( \frac{1}{3} \). As we showed in Chapter 5, this corresponds to a non-perturbative \( D = 6 \) vacuum in which twisted sectors have just one singlet and nine tensor multiplets appear. Non-perturbative \( D = 6 \) transitions exist between the modular invariant \( E_B = 0 \) and the \( E_B = \frac{1}{3} \) models. Thus, the corresponding \( D = 4 \) non-perturbative orbifold would just have some singlets in the twisted \( A, B, A - B \) sectors and there should be non-perturbative transitions in which all the states
transforming as $6$-plets in those twisted sectors disappear from the massless spectrum. The untwisted and $A + B$-twisted sectors will remain unchanged. This would just be a 4-dimensional version of the $D = 6$ transitions in which twenty nine hypermultiplets are converted into one tensor multiplet. Notice how the non-Abelian $SU(4)^3 \times SO(8)$ anomalies cancel since the untwisted sector is (non-trivially) anomaly-free by itself. The existence of $U(1)$ anomalies is expected, but notice that now a generalized 4-dimensional Green-Schwarz mechanism can take place since there will be extra singlet fields (associated to the $9 \times 3$ tensors of the $D = 6$ twisted sectors) which will couple to the gauge groups in a non-universal manner.

Let us finally remark that no special non-perturbative effects are expected to arise from the $A + B$ sector. Indeed this sector is identical to the twisted sector of a $Z_3$, $D = 4$ orbifold considered in ref. [29]. These authors showed this orbifold to be dual to a certain Type IIB $Z_3$, $D = 4$ orientifold which has the relevant characteristic of having no five-branes (very much like the $Z_3^A$ GJ model). This implies that one does not expect new non-perturbative charged hypermultiplets nor enhanced gauge group coming from this sector.

Example 2

This second example illustrates how there can be non-perturbative transitions in which chiral generations disappear from the massless spectrum. In this specific case the final non-perturbative $D = 4$ model will have three $E_6$ generations. Consider the $Z_3 \times Z_3$ orbifold on $E_8 \times E_8$ with gauge shifts

$$A = \frac{1}{3}(1,1,0,\cdots,0) \times (0,\cdots,0)$$
$$B = \frac{1}{3}(0,1,1,0,\cdots,0) \times (0,\cdots,0)$$

This leads to a perfectly modular invariant orbifold with gauge group $E_6 \times U(1)^2 \times E_8$. However, we are going to consider the particular version of this orbifold with discrete torsion first considered in ref. [43]. This model has the particular property that there is no $(A+B)$ massless twisted sector, all particles are projected out. In this way we get rid of the sector which is purely 4-dimensional. The model has now three $27$’s in the untwisted sector and nine $\overline{27}$’s in each of the sectors $A, B$ and $A - B$. Hence, altogether the model
has twenty four net antigenerations. Very much like in the previous example, we can now consider a non-perturbative orbifold in which the $D = 6$ subsectors $A, B$ and $A - B$ have a left-handed vacuum energy shifted by $\frac{1}{3}$. As we showed in Chapter 3, this corresponds to a non-perturbative $D = 6$ vacuum with just singlets in the twisted sectors and eighteen five-branes (leading to tensor multiplets) in each of the three twisted sectors. Therefore, all the twenty seven antigenerations of the twisted sectors disappear from the spectrum and we are only left with three $E_6$ generations coming from the untwisted sector, plus singlets. As in the previous example the $U(1)$’s will now be anomalous but there will be extra chiral singlets, coming from the tensors, with non-universal couplings to the gauge fields which will lead to a generalized version of the GS mechanism in $D = 4$.

This example shows that the number of chiral generations is not invariant under non-perturbative effects. Vacua with different number of generations are connected.

**Example 3**

We also expect chirality changing transitions in compactifications of heterotic $SO(32)$ theory. Consider the $Z_3 \times Z_3$ orbifold with gauge shifts

\[
A = \frac{1}{3}(1, 1, 1, 1, -2, 2, -1, 0, \cdots, 0) \\
B = \frac{1}{3}(0, 0, 0, 0, 0, 1, 1, 0, \cdots, 0)
\]  

(7.5)

This leads to a modular invariant $D = 4$ model with gauge group $SO(16) \times SU(6) \times SU(2) \times U(1)^2$ and chiral multiplet spectrum shown in Table 11.

Now, the $A$ and $A - B$ shifts by themselves would have given rise to the $U(8) \times SO(16)$ model in $D = 6$. We know that that model has transitions to a model with vacuum shift $E_B = \frac{1}{3}$ in which only singlets (and tensor multiplets) appear in the twisted sector. Thus, one would expect the existence of $D = 4$ non-perturbative transitions in which the $SU(6)$ chiral generations appearing in the $A$ and $A - B$ twisted sectors disappear from the massless spectrum. This would be analogous to the transitions discussed in the previous example. Unfortunately, in the present example we do not know if extra non-perturbative effects associated to the twisted $A + B$ sector exist or not, but it seems reasonable to expect the existence of these chirality changing transitions coming from the $A$ and $A - B$ sectors.

