Commutative Jacobi Fields in Fock Space

Y. M. Berezansky

Sfb 288 Preprint No. 258

Swg 729

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 288 entstanden und als Manuskript vervielfältigt worden.

Berlin, April 1997

The list of preprints of the Sonderforschungsbereich 288 is available at:
<http://www-sfb288.math.tu-berlin.de>

COMMUTATIVE JACOBI FIELDS IN FOCK SPACE

YURIJ M. BEREZANSKY

A Jacobi field is understood to be a family $(\tilde{A}(\varphi))_{\varphi \in \Phi}$ of commuting selfadjoint operators $\tilde{A}(\varphi)$ acting in a Fock space, having a Jacobi structure, and depending linearly on the test functions φ . In this article, we give a spectral representation of such a family and outline its applications to the theory of distributions on an infinite dimensional space.

0 Introduction

In articles [3, 9, 10] there was proposed a generalization of the white noise analysis (see [12, 7]), i.e. a theory of distributions on an infinite dimensional space was constructed in the case of a non-Gaussian measure. In this generalization, the role of the measure ρ , which gives the pairing, was played by the spectral measure of some commuting family $A = (\tilde{A}(\varphi))_{\varphi \in \Phi}$ of selfadjoint operators $\tilde{A}(\varphi)$ acting in a Fock space and depending linearly on the test function φ . The operators of this family are constructed by giving a sequence of Jacobi matrices, and they themselves have a Jacobi structure with respect to the Fock space; the measure ρ is equal to the infinite product of the spectral measures of these Jacobi matrices.

The Fourier transform connected with the family A maps the Fock space onto the space L^2 on the spectrum of this family with respect to the measure ρ . Moreover, the rigging of the Fock space is transferred to some rigging of the space L^2 , and the latter rigging generates the corresponding theory of generalized functions on this spectrum.

In works [14, 15], an important generalization of this approach was suggested and a theory was constructed by using a family $(\tilde{A}(\varphi))_{\varphi \in \Phi}$ of commuting selfadjoint operators acting in a Fock space, having a Jacobi structure, and depending linearly on φ , i.e. by prescribing a commutative Jacobi field (for another purpose, the case of non-commutative fields was introduced and investigated in [8, 11]). In this scheme, the measure ρ is not necessarily a product measure.

In this article we propose a somewhat another axiomatics of a similar Jacobi field which is, in fact, more simple and is connected with a Hilbert rigging. Its spectral representation is constructed and applications to the theory of the above mentioned distributions are outlined.

In what follows, the Jacobi field of operators appears as an independent object that generalizes the classical theory of Jacobi matrices. Apparently, for such objects, it would be interesting to develop another aspects of the theory of Jacobi matrices and the moment problem, which is connected, in particular, with works of M. G. Krein (for references see [13, 1]). One of such problems would be, for example, to investigate the inverse problem of reconstructing a field from its spectral measure.

This paper was written at the Humboldt University of Berlin during my visit there with a support from Volkswagen-Stiftung. I express my deep gratitude to the Humboldt University and to Professor J. Brüning for the hospitality.

1 Preliminaries

1.1 Projection spectral theorem

Let $\mathcal{H}$ be a separable Hilbert space, Φ be a topological real linear space, and let $A = (\tilde{A}(\varphi))_{\varphi \in \Phi}$ be a family of commuting selfadjoint operators in $\mathcal{H}$ (i.e. these operators commute in the sense of their resolutions of the identity). Let

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D} \quad (1.1)$$

be a rigging of the zero space $\mathcal{H}$ such that $\mathcal{H}_+$ is a Hilbert space, which is topologically (i.e. densely and continuously) and quasinuclearly (i.e. the inclusion operator is of the Hilbert–Schmidt type) embedded into $\mathcal{H}$, $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$ with respect to the zero space $\mathcal{H}$ (with the pairing $(\cdot, \cdot)_{\mathcal{H}}$), and $\mathcal{D}$ is a separable projective limit of Hilbert spaces which is topologically embedded into $\mathcal{H}_+$. Then we have [2, 7]

Theorem 1.1 *Let the family A satisfy the following conditions.*

- a) $\mathcal{D} \subset \text{Dom } \tilde{A}(\varphi)$, $\varphi \in \Phi$, and the closure of the operator $A(\varphi) = \tilde{A}(\varphi) \upharpoonright \mathcal{D}$ coincides with $\tilde{A}(\varphi)$ (in the sequel we always denote the closure by $\tilde{}$).
- b) $A(\varphi)$ is a linear continuous operator from $\mathcal{D}$ into $\mathcal{H}_+$.
- c) For every $f \in \mathcal{D}$, the mapping $\Phi \ni \varphi \mapsto A(\varphi)f \in \mathcal{H}_+$ is linear and continuous.
- d) For the family A there exists a strong cyclic vector Ω , i.e. there exist products $A(\varphi_1) \cdots A(\varphi_n)$ such that $\Omega \in \text{Dom}(A(\varphi_1) \cdots A(\varphi_n))$ and the set of all the vectors Ω , $A(\varphi_1) \cdots A(\varphi_n)\Omega$, $n \in \mathbb{N}$, is total in $\mathcal{H}_+$ (i.e. their linear span is dense in $\mathcal{H}_+$).

Then there exists a probability measure ρ on $(\Phi', \mathcal{C}_\sigma(\Phi'))$, where Φ' is the dual of Φ and $\mathcal{C}_\sigma(\Phi')$ is the σ -algebra generated by cylinder sets in Φ' such that, for ρ -almost every $x \in \Phi'$ there is a generalized joint eigenvector $\xi(x) \in \mathcal{H}_-$ of the family A corresponding to the joint eigenvalue x , i.e.

$$(\xi(x), A(\varphi)f)_{\mathcal{H}} = \langle x, \varphi \rangle (\xi(x), f)_{\mathcal{H}}, \quad \varphi \in \Phi, f \in \mathcal{D}; \xi(x) \neq 0, \quad (1.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between Φ and Φ' .

The mapping

$$\mathcal{H}_+ \ni f \mapsto (f, \xi(x))_{\mathcal{H}} = \hat{f}(x) \in \mathbb{C}^1, \quad x \in \Phi', \quad (1.3)$$

can be continuously extended to a unitary operator F from $\mathcal{H}$ onto the whole $L^2(\Phi', \mathcal{C}_\sigma(\Phi'), \rho)$: $\mathcal{H} \ni f \mapsto (Ff)(x) := \hat{f}(x) \in L^2(\Phi', \mathcal{C}_\sigma(\Phi'), \rho)$, — the so-called Fourier transform corresponding to the family A .

The image of the operator $\tilde{A}(\varphi)$, $\varphi \in \Phi$, under F is the operator of multiplication by the function $\Phi' \ni x \mapsto \langle x, \varphi \rangle \in \mathbb{C}^1$.

In the sequel we will apply this theorem to the case, where the spaces Φ and $\mathcal{H}$ are closely connected.

1.2 A rigging of a Fock space

Let H be a real separable Hilbert space, and let $\Phi = \text{prlim}_{\tau \in T} H_\tau$ be the projective limit of real separable Hilbert spaces H_τ , indexed by a set T of an arbitrary cardinality. Suppose that all the spaces H_τ are embedded into H topologically and, moreover, $\|\varphi\|_{H_\tau} \geq \|\varphi\|_H$, $\varphi \in H_\tau$. Let $H_{-\tau}$ be the dual of H_τ , $t \in T$, with respect to the zero space H . Then, $\Phi' = \text{indlim}_{\tau \in T} H_{-\tau}$ is the dual of Φ . Thus we have the following chain

$$\Phi' = \text{indlim}_{\tau \in T} H_{-\tau} \supset \bigcup_{\tau \in T} H_{-\tau} \supset H_{-\tau} \supset H \supset H_\tau \supset \bigcap_{\tau \in T} H_\tau \supset \text{prlim}_{\tau \in T} H_\tau = \Phi, \quad (1.4)$$

$\tau \in T.$

By taking the n -th symmetric tensor power of the spaces in (1.4), we get, after complexification, the following chain of n -particle Fock spaces:

$$\begin{aligned} \mathcal{F}_n(\Phi') &= \text{indlim}_{\tau \in T} \mathcal{F}_n(H_{-\tau}) \supset \mathcal{F}_n(H_{-\tau}) \supset \mathcal{F}_n(H) \supset \\ &\supset \mathcal{F}_n(H_\tau) \supset \text{prlim}_{t \in T} \mathcal{F}_n(H_\tau) = \mathcal{F}_n(\Phi), \tau \in T, \end{aligned} \quad (1.5)$$

with the assumption that $\mathcal{F}_0(\cdot) = \mathbb{C}^1$ [7]. Hence, $\mathcal{F}_n(H) = H_{\mathbb{C}} \hat{\otimes} \dots \hat{\otimes} H_{\mathbb{C}}$ (n times), where $\hat{\otimes}$ denotes the symmetric tensor product and $H_{\mathbb{C}}$ is the complexification of H . The complex conjugation of vectors f will be denoted by $f \mapsto \bar{f}$.

