A new class of unstable modes of rotating relativistic stars

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The first numerical study of axial (toroidal) pulsation modes of a slowly rotating relativistic star is presented. The calculation includes terms of first order in $\epsilon \equiv \Omega \sqrt{R^3/M} << 1$ ($R$ is the radius, $M$ is the mass and $\Omega$ is the rotation frequency of the star), and accounts for effects due to the coriolis force. Effects due to the centrifugal flattening of the star enter at order $\epsilon^2$ and are not included in the analysis. It is shown that increased rotation tends to decrease the damping times for prograde modes, while retrograde become longer lived. Specifically, we show that rotation affects the axial gravitational-wave $w$-modes in this way. We also present the first relativistic calculation of the so-called $r$-modes (analogous to Rossby waves in the Earth’s oceans). These have frequencies of the same order of magnitude as the rotation frequency of the star. The presented results indicate that the $r$-modes are unstable due to the emission of gravitational radiation for all rotating perfect fluid stars. This is interesting since the previously considered gravitational-wave instability associated with (for example) the $f$-mode of the star sets in at a critical rotation rate. Because they are unstable also for the slowest rotating stars the $r$-modes may well be of considerable astrophysical importance.
1. Introduction

This paper is the first in a series of investigations of perturbations of slowly rotating relativistic stars. The original intention was to present an exhaustive description of the problem, together with a detailed discussion of the effect that rotation has on the various pulsation modes of a relativistic star. However, a preliminary investigation into the axial (also referred to as toroidal, or odd-parity) mode-problem unveiled an interesting and rather surprising result: There exist pulsation modes of a rotating relativistic star that are unstable due to the emission of gravitational radiation at all rates of rotation. That is, these modes — the relativistic analogue of the Newtonian $r$-modes, cf. Papaloizou and Pringle (1978) and Saio (1982) — are unstable even for very slowly rotating perfect fluid stars. The instability of the $r$-modes is thus different to previously considered mode-instabilities (e.g. for the $f$-mode of the star) that set in at a certain rate of rotation (Comins 1979; Friedman 1983; Managan 1985; Imamura, Durisen and Friedman 1985; Lindblom 1986; Lindblom and Mendell 1995; Stergioulas and Friedman 1997). But the fact that the $r$-modes are unstable can still be understood in terms of the mechanism that was first discussed by Chandrasekhar (1970) and Friedman and Schutz (1978).

This paper describes the calculation that led to the discovery of the unstable $r$-modes without many of the technical details. These will be reported elsewhere.

2. Slowly rotating relativistic stars

This paper concerns pulsation modes of a slowly rotating relativistic star. The modes follow from a study of linear perturbations of the stellar fluid and the associated spacetime metric, for a star that rotates sufficiently slowly that the effect of rotation can be considered as a small perturbation of a non-rotating configuration. The analysis is based on a double perturbation expansion. To lowest order the star is static and non-rotating, and the perturbations are such that $|h_{\mu\nu}| << 1$, where $g_{\mu\nu} = g_{\mu\nu}^{\text{background}} + h_{\mu\nu}$, and also $\epsilon \equiv \Omega \sqrt{R^3/M} << 1$. Here $R$ is the radius of the star, $M$ is the mass, and $\Omega$ is the rotation frequency of the star according to a distant observer (the star rotates uniformly so $\Omega$ is a constant). Geometrized units $c = G = 1$ are used throughout the paper.

Before presenting the details of the problem it is meaningful to discuss whether a “slow-rotation expansion” is likely to provide results of physical relevance. It is rather straightforward to show that this is the case. The rotation of a star is absolutely limited by the Kepler frequency, at which mass shedding at the stellar equator makes the star unstable. The following empirical formula has proved to be a reasonable approximation of this limiting frequency

$$\Omega_K \approx C \sqrt{\frac{M}{M_\odot} \left(\frac{10 \text{ km}}{R}\right)^3},$$

where $C \approx 7.8 \times 10^3 \text{s}^{-1}$ (Haensel and Zdunik 1989; Friedman, Ipser and Parker 1989; Cook, Shapiro and Teukolsky 1994). $M$ and $R$ refer to the mass and the radius of the corresponding non-rotating star. By rewriting the definition of the expansion parameter $\epsilon$ in the spirit of this relation we get

