A timing formula for main-sequence star binary pulsars

Norbert Wex★
Max-Planck-Institut für Radioastronomie, Auf dem Hügel 69, D-53121 Bonn, Germany

ABSTRACT
In binary radio pulsars with a main-sequence star companion, the spin-induced quadrupole moment of the companion gives rise to a precession of the binary orbit. As a first approximation one can model the secular evolution caused by this classical spin-orbit coupling by linear-in-time changes of the longitude of periastron and the projected semi-major axis of the pulsar orbit. This simple representation of the precession of the orbit neglects two important aspects of the orbital dynamics of a binary pulsar with an oblate companion. First, the quasiperiodic effects along the orbit, due to the anisotropic $1/r^3$ nature of the quadrupole potential. Secondly, the long-term secular evolution of the binary orbit which leads to an evolution of the longitude of periastron and the projected semi-major axis which is non-linear in time.

In this paper a simple timing formula for binary radio pulsars with a main-sequence star companion is presented which models the short-term secular and most of the short-term periodic effects caused by the classical spin-orbit coupling. I also give extensions of the timing formula which account for long-term secular changes in the binary pulsar motion. It is shown that the short-term periodic effects are important for the timing observations of the binary pulsar PSR B1259–63. The long-term secular effects are likely to become important in the next few years of timing observations of the binary pulsar PSR J0045–7319. They could help to restrict or even determine the moments of inertia of the companion star and thus probe its internal structure.

Finally, I reinvestigate the spin-orbit precession of the binary pulsar PSR J0045–7319 since the analysis given in the literature is based on an incorrect expression for the precession of the longitude of periastron. A lower limit of 20° for the inclination of the B star with respect to the orbital plane is derived.

Key words: pulsar timing – binary pulsars – classical spin-orbit coupling – pulsars: individual: PSR J0045–7319, PSR B1259–63

1 INTRODUCTION
Timing observations of pulsars, i.e. the measurement of the time-of-arrival (TOA) of pulsar signals at a radio telescope, is one of the few high-precision experiments in astronomy and therefore has a wide-ranging field of interesting applications (Bell 1996). The first evidence for the existence of gravitational waves as predicted by Einstein’s theory of gravity (Taylor & Weisberg 1989) and the first discovery of extrasolar planets (Wolszczan & Frail 1992) are just the most striking examples for the achievements of high-precision pulsar-timing observations. Approximately 10% of the known pulsars are members of binary star system, i.e. in orbit around a white dwarf, neutron star or main-sequence star companion. Timing observations of these binary pulsars is a powerful tool to study various physical and astrophysical effects related to binary star motion and stellar evolution. To extract the maximum possible information from pulsar timing observations, one needs an appropriate model (timing formula) for transforming each measured topocentric TOA, $t_{\text{obs}}$, to the corresponding time of emission, $T$, measured in the reference frame of the pulsar. Various timing formulae, particularly for relativistic binary pulsars, have been developed to describe radio-pulsar timing observations.

With the discovery of PSR B1259–63 during a high-frequency survey of the Galactic plane by Johnston et al. (1992) the first radio pulsar with a massive, non-degenerate companion was found. PSR B1259–63 is a 48-ms pulsar in a highly eccentric orbit with the main-sequence Be star SS 2883. The second known radio pulsar with a massive, non-degenerate companion is PSR J0045–7319, discovered in a systematic search of the Magellanic Clouds for radio pulsars (McConnell et al. 1991, Kaspi et al. 1994). Some of the parameters of these two main-sequence star binary pulsars are...
listed in Table 1. For both binary systems significant deviations from a Keplerian orbit have been detected which are most easily explained by classical spin-orbit coupling (Lai et al. 1995, Kaspi et al. 1996, Wex et al. 1997). Due to their high proper rotation the main-sequence star companions of PSRs B1259–63 and J0045–7319 show rotational deformation and thus give rise to a gravitational quadrupole field. As a result of this a coupling between the orbital angular momentum and the spin of the companion takes place and leads to a precession of the binary orbit.

In this paper I will show that the present timing formulae represent only crude approximations to the orbital dynamics caused by the classical spin-orbit coupling and that there is a need for a new timing formula to model the timing observations. Before I focus on the construction of a new timing formula for main-sequence star binary pulsars a short presentation of the most important timing models is given.

In a simple spin-down law the pulsar proper time is related to the phase, $\phi$, of the pulsar by

$$\phi(T) = \phi_0 + \nu T + \frac{1}{2} \nu^2 T^2 + \frac{3}{8} \nu^3 T^3,$$  

(1)

where $\nu$, $\dot{\nu}$, and $\ddot{\nu}$ are the rotation frequency of the pulsar, its first and second time derivative, respectively (spin parameters).

For a single pulsar the timing formula includes terms related to the position ($\alpha, \delta$), proper motion ($\mu_\alpha, \mu_\delta$) and parallax ($\pi$) of the pulsar. Moreover, it contains terms related to relativistic time dilation and light propagation effects in the solar system and also corrects for propagation effects in the interstellar medium (Backer 1989, Taylor 1989, Doroshenko & Kopeikin 1990):

$$T = t_{\text{obs}} + \Delta C - D/f_b^2$$

$$+ \Delta_{R\odot}(\alpha, \delta, \mu_\alpha, \mu_\delta, \pi) + \Delta_{E\odot} + \Delta_{S\odot}(\alpha, \delta).$$  

(2)

$\Delta C$ corrects for the offset between the observatory clock and the 'Terrestrial Dynamical Time' represented by the best terrestrial standard of time. $D/f_b^2$ corrects for the dispersive delay in the interstellar medium at the (barycentric) frequency $f_b$ where $D$ is proportional to the column density of free electrons between the pulsar and the observer. $\Delta_{R\odot}$ describes the so called Roemer delay, a change in the time of flight of the radio signal caused by the motion of the observer in the solar system reference frame. $\Delta_{E\odot}$ represents the transformation between 'Terrestrial Dynamical Time' and 'Barycentrical Dynamical Time' (Fairhead & Bregtagnon 1990). Finally, $\Delta_{S\odot}$ describes the Shapiro delay in the gravitational field of the Sun (Shapiro 1964).

For binary pulsars the timing formula (2) has to be extended by terms representing orbital motion and light propagation effects in the binary system (Blandford & Teukolsky 1976, Damour & Deruelle 1986, Damour & Taylor 1992):

$$T = t_{\text{obs}} + \Delta C - D/f_b^2 + \Delta_{R\odot} + \Delta_{E\odot} + \Delta_{S\odot}$$

$$+ \Delta_R + \Delta_E + \Delta_S,$$  

(3)

where the major effect is the Roemer delay $\Delta_R$ which depends on the orbital motion of the pulsar and the orientation of the pulsar orbit with respect to the line of sight. If the binary motion is purely Keplerian then the Roemer delay depends on 5 (Keplerian) parameters:

- $P_b$, the orbital period of the binary system,
- $x = a_p \sin i/c$, the projected semi-major axis,
- $e$, the eccentricity of the orbit,
- $\omega$, the longitude of periastron,
- $T_0$, the time of periastron passage.