Note that, if there exists $\tau_0 \in T$ such that the embedding $H_{\tau_0} \hookrightarrow H$ is quasinuclear, then the embedding $\mathcal{F}_n(H_{\tau_0}) \hookrightarrow \mathcal{F}_n(H)$ is also quasinuclear for all $n \in \mathbb{N}$. Fix this τ_0 , and denote it by 1.

Given a real Hilbert space G and a weight $p = (p_n)_{n=0}^\infty$, $p_n > 0$, a weighted Fock space

$$\mathcal{F}(G, p) = \bigoplus_{n=0}^\infty \mathcal{F}_n(G) p_n$$

is defined to be the Hilbert space consisting of sequences $(f_n)_{n=0}^\infty$, $f_n \in \mathcal{F}_n(G)$, such that

$$\sum_{n=0}^\infty \|f_n\|_{\mathcal{F}_n(G)}^2 p_n < \infty$$

with the scalar product

$$\left((f_n)_{n=0}^\infty, (g_n)_{n=0}^\infty \right)_{\mathcal{F}(G,p)} = \sum_{n=0}^\infty (f_n, g_n)_{\mathcal{F}_n(G)} p_n.$$

In particular, if all $p_n = 1$, then $\mathcal{F}(G, p) = \mathcal{F}(G)$ is the usual (symmetric) Fock space. The vector $\Omega = (1, 0, 0, \dots) \in \mathcal{F}(G, p)$ is called a vacuum of the space $\mathcal{F}(G, p)$.

So, for any $\tau \in T$ and $p = (p_n)_{n=0}^\infty$, $p_n \geq 1$, we can construct the space $\mathcal{F}(H_\tau, p)$. It is easy to see that this space is topologically imbedded into $\mathcal{F}(H)$ and the dual of $\mathcal{F}(H_\tau, p)$ with respect to $\mathcal{F}(H)$ is the space $\mathcal{F}(H_{-\tau}, p^{-1})$, $p^{-1} = (p_n^{-1})_{n=0}^\infty$.

Let $\mathcal{F}_{\text{fin}}(\Phi)$ be the topological direct sum $\bigoplus_{n=0}^\infty \mathcal{F}_n(\Phi)$ of the spaces $\mathcal{F}_n(\Phi)$; this space consists of finite sequences $(f_n)_{n=0}^\infty$, $f_n \in \mathcal{F}_n(\Phi)$ (i.e. $f_n = 0$ for all $n \geq \text{some } n \in \mathbb{N}$). The convergence in this space is equivalent to the uniform finiteness and coordinate-wise convergence. Note that, if $\Phi = H_\tau$, we get the space $\mathcal{F}_{\text{fin}}(H_\tau)$ consisting of finite sequences $(f_n)_{n=0}^\infty$, $f_n \in \mathcal{F}_n(H_\tau)$.

The dual $\mathcal{F}'_{\text{fin}}(\Phi)$ of $\mathcal{F}_{\text{fin}}(\Phi)$ with respect to $\mathcal{F}(H)$ is the topological product of the spaces $\mathcal{F}_n(\Phi')$. It consists of all sequences of the form $(\xi_n)_{n=0}^\infty$, $\xi_n \in \mathcal{F}_n(\Phi')$.

Thus, for any $\tau \in T$ and any sequence $p = (p_n)_{n=0}^\infty$, $p_n \geq 1$ (it is convenient to suppose that p is a non-decreasing sequence), we get the following chain

$$\begin{aligned} \mathcal{F}'_{\text{fin}}(\Phi) &= \times_{n=0}^\infty \mathcal{F}_n(\Phi') \supset \mathcal{F}(H_{-\tau}, p^{-1}) \supset \mathcal{F}(H) \supset \mathcal{F}(H_\tau, p) \supset \\ &\supset \bigoplus_{n=0}^\infty \mathcal{F}_n(\Phi) = \mathcal{F}_{\text{fin}}(\Phi). \end{aligned} \tag{1.6}$$

It is easy to see that the space $\mathcal{F}_{\text{fin}}(\Phi)$ in the right-hand side of (1.6) is equal to the projective limit of the spaces $\mathcal{F}(H_\tau, p)$ as p runs over the set of all non-decreasing sequences with $p_n \geq 1$ for arbitrarily changing $\tau \in T$. Therefore, the left-hand side of (1.6) is equal to the corresponding inductive limit of $\mathcal{F}(H_{-\tau}, p^{-1})$.

One can easily verify that the above mentioned quasinuclearity of the embedding $H_1 \hookrightarrow H$ allows to find a sequence $(p_n)_{n=0}^\infty$, $p_n \geq 1$, such that the embedding $\mathcal{F}(H_1, p) \hookrightarrow \mathcal{F}(H)$ is also quasinuclear [7, 9]. The condition on p is the following:

$$\sum_{n=0}^\infty \|O_1\|_{\text{HS}}^{2n} p_n^{-1} < \infty, \tag{1.7}$$

where O_1 is the embedding operator $H_1 \hookrightarrow H$ and the index HS denotes its Hilbert-Schmidt norm. If the space Φ is nuclear, then $\mathcal{F}_{\text{fin}}(\Phi)$ is also a nuclear space.

1.3 Basis of occupation numbers of a Fock space

Into the Fock space $\mathcal{F}(H)$ from chain (1.6), we introduce the following basis, which is called the basis of occupation numbers (BON) [7, 9].

Fix, in a real Hilbert space H , an arbitrary orthonormal basis $(e_j)_{j=1}^\infty$ and construct the vectors

$$\begin{aligned} e_\alpha &= \left(\frac{|\alpha|!}{\alpha_1! \alpha_2! \dots} \right)^{1/2} e_1^{\otimes \alpha_1} \widehat{\otimes} e_2^{\otimes \alpha_2} \widehat{\otimes} \dots, \\ \alpha &= (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_{+,0}^\infty, \quad |\alpha| = \alpha_1 + \alpha_2 + \dots, \end{aligned} \tag{1.8}$$

where $\mathbb{Z}_{+,0}^\infty$ denotes the set of all finite sequences from $\mathbb{Z}_+^\infty = \mathbb{Z}_+ \times \mathbb{Z}_+ \times \dots$, $\mathbb{Z}_+ = \{0, 1, \dots\}$. Thus the last sum in (1.8) is finite; we omit also the expression $e_j^{\otimes 0}$, and therefore, every tensor product in (1.8) consists of a finite number of factors.

It is easy to prove that vectors (1.8), with $|\alpha| = n$, form an orthonormal basis in the space $\mathcal{F}_n(H) = H_{\mathbb{C}} \widehat{\otimes} \dots \widehat{\otimes} H_{\mathbb{C}}$ (n times) and, therefore, vectors (1.8), with an arbitrary $\alpha \in \mathbb{Z}_{+,0}^\infty$, form an orthonormal basis in $\mathcal{F}(H)$, — the so-called BON.

In what follows, we will choose the vectors e_j for the orthonormal basis from the space Φ . Therefore, vectors (1.8) belong to every space from chain (1.6) and form a total set in it.

1.4 Infinite tensor product of spaces and some operators in it

Let $(H_j)_{j=1}^\infty$ be a sequence of separable Hilbert spaces. In every space H_j , we fix an arbitrary orthonormal basis $(k_i^{(j)})_{i=0}^\infty$. The infinite tensor product $\mathcal{H}^\otimes = \bigotimes_{j=1,k}^\infty H_j$ of the spaces H_j with the stabilizing sequence $k = (k_0^{(j)})_{j=1}^\infty$ is defined to be the Hilbert space spanned by the formal products

$$k_{\alpha_1}^{(1)} \otimes k_{\alpha_2}^{(2)} \otimes \dots = k_\alpha, \quad \alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_{+,0}^\infty, \quad (1.9)$$

which are taken to form an orthonormal basis. For every α from (1.9), starting with some n , we have $\alpha_{n+1} = \alpha_{n+2} = \dots = 0$, and so the “tail” of vectors (1.8) have the form $k_0^{(n+1)} \otimes k_0^{(n+2)} \otimes \dots$, which depends on k .