$$\epsilon = \Omega D \sqrt{\frac{M_\odot}{M} \left(\frac{R}{10 \text{ km}}\right)^3},$$
where $D = 8.6 \times 10^{-5}$ s. That is, the limiting Kepler frequency corresponds to a value of $\epsilon$ that is slightly smaller than 0.7. This is the largest value of $\epsilon$ that should ever be relevant. A calculation including only terms of order $\epsilon$ will probably not provide an accurate description of extremely rapidly rotating stars, but it is interesting to note that an approach that is consistent to $O(\epsilon^2)$ may prove useful also for such cases. It has, in fact, been shown that a second-order calculation ($\epsilon^2$) of the upper mass limit (and other properties) for rotating neutron stars is accurate to within a few percent (Weber and Glendenning 1992).

As a second example, consider a millisecond pulsar with a period of 2 ms (cf. PSR 1937+21 that has a period of 1.6 ms). This corresponds to $\Omega = 3100s^{-1}$ and if we for simplicity assume that $R = 10$ km and $M = M_\odot$ the corresponding value of $\epsilon$ is roughly 0.3. This value should be within reach of a calculation to first order in the rotation expansion. Consequently, such a calculation has the potential to provide considerable information about the fastest spinning pulsars that have been observed.

There are practical reasons why the perturbation approach may be preferred to a fully nonlinear calculation. There are by now several reliable methods for numerically solving the Einstein equations to construct fully relativistic rotating stellar models (see, for example, Friedman et al. 1986; Stergioulas and Friedman 1995), but to calculate the neutral modes (that signal the onset of gravitational-wave instability) for these models is still a difficult task (Stergioulas and Friedman 1997). And it will be considerably harder to calculate the actual pulsation modes of a rapidly rotating, relativistic star. A full consideration of the pulsation properties of rapidly rotating stars will require new computational techniques and considerable computer power. In contrast, the slow-rotation approach should not require any conceptually new steps. The relevant perturbation equations can be reduced to wave equations, and the main difference from the non-rotating case is that one must consider several coupled equations. That the equations take a standard form is an enormous advantage when one is interested in extracting the complex frequencies of the pulsation modes. Moreover, the slow-rotation calculation serves as a useful benchmark test for future work on rapidly rotating stellar models.

### 3. Axial perturbations of a slowly rotating star

As was shown by, for example, Hartle (1967), the centrifugal force affects the shape of a rotating relativistic star through terms of order $\epsilon^2$. Hence, one can assume that a slowly rotating star remains spherical as a first approximation. The corresponding metric can be written

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 - 2\omega \sin^2 \theta dt d\phi + \sin^2 \theta d\phi^2)$$

For a given equation of state $p = p(\rho)$, where $\rho$ is the energy density, the pressure $p$ and the two metric coefficients $\nu$ and $\lambda$ are determined by the standard TOV equations [I use standard notation and the relevant equations are listed by, for example, Kojima (1992)].

When terms of order $\epsilon$ are included in the slow-rotation expansion one must account for the “dragging of inertial frames”, that is represented by $\omega$ in (1). After defining $\tilde{\omega} = \Omega - \omega$ we have (Hartle 1967)

$$\frac{d^2 \tilde{\omega}}{dr^2} - \left[4\pi(\rho + p)e^\lambda r - \frac{4}{r}\right] \frac{d\tilde{\omega}}{dr} - 16\pi(\rho + p)e^\lambda \tilde{\omega} = 0 .$$

The desired solution to this equation is well behaved at the centre of the star;

$$\tilde{\omega} \approx \tilde{\omega}_0 \left[1 + \frac{8\pi}{5}(\rho_0 + p_0)r^2 + O(r^4)\right], \quad \text{as } r \to 0 ,$$
where $p_0$ and $\rho_0$ are the central values of the pressure and the density. In the vacuum outside the star we have
\[
\omega = \frac{2J}{r^3}
\]  
where $J$ is the total angular momentum of the star. The constants $\tilde{\omega}_0$ and $J$ are fixed by the requirement that $\tilde{\omega}$, and its derivative, be continuous across $r = R$.