$a_p$ is the semi-major axis of the pulsar orbit, $i$ is the inclination of the orbital plane with respect to the line of sight, where $i = 90^\circ$ implies edge on, and $c$ is the speed of light. The Roemer delay caused by the Keplerian motion of a binary system is given by

$$\Delta_R = x[(\cos U - e) \sin \omega + (1 - e^2)^{1/2} \sin U \cos \omega],$$  

(4)

where $U$, the eccentric anomaly, is related to time, $T$, by the Kepler equation

$$U - e \sin U = 2\pi - T/T_0.$$  

(5)

Soon after the discovery of PSR B1913+16 (Hulse & Taylor 1975) it was clear that a pure Keplerian timing model is not appropriate to analyse the timing observation of this 7.8-hour orbital-period binary pulsar. For this purpose Blandford & Teukolsky (1976) derived a timing model (BT model) which contains the largest short-period relativistic effect, the 'Einstein delay' $\Delta_E$, a combination of special-relativistic time dilation and gravitational redshift. They also included secular drifts of the main orbital parameters by following linear-in-time expressions:

$$P_b = P_{b0} + \dot{P}_b(T - T_0),$$  

(6)

$$x = x_0 + \dot{x}(T - T_0),$$  

(7)

$$e = e_0 + \dot{e}(T - T_0),$$  

(8)

$$\omega = \omega_0 + \dot{\omega}(T - T_0).$$  

(9)

Based on a remarkably simple analytic solution of the post-Newtonian two-body problem (Damour & Deruelle 1985) Damour & Deruelle (1986) derived an improved timing formula (DD model) for relativistic binary pulsars. The DD model differs from the BT model in two ways: it contains the Shapiro delay $\Delta_S$, which is of particular importance for $i$ close to 90°, and it allows for periodic effects in the orbital motion, e.g. in the BT model only the secular drift of the longitude of periastron is taken into account (equation (9)), whereas the DD model describes both the secular and quasi-periodic changes in $\omega$ according to

<table>
<thead>
<tr>
<th>B1259–63</th>
<th>J0045–7319</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_b$ (days)</td>
<td>1237</td>
</tr>
<tr>
<td>$x$ (sec)</td>
<td>1296</td>
</tr>
<tr>
<td>$e$</td>
<td>0.870</td>
</tr>
<tr>
<td>$\omega$ (deg)</td>
<td>138.7</td>
</tr>
<tr>
<td>$i$ (deg)</td>
<td>36 or 144</td>
</tr>
<tr>
<td>$m_*$ ($M_\odot$)</td>
<td>$\sim 10$</td>
</tr>
<tr>
<td>$R_*$ ($R_\odot$)</td>
<td>$\sim 6$</td>
</tr>
</tbody>
</table>
\[ \omega = \omega_0 + kA_e(U), \quad \text{with} \quad \dot{\omega} = 2\pi k/P_b, \quad (10) \]

where

\[ A_e(U) = 2 \arctan \left[ \frac{(1 + e)^{1/2}}{1 - e} \tan \frac{U}{2} \right], \quad (11) \]

and \( U \) is a function of \( T \) given by the solution of the (generalised) Kepler equation

\[ U - e \sin U = 2\pi \left( \frac{T - T_0}{P_b} \right) - \frac{P_b}{2} \left( \frac{T - T_0}{P_b} \right)^2. \quad (12) \]

\( \dot{P}_b \) accounts for any secular change in the orbital period, like the damping caused by tidal dissipation or the emission of gravitational waves. The post-Keplerian Roemer delay is given by

\[ \Delta_R = x \{ [\cos U - e(1 + \delta_t)] \sin \omega(U) \\
+ [1 - e^2(1 + \delta_t)^2]^{1/2} \sin U \cos \omega(U) \}, \quad (13) \]

where \( \omega(U) \) is given by equation (10). The post-Keplerian parameters \( \delta_t \) and \( \delta_p \) represent periodic post-Newtonian changes in the orbital motion, i.e. periodic changes of order \( (v/c)^2 \), where \( v \) is a typical orbital velocity of the binary star system.

To date, more than 50 radio pulsars are known which are members of a binary star system. The vast majority of these binary pulsars have a compact degenerate companion which is either a helium white dwarf or a neutron star. Timing observations for most of these binary pulsars can be fully explained by the timing models above. In a first approximation these models can be used for timing observations of the two main-sequence star binary pulsars, PSRs B1259–63 and J0045–7319, by fitting for the five Keplerian parameters, and for \( \omega \) and \( x \). However, this approximation models only the short-term secular changes of the binary orbit correctly.

In this paper an improved timing model for binary radio pulsars with main-sequence star companions is presented. The new timing formula accounts for the short-term secular precessional effects, for most of the short-term periodic orbital effects and for the long-term secular effects which are caused by classical spin-orbit coupling. In Section 2 a detailed investigation of the orbital dynamics of binary systems with classical spin-orbit coupling is given. First, a simple analytic solution is presented for the case that the orbital motion takes place in the equatorial plane of the massive companion. This solution will be helpful when developing the new timing formula. Then, the orbital dynamics of the general case, i.e. arbitrary orientation of the orbit with respect to the companion, is studied. In Section 3 it is shown how the orbital dynamics influences the timing observation of the main-sequence star binary pulsars PSRs B1259–63 and J0045–7319. As a consequence the new timing formula is developed. In Section 4 the long-term validity of the old and new timing formulae is studied. Extensions which are quadratic in time and take into account long-term precessional effects in the binary orbit are presented. It is shown that observations of such long-term secular effects have the potential to probe the internal structure of the companion. In Section 5 I investigate the orbital precession of PSR J0045–7319 since results presented so far in the literature were based on an incorrect formula for the precession of the longitude of periastron. In Section 6 the conclusions are given.

2 THE ORBITAL MOTION OF MAIN-SEQUENCE STAR BINARY PULSARS

The quadrupole of a main-sequence star companion is given by the difference between the moments of inertia about the spin-axis, \( I_0^* \), and an orthogonal axis, \( I_1^* \). It is proportional to the mass of the companion, \( m_* \), the (polar) radius of the rotating star, \( R_* \), and to the spin squared (Cowling 1938, Schwarzschild 1958):

\[ I_3^* - I_1^* = \frac{3}{2} k m_* R_*^2 \Omega_*^2 \equiv m_* q, \quad (14) \]

where \( k \) is the apsidal motion constant and \( \Omega_* \) is the dimensionless proper rotation of the companion. \( \Omega_* \) is the angular velocity of the companion’s proper rotation. A main-sequence star of 10\( M_\odot \) has \( k \lesssim 0.03, R_* \sim 6R_\odot \). If the star is rotating at 70% of its break-up velocity, which appears to be a typical value for Be stars (Porter 1996), one finds for its quadrupole moment

\[ q = \frac{3}{2} k R_*^2 \Omega_*^2 \lesssim 0.35 R_\odot^2 = 7.6 \times 10^{-6} \text{AU}^2. \quad (15) \]

The spin-induced quadrupole moment of the companion leads to an additional \( 1/r^3 \) potential term in the gravitational interaction between the two components which implies an apsidal motion and, when the spin of the companion is not aligned with the orbital angular momentum, to a precession of the orbital plane. The general expressions for the rates of apsidal motion and orbital precession can be found in Smarr & Blandford (1976) and Kopal (1978), (see Section 4 in this paper). The expressions in Smarr & Blandford (1976) and Kopal (1978) are derived by averaging the orbital dynamics over a full orbital period, in order to get the secular changes in the orbit. This procedure, by definition, neglects all short-term periodic effects. But, as will be shown in this paper, for main-sequence star binary pulsars with a long orbital period and a high eccentricity, like PSRs B1259–63 and J0045–7319, these short-term periodic effects are important.