In conclusion, our lack of a better knowledge of non-perturbative dynamics in $D = 4$, $N = 1$ does not allow us to make a straightforward generalization to this case. However,
there are certain classes of $Z_N \times Z_M$ $D = 4$ non-perturbative orbifolds in which interesting conclusions can be obtained on the basis of $D = 6$ information. Among the most relevant ones is the observation that there are non-perturbative transitions which change the number of chiral generations.

### 8 Final Comments and outlook

In this paper we have studied a class of non-perturbative $D = 6$, $N = 1$ orbifolds of $Spin(32)/Z_2$ and $E_8 \times E_8$ heterotic strings. They are obtained by modding the gauge degrees of freedom of these theories by $Z_M$ actions not obeying the usual (perturbative) modular invariance constraints. These models have perturbative and non-perturbative pieces. The massless spectrum of the perturbative sector is obtained by the usual string mass formula, with the $Z_M$ twisted sector vacua subject to a shift in the vacuum energy. This is due to the presence of a non-vanishing flux of the antisymmetric field $H$ at the fixed points. The number of hypermultiplets found in this way matches with the index theorem formulae for instantons on $Z_M$ ALE spaces. The non-perturbative sector is obtained from the knowledge of small instanton dynamics. Some of the models are orbifold equivalents of the vacua obtained from smooth $K3$ compactifications in the presence of small instantons.
For those orbifold models the non-perturbative sector is provided by the world-volume theories of wandering five-branes with the world-volume in the uncompactified dimensions. In some other cases the non-perturbative sector includes five-branes which are stuck at the $Z_M$ fixed points giving more exotic physics including tensor multiplets as described in ref. [5]. The heterotic duals of the $Z^{A_M}_M$ Type IIB orientifolds can be understood in this way but there are many other models which can be constructed.

We think that the present class of non-perturbative vacua are of practical interest not only because it provides the heterotic duals of a number of models but also because of its simplicity. One just uses familiar orbifold techniques supplemented by information about small instanton physics. Many models can be obtained for the different $Z_M$’s by using the diverse possible embeddings in the gauge degrees of freedom. In contrast, notice that in Type IIB orientifolds the gauge embedding (up to the addition of Wilson lines and/or discrete torsion) is essentially uniquely fixed by tadpole cancelation. Also, a technique like F-theory is by far more general and powerful but is slightly cumbersome if one is interested in knowing in detail the hypermultiplet spectra and the charges under the gauge groups, especially $U(1)$’s, for particular points of F-theory moduli space.

As we said above, the non-perturbative sector of these orbifold models is obtained from small instanton dynamics. In orbifolds like these, information about five-branes wandering in the bulk and five-branes stuck to the fixed points of the orbifold are both needed. Our knowledge about the latter is only partial. When a large enough number of point-like instantons are piled on a $Z_M$ singularity in a $Spin(32)/Z_2$ compactification, there is a Coulomb phase in which a known spectrum of tensor, gauge and hypermultiplets appear. Such complete information (particularly for the hypermultiplet sector) is not yet available for the $E_8 \times E_8$ case. Furthermore we do not know, except for some cases, the expected non-perturbative spectrum when the number of small instantons on the singularity is smaller than the critical value. It is reasonable to expect that this information will soon become available and a larger class of non-perturbative orbifolds along the lines considered in this paper will be constructed.

In addition to the previous comment, there are several interesting questions to pose. For example, what happens if the perturbative sector of our model is not a simple geometrical orbifold but a (non-modular invariant) asymmetric orbifold or, in general, an
arbitrary CFT. Most likely, there are similar non-perturbative effects that render the theory consistent, and it would be interesting to determine their nature. Another interesting point is the extension of these ideas to $D = 4, N = 1$ vacua, which is of more direct phenomenological interest. We have seen that in certain $N = 1, D = 4, Z_N \times Z_M$ orbifolds one can obtain some non-perturbative vacua by using $D = 6$ information. One finds that the existence of $D = 6$ transitions in which one tensor multiplet transmutes into twenty nine charged hypermultiplets, implies in $D = 4$ the existence of transitions in which chiral generations (e.g. $27$’s of $E_6$ or $15 + 2\cdot \bar{6}$ of $SU(6)$) change into singlets. Thus $D = 4, N = 1$ vacua with different number of chiral generations are non-perturbatively connected. Of course, this is of relevance if one is eventually interested in describing the observed physics in terms of string (or M-theory) dynamics.

Acknowledgments

We thank F. Quevedo for useful discussions. G.A. thanks ICTP and the Departamento de Física Teórica at UAM for hospitality and financial support. A.M.U, L.E.I. and G.V. thank CICYT (Spain) for financial support. A.F. thanks CONICIT (Venezuela) for a research grant S1-2700. A.M.U. thanks the Governement of the Basque Country for financial support.

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