The equations that describe perturbations of a slowly rotating star in general relativity have previously been considered by Chandrasekhar and Ferrari (1991a) and Kojima (1992). I will use the notation of Kojima (1992), and refer the interested reader to his paper for a listing of the original perturbation equations.

The perturbations of a non-rotating star can be divided into two classes: axial and polar (often referred to as odd- and even parity perturbations, respectively). The equations that describe these two classes decouple and can be considered separate from each other. Specifically, one can show that axial perturbations are governed by a single wave equation (in Regge-Wheeler gauge)
\[
D_l^X(X_{lm}) = \frac{d^2X_{lm}}{dr_*^2} + (\sigma^2 - V^{X}_{l})X_{lm} = 0 .
\]
(5)

Once the function $X_{lm}$ is known one can infer the metric perturbations $h_{\theta\theta}$, $h_{\phi\phi}$, $h_{r\theta}$ and $h_{r\phi}$ as well as the fluid displacement $\xi_\phi$. The angular dependence of all perturbed variables is expressed in terms of the spherical harmonics $Y_{lm}$ [for details, see Kojima (1992)]. In (5) it is assumed that the time dependence of the perturbation is $e^{-\sigma t}$. The assumption of a harmonic time-dependence is adequate as long as the focus is on the spectral properties of the star. The tortoise coordinate $r_*$, which is defined by
\[
\frac{d}{dr_*} = e^{(\nu-\lambda)/2} \frac{d}{dr} ,
\]
(6)
is familiar from studies of perturbed black holes, and the effective potential is
\[
V^{X}_{l}(r) = e^{\nu} \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} + 4\pi(\rho - p) \right] .
\]
(7)

This expression is valid both inside the star and in the exterior vacuum [$M(r)$ is the mass inside radius $r$].

Let us now assume that the star is slowly rotating. To incorporate the first order effects of rotation on the perturbations we consider
\[
X_{lm} = X^0_{lm} + X^{(1)}_{lm} + ... 
\]
(8)

where $X^0_{lm}$ is a solution to the axial equation for a nonrotating star and $X^{(1)}_{lm}$ corresponds to terms of order $\epsilon$. This results in the coupled equations
\[
D_l^X(X^{0}_{lm}) = 0 ,
\]
(9)
and
\[
D_l^X(X^{(1)}_{lm}) = A^X_{lm} + P^{X}_{l+1m} + P^{X}_{l-1m} .
\]
(10)

In this schematic description, $A^X_{lm}$ represents the coupling to the zero order axial equation, and $P^{X}_{l\pm 1m}$ couple the first order axial perturbations to the zeroth order polar ones.
The explicit form of the coupling to the lowest order axial solution is (Kojima 1992)

\[ A^{X}_{lm} = \frac{2m}{l(l+1)\sigma} \left\{ \left[ l(l+1)\sigma^2(\Omega - \tilde{\omega}) + [l(l+1) - 2]8\pi(p + \rho)e^{\nu} \tilde{\omega} - \frac{e^{\nu}}{r^2} \frac{d\tilde{\omega}}{dr} \left[ l(l+1) - 2(2\nu - 5M - 4\pi\rho r^3) - 2re^{-\lambda} \right] \right] X^{(0)}_{lm} + l(l+1)e^{(\nu - \lambda)/2} \frac{d\tilde{\omega}}{dr} \frac{dX^{(0)}_{lm}}{dr} \right\}. \tag{11} \]

In the exterior vacuum this expression reduces to

\[ A^{X}_{lm} = \frac{2mJ}{l(l+1)\sigma} \left\{ \left[ \frac{2l(l+1)\sigma^2}{r^3} - 6e^{\nu} \frac{/[l(l+1) - 3](2\nu - 5M - M)}{r^4} - \frac{6l(l+1)e^{\nu}}{r^4} \frac{dX^{(0)}_{lm}}{dr} \right] X^{(0)}_{lm} + 6l(l+1)e^{\nu} \frac{dX^{(0)}_{lm}}{dr} \right\}. \tag{12} \]

The explicit form for \( P_{\pm l}^{X} \) was given by Kojima (1992).