For a study of the short-term periodic effects one needs the orbital motion in full detail. In the centre-of-mass system the (Newtonian) orbital dynamics of a binary pulsar with an oblate companion star is given by the Hamiltonian (Barker & O’Connell 1975)

\[ \mathcal{H} = \frac{\mathbf{p}^2}{2\mu} - \frac{GM\mu}{r} \left( 1 + \frac{q}{2r^2} \left[ 1 - 3(\mathbf{s} \cdot \hat{n})^2 \right] \right), \quad (16) \]

where the linear momentum \( \mathbf{p} \) is related to the linear moment of pulsar and companion by \( \mathbf{p} = \mathbf{p}_p = -\mathbf{p}_c \). \( \mathbf{r} \) is a vector pointing from the companion to the pulsar, \( r \equiv |\mathbf{r}| \), \( \hat{n} \equiv \mathbf{r}/r, M = m_p + m_\ast \) is the total mass of the system, \( \mu \equiv m_p m_\ast/M \) is the reduced mass and \( \mathbf{s} \) is the unit vector in direction of the spin of the companion.

Before studying the full dynamics, I will investigate the special case of motion in the equatorial plane. Although the
two known main-sequence star binary radio pulsars do not orbit their companion in the equatorial plane the comparably simple solution of this problem will be helpful in developing a new timing formula in the next section.

2.1 The equatorial motion

For motion in the equatorial plane ($\hat{s} \cdot \hat{n} = 0$) the Hamiltonian (16) reduces to

\[
\mathcal{H}_\perp = \frac{p^2}{2\mu} - \frac{GM\mu}{r} \left(1 + \frac{q}{2r^2}\right).
\]

(17)

The invariance of this Hamiltonian under time translation and spatial rotations implies the conservation of the (reduced) energy, $E \equiv \mathcal{H}_\perp/\mu$, and the (reduced) total angular momentum, $J \equiv \mathbf{r} \times p/\mu$. If one introduces polar coordinates, $r = r(\cos f, \sin f, 0)$, and makes use of the conserved quantities one finds the following equations of motion:

\[
r^2 = 2E + \frac{2GM}{r} - \frac{J^2}{r^2} + \frac{GMq}{r^3},
\]

(18)

\[
\dot{f} = \frac{J}{r^2},
\]

(19)

where $J \equiv |\mathbf{J}|$. These equations of motion can be solved to first order in $q$ by a quasi-Keplerian trigonometric parametrisation (cf. Damour & Deruelle 1985). For bound orbits, $E < 0$, one finds

\[
2\pi \frac{t - t_0}{P_b} = U - e_t \sin U,
\]

(20)

\[
r = a(1 - e \cos U),
\]

(21)

\[
f - f_0 = (1 + k)e_t(U),
\]

(22)

where $A_{e_t}(U)$ is given by equation (11) and

\[
P_b = \frac{2\pi GM (-2E)^{3/2}}{a^{3/2}},
\]

(23)

\[
e_t \equiv \left(1 + \frac{2EJ^2}{G^2M^2} - \frac{2Eq}{J^2}\right)^{1/2},
\]

(24)

\[
a = \frac{GM}{2E} \left(1 + \frac{Eq}{J^2}\right),
\]

(25)

\[
e_t \equiv (1 + \delta_t)e_t = \left(1 - \frac{Eq}{J^2}\right)e_t,
\]

(26)

\[
k = \frac{3}{2} \left(\frac{GM}{J^2}\right)^2 q,
\]

(27)

\[
e_f \equiv (1 + \delta_f)e_t = \left(1 - \frac{2Eq}{J^2}\right)e_t.
\]

(28)

The motion of the pulsar is

\[
r_\rho = \frac{m_\rho}{M} r(\cos f, \sin f, 0).
\]

(29)

The full dynamics given by the Hamiltonian (16) is a well known problem in the theory of Earth satellite motion. Due to the anisotropic nature of the quadrupole potential one does not expect any simple analytic solution as in the previous section. Various methods have been developed to solve this problem (see e.g. Hagiwara 1970, Roy 1978). To first order in $q$ the dynamics given by equation (16) can be solved by the method of the variation of the elements (osculating orbits). The following six elements are used to represent the osculating orbit:

- $a$, the semi-major axis of the (relative) orbit
- $e$, the eccentricity of the orbit
- $\theta$, the inclination of the orbital plane
- $\phi$, the longitude of the ascending node
- $\psi$, the longitude of periastron
- $M$, the mean anomaly

The angles are defined with respect to the equatorial plane of the oblate companion (see Fig. 1). Lagrange’s planetary equations for this problem can be written in the form

\[
\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M},
\]

(30)

\[
\frac{dc}{dt} = \frac{(1 - e^2)^{1/2}}{na^2e} \left(1 - e^2\right)^{1/2} \frac{\partial R}{\partial M} - \frac{\partial R}{\partial \psi},
\]

(31)

\[
\frac{d\theta}{dt} = \frac{1}{na^2(1 - e^2)^{1/2}\sin \theta} \left(\cos \theta \frac{\partial R}{\partial \psi} - \frac{\partial R}{\partial \phi}\right),
\]

(32)

\[
\frac{d\psi}{dt} = \frac{1}{na^2(1 - e^2)^{1/2}\sin \theta} \frac{\partial R}{\partial \phi},
\]

(33)

\[
\frac{dM}{dt} = n - \frac{2}{na^2} \left(1 - e^2\right) \frac{\partial R}{\partial e} + \frac{\partial R}{\partial a}.
\]

(34)

The relevant parts of the disturbing function $R \equiv R_s + R_\rho$ are
To derive the first-order short-period perturbations, the disturbances \( \mathcal{R}_s \) and \( \mathcal{R}_p \) are used. The zero subscript indicates evaluation at the initial condition. \( n_0 \) is related to the initial semi-major axis, \( a_0 \), by

\[ n_0^2 a_0 = GM. \]  

To simplify the representation of the first-order short-period perturbations I define the functions

\[ S_{e,\theta,\phi} \equiv \left( \frac{1 - \frac{3}{2} \sin^2 \theta}{2} \right) \left( \frac{1 - e^2}{4} \right) \sin f + \frac{e}{2} \sin 2f + \frac{e^2}{12} \sin 3f, \]

\[ C_{e,\theta,\phi} \equiv \left( 1 + \frac{3}{2} e \right) \left( \cos f + \frac{e}{2} \cos 2f + \frac{e^2}{12} \cos 3f \right), \]

and

\[ S_{\psi,f}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \equiv \alpha_0 \sin(2\psi - f) + \sum_{m=1}^5 \alpha_m \sin(2\psi + mf), \]

\[ C_{\psi,f}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \equiv \alpha_0 \cos(2\psi - f) + \sum_{m=1}^5 \alpha_m \sin(2\psi + mf), \]

and

\[ W_{e,f} \equiv f - \mathcal{M} + e \sin f. \]