Orthonormal bases (1.8) and (1.9) are indexed by the same set $\mathbb{Z}_{+,0}^\infty$. Therefore, there exists a natural unitary isomorphism K between $\mathcal{H}^\otimes$ and $\mathcal{F}(H)$ and it is given by the formula

$$K k_\alpha = e_\alpha, \quad \alpha \in \mathbb{Z}_{+,0}^\infty. \quad (1.10)$$

Note that, by using this unitary K , rigging (1.6) of the space $\mathcal{F}(H)$ can be transferred to make a rigging of the space $\mathcal{H}^\otimes$.

In every space H_j , we consider an essentially selfadjoint operator B_j , the domain of which, $\text{Dom}(B_j)$, contains $(k_i^{(j)})_{i=0}^\infty$. Every B_j generates an operator

$$B_j = 1 \otimes \dots \otimes 1 \otimes B_j \otimes 1 \otimes 1 \otimes \dots \quad (1.11)$$

(B_j is at the j -th place), which acts in the Hilbert space $\bigotimes_{j=1,k}^\infty H_j$ as follows:

$$B_j(f_1 \otimes \dots \otimes f_{j-1} \otimes f_j \otimes f_{j+1} \otimes \dots) = f_1 \otimes \dots \otimes f_{j-1} \otimes (B_j f_j) \otimes f_{j+1} \otimes \dots, \quad (1.12)$$

where $f_i \in \text{Dom}(B_j)$ and, for i large enough, $f_i = k_0^{(i)}$, $i \in \mathbb{N}$ (the products $f_1 \otimes f_2 \otimes \dots$ belong to $\bigotimes_{j=1,k}^\infty H_j$). The domain of B_j is given by

$$\begin{aligned} \text{Dom}(B_j) &= \text{a. } \bigotimes_{j=1,k}^\infty \text{Dom}(B_j) := \\ &= \text{l.s. } \{f_1 \otimes \dots \otimes f_n \otimes k_0^{(n+1)} \otimes k_0^{(n+2)} \otimes \dots, \\ &\quad f_i \in \text{Dom}(B_i), \ n \in \mathbb{Z}_+\}, \end{aligned} \quad (1.13)$$

where l.s. denotes the linear span.

We have [2]

Theorem 1.2 *The operators $\mathcal{B}_j$, $j \in \mathbb{N}$, are essentially selfadjoint, and their respective closures, $\tilde{\mathcal{B}}_j$, commute.*

1.5 Operator Jacobi matrices in a Hilbert space

Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ be a fixed orthogonal decomposition of the Hilbert space $\mathcal{H}$ with respect to its subspaces, and let A be some operator in $\mathcal{H}$. Suppose that, for every $\mathcal{H}_n \subset \text{Dom}(A)$, $A \upharpoonright \mathcal{H}_n$ is bounded and $\text{Ran}(A \upharpoonright \mathcal{H}_n) \subset \mathcal{H}_{n-1} \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1}$ (for $n = 0$, $\text{Ran}(A \upharpoonright \mathcal{H}_0) \subset \mathcal{H}_0 \oplus \mathcal{H}_1$). Thus we can introduce the Jacobi matrix

$$\begin{pmatrix} A_{00} & A_{01} & 0 & 0 & 0 & \dots \\ A_{10} & A_{11} & A_{12} & 0 & 0 & \dots \\ 0 & A_{21} & A_{22} & A_{23} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (1.14)$$

which consists of bounded operators A_{jk} acting from the space $\mathcal{H}_k$ into the space $\mathcal{H}_j$, and the action of A on finite vectors $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$, is given by: $(Af)_j = \sum_{k=0}^{\infty} A_{jk} f_k$, $j \in \mathbb{Z}_+$.

In the sequel, we will use the operators A of this type. They are defined on all finite sums $\mathcal{H}_0 \oplus \dots \oplus \mathcal{H}_m$, $m \in \mathbb{Z}_+$, and this set forms $\text{Dom}(A)$. Now it will be convenient, instead of matrix (1.14), to introduce the matrix

$$\begin{pmatrix} b_0 & c_0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & c_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & c_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (1.15)$$

where the operators $a_n : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$, $b_n : \mathcal{H}_n \rightarrow \mathcal{H}_n$, $c_n : \mathcal{H}_{n+1} \rightarrow \mathcal{H}_n$, $n \in \mathbb{Z}_+$, are bounded. The action of the operator A on finite vectors is given by:

$$\begin{aligned} (Af)_n &= a_{n-1} f_{n-1} + b_n f_n + c_n f_{n+1}, & n \in \mathbb{N}, \\ (Af)_0 &= b_0 f_0 + c_0 f_1, \\ f &= (f_n)_{n=0}^{\infty}, & f_n \in \mathcal{H}_n. \end{aligned} \quad (1.16)$$

If the operator A is essentially selfadjoint, then $c_n = a_n^*$, $n \in \mathbb{Z}_+$. The spectral theory for such Jacobi selfadjoint operators $\tilde{A}$ is well developed in the regular case, where $\forall n \in \mathbb{Z}_+$ a (continuous) inverse of the operator a_n exists [1]. Henceforth we will suppose that $a_{-1} = c_{-1} = 0$.

1.6 Vector-valued multilinear forms

Let $H_1, \mathcal{H}$ be two complex or real separable Hilbert spaces, $n \in \mathbb{N}$, and

$$H_1 \times \dots \times H_1 \ni \{\varphi_1, \dots, \varphi_n\} \mapsto A(\varphi_1, \dots, \varphi_n) \in \mathcal{H} \quad (1.17)$$

be a symmetric n -linear form with values in $\mathcal{H}$. For such a form, the following inequality holds true [16]:

$$\begin{aligned} \sup_{\|\varphi\|_{H_1}=1} \|A(\varphi, \dots, \varphi)\|_{\mathcal{H}} &\leq \sup_{\|\varphi_1\|_{H_1}=\dots=\|\varphi_n\|_{H_1}=1} \|A(\varphi_1, \dots, \varphi_n)\|_{\mathcal{H}} \\ &\leq \frac{n^n}{n!} \sup_{\|\varphi\|_{H_1}=1} \|A(\varphi, \dots, \varphi)\|_{\mathcal{H}} = C_n. \end{aligned} \quad (1.18)$$

Note that the finiteness of the $\sup$ in the middle of (1.18) means that mapping (1.17) is continuous; the finiteness of the first $\sup$ is equivalent to continuity of the diagonal mapping $H_1 \ni \varphi \mapsto A(\varphi, \dots, \varphi) \in \mathcal{H}$, so that inequality (1.18) implies that continuity of form (1.17) is equivalent to continuity of the corresponding diagonal mapping.

Let $H_1^{\widehat{\otimes} n} = H_1 \widehat{\otimes} \dots \widehat{\otimes} H_1$ be a symmetric tensor product of n copies of the space H_1 . The form given by (1.17) allows to construct some linear mapping $V_n : H_1^{\widehat{\otimes} n} \rightarrow \mathcal{H}$. Namely, let $(e_j)_{j=1}^\infty$ be some (nonorthonormal, in general) basis in the space H_1 . Then $(e_{j_1} \widehat{\otimes} \dots \widehat{\otimes} e_{j_n})_{j_1, \dots, j_n=1; j_1 \leq \dots \leq j_n}^\infty$ is a basis in $H_1^{\widehat{\otimes} n}$, and we set

$$\begin{aligned} H_1^{\widehat{\otimes} n} \ni f &= \sum_{j_1, \dots, j_n=1; j_1 \leq \dots \leq j_n}^\infty f_{j_1, \dots, j_n} e_{j_1} \widehat{\otimes} \dots \widehat{\otimes} e_{j_n} \mapsto \\ V_n f &= \sum_{j_1, \dots, j_n=1; j_1 \leq \dots \leq j_n}^\infty f_{j_1, \dots, j_n} A(e_{j_1}, \dots, e_{j_n}) \in \mathcal{H}. \end{aligned} \quad (1.19)$$

For f having a finite sequence of coordinates, the right-hand side of (1.19) is well defined and does not depend on the choice of the basis; V_n is, of course, linear.