The apparent coupling between the axial and polar perturbations in (10) potentially complicates the definition of a stellar pulsation mode, but these complications enter only at order \( \epsilon^2 \) and higher. As in the non-rotating case, the asymptotic behaviour of a general solution is

\[ X_{lm} \sim x_{\text{out}}e^{i\sigma r} + x_{\text{in}}e^{-i\sigma r}. \tag{13} \]

for a general frequency \( \sigma \). Here \( x_{\text{out}} = x^{(0)}_{\text{out}} + x^{(1)}_{\text{out}} \), and similar for \( x_{\text{in}} \). In analogy with the non-rotating case a quasinormal mode of the star should correspond to a frequency \( \sigma_n \) such that \( x_{\text{in}}(\sigma_n) = 0 \). In principle, one can find such solutions to the axial slow-rotation equations for any magnitude of \( P_{l}^{X} \) (Chandrasekhar and Ferrari 1991a). But in this paper the primary interest concerns modes that limit to the axial modes of a nonrotating star as \( \epsilon \to 0 \). For such modes the zero order (nonrotating) polar perturbations should vanish, and hence one should use \( P_{l+1}^{X} = P_{l-1}^{X} = 0 \) in (10). The coupling between the two classes of perturbations will affect the mode-frequencies only at order \( \epsilon^2 \) and higher (Kojima 1993a). In principle, an axial mode generates a polar perturbation of order \( \epsilon \) that feeds back and affects the axial mode-spectrum at second order. Since this paper is restricted to the first order axial mode-spectrum the effect of the coupling to the polar perturbations will not be included. An axial pulsation mode is defined by a solution to (9) and (10), with \( P_{l+1}^{X} = P_{l-1}^{X} = 0 \) for which \( x_{\text{in}}(\sigma_n) = 0 \). This approach is identical to that used by Kojima for the polar problem (Kojima 1993b). The idea seems consistent, but the definition of modes for a rotating star is a complicated issue that requires further discussion.

The integration of (9) and (10) is initiated at a small value of \( r \) using the regular power series solutions. In the case of (10) I use only the particular solution that corresponds to the coupling term \( A^{X}_{lm} \). The equations are then integrated to the surface of the star \( r = R \). This part of the numerical calculation is straightforward. The main difficulty arises in the exterior of the star, and is associated with the well-known fact that the eigenfunction of a pulsation mode diverges as \( r \to \infty \). Specifically, a purely outgoing-wave solution will behave as \( X_{lm} \sim e^{i\sigma r} \) for large \( r \). A pulsation mode that is damped with time according to an observer at fixed \( r \) corresponds to \( \text{Im} \sigma_n < 0 \), and the desired solution to (9) and (10) will therefore diverge as \( r \) increases. For a general complex frequency the decaying solution — that represents ingoing waves — will drown...
in the growing one and, unless $\Im \sigma_n$ is relatively small, one cannot extract the asymptotic amplitudes $x_{\text{out}}$ and $x_{\text{in}}$ from the numerical data.

Several methods have been devised to deal with this problem (Kokkotas and Schutz 1992; Leins et al 1993; Andersson et al 1995). Most of these were taken over from the study of black-hole quasinormal modes. However, none of these methods is suitable for coupled equations. For example, while the WKB method of Kokkotas and Schutz (1992) can readily be used for a single second order ODE it is very difficult to use it for two coupled equations. In the slow-rotation problem one must deal with coupled equations. As long as the focus is on the spectral properties of the axial perturbations there is a way around this: By moving the term in the right hand side of (10) over to the left hand side and replacing $X^{(0)}$ with $X^{(1)}$ we only introduce an error of order $\epsilon^2$. Thus, the new equation — that can be approached in the standard way — is consistent to $O(\epsilon)$. This trick was used by Kojima in his study of the polar problem (Kojima 1993b), but I have chosen not to use it here.