To derive the first-order short-period perturbations, the disturbing function \( \mathcal{R} \) in equations (30) to (35) is replaced by \( \mathcal{R}_p \). Integration of the resulting equations leads to the following expressions for the six elements (Kozai 1959, Fitzpatrick 1970):

\[ \Delta a_p = \frac{q a_0}{2 (1 - e^2)} \left\{ C_{e_0,0,f} \left( \frac{\mu^2}{2}, \frac{12+3q^2}{16}, \frac{2+3q^2}{4e_0^2}, \frac{12+3q^2}{16}, \frac{3q^2}{8}, \frac{q^2}{16} \right) \right\}, \]

\[ \Delta e_p = Q \{ C_{e_0,0,f} \}

\[ + \sin^2 \theta_0 C_{\psi_0,f} \left( \frac{\mu^2}{16}, \frac{2+3q^2}{16}, 3q_0, \frac{12+3q_0^2}{4e_0^2}, \frac{28+17q_0^2}{48}, 3q_0, \frac{q_0^2}{16} \right) \}, \]

\[ \Delta \theta_p = Q \cos \theta_0 \sin \theta_0 C_{\psi_0,f} \left( 0, \frac{q_0}{2}, \frac{1}{2}, 0, 0 \right), \]

\[ \Delta \phi_p = Q \cos \theta_0 \left\{ -W_f + S_{\psi_0,f} \left( 0, \frac{q_0}{2}, \frac{1}{2}, 0, 0 \right) \right\}, \]

\[ \Delta \psi_p = \frac{Q}{e_0} \left\{ \left( \frac{2 - \frac{5}{2} \sin^2 \theta_0}{1} \right) \sin^2 \theta_0 W_f + S_{\psi_0,f} \right\}, \]

\[ \Delta M_p = -\frac{Q}{e_0} \left\{ S_{\psi_0,f} \right\}, \]

and

\[ \Delta \psi = \frac{Q}{e_0} \left\{ \left( \frac{2 - \frac{5}{2} \sin^2 \theta_0}{1} \right) \sin^2 \theta_0 W_f + S_{\psi_0,f} \right\}, \]

The mean and true anomaly are connected by the equations

\[ M = U - e \sin U, \]

\[ f = 2 \arctan \left( \frac{1 + e}{1 - e} \tan \frac{U}{2} \right). \]

The distance between the main-sequence star and the pulsar is

\[ r = a(1 - e \cos U) = \frac{a(1 - e^2)}{1 + e \cos f}. \]

The position vector of the pulsar at the time \( t \) with respect to the equatorial coordinate system is

\[ r = r \left( \begin{array}{ccc} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} \cos(\psi + f) \\ \sin(\psi + f) \cos \theta \\ \sin(\psi + f) \sin \theta \end{array} \right). \]

where the (variable) parameters \( a, e, \theta, \psi, \mathcal{M} \) are given by

\[ \xi(t) = \xi_0 + [\Delta \xi_f(t) - \Delta \xi_s(0)] + [\Delta \xi_p(t) - \Delta \xi_p(0)] \]

(\( \xi \) stands for one of these six parameters). To calculate \( \Delta \xi_f \) one uses \( \mathcal{M} = \mathcal{M}_0 + n_0 t, e = e_0 \) in equations (53) and (54) to obtain an approximated value for \( f \) which then is used in evaluating equations (47) to (52).

As an example I substitute the orbital elements of the main-sequence star binary pulsar PSR B1259–63 (Table 1) into equations (47) to (52). For the initial value of \( \theta (\theta_0) \) I take 60°, which is a realistic value for this binary system (see Wex et al. 1997). The values for \( \phi_0 \) and \( \psi_0 \) can be estimated from \( \omega, i, \lambda_\star \) and \( \lambda_\star \) can be derived from optical observations of the projected proper rotation of the Be star. Johnston et al. (1994) found \( \lambda_\star \sim 30° \) (or 150°). For more details see Appendix A, where the transformations between the angles \( \omega, i, \lambda_\star \) and the angles of the equatorial system, \( \theta, \psi, \phi \) are given. For the ‘strength’ of the quadrupole I use \( q = 7.6 \times 10^{-6} \) AU² (see equation (15)).

The results for two full orbits of PSR B1259–63 are given in Fig. 2 and Fig. 3. Fig. 2 presents the changes of the four ‘periodic’ parameters, \( a, e, \theta, \mathcal{M} \), which take their initial value after every full period. Fig. 3 presents the changes of the \( \phi \) and \( \psi \). I call \( \phi \) and \( \psi \) ‘secular’ parameters since they show both periodic and secular changes. It is obvious that for all parameters the major changes take place very close to the periastron passages, as expected from the \( 1/r^2 \) nature of the quadrupole potential.
Figure 2. Calculated changes of the ‘periodic’ parameters of PSR B1259–63. I used the parameters given in Table 1, the first row of Table A2 in Appendix A and $q = 7.6 \times 10^{-6}$ AU$^2$.

\[ \Delta R = \frac{1}{c} \hat{K}_0 \cdot r_p, \]  

(58)

where $c$ is the speed of light and

\[ r_p = \frac{m_\star}{M} r \]  

(59)

is the position vector of the pulsar originating in the centre of mass of the binary system. Using equation (56) leads to

\[ \Delta R = \frac{m_\star}{M} \frac{r}{c} \left\{ -\sin \lambda_\star \sin \phi \cos(\psi + f) + \cos \phi \sin(\psi + f) \cos \theta \right\} \]  

(60)

As mentioned in the introduction, the simplest timing model for binary pulsars is the BT model where changes in the longitude of periastron, $\omega$, and changes in the projected semi-major axis of the pulsar orbit, $x$, are assumed to be linear in time. The discussion in the previous section clearly showed that the (osculating) parameters of a binary system with an oblate companion do not change linearly in time (cf. Fig. 2 and Fig. 3). Thus one does not expect that the application of the BT model leads to a perfect fit. Fig. 4 shows the difference between equations (4),(7),(9) and the actual Roemer delay expected for the binary pulsars PSRs B1259–63 and J0045–7319. (Using the DD model instead leads to similar results). The typical precision in the measurement of the arrival time of pulsar signals is of the order of 100 $\mu$s for PSR B1259–63 (Johnston et al. 1994) and a few ms for PSR J0045–7319 (Kaspi et al. 1994). For both pulsars the deviations given in Fig. 4 are larger than the error in the TOAs. For PSR B1259–63 it is more than a factor of ten. Therefore the BT model is only a very crude approximation for these two binary pulsars and there is the need for an improved timing model, for PSR B1259–63 in particular.