We will use mapping (1.19) in a more special case when it will be defined on some linear subset of $H_1^{\widehat{\otimes} n}$.

Lemma 1.1 *Let $H_2 \subset H_1$ be some Hilbert spaces, for which the embedding $H_2 \hookrightarrow H_1$ is quasinuclear. Then mapping (1.19) defined on $H_2^{\widehat{\otimes} n}$ is a linear continuous operator from $H_2^{\widehat{\otimes} n}$ into $\mathcal{H}$ (we suppose that form (1.17) is continuous with respect to H_1).*

Proof. Let $(e_j)_{j=1}^\infty$ be an orthonormal basis in the space H_2 . By using (1.19) and (1.18), we have

$$\begin{aligned} \|V_n f\|_{\mathcal{H}}^2 &= \left\| \sum_{j_1, \dots, j_n=1; j_1 \leq \dots \leq j_n}^\infty f_{j_1, \dots, j_n} A(e_{j_1}, \dots, e_{j_n}) \right\|_{\mathcal{H}}^2 \\ &\leq \sum_{j_1, \dots, j_n=1; j_1 \leq \dots \leq j_n}^\infty |f_{j_1, \dots, j_n}|^2 \sum_{j_1, \dots, j_n=1}^\infty \|A(e_{j_1}, \dots, e_{j_n})\|_{\mathcal{H}}^2 \\ &= C_n^2 \|f\|_{H_2^{\widehat{\otimes} n}}^2 \sum_{j_1, \dots, j_n=1}^\infty \|e_{j_1}\|_{H_1}^2 \dots \|e_{j_n}\|_{H_1}^2 \\ &= C_n^2 \|O_{2,1}\|_{\text{HS}}^{2n} \|f\|_{H_2^{\widehat{\otimes} n}}^2, \end{aligned}$$

where $O_{2,1}$ is the embedding operator $H_2 \hookrightarrow H_1$. □

2 Commuting Jacobi field and its spectral representation

Let H be a real separable Hilbert space and let (1.4) be its real rigging with a nuclear space Φ . Fix $1, 2 \in T$ so that the imbeddings $H_1 \hookrightarrow H$, $H_2 \hookrightarrow H_1$ are quasinuclear. So, we have the chain

$$\Phi' \supset H_{-2} \supset H_{-1} \supset H \supset H_1 \supset H_2 \supset \Phi \quad (2.1)$$

with the pairing $(\cdot, \cdot)_H = \langle \cdot, \cdot \rangle$. By using H , construct $\mathcal{F}(H)$ and its rigging (1.6).

We will investigate the family $(A(\varphi))_{\varphi \in \Phi}$ of essentially selfadjoint operators $A(\varphi)$ acting in the Fock space $\mathcal{F}(H)$ with $\text{Dom}(A(\varphi)) = \mathcal{F}_{\text{fin}}(\Phi)$. This family is called a commuting Jacobi field if the following axioms are satisfied.

[F.1] Every operator $A(\varphi)$, $\varphi \in \Phi$, can be represented as an operator Jacobi matrix in the space $\mathcal{F}(H_1) \subset \mathcal{F}(H)$ of form (1.15) with respect to the orthogonal decomposition $\mathcal{F}(H_1) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H_1)$:

$$A(\varphi) = \begin{pmatrix} b_0(\varphi) & c_0(\varphi) & 0 & 0 & 0 & \dots \\ a_0(\varphi) & b_1(\varphi) & c_1(\varphi) & 0 & 0 & \dots \\ 0 & a_1(\varphi) & b_2(\varphi) & c_2(\varphi) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (2.2)$$

The operators $a_n(\varphi) : \mathcal{F}_n(H_1) \rightarrow \mathcal{F}_{n+1}(H_1)$, $b_n(\varphi) : \mathcal{F}_n(H_1) \rightarrow \mathcal{F}_n(H_1)$, $c_n(\varphi) : \mathcal{F}_{n+1}(H_1) \rightarrow \mathcal{F}_n(H_1)$, $n \in \mathbb{Z}_+$, are assumed to be bounded and real (i.e. they map real vectors into real ones).

[F2] For every $n \in \mathbb{Z}_+$ and $f_n \in \mathcal{F}_n(H_1)$, $f_{n+1} \in \mathcal{F}_{n+1}(H_1)$, the mappings

$$\Phi \ni \varphi \mapsto a_n(\varphi)f_n \in \mathcal{F}_{n+1}(H_1), \quad b_n(\varphi)f_n \in \mathcal{F}_n(H_1), \quad c_n(\varphi)f_{n+1} \in \mathcal{F}_n(H_1) \quad (2.3)$$

are linear and continuous when φ changes in the norm of the space H_1 .

It follows from conditions [F.1], [F.2] that the mapping $\Phi \ni \varphi \mapsto A(\varphi)$ can be extended to the whole space H_1 by continuity. Properties [F.1], [F.2] are also satisfied by the matrices $A(\varphi)$ with $\varphi \in H_1$.

To formulate the last and very essential axiom, it is necessary, at first, to introduce some constructions. Let $\Omega = (1, 0, 0, \dots)$ be a vacuum vector of the Fock space $\mathcal{F}(H_1)$. Fix $n \in \mathbb{N}$ and introduce the real form of type (1.17):

$$H_1 \times \dots \times H_1 \ni \{\varphi_1, \dots, \varphi_n\} \mapsto A(\varphi_1, \dots, \varphi_n) := A(\varphi_1) \dots A(\varphi_n)\Omega \in \mathcal{F}_n(H_1) \quad (2.4)$$

(its n -linearity, continuity, and symmetry follows from [F.1], [F.2] and from the commutativity of the operators $A(\varphi)$). By applying the real version of Lemma 1.1 to form (2.4), and then taking the complexification, we get the linear continuous operator of type (1.19):

$$\mathcal{F}_n(H_2) \ni f_n \mapsto V_n f_n \in \bigoplus_{j=0}^n \mathcal{F}_j(H_1) \subset \mathcal{F}(H_1),$$

where

$$\forall \varphi_1, \dots, \varphi_n \in H_2 : \quad V_n(\varphi_1 \hat{\otimes} \dots \hat{\otimes} \varphi_n) = A(\varphi_1) \dots A(\varphi_n) \Omega \in \bigoplus_{j=0}^n \mathcal{F}_j(H_1). \quad (2.5)$$

By using V_n , we can introduce the corresponding continuous “coordinate” operators:

$$\mathcal{F}_n(H_2) \ni f_n \mapsto V_{j,n} f_n := (V_n f_n)_j \in \mathcal{F}_j(H_1), \quad j = 0, \dots, n. \quad (2.6)$$

Note that the operator $V_{n,n}$ has the following property (see (2.2)):

$$\begin{aligned} \mathcal{F}_n(H_2) \ni \varphi_1 \hat{\otimes} \dots \hat{\otimes} \varphi_n &\mapsto V_{n,n}(\varphi_1 \hat{\otimes} \dots \hat{\otimes} \varphi_n) = (A(\varphi_1) \dots A(\varphi_n) \Omega)_n \\ &= a_{n-1}(\varphi_1) \dots a_0(\varphi_n) 1 \in \mathcal{F}_n(H_1). \end{aligned} \quad (2.7)$$

The last axiom is the following:

[F.3] (joint regularity of the field) For every $n \in \mathbb{N}$, the operator $\mathcal{F}_n(H_2) \ni f_n \mapsto V_{n,n} f_n \in \mathcal{F}_n(H_1)$ is invertible, i.e. $\text{Ran } V_{n,n} = \mathcal{F}_n(H_1)$ and the operator has a bounded inverse.

Note that we do not demand that $\forall n \in \mathbb{Z}_+$, the operators $a_n(\varphi)$, $b_n(\varphi)$, $c_n(\varphi)$ be extendable to bounded operators between the corresponding spaces $\mathcal{F}_j(H)$ although often such an extension exists. In such a case, we have, of course, that $c_n(\varphi) = (a_n(\varphi))^* : \mathcal{F}_{n+1}(H) \rightarrow \mathcal{F}_n(H)$, $n \in \mathbb{Z}_+$. Also the operator $V_{n,n}$ can often be extended to a similar operator acting in the space $\mathcal{F}_n(H)$, $n \in \mathbb{Z}_+$.