Instead, I have devised a "new" method to suppress the exponential divergence of the eigenfunctions. This method has the advantage that it will remain useful also for manifestly coupled equations. The integration of (9) and (10) is simply done using complex-valued $r$ in the exterior spacetime. One would expect the exponential growth in $X \sim e^{\pm i \sigma r} \approx e^{\pm i \sigma r}$ to be suppressed if the integration is performed along a straight line such that $\text{Arg}(r) = -\text{Arg}(\sigma)$. Numerical tests show that this is indeed the case, and what remains is an (almost) purely oscillating function of $r$. This function can be integrated with satisfactory precision from the surface of the star out to a large value of $|r|$. The numerical solution is then matched to an expansion of form $e^{\pm i \sigma r} \sum a_n (1/r)^n$ at the final point of integration. This yields the values of the asymptotic amplitudes $x_{\text{out}}$ and $x_{\text{in}}$ for any given complex frequency. Given these amplitudes one can iterate to find a frequency such that $x_{\text{in}}(\sigma_n) = 0$, and identify a quasinormal mode of the star.

To test the reliability of the new integration method I repeated the polar-mode calculations of Andersson, Kokkotas and Schutz (1995). Most of their mode-frequencies for a non-rotating polytrope could be reproduced with satisfactory precision. However, the new scheme is not able to get to the few $w$-modes that are extremely rapidly damped. Moreover, the new integration approach is not accurate enough to enable iteration for very slowly damped modes, like the $f$-mode, in contrast to the method used by Andersson, Kokkotas and Schutz (1995). This is not a serious drawback, however. Slowly damped modes can be identified by a simple technique based on results for real frequencies [the details of this method, which is an alternative to the standard resonance method (Chandrasekhar and Ferrari 1991b), will be described elsewhere].

4. The effect of rotation on the axial modes

In the non-rotating case the symmetries of the perturbation equations dictate that the stellar pulsation modes be symmetric with respect to the imaginary $\sigma$ axis. That is, if $\sigma_n$ corresponds to a mode then another mode is associated with $-\bar{\sigma}_n$ (the bar denotes complex conjugation). When the star is rotating this symmetry is broken. Each of the modes for a non-rotating star splits into $2l + 1$ distinct modes, each corresponding to a different value of $m$ (cf. the Zeeman splitting in quantum problems), and the symmetry changes in such a way that $\sigma_{nm}$ is now associated with $-\bar{\sigma}_{n-m}$.

The angular dependence of the perturbation functions is expressed in terms of spherical
harmonics (cf. Kojima 1992). Specifically, the dependence on the azimuthal angle is $e^{im\phi}$. Since the time-dependence is assumed to be $e^{-i\sigma t}$ positive frequency ($\text{Re} \, \sigma > 0$) modes for $m > 0$ will be prograde relative to the stellar fluid and $m < 0$ modes will be retrograde. Typically, retrograde modes will be destabilized by increased rotation, and the corresponding damping rates should decrease. For the prograde modes rotation will have a stabilizing effect, and their damping rate should therefore increase. The reason for this is the central mechanism of the Chandrasekhar-Friedman-Schutz instability (Chandrasekhar 1970; Friedman and Schutz 1978): Suppose that a mode has a frequency such that it is retrograde according to an observer in the fluid, but because of the dragging of inertial frames the mode appears prograde according to a distant observer. The distant observer — who measures energy according to his own time-line — finds that the mode carries positive angular momentum away from the star. But since the perturbed fluid actually rotates slower than it would in absence of theperturbation the angular momentum of the retrograde mode is negative. The emission of gravitational waves makes the angular momentum increasingly negative and leads to an instability. The same argument can be used to explain the destabilization of retrograde modes in general.

A nonrotating perfect fluid star has no (non-trivial) axial pulsation modes in Newtonian theory. All the familiar fluid pulsation modes (the $f$, $p$ and $g$ modes) are associated with polar perturbations. But in the relativistic picture an infinite set of axial modes exist (Chandrasekhar and Ferrari 1991c; Kokkotas 1994). These are gravitational wave $w$-modes that exist because of the dynamic spacetime (see, for example, Kokkotas and Schutz 1992; Andersson et al 1996). The fundamental $w$-mode of a non-rotating neutron star typically has a pulsation frequency of the order of 10 kHz and damps out in a fraction of a millisecond. The modes become slower damped for more relativistic models. Furthermore, it is possible that an increased rotation rate will lead to relatively slowly damped (or even unstable) $w$-modes since retrograde modes should become slower damped as the rotation of the star increases.