3 TIMING MODELS FOR MAIN-SEQUENCE STAR BINARY PULSARS I. SHORT-TERM PERIODIC EFFECTS

Let $\hat{K}_0 = (0, -\sin \lambda_\star, \cos \lambda_\star)$ be the unit vector which indicates the direction of sight (see Fig. 1). The Roemer delay measured by an observer on Earth is then given by...
A timing formula for main-sequence star binary pulsars

Figure 4. Roemer delay as used in the BT timing model minus Roemer delay as expected for PSRs B1259–63 (upper figure) and J0045–7219 (lower figure). For the initial values of the angles $i, \lambda^*, \theta, \phi$ and $\psi$ I used the numerical values given in appendix A (first row of Table A1 and A2).

To extract reliable information from timing observations of main-sequence star binary pulsars one could construct a timing model that contains the full orbital dynamics given in the previous section. In the (unlikely) case that the orbital motion takes place in the equatorial plane of the Be star one can use the DD timing model as shown by the solution in section 2.1. For the general case one has to use the solution of Section 2.2 which leads to a rather complicated timing formula with a comparably high number of parameters to fit for where some of these parameters are only indirectly related with observable effects. Therefore, given the limited number of TOAs and the finite size of their measurement errors, one sees that in general a timing formula based on the equations of Section 2.2 is not a practical procedure. What one is looking for is a simple timing model which is a very good approximation to reality. In the ideal case the number of parameters should be the same as in the BT model.

The main deficit of the BT model is the use of equations (7) and (9) to describe the precession of the orbit. Even for a purely equatorial motion (see Section 2.1) equation (9) has to be replaced by (10) according to the DD timing model. If the orbit is tilted with respect to the equatorial plane the precession of the orbit leads to a change in the inclination of the orbital plane with respect to the line of sight. Fig. 5 gives this change of $i$ for PSR B1259–63 as a function of the time, $t$, and as a function of the true anomaly, $f$. The change of $i$ is neither linear in $t$ nor linear in $f$. But the assumption of linearity in $f$ is obviously much closer to reality than the assumption of linearity in $t$. The same is true for the change of $x$ which is a function of $i$.

Therefore I define a new timing model which I call BT++ model. Analogue to the construction of the BT+ model by Damour and Taylor (1992) the BT++ model is defined by replacing equations (7) and (9) in the BT model by

$$x = x_0 + \xi A_e(U), \quad \dot{x} \equiv 2\pi \xi/P_b,$$

and

$$\omega = \omega_0 + k A_e(U), \quad \dot{\omega} \equiv 2\pi k/P_b.$$  \hspace{1cm} (61)

A comparison of the expected Roemer delay and the Roemer delay as used in the BT++ model is given in Fig. 6. For both main-sequence star binary pulsars, PSRs B1259–63 and J0045–7319, the BT++ model is off by clearly less than the typical error in the TOAs. Only very close to periastron the deviations for PSR B1259–63 show a sharp peak of 200 $\mu$s, a value which is slightly larger than the typical measurement error. On the other hand, so far there are no timing observation of this pulsar close to periastron. The reason is the occultation by the circumstellar disk which lasts from $\sim 20$ days before until $\sim 20$ days after periastron (Johnston et al. 1996).

I conclude that one should certainly use the BT++ model instead of the BT model to fit the TOAs of PSR B1259–63. The BT++ model has the same number of parameters as the BT model, but is able to account for the fact that changes of the binary parameters happen mainly close to periastron. The BT++ model was already applied successfully to fit the TOAs of PSR B1259–63 (Wex et al. 1996).
Figure 6. Roemer delay as used in the BT++ timing model minus Roemer delay as expected for PSRs B1259–63 (upper figure) and J0045–7219 (lower figure). For the initial values of the angles $i$, $\lambda^*$, $\theta$, $\phi$ and $\psi$ I used the numerical values given in appendix A (first row of Table A1 and A2).

As concluded in Section 2.1, the correct timing model for equatorial orbits is the DD model. The DD model takes into account all the periodic effects of the (equatorial) orbital motion. One can now try to construct an even better timing model for main-sequence star binary pulsars, say DD+, by replacing equation (7) in the DD model by equation (61). The result is a timing model which combines the advantages of the BT++ model in describing the precession of the orbital plane and the DD model in describing periodic orbital effects. The representation of the Roemer delay in the DD+ model contains one more parameter than in the BT++ model. The DD+ model has the same number of parameters as the DD model. From Fig. 7 one sees that the DD+ model represents a nearly perfect fit for most parts of the orbit and close to periastron it is an improvement by a factor of 2 with respect to the BT++ model. At present the measurement precision for the TOAs for PSRs B1259–63 and J0045–7319 does not allow to fit for the (full) DD+ model.

Figure 7. Roemer delay as used in the DD+ timing model minus Roemer delay as expected for PSRs B1259–63. For the initial values of the angles $i$, $\lambda^*$, $\theta$, $\phi$ and $\psi$ I used the numerical values given in appendix A (first row of Table A2).

4 TIMING MODELS FOR MAIN-SEQUENCE STAR BINARY PULSARS II. LONG-TERM SECULAR EFFECTS

So far only the short-term effects in the orbital motion have been dealt with. I have shown the advantage of the BT++ (and DD+) model in taking into account the short-term periodic effects of the orbital motion. In the BT, BT+, DD, BT++ and DD+ model the secular changes in $\omega$ and $x$ are assumed to be linear in time. This approximation will hold as long as there are only small changes in $\omega$ and $x$. In this section I will focus on the long term precession of the binary orbit and its influence on pulsar timing and will investigate the limits of the present timing models.

In the following discussion I neglect periodic effects and focus only on the secular changes in the orbit of the binary system caused by the spin induced quadrupole of the main-sequence star companion. The solution presented in Section 2.2 does not give the long term behaviour correctly. It does not take into account the change in the orientation of the main-sequence star due to spin-orbit coupling. The change of the orientation of the main-sequence star is of order $q$ and thus appears in the equations of motion at order $q^2$ which was neglected in Section 2.2. On long time scales the orientation of the main-sequence star changes by a comparably large amount and therefore the contribution, although of order $q^2$, becomes numerically significant. The solution in Section 2.2 is perfectly suited for a discussion of periodic effects during a few orbital turns, but for the study of the long-term behaviour one should focus on the conserved quantities, which are the total energy and the total angular momentum. The total angular momentum, $\mathbf{J}$, is the sum of the orbital angular momentum, $\mathbf{L}$, and the spin of the main-sequence star, $\mathbf{S}$. On average, over one full period, the length of $\mathbf{S}$, $|\mathbf{S}|$, and $\mathbf{L}$, $|\mathbf{L}|$, are conserved (Barker & O’Connel 1975). Fig. 8 illustrates the resulting orbital dynamics.