By using $A(\varphi)$, $\varphi \in \Phi$, we introduce the “creation”, “neutral”, and “annihilation” operators $A_+(\varphi)$, $B(\varphi)$, and $A_-(\varphi)$, which will be given by matrix (2.2) if the lower, main, or upper diagonal, respectively, is preserved and other entries are filled with zeros. The domains of these operators coincide with $\mathcal{F}_{\text{fin}}(H) = \text{Dom}(A(\varphi))$. We have that

$$A(\varphi) = A_+(\varphi) + B(\varphi) + A_-(\varphi), \quad \varphi \in \Phi. \quad (2.8)$$

For an arbitrary $n \in \mathbb{Z}_+$, the operator $A_+(\varphi)$ maps $\mathcal{F}_n(\varphi)$ into $\mathcal{F}_{n+1}(\varphi)$, the operator $B(\varphi)$ preserves this space, and the operator $A_-(\varphi)$ maps it into $\mathcal{F}_{n-1}(\varphi)$ (or into 0 if $n = 0$).

Take the introduced family $A = (\tilde{A}(\varphi))_{\varphi \in \Phi}$ of selfadjoint commuting operators acting in the space $\mathcal{H} = \mathcal{F}(H)$ and apply Theorem 1.1 to this family. Choose rigging (1.1) with the spaces from chain (1.6) in the form

$$\begin{array}{ccccccc} \mathcal{F}(H_{-1}, p^{-1}) & \supset & \mathcal{F}(H) & \supset & \mathcal{F}(H_1, p) & \supset & \mathcal{F}_{\text{fin}}(\Phi) \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathcal{H}_- & & \mathcal{H} & & \mathcal{H}_+ & & \mathcal{D} \end{array} \quad (2.9)$$

where the weight p is fixed and taken to satisfy condition (1.7), which gives the nuclearity of the imbedding $\mathcal{H}_+ \hookrightarrow \mathcal{H}$. The pairing in (2.9) will often be denoted by $\langle \cdot, \cdot \rangle$. So, $(\cdot, \cdot)_H = \langle \cdot, \cdot \rangle$, $(\cdot, \cdot)_{\mathcal{F}(H)} = \langle \cdot, \cdot \rangle$, $(\cdot, \cdot)_{\mathcal{F}_n(H)} = \langle \cdot, \cdot \rangle$, $n \in \mathbb{Z}_+$.

Let us check conditions of Theorem 1.1.

Condition a) is implied by the construction. Condition b) follows from formula (1.16), conditions (2.3), and from the observation that, if a sequence $(f^{(\nu)})_{\nu=1}^{\infty}$ of vectors $f^{(\nu)} \in \mathcal{F}_{\text{fin}}(\Phi)$ converges in the sense of the space $\mathcal{F}_{\text{fin}}(\Phi)$, then, for each $n \in \mathbb{Z}_+$ fixed, the sequence of coordinates $f_n^{(\nu)}$ converges in the sense of the space $\mathcal{F}_n(H_1)$. Condition c) follows from (2.3).

Lemma 2.1 *The vacuum Ω is a strong cyclic vector for the family $(\tilde{A}(\varphi))_{\varphi \in \Phi}$ and rigging (2.9), i.e. condition d) is satisfied.*

Proof. Fix $n \in \mathbb{N}$, the set $\{\varphi^{\otimes n}, \varphi \in \Phi\}$ is total in $\mathcal{F}_n(H_2)$. Because the operator $V_{n,n}$ is invertible, this totality implies the totality, in $\mathcal{F}_n(H_1)$, of the set

$$\{V_{n,n}(\varphi^{\otimes n}) = a_{n-1}(\varphi) \dots a_0(\varphi)1 = ((A_+(\varphi))^n \Omega)_n, \varphi \in \Omega\}. \quad (2.10)$$

From the totality of set (2.10), we can conclude that the set

$$\{\Omega, A_+(\varphi_1) \dots A_+(\varphi_n) \Omega \mid \varphi_1, \dots, \varphi_n \in \Phi, n = 1, \dots, m\} \quad (2.11)$$

is total in $\bigoplus_{n=0}^m \mathcal{F}_n(H_1) = F_m$, $m \in \mathbb{Z}_+$ (it is more convenient to use different φ in (2.10)).

Let us prove that the set obtained from (2.11) by replacing the operators $A_+(\varphi_j)$ with $A(\varphi_j)$ is total. So, suppose by induction that the set

$$\{\Omega, A(\varphi_1) \dots A(\varphi_n) \Omega \mid \varphi_1, \dots, \varphi_n \in \Phi, n = 1, \dots, m\} \quad (2.12)$$

is total in F_m for $m = 0, \dots, k$ and prove its totality for $m = k + 1$.

Let f be a vector from F_{k+1} and $\epsilon > 0$. Since set (2.11) is total for $m = k + 1$, we have that there is a vector g , which is a linear combination of the vectors $\Omega, A_+(\varphi_1^{(1)})\Omega, \dots, A_+(\varphi_1^{(k+1)}) \dots A_+(\varphi_{k+1}^{(k+1)})\Omega$, such that $\|f - g\|_{F_{k+1}} < \epsilon$.

By using the equality $A_+(\varphi) = A(\varphi) - B(\varphi) - A_-(\varphi)$ (see (2.8)), we construct a corresponding to g vector h by replacing, in its representation, the operators $A_+(\varphi)$ by $A(\varphi)$. Thus we can write that $g = h + r$, where h is a linear combination of $\Omega, A(\varphi_1^{(1)})\Omega, \dots, A(\varphi_1^{(k+1)}) \dots A(\varphi_{k+1}^{(k+1)})\Omega$ and r is constructed by using the operators $B(\varphi)$ and $A_-(\varphi)$ instead of $A(\varphi)$. Therefore, $r \in F_k$ (see the action of the operators $B(\varphi)$ and $A_-(\varphi)$) and, by the inductive hypothesis, the vector r can be approximated by a linear combination s of vectors from (2.12) for $m = k$: $\|r - s\|_{F_k} < \epsilon$.

As a result, $\|f - (h + s)\|_{F_{k+1}} \leq \|f - (h + r)\|_{F_{k+1}} + \|r - s\|_{F_k} < 2\epsilon$ and $h + s$ is a linear combination of vectors (2.12) for $m = k + 1$. This yields the totality of vectors (2.12) for $m = k + 1$.

Since the assumption is true for $k = 1$, set (2.12) is total in F_m for arbitrary $m \in \mathbb{Z}$. From this and the construction of the space $\mathcal{H}_+ = \mathcal{F}(H_1, p)$, it follows that set (2.12), with $m = \infty$, is total in $\mathcal{H}_+$, i.e. Ω is a strong cyclic vector. $\square$

Now we are ready to use Theorem 1.1 in our case. But it will be more convenient to investigate properties of the corresponding generalized joint eigenvector $\xi(x)$, which will

be denoted by $P(x) = \xi(x)$. For the corresponding $x \in \Phi'$, this vector belongs to the space $\mathcal{H}_- = \mathcal{F}(H_{-1}, p^{-1})$, and therefore,

$$\begin{aligned} P(x) &= (P_0(x), P_2(x), \dots), \quad P_n(x) \in \mathcal{F}_n(H_{-1}), \\ \|P(x)\|_{\mathcal{F}(H_{-1}, p^{-1})}^2 &= \sum_{n=0}^{\infty} \|P_n(x)\|_{\mathcal{F}_n(H_{-1})}^2 p_n^{-1} < \infty. \end{aligned} \quad (2.13)$$

Equality (1.2) now has the form

$$\begin{aligned} (P(x), A(\varphi)f)_{\mathcal{F}(H)} &= (x, \varphi)_H (P(x), f)_{\mathcal{F}(H)}, \\ \varphi &\in \Phi, \quad f \in \mathcal{D} = \mathcal{F}_{\text{fin}}(\Phi); \quad x \in \Phi'. \end{aligned} \quad (2.14)$$

By using [F.1], we can rewrite (2.14) in the following way:

$$\begin{aligned} (x, \varphi)_H \sum_{n=0}^{\infty} (P_n(x), f_n)_{\mathcal{F}_n(H)} \\ &= \sum_{n=0}^{\infty} (P_n(x), a_{n-1}(\varphi)f_{n-1} + b_n(\varphi)f_n + c_n(\varphi)f_{n+1})_{\mathcal{F}_n(H)} \\ &= \sum_{n=0}^{\infty} (c_{n-1}^+(\varphi)P_{n-1}(x) + b_n^+(\varphi)P_n(x) + a_n^+(\varphi)P_{n+1}(x), f_n)_{\mathcal{F}_n(H)}. \end{aligned} \quad (2.15)$$