I have performed a sample of calculations for the polytropic equation of state

$$p = 100\text{km} \rho^2.$$  \hfill (14)

Figure 1 shows results for the slowest damped axial $w$-mode of a stellar model with central density $\rho_c = 3 \times 10^{15}\text{g/cm}^3$. This leads to $R = 8.861$ km and $M = 1.869$ km = 1.27$M_{\odot}$, i.e. reasonable parameters for a neutron star. The results shown in Figure 1 nicely illustrate the effects of rotation on the stellar pulsation modes. As anticipated, each mode for the non-rotating star splits into $2l + 1$ distinct ones. The prograde modes become shorter lived as the rotation rate increases, while the damping rate of the retrograde modes decreases.

As already mentioned there are no non-trivial axial pulsation modes of a non-rotating star in Newtonian theory. But there is a set of Newtonian axial modes that are degenerate at zero frequency. As soon as the star rotates they become non-zero. These are the so-called $r$-modes — that are analogous to Rossby waves in the Earth’s oceans — and Newtonian calculations to first order in rotation show that they have frequencies of order $\Omega$ (Papaloizou and Pringle 1978; Saio 1982). Newtonian calculations also show that the $r$-modes remain degenerate at first order in $\Omega$. To unveil the detailed behaviour or the radial eigenfunctions and break the degeneracy one must proceed to higher orders (Smeyers 1980; Provost et al. 1981; Smeyers et al. 1981; Saio 1982).

The $r$-modes have not previously been calculated in relativity, and the result of the present calculation is perhaps surprising: The relativistic analogue of the $r$-modes are unstable for all rotating stars. This result does, however, make perfect sense. A Newtonian calculation to first
order in $\Omega$ yields the $r$-mode frequency (Papaloizou and Pringle 1978; Saio 1982);

$$\sigma_r = m\Omega \left( 1 - \frac{2}{l(l+1)} \right), \quad (15)$$

according to a distant observer. That is, these modes are retrograde relative to the fluid but will appear prograde at infinity. Such modes should (as already discussed) be destabilized by the emission of gravitational waves, and since the $r$-modes are initially (for a non-rotating star) situated at $\sigma_r = 0$ they should become unstable as soon as the star spins up.

A sample of results for the $r$-modes and the polytropic equation of state (14) are shown in Table 1. The tabulated data are for the same stellar model that was used to generate the $w$-mode data in Figure 1 (central density $\rho_c = 3 \times 10^{15} \text{g/cm}^3$). It is interesting to note that the growth-times for these modes (represented by the “e-unfolding” time $\tau_e = 1/\text{Im} \sigma_n$) are rather short. For example, for a millisecond pulsar the amplitude of the modes listed in Table 1 would multiply by a factor of three in a few seconds.

The $r$-mode growth rates given in Table 1 should, however, be viewed with some caution. The reason for this is the simple fact that, for small frequencies $\sigma \sim \Omega$, the perturbation equations (9) and (10) explicitly include terms that are formally of second order in $\epsilon \sim \Omega$. As one can readily deduce from the data in Table 1 the numerical results scale with the rotation parameter as $\text{Im} \sigma_r \sim \epsilon^2$ as $\epsilon \to 0$. Hence, the estimated growth rates for the $r$-modes are due to the included second order terms — the eigenfrequency is real to first order in $\epsilon$. That is, the presented results for the imaginary parts of the eigenfrequencies are not obtained through a consistent calculation. It is possible (likely) that the results will change considerably once the calculation is extended to order $\epsilon^2$.

The numbers listed in Table 1 are nevertheless interesting. They provide the first suggestion that the $r$-modes are unstable in relativity. An argument based on the oscillation frequencies of the modes (which are of order $\epsilon$ and should be obtained correctly in the calculation) and the mechanism behind the Chandrasekhar-Friedman-Schutz instability clearly establishes the existence of this instability. It is also far from clear how “wrong” the growth rates listed in Table 1 are. The error seems to be of the same order as the result itself, but this does not necessarily mean that the obtained result is useless as an order-of-magnitude estimate. What is clear at the moment is that a calculation that is consistent to second order in $\epsilon$, or an alternative estimate of the growth rate of the instability, is urgently needed. It will either show that $\text{Im} \sigma_r \sim \epsilon^2$ or establish that the true damping enters at higher orders. If the latter is the case a slow-rotation calculation of these growth rates seems very difficult.