Averaged over one orbital period one finds for the change of $\Phi$ and $\Psi$ (Smarr & Blandford 1976, Kopal 1978)

$$\dot{\Phi} = -\bar{n}\bar{Q}\left(\frac{\sin \bar{\theta} \cos \bar{\theta}}{\sin \bar{\theta}_j}\right) = \text{const.}$$

and

$$\dot{\Psi} = -\bar{n}\bar{Q}\left(\frac{\sin \bar{\theta} \cos \bar{\theta}}{\sin \bar{\theta}_j}\right) = \text{const.}$$

© 0000 RAS, MNRAS 000, 000–000
Equations (66) to (69) give the full (secular) evolution of the
for the rest of this section. Equations (63) and (64) can be
averaged over a full orbital period (c.f. Section 2.2). For sim-
erized I will skip the bar on top of the averaged quantities
thesized. Thus $\bar{x}$ and $\bar{\Omega}$ evolve linearly in time,
perpendicular to the total angular momentum plane. The invariable $(X-Y)$ plane is per-
line-of-sight ($K_0$) is in the $Y-Z$ plane. $J$ is a conserved quantity and,
if averaged over one full period, $|L|$ and $|S|$ are conserved quanti-
ties. Thus $i, \theta_j, \omega$, and $\theta$ are conserved. The angles $\Phi$ and $\Psi$ change
linearly with time.

$$\Psi = \bar{n}Q \left(1 - \frac{3}{2} \sin^2 \bar{\theta} + \frac{1}{2} \sin 2\bar{\theta} \cot \bar{\theta_j}\right) = \text{const.},$$  
(64)

where

$$Q = \frac{3(I_i^0 - I_j^0)/m_e}{2a^2(1 - e^2)^2} = \frac{kR_i^2\dot{\varphi}^2}{a^2(1 - e^2)^2}.$$  
(65)

The bar on top of the quantities indicates that they are aver-
eraged over a full orbital period (c.f. Section 2.2). For sim-
plicity I will skip the bar on top of the averaged quantities for
the rest of this section. Equations (63) and (64) can be
derived directly from equations (39) and (40).

Analogously to the calculations performed in Appendix A one finds the relations

$$\cos i = \cos i \cos \Psi - \sin i \sin \Psi \cos \Phi,$$  
(66)

$$\sin \omega = \frac{1}{\sin i} \left[\sin \theta_j \cos i \sin \theta_j \cos \Psi \right] \sin \Psi,$$  
(67)

$$\cos \omega = \frac{1}{\sin i} \left[\sin \theta_j \cos i \sin \theta_j \cos \Psi \right] \cos \Psi,$$  
(68)

$$(\text{see Fig. 8 for the definition of } i, \theta_j, \Phi \text{ and } \Psi).$$

The angles $i, \theta_j$ and $\Phi$ are conserved quantities. The angles $\Phi$ and $\Psi$
evolve linearly in time, $t$,

$$\Phi = \Phi_0 + \dot{\Phi}(t - t_0) \text{ and } \Psi = \Psi_0 + \dot{\Psi}(t - t_0).$$  
(69)

Equations (66) to (69) give the full (secular) evolution of the
projected semi-major axis $x = a_\text{e} \sin i/c$, and the longitude of
periastron, $\omega$. This evolution is clearly non-linear in time
since changes in $\Psi$ couple in a complicated way with changes
in $\Phi$ to produce the secular changes in $i$ and $\omega$.

A first approximation of this non-linear behaviour can be
given by $^\dagger$

$^\dagger$ To include short-term periodic effects, as discussed in the previ-
ous section, one has to replace $\dot{x}_0(t - t_0)$ by $\xi_A(U)$ and $\dot{\omega}_0(t - t_0)$
by $kA_e(U)$; see equations (61), (62).
get $Q$ from equation (72) or (74) and the spin of the companion using $|L|$, $\theta_j$ and $\theta$ (see Fig. 8). This allows to determine the moments of inertia $I^x_\ast$ and $I^y_\ast$ of the companion, which are related with its internal structure.

For PSR B1259–63 the changes of $i$ and $\omega$ are typically of the order of one second of arc per orbit (see Fig. 5). Since the orbital period is 3.4 years, the quadratic-in-time changes of these angles will be absolutely negligible for the next few decades.

For PSR J0045–7319 the situation is different. The changes in $x$ and $\omega$ are two orders of magnitude larger than for PSR B1259–63. Fig. 9 shows the result of fitting for simulated pulse arrival times for PSR J0045–7319. The first figure (9a) gives the pre-fit residuals for a BT model that leads to a good fit for the first few orbits. After four years the model is off by about 40 ms. Fitting for the whole time span of 1500 days using the BT model one finds the post-fit residuals given in the second figure (9b). Most of the deviations of Fig. 9a are absorbed in the spin parameters by changing them according to

$$\Delta P = -2.1 \times 10^{-10} \, \text{s}, \quad \Delta \dot{P} = -3.3 \times 10^{-18}.$$  

The residuals are in the order of the present measurement precision. For future observations the BT model (and therefore the BT++ and DD+ models) will fail to explain the observations and one has to fit for higher derivatives in $x$ and $\omega$. The result of such a fit is presented in the last figure (9c).

Finally, it should be mentioned that fitting for a $\dot{P}_b$ instead of $\dot{x}$ and $\dot{\omega}$ improves the residuals only marginally and gives a $\dot{P}_b$ which is two orders of magnitudes smaller than the one observed in this system. Therefore the nonlinear drifts of $x$ and $\omega$ cannot explain the observed $\dot{P}_b$.

5 PSR J0045–7319 — EVIDENCE FOR A NEUTRON-STAR BIRTH KICK?

The orbital precession of the binary pulsar PSR J0045–7319 was seen as a direct evidence that the neutron star of this system received a kick of at least 100 km/s at the moment of birth (Kaspi et al. 1996). Since the theoretical analysis in this paper is based on the calculations of Smarr and Brandford (1976) and Lai et al. (1995) an incorrect formula for $\dot{\omega}$, the precession of the longitude of periastron, was used (equation (1) in Kaspi et al. 1996). Therefore their limits on $\dot{\omega}$ and $\dot{\theta}$ for B1259–63 and Lai et al. (1995) an incorrect formula for $\dot{\omega}$ and $\dot{\theta}$, for PSR J0045–7319 the situation is different. The changes of $i$ and $\omega$ are two orders of magnitude larger than the one observed in this system. Therefore the nonlinear drifts of $x$ and $\omega$ cannot explain the observed $\dot{P}_b$.

Using the values of Table 1 in Kaspi et al. (1996) one finds for PSR J0047–7319

$$\frac{\dot{\omega}_0 x_0}{x_0} = 1.80 \pm 0.05.$$  

$\dot{\omega}$ was corrected for the general relativistic contribution of 0.004$^\circ$/yr. The uncertainty in the masses leads to $i = 44^\circ \pm 5^\circ$ or $i = 136^\circ \pm 5^\circ$.

Equation (76) restricts the values of $\Phi_0$ and $\theta$ as shown in Fig. 10 and one finds as a lower limit on the inclination of the B star with respect to the orbital plane

$$\theta > 20^\circ.$$  

This limit is only slightly smaller than the one given in Kaspi et al. (1996) and so does not change their major conclusion, i.e. that the binary system PSR J0045–7319 provides direct evidence for a neutron-star birth kick of at least 100 km/s. However, their conclusion that $\dot{\omega}_0 > 0$ implies $\theta < 55^\circ$ is incorrect. In principle one can have $\theta$ up to $70^\circ$, although $\theta$ being close to $70^\circ$ requires an unphysically high apsidal motion constant $k$ for the B star. Numerical simulations show that $\theta > 65^\circ$ is excluded by the fact that present timing.
observations are still in agreement with a simple $\dot{x} - \dot{\omega}$ model. Similar arguments apply for the case $\theta > 90^\circ$.