Here we used the adjoint coefficients with respect to the spaces $\mathcal{F}_n(H)$:

$$\begin{aligned} a_n^+(\varphi) : \mathcal{F}_{n+1}(H_{-1}) &\rightarrow \mathcal{F}_n(H_{-1}), \quad b_n^+(\varphi) : \mathcal{F}_n(H_{-1}) \rightarrow \mathcal{F}_n(H_{-1}), \\ c_n^+(\varphi) : \mathcal{F}_n(H_{-1}) &\rightarrow \mathcal{F}_{n+1}(H_{-1}). \end{aligned} \quad (2.16)$$

These operators are introduced as follows: for $n \in \mathbb{Z}_+$ and an operator $A : \mathcal{F}_k(H_1) \rightarrow \mathcal{F}_j(H_2)$, the adjoint operator $A^+ : \mathcal{F}_j(H_{-1}) \rightarrow \mathcal{F}_k(H_{-1})$ is defined by the equality

$$\begin{aligned} (Af_k, g_j)_{\mathcal{F}_j(H)} &= (f_k, A^+g_j)_{\mathcal{F}_k(H)}, \\ f_k &\in \mathcal{F}_k(H_1), \quad g_j \in \mathcal{F}_j(H_{-1}); \quad j, k \in \mathbb{Z}_+. \end{aligned} \quad (2.17)$$

It follows from (2.15) that $P(x)$ in the form (2.13) is a solution of the difference equation with operator-valued coefficients:

$$\begin{aligned} c_{n-1}^+(\varphi)P_{n-1}(x) + b_n^+(\varphi)P_n(x) + a_n^+(\varphi)P_{n+1}(x) &= (x, \varphi)_H P_n(x), \\ n &\in \mathbb{Z}_+, \quad \varphi \in \Phi, \quad x \in \Phi'. \end{aligned} \quad (2.18)$$

Note that using selfadjointness of $A(\varphi)$, [F.1], and (2.16) it is easy to conclude that matrix (2.2) is Hermitian in the following sense:

$$\begin{aligned} c_n^+(\varphi) \upharpoonright \mathcal{F}_n(H_1) &= a_n(\varphi), \quad b_n^+(\varphi) \upharpoonright \mathcal{F}_n(H_1) = b_n(\varphi), \quad a_n^+(\varphi) \upharpoonright \mathcal{F}_{n+1}(H_1) = c_n(\varphi), \\ \varphi &\in \Phi, \quad n \in \mathbb{Z}_+. \end{aligned} \quad (2.19)$$

In the case where the operator Jacobi matrix is scalar or regular, it is possible, by using recurrence relation (2.18) and the initial data $P_0(x)$, to find recurrently a solution (2.13) of the equation. In the case under consideration, such a procedure is impossible because the operator $a_n^+(\varphi)$ at $P_{n+1}(x)$ is not invertible. Nevertheless, using condition [F.3], it is still possible to find a solution of (2.18).

To do that, we apply equality (2.14) n times. Then we get $\forall \varphi \in \Phi$,

$$\begin{aligned} (P(x), (A(\varphi))^n \Omega)_{\mathcal{F}(H)} &= (x, \varphi)_H (P(x), (A(\varphi))^{n-1} \Omega)_{\mathcal{F}(H)} = \dots \\ &= (x, \varphi)_H^n (P(x), \Omega)_{\mathcal{F}(H)} = (x^{\otimes n}, \varphi^{\otimes n})_{\mathcal{F}_n(H)} P_0(x), \\ x &\in \Phi', \quad n \in \mathbb{N}. \end{aligned} \quad (2.20)$$

On the other hand, $(A(\varphi))^n \Omega \in \bigotimes_{j=0}^n \mathcal{F}_j(H_1)$ and so, by using (2.20), and (2.5), (2.6), we can write

$$\begin{aligned} (x^{\otimes n}, \varphi^{\otimes n})_{\mathcal{F}_n(H)} P_0(x) &= (P(x), (A(\varphi))^n \Omega)_{\mathcal{F}(H)} \\ &= (P(x), V_n(\varphi^{\otimes n}))_{\mathcal{F}(H)} \\ &= \sum_{j=0}^n (P_j(x), (V_n(\varphi^{\otimes n}))_j)_{\mathcal{F}_j(H)} \\ &= (P_n(x), V_{n,n}(\varphi^{\otimes n}))_{\mathcal{F}_n(H)} + \sum_{j=0}^{n-1} (P_j(x), V_{j,n}(\varphi^{\otimes n})). \end{aligned} \quad (2.21)$$

The vectors $\varphi^{\otimes n}$, $\varphi \in \Phi$, form a total set in the space $\mathcal{F}_n(H_2)$, and the operators $V_{j,n} : \mathcal{F}_n(H_2) \rightarrow \mathcal{F}_j(H_1)$ act continuously. Therefore, it follows from (2.21) that $\forall f_n \in \mathcal{F}_n(H_2)$,

$$(x^{\otimes n}, f_n)_{\mathcal{F}_n(H)} P_0(x) = (P_n(x), V_{n,n} f_n)_{\mathcal{F}_n(H)} + \sum_{j=0}^{n-1} (P_j(x), V_{j,n} f_n)_{\mathcal{F}_j(H)}. \quad (2.22)$$

As it follows from [F.3], $V_{n,n} f_n = g_n$ can be any vector from $\mathcal{F}_n(H_1)$ as the vector f_n runs over $\mathcal{F}_n(H_2)$. Thus (2.22) can be rewritten as

$$(x^{\otimes n}, V_{n,n}^{-1} g_n)_{\mathcal{F}_n(H)} P_0(x) = (P_n(x), g_n)_{\mathcal{F}_n(H)} + \sum_{j=0}^{n-1} (P_j(x), V_{j,n} V_{n,n}^{-1} g_n)_{\mathcal{F}_j(H)},$$

or, if we use the adjoint operator of type (2.17), we can write

$$P_n(x) = P_0(x) (V_{n,n}^+)^{-1} x^{\otimes n} - \sum_{j=0}^{n-1} (V_{n,n}^+)^{-1} V_{j,n}^+ P_j(x), \quad n \in \mathbb{N}. \quad (2.23)$$

In (2.23) it is necessary to take $x \in H_{-2}$ because $(V_{n,n}^+)^{-1} : \mathcal{F}_n(H_{-2}) \rightarrow \mathcal{F}_n(H_{-1})$. Such a choice of x is sufficient since the set $H_{-2} \subset \Phi'$ also has the full measure ρ (the

latter assertion follows from the possibility to construct ρ as a spectral measure of the family $(\tilde{A}(\varphi))_{\varphi \in H_2}$.

By setting $n = 1, 2, \dots$ in (2.23), we can find $P_n(x)$ from $P_0(x)$. This procedure implies that if, for some x , $P_0(x) = 0$, then $P_n(x) = 0$ for all $n = 1, 2, \dots$. Because $P(x) = \xi(x) \neq 0$, we have $P_n(x) \neq 0$ and, by multiplying $P_n(x)$ by the constant $C = P_0^{-1}(x)$, we can get that $P_0(x) = 1$ (equations (2.18), (2.23) are homogeneous). The obtained now vectors $P_n(x)$, $n \in \mathbb{Z}_+$, are real.

From bruening Mon Apr 21 08:28:07 1997 Received: by spectrum.mathematik.hu-berlin.de (AIX 4.1/UCB 5.64/4.03) id AA21184; Mon, 21 Apr 1997 08:28:06 +0200 Date: Mon, 21 Apr 1997 08:28:06 +0200 From: bruening (Jochen Bruening) Message-Id: j9704210628.AA21.berlin.de; To: bruening@mathematik.hu-berlin.de Cc: bruening Status: R

So, as a result we have the following theorem.

Theorem 2.1 *For a commuting Jacobi field $A = (A(\varphi))_{\varphi \in \Phi}$ with riggings (2.1) and (2.9) there exists a probability Borel measure ρ on the space H_{-2} such that the (Fourier or Wiener-Itô-Segal) transform*

$$\begin{aligned} \mathcal{F}(H_1, \rho) \ni f = (f_n)_{n=0}^\infty &\mapsto \hat{f}(x) = (If)(x) = \sum_{n=0}^\infty (f_n, P_n(x))_{\mathcal{F}_n(H)} \\ &\in L^2(H_{-2}, d\rho(x)) = L^2(\Phi', d\rho(x)), \end{aligned} \quad (2.24)$$

being extended by continuity, is unitary from the space $\mathcal{F}(H)$ onto the space $L^2(\Phi', d\rho(x))$.