A second-order calculation would also be interesting since the degeneracy of the $r$-modes should be broken at order $\epsilon^2$. Then $r$-modes with a different number of nodes in the radial eigenfunctions should lead to distinct eigenfrequencies (Saio 1982).

5. Discussion

I have presented the first numerical results for axial pulsation modes of a slowly rotating relativistic star. The calculation includes all terms of first order in $\epsilon \equiv \Omega \sqrt{R^3/M}$. It includes effects due to the coriolis force, but neglects effects due to the centrifugal flattening of the star. Such effects will appear at order $\epsilon^3$ and higher, and may well have a considerable effect on the various stellar pulsation modes.
It has been shown that increased rotation has a stabilizing effect on prograde modes, while retrograde modes are destabilized. This was expected from the mechanism behind the familiar Chandrasekhar-Friedman-Schutz instability (Chandrasekhar 1970; Friedman and Schutz 1978). Specifically, it was demonstrated that rotation affects the axial gravitational wave $w$-modes in the anticipated way.

I have also presented the first relativistic results for the so-called $r$-modes. The somewhat surprising result that these modes are unstable due to the emission of gravitational radiation even for the slowest rotating stars is interesting for many reasons. First of all, axial modes have not been considered in previous studies of stellar mode-instabilities (see, for example, Friedman 1983; Lindblom 1986; Stergioulas and Friedman 1997). The present result shows that they are relevant, and that more detailed work is required in order for us to determine whether the unstable $r$-modes are astrophysically important.

We need to establish if this new class of instabilities is stronger or weaker than the familiar instability associated with (say) the $f$-mode of a rapidly rotating star. The answer depends on the relative growth times of the various unstable modes. As already mentioned, the numerical results presented here (cf. Table 1) should be used with considerable caution, but they will have to serve as a guideline until more reliable (higher order) data become available.

If the unstable $r$-modes are to be of astrophysical interest they must grow on a time-scale much shorter than that associated with the damping due to viscosity, cf. the instability associated with the $f$-mode that is estimated to limit the rotation period of a $10^7$ K neutron star at the level of 1-2 ms (Cutler and Lindblom 1987). To make a similar estimate based on the instability growth rates obtained in this paper is probably meaningless, especially since these results are expected to change (probably by several orders of magnitude) once more detailed studies are done.

The $r$-mode instability is conceptually interesting, especially since it is active for all, even the slowest, rotating stars. It is also possible that this new class of unstable modes of rotating relativistic stars is of considerable astrophysical relevance. The $r$-mode instability could, for example, limit the rotation period of nascent neutron stars. Whether this is the case or not is obviously unclear at present. It is a challenge for future work to provide an answer to such questions.

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Table 1: r-modes for a $\Gamma = 2$ polytrope. This specific star has central density $3 \times 10^{15}$g/cm$^3$, which leads to $R = 8.861$ km and $M = 1.869$ km = $1.27M_\odot$. All these modes are unstable due to the emission of gravitational waves. The dimensionless rotation parameter $\epsilon$, the rotation period $2\pi/\Omega$, the pulsation frequency $f$ and the “e-unfolding” time $\tau_e$ of the modes are listed. The data are for $l = 2$ and $|m| = 2$.

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Fig. 1.— The effect of rotation on the slowest damped axial gravitational wave mode for a $\Gamma = 2$ polytrope. This specific star has central density $3 \times 10^{15} \text{g/cm}^3$, which leads to $R = 8.861 \text{ km}$ and $M = 1.869 \text{ km} = 1.27M_\odot$. The left panel shows the pulsation frequency of the star in kHz as a function of the dimensionless rotation parameter $\epsilon \equiv \Omega \sqrt{R^3/M}$. For this stellar model $\epsilon \approx 0.2$ would correspond to a pulsar with period 2 ms. The data corresponds to $l = 2$ and (from bottom to top) $m = -2, -1, 0, 1, 2$. The $m < 0$ modes are retrograde while the $m > 0$ modes are prograde according to an observer in the stellar fluid.