6 CONCLUSIONS

In this paper I have presented a timing formula for main-sequence star binary pulsars which takes into account most of the periodic variations along the orbit caused by the anisotropic nature of the $1/r^3$ potential of the spin-induced quadrupole moment of the companion star. The new timing formula contains the same number of parameters as the Blandford-Teukolsky timing formula. I have shown by numerical simulations, that the new timing formula leads to much better results in case of the long-orbital period binary pulsar PSR B1259–63 than the Blandford-Teukolsky or the Damour-Deruelle timing formula. Only very close to periastron the new timing formula shows deviations are slightly greater than the typical measurement error in the time-of-arrival of the pulsar signals. But so far there are no timing observations of PSR B1259–63 close to periastron, due to the eclipse of the pulsar caused by the circumstellar material. For PSR J0045–7319 these periodic variations are of the order of the measurement precision.

I have given quadratic-in-time extensions of the timing formula which account for long-term secular changes in the orientation of the binary-pulsar orbit. In particular for the binary pulsar PSR J0045–7319 these extensions might be important in the next few years of timing observations depending on the, so far unknown, orientation of the B star spin and the total angular momentum of the binary system. I have concluded that the measurement of these long-term secular effects has the potential to probe the internal structure of the companion.

Finally I have reinvestigated the classical spin-orbit precession of the binary pulsar PSR J0045–7319 since the theoretical analysis of this binary system given in Lai et al. (1995) and Kaspi et al. (1996) is based on an incorrect expression for the precession of the longitude of periastron. I have found $20^\circ$ as a lower limit for the inclination of the B star with respect to the orbital plane which does not change the conclusions concerning the neutron-star birth kick in this system given in Kaspi et al. (1996).

ACKNOWLEDGMENTS

I thank Peter Müller for many stimulating discussions and Simon Johnston for carefully reading the manuscript.

REFERENCES

Doroshenko O., Kopeikin S., 1990, SvA, 34, 496
Kozai Y., 1959, AJ, 64, 367
Roy A.E., 1978, Orbital Motion, Adam Hiller Ltd, Bristol

© 0000 RAS, MNRAS 000, 000–000
APPENDIX A: TRANSFORMATIONS BETWEEN $\omega$, $i$, $\lambda_\ast$, $\phi$, $\psi$, AND $\psi$

There are three angles in a main-sequence-star binary pulsar system which in principle can be derived from timing and optical observations. From timing observations one can derive the longitude of periastron, $\omega$, which is one of the five Keplerian parameters. If one knows the masses of pulsar and companion, then timing observations can be used to derive $\sin i$ where $i$ is the inclination of the orbit with respect to the line-of-sight. From optical observations and assumptions on the actual rotational velocity of the main-sequence star companion one can derive $\sin \lambda_\ast$, where $\lambda_\ast$ is the angle between the line of sight and the spin of the companion. In the calculations performed in Section 2.2 I make use of the three angles, $\theta$, $\phi$, and $\psi$, to describe the orientation of the binary orbit with respect to the equatorial plane of the main-sequence star companion and the line of sight (see Fig. 1). In this appendix I shall give analytical relations between these two different sets of angles which are needed in this paper.

The orientation of the orbit of the binary system is fixed by two unit vectors, a unit vector, $\mathbf{k}$, which is perpendicular to the orbital plane, and a unit vector, $\mathbf{a}$, which lies in the orbital plane and points in direction to periastron:

$$\mathbf{k} = (\sin \theta \sin \phi, -\sin \theta \cos \phi, \cos \theta)$$  \hspace{1cm} (A1)

and (cf. equation (56))

$$\mathbf{a} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \psi \\ \sin \psi \cos \theta \\ \sin \psi \sin \theta \end{pmatrix}. \hspace{1cm} (A2)$$

Given $\theta$, $\phi$, $\psi$, and $\lambda_\ast$, one finds for the orbital inclination $i$ with respect to the line-of-sight

$$\cos i = \mathbf{k} \cdot \mathbf{K}_0 = \sin \lambda_\ast \sin \theta \cos \phi + \cos \lambda_\ast \cos \theta. \hspace{1cm} (A3)$$

The longitude of periastron, $\omega$, which is the angle between $\mathbf{K}_0 \times \mathbf{k}$ and $\mathbf{a}$, is determined by

$$\sin \omega = \frac{(\mathbf{K}_0 \times \mathbf{k}) \cdot \mathbf{a}}{\sin i} = \frac{\mathbf{a} \cdot \mathbf{K}_0}{\sin i}, \hspace{1cm} (A4)$$

$$\cos \omega = \frac{\mathbf{a} \cdot (\mathbf{K}_0 \times \mathbf{k})}{\sin i}, \hspace{1cm} (A5)$$

and thus

$$\sin \omega = \frac{1}{\sin i} \left[ (\sin \phi \cos \psi + \cos \phi \sin \psi \cos \theta) \sin \lambda_\ast + \sin \psi \sin \theta \cos \lambda_\ast \right], \hspace{1cm} (A6)$$

$$\cos \omega = \frac{1}{\sin i} \left[ (\sin \phi \sin \psi - \cos \phi \cos \psi \cos \theta) \sin \lambda_\ast + \cos \psi \sin \theta \cos \lambda_\ast \right]. \hspace{1cm} (A7)$$

One needs both, $\sin \omega$ and $\cos \omega$, to determine $\omega$ uniquely, since $\omega$ can have any value between $0$ and $2\pi$.

Given $\omega$, $i$, $\lambda_\ast$, and $\theta$ one finds from equation (A3)

$$\cos \phi = \frac{\cos i - \cos \theta \cos \lambda_\ast}{\sin \theta \sin \lambda_\ast}. \hspace{1cm} (A8)$$

Equation (A8) does not determine $\phi$ uniquely, since $\phi$ runs between $0$ and $2\pi$. Using equations (A6) and (A7) one obtains two equations for $\phi$:

$$\sin (\omega - \psi) = -\frac{\sin \lambda_\ast \sin \phi}{\sin i}, \hspace{1cm} (A9)$$

Table A1. Values for $\phi$ and $\psi$ for PSR J0045–7319 assuming $\theta = 30^\circ$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_\ast$</th>
<th>$\phi$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$44^\circ$</td>
<td>$15^\circ$</td>
<td>$156^\circ$</td>
<td>$106^\circ$</td>
</tr>
<tr>
<td>$136^\circ$</td>
<td>$165^\circ$</td>
<td>$25^\circ$</td>
<td>$304^\circ$</td>
</tr>
<tr>
<td>$335^\circ$</td>
<td>$286^\circ$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table A2. Values for $\phi$ and $\psi$ for PSR B1259–63 assuming $\theta = 60^\circ$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_\ast$</th>
<th>$\phi$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$36^\circ$</td>
<td>$30^\circ$</td>
<td>$30^\circ$</td>
<td>$114^\circ$</td>
</tr>
<tr>
<td>$144^\circ$</td>
<td>$150^\circ$</td>
<td>$131^\circ$</td>
<td>$351^\circ$</td>
</tr>
<tr>
<td>$229^\circ$</td>
<td>$286^\circ$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$\cos(\omega - \psi) = \frac{\sin \theta \cos \lambda_\ast - \cos \theta \sin \lambda_\ast \cos \phi}{\sin i}. \hspace{1cm} (A10)$$

As an example for these transformations Tables 2 and 3 give all possible values of the angles $\phi$ and $\psi$ for PSR J0045–7319 and PSR B1259–63 for a given angle $\theta$. For the numerical simulations in the main part of this paper I make only use of the values given in the first row of each of the tables.