Here, for ρ -almost every $x \in H_{-2}$, the generalized real eigenvector $P(x) = (P_n(x))_{n=0}^\infty \in \mathcal{F}(H_{-1}, p^{-1})$ is a joint solution of the family of equations

$$\begin{aligned} c_{n-1}^+(\varphi)P_{n-1}(x) + b_n^+(\varphi)P_n(x) + a_n^+(\varphi)P_{n+1}(x) &= (x, \varphi)_H P_n(x), \\ \varphi \in \Phi, \quad n \in \mathbb{Z}_+, \quad P_0(x) &= 1, \end{aligned} \quad (2.25)$$

and has the following form (with bounded real operators $W_{n,j}$):

$$\begin{aligned} P_n(x) &= W_{n,n}x^{\otimes n} + \sum_{j=0}^{n-1} W_{n,j}P_j(x), \\ W_{n,n} &= (V_{n,n}^+)^{-1} : \mathcal{F}_n(H_{-2}) \rightarrow \mathcal{F}_n(H_{-1}), \\ W_{n,j} &= -(V_{n,n}^+)^{-1}V_{j,n}^+ : \mathcal{F}_j(H_{-1}) \rightarrow \mathcal{F}_n(H_{-1}), \\ j &= 0, \dots, n-1; \quad n \in \mathbb{N}. \end{aligned} \quad (2.26)$$

Here the operators $V_{j,n}$ are defined by (2.6), (2.7).

The unitary transform (2.24) maps operators $A(\varphi)$ into the operators of multiplication by $(x, \varphi)_H$, series (2.24) is absolutely convergent for ρ -almost all $x \in H_{-2}$.

It should be explained that the σ -algebra generated by cylindrical sets in Φ' is equal to the σ -algebra generated by Borel sets in Φ' because Φ is nuclear [7]. Therefore, the measure ρ is a Borel measure. The set H_{-2} is a set of full measure ρ , and hence $L^2(H_{-2}, d\rho(x)) = L^2(\Phi', d\rho(x))$.

Let us look more closely at the set of functions $\hat{f}(x) = (If)(x)$, $f \in \mathcal{F}(H_1, p)$. By inductive use of (2.26), we can conclude that $\forall x \in H_{-2}$,

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= W_{1,1}x + W_{1,0}1, \\
P_2(x) &= W_{2,2}x^{\otimes 2} + W_{2,1}P_1(x) + W_{2,0}1 \\
&= W_{2,2}x^{\otimes 2} + W_{2,1}(W_{1,1}x + W_{1,0}1) + W_{2,0}1 \\
&= W_{2,2}x^{\otimes 2} + W_{2,1}W_{1,1}x + (W_{2,1}W_{1,0} + W_{2,0})1, \\
P_3(x) &= W_{3,3}x^{\otimes 3} + W_{3,2}P_2(x) + W_{3,1}P_1(x) + W_{3,0}1 = \dots \\
&\dots \dots \dots
\end{aligned} \tag{2.27}$$

In the scalar case when $H = \mathbb{R}^1$, the introduced polynomials $P_n(x)$ are polynomials of the first kind connected with the Jacobi matrix A . In a general case when field (2.2) is considered, we will still use this name and call objects (2.27) polynomials of the first kind connected with the field.

Denote by $\mathcal{P}_n(H_{-2})$, $n \in \mathbb{Z}$, the set of all continuous polynomials on H_{-2} of the degree less or equal to n , i.e. the functions

$$H_{-2} \ni x \mapsto p_n(x) = \sum_{j=0}^n (x^{\otimes j}, a_n)_{\mathcal{F}_n(H)} = \sum_{j=0}^n \langle x^{\otimes j}, a_n \rangle \in \mathbb{C}^1, \tag{2.28}$$

with the coefficients $a_n \in \mathcal{F}_n(H_2)$. Then $\mathcal{P}(H_{-2}) = \bigcup_{n=0}^{\infty} \mathcal{P}_n(H_{-2})$ is the set of all such polynomials.

By applying formulas (2.27) to $P_n(x)$ in (2.24) and moving the coefficients $W_{n,n}, W_{n,n-1}W_{n-1,n-1}, \dots$ from $x^{\otimes n}, x^{\otimes(n-1)}, \dots$ to f_n , we see that every term $\langle f_n, P_n(x) \rangle$ in (2.24) has the form of (2.28). Therefore, the Fourier transform of any finite sequence $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(H_1, p)$ belongs to $\mathcal{P}(H_{-2})$ (and belongs to $\mathcal{P}_n(H_{-2})$ if $f_{n+1} = f_{n+2} = \dots = 0$). Moreover, the following assertion holds true.

Proposition 2.1 *We have*

$$I(\{f = (f_j)_{j=0}^{\infty}, f_j \in \mathcal{F}_j(H_1) | f_j = 0, j > n\}) = \mathcal{P}_n(H_{-2}), \quad n \in \mathbb{Z}_+, \tag{2.29}$$

and therefore, $I(\mathcal{F}_{\text{fin}}(H_1)) = \mathcal{P}(H_{-2})$.

Proof. It is necessary to prove that the set in the right-hand side of (2.29) is embedded into the set in the left-hand side. For $n = 0$, this assertion is true; suppose it holds for $n \leq k$ and let us prove it for $n = k + 1$.

Let $p_{k+1}(x)$ be a polynomial of type (2.28) with coefficients $a_0, \dots, a_{k+1}$; $f = (\delta_{jk+1}(W_{k+1,k+1}^+)^{-1}\bar{a}_{k+1})_{j=0}^{\infty} \in \mathcal{F}(H_1, p)$ (note that $(W_{k+1,k+1}^+)^{-1}: \mathcal{F}_{k+1}(H_2) \rightarrow \mathcal{F}_{k+1}(H_1)$). Then by (2.24) and (2.27), we have

$$\begin{aligned}
\hat{f}(x) &= \langle (W_{k+1,k+1}^+)^{-1}\bar{a}_{k+1}, P_{k+1}(x) \rangle \\
&= \langle (W_{k+1,k+1}^+)^{-1}\bar{a}_{k+1}, W_{k+1,k+1}x^{\otimes(k+1)} + W_{k+1,k}W_{k,k}x^{\otimes k} + \dots \rangle \\
&= \langle \bar{a}_{k+1}, x^{\otimes(k+1)} \rangle + q_k(x) = \langle x^{\otimes(k+1)}, a_{k+1} \rangle + q_k(x),
\end{aligned}$$

where $q_k(x) = \langle W_{k,k}^+ W_{k+1,k}^+ (W_{k+1,k+1}^+)^{-1} \bar{a}_{k+1} \rangle + \dots$. The polynomials $q_k(x)$ are of form (2.28) and have the degree $\leq k$. Hence the difference $p_{k+1}(x) - \hat{f}(x)$ belongs to $\mathcal{P}_k(H_{-2})$ and, by the induction hypothesis, it belongs to the set in the left-hand side of (2.29) with $n = k$. So, $p_{k+1}(x)$ belongs to this set for $n = k + 1$. $\square$

Note that this proposition implies that every power $\langle x, a \rangle^n$, $a \in H_{-2}$, belongs to the space $L^2(H_{-2}, d\rho(x))$, in other words, for the measure ρ "all moments exist".