APPENDIX B: FIRST AND SECOND TIME DERIVATIVE OF $x$ AND $\omega$ FOR SMALL $\theta_j$

The secular precession of the orbit of a binary pulsar with an oblate companion star is described by the change of the angles $\Phi$ and $\Psi$ as given in Fig. 8. The change of these angles is linear in time with the following constant angular velocities (see equations (63) and (64))

$$\dot{\Phi} = -nQ \left( \frac{\sin \theta \cos \theta}{\sin \theta_j} \right), \hspace{1cm} (B1)$$

and

$$\dot{\Psi} = nQ \left( 1 - \frac{3}{2} \sin^2 \theta + \frac{1}{2} \sin 2\theta \cot \theta_j \right), \hspace{1cm} (B2)$$

where I omitted the bar on the top of the quantities.

The angles $\Phi$ and $\Psi$ are not directly observable in pulsar timing measurements. The change of these quantities is seen indirectly through a change of the inclination of the orbit with respect to the line of sight, $i$, i.e. a change of the projected semi-major axis, $x$, and a change of the longitude of periastron, $\omega$. These observable quantities are related to the precessing angles $\Phi$ and $\Psi$ by equations (66) to (68):

$$\cos i = \cos i_j \cos \theta_j - \sin i_j \sin \theta_j \cos \Phi \hspace{1cm} (B3)$$

$$\sin \omega = \frac{1}{\sin i} \left[ (\sin \theta_j \cos i_j + \cos \theta_j \sin i_j \cos \Phi) \sin \Psi + \sin i_j \sin \Phi \cos \Psi \right], \hspace{1cm} (B4)$$

$$\cos \omega = \frac{1}{\sin i} \left[ (\sin \theta_j \cos i_j + \cos \theta_j \sin i_j \cos \Phi) \cos \Psi - \sin i_j \sin \Phi \sin \Psi \right]. \hspace{1cm} (B5)$$

Therefore, in general, the linear-in-time precession of the angles $\Phi$ and $\Psi$ will cause changes of the observables $x$ and $\omega$. 

© 0000 RAS, MNRAS 000, 000–000
which are non-linear in time. If the time span of observations is much shorter than the period of precession, one can still use a linear approximation of the form

\[ x(t) = x_0 + \dot{x}_0(t - t_0), \quad (B6) \]

\[ \omega(t) = \omega_0 + \dot{\omega}_0(t - t_0). \quad (B7) \]

Including terms which are quadratic in time

\[ x(t) = x_0 + \dot{x}_0(t - t_0) + \frac{1}{2} \ddot{x}_0(t - t_0)^2, \quad (B8) \]

\[ \omega(t) = \omega_0 + \dot{\omega}_0(t - t_0) + \frac{1}{2} \ddot{\omega}_0(t - t_0)^2, \quad (B9) \]

is the next natural step once the linear approximation deviates more than the measurement precision. The quantities \( \dot{x}_0, \ddot{x}_0, \omega_0, \ddot{\omega}_0 \), if measured in timing observations, contain information about the orientation and the quadrupole moment of the companion star.

In general the spin of the oblate companion is much smaller than the orbital angular momentum and \( \theta_J \ll 1 \). If \( \theta_J \ll i_J, \pi - i_J \) one can do an (Laurent) expansion of equations (B3) to (B5) in \( \theta_J \) to get simplified expressions for \( \dot{x}_0, \ddot{x}_0, \omega_0, \ddot{\omega}_0 \). While doing these expansions it is important to keep in mind that \( \dot{\Phi} \) and the leading term of \( \dot{\Psi} \) are of order \( \theta_J^1 \), but \( \dot{\Phi} + \dot{\Psi} \) is only of order \( \theta_J^2 \). A fact which has been overlooked by Smarr and Blandford (1976) and Lai et al. (1995) leading to a wrong result for \( \ddot{\omega}_0 \).

Equation (B3) leads to the approximation for the inclination of the orbit with respect to the line-of-sight

\[ i = i_J + \theta_J \cos \Phi + O(\theta_J^2). \quad (B10) \]

Therefore using

\[ \dot{x} = \frac{a \cos i \ di}{c} = x \cot i \ di \quad (B11) \]

and equations (B1), (B2) one finds to leading order in \( \theta_J \)

\[ \dot{x}_0 = nQx_0 \cos i \sin \theta \cos \theta \sin \Phi_0 + O(\theta_J) \quad (B12) \]

and

\[ \ddot{x}_0 = -n^2Q^2x_0 \cos i \left( \frac{\sin^2 \theta \cos^2 \theta \cos \Phi_0}{\sin \theta_J} \right) + O(\theta_J^3). \quad (B13) \]

Equations (B4) and (B5) imply that the longitude of periastron can be approximated by

\[ \omega = \Psi + \Phi - \theta_J \cot i \sin \Phi + O(\theta_J^2). \quad (B14) \]

Keeping in mind that \( \dot{\Phi} = O(\theta_J^{-1}) \) and \( \dot{\Phi} + \dot{\Psi} = O(\theta_J^0) \) one finds to leading order in \( \theta_J \)

\[ \dot{\omega}_0 = nQ \left( 1 - \frac{3}{2} \sin^2 \theta + \cot i \sin \theta \cos \theta \cos \Phi_0 \right) + O(\theta_J) \quad (B15) \]

and

\[ \ddot{\omega}_0 = n^2Q^2 \cot i \left( \frac{\sin^2 \theta \cos^2 \theta \sin \Phi_0}{\sin \theta_J} \right) + O(\theta_J^0). \quad (B16) \]

The third term in the brackets of equation (B15) was overlooked by Smarr and Blandford (1976) and Lai et al. (1995) by keeping only the highest order in \( \theta_J \) in equation (B14) but then (inconsistently) keeping also the term of order \( \theta_J^0 \) when doing the time derivative since the highest term in the time derivative (order \( \theta_J^{-1} \)) vanished.

In Fig. B1 I compare the exact values of \( \dot{\omega}_0, \ddot{\omega}_0, \dot{x}_0, \) and \( \ddot{x}_0 \) and their approximations given by the equations above for a PSR J0045–7319 like binary system. As a typical value for \( nQ \) I used \( 2 \times 10^{-3} \) rad/yr (c.f. Lai et al. 1995).

![Figure B1](image-url)