In what follows, we keep the notations $f \mapsto \hat{f}(x) = (If)(x)$ for the extension, by continuity, of mapping (2.24) to the whole space $\mathcal{F}(H)$. The Parseval equality, which is connected with this unitary transform, has the following form: $\forall f, g \in \mathcal{F}(H)$,

$$\begin{aligned} (f, g)_{\mathcal{F}(H)} &= \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{F}_n(H)} \\ &= \int_{H_{-2}} \hat{f}(x) \overline{\hat{g}(x)} d\rho(x) = (\hat{f}, \hat{g})_{L^2(H_{-2}, d\rho(x))}. \end{aligned} \quad (2.30)$$

In the Fock space $\mathcal{F}(H)$, we take a BON $(e_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty}$ of type (1.8) constructed from an orthonormal basis $(e_j)_{j=1}^\infty$, where $e_j \in \Phi$ (or, at least, $e_j \in H_2$). Images of the vectors e_α with respect to the unitary transform I form a real orthonormal basis in the space $L^2((H_{-2}, d\rho(x)))$ which, according to (2.24), has the form

$$(\hat{e}_\alpha)(x) = (I e_\alpha)(x) = \langle e_\alpha, P_{|\alpha|}(x) \rangle, \quad \alpha \in \mathbb{Z}_{+,0}^\infty. \quad (2.31)$$

By replacing g with e_α in (2.30), we get a formula for finding the vector f , in terms of its BON-coordinates, from its Fourier transform $\hat{f}$:

$$f_\alpha = \int_{H_{-2}} \hat{f}(x) (\hat{e}_\alpha)(x) d\rho(x), \quad f = (f_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty}. \quad (2.32)$$

The orthogonality relations for $(\hat{e}_\alpha)(x)$, which generalize classical relations for orthogonal polynomials, have the form

$$\int_{H_{-2}} (\hat{e}_\alpha)(x) (\hat{e}_\beta)(x) d\rho(x) = \delta_{\alpha,\beta}; \quad \alpha, \beta \in \mathbb{Z}_{+,0}^\infty. \quad (2.33)$$

In some cases, using representation (2.27) for $P_n(x)$, it is possible to give more detail formulas for orthonormal basis (2.31). Note also that the Fourier transform (2.24) can be rewritten in BON as follows:

$$\begin{aligned} \mathcal{F}(H_1, p) \ni f &= (f_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty} \mapsto \hat{f}(x) = (If)(x) \\ &= \sum_{\alpha \in \mathbb{Z}_{+,0}^\infty} f_\alpha (\hat{e}_\alpha)(x) \in L^2(H_{-2}, d\rho(x)). \end{aligned} \quad (2.34)$$

Finally, let us introduce some simple but essential notions and prove corresponding facts.

The orthogonal decomposition $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$ of the Fock space induces, via the unitary I , an orthogonal decomposition (chaos decomposition) of the space $L^2(H_{-2}, d\rho(x)) = L^2(\Phi', d\rho(x)) =: (L^2)$

$$(L^2) = \bigoplus_{n=0}^{\infty} (L_n^2), \quad (L_n^2) = I(\mathcal{F}_n(H)); \quad (2.35)$$

let Pr_n be the orthogonal projector in (L^2) onto (L_n^2) , $n \in \mathbb{Z}_+$.

According to Proposition 2.1, $\mathcal{P}(H_{-2}) = I(\mathcal{F}_{\text{fin}}(H_1))$ is dense in (L^2) because $\mathcal{F}_{\text{fin}}(H_1)$ is dense in $\mathcal{F}(H)$. From (2.29) and density of $\mathcal{F}_j(H_1)$ in $\mathcal{F}_j(H)$, it follows that $\mathcal{P}_n(H_{-2})$ is dense in $\bigoplus_{j=0}^n (L_j^2) = I(\bigoplus_{j=0}^n \mathcal{F}_j(H))$, i.e. for its closure $\tilde{\mathcal{P}}_n(H_{-2})$ in (L^2) , the following equality holds:

$$\tilde{\mathcal{P}}_n(H_{-2}) = \bigoplus_{j=0}^n (L_j^2), \quad \text{or} \quad (L_n^2) = \tilde{\mathcal{P}}_n(H_{-2}) \ominus \tilde{\mathcal{P}}_{n-1}(H_{-2}), \quad n \in \mathbb{N}. \quad (2.36)$$

To calculate the vector $f \in \mathcal{F}(H)$ from the function $F \in \mathcal{F}(H)$ such that $\hat{f} = F$, it is necessary to use formula (2.32) with $\alpha \in \mathbb{Z}_{+,0}^\infty$, $|\alpha| = n$; $f \in \mathcal{F}_n(H)$.

Proposition 2.2 *The projection onto (L_n^2) of continuous monomial $H_{-2} \ni x \mapsto \langle x^{\otimes n}, a_n \rangle \in \mathbb{C}^1$, $a_n \in \mathcal{F}_n(H_2)$, $n \in \mathbb{Z}_+$, regarded as a function from the space (L^2) , is given by the following formula:*

$$\text{Pr}_n \langle x^{\otimes n}, a_n \rangle = \langle P_n(x), V_{n,n} a_n \rangle. \quad (2.37)$$

Proof. It is quite similar to the proof of Proposition 2.1. Namely, note at first that, according to (2.24), the function $\langle P_n(\cdot), V_{n,n} a_n \rangle = \langle V_{n,n} \bar{a}_n, P_n(\cdot) \rangle \in (L_n^2)$. Therefore to prove (2.37), it is sufficient to check that the difference $\langle x^{\otimes n}, a_n \rangle - \langle P_n(x), V_{n,n} a_n \rangle$ is orthogonal to the space (L_n^2) . But using formulas (2.27) and equality $W_{n,n} = (V_{n,n}^+)^{-1}$, we get:

$$\begin{aligned} & \langle x^{\otimes n}, a_n \rangle - \langle P_n(x), V_{n,n} a_n \rangle \\ &= \langle x^{\otimes n}, a_n \rangle - \langle W_{n,n} x^{\otimes n} + W_{n,n-1} W_{n-1,n-1} x^{\otimes(n-1)} + \dots, V_{n,n} a_n \rangle \\ &= \langle x^{\otimes n}, a_n \rangle - \langle W_{n,n} x^{\otimes n}, V_{n,n} a_n \rangle - q_{n-1}(x) = -q_{n-1}(x). \end{aligned}$$

Here $q_{n-1}(x) = \langle W_{n,n-1} W_{n-1,n-1} x^{\otimes(n-1)} + \dots, V_{n,n} a_n \rangle$ is some polynomial of type (2.28) and degree $\leq n-1$, i.e. $q_{n-1} \in \mathcal{P}_{n-1}(H_{-2}) = \bigoplus_{j=0}^{n-1} (L_j^2) \perp (L_n^2)$ (see (2.36)). $\square$

Formula (2.37) generalizes the classical relation between orthogonal polynomials in variable $x \in \mathbb{R}^1$ and its powers x^n .

Projection (2.37) is called a Wick monomial and denoted by

$$: \langle x^{\otimes n}, a_n \rangle := \langle P_n(x), V_{n,n} a_n \rangle, \quad x \in H_{-2}, \quad a_n \in \mathcal{F}_n(H_2), \quad n \in \mathbb{Z}_+ \quad (2.38)$$

(we stress that the colons used here are not to designate Pr_n but only to denote the action of this projector on $\langle x^{\otimes n}, a_n \rangle$; we do not use the index n with the colons).

According to (2.26), $V_{n,n} = (W_{n,n}^{-1})^+$ and, therefore, (2.38) can be rewritten in the following way: $\forall a_n \in \mathcal{F}_n(H_2), n \in \mathbb{Z}_+$,

$$: \langle x^{\otimes n}, a_n \rangle := \langle : x^{\otimes n} :, a_n \rangle, \quad : x^{\otimes n} := W_{n,n}^{-1} P_n(x), \quad x \in H_{-2}. \quad (2.39)$$

The mapping

$$H_{-2} \ni x \mapsto x^{\otimes n} := W_{n,n}^{-1} P_n(x) \in H_{-2} \quad (2.40)$$

is said to be the Wick power.

In terms of $: x^{\otimes n} :$ expression (2.24) can be rewritten in the following way:

$$\begin{aligned} \mathcal{F}(H_1, p) \ni f = (f_n)_{n=0}^\infty &\mapsto \hat{f}(x) = (If)(x) = \sum_{n=0}^\infty \langle f_n, W_{n,n} : x^{\otimes n} : \rangle = \\ &= \sum_{n=0}^\infty \langle V_{n,n}^{-1} f_n, : x^{\otimes n} : \rangle = \sum_{n=0}^\infty : \langle V_{n,n}^{-1} f_n, x^{\otimes n} \rangle : \in L^2(H_{-2}, d\rho(x)). \end{aligned} \quad (2.41)$$

The extension of I onto whole $\mathcal{F}(H)$ is also given by formula (2.41) if the expression $: \langle V_{n,n}^{-1} f_n, x^{\otimes n} \rangle :$ is reasonably defined for $f_n \in \mathcal{F}_n(H)$.

3 About spaces of test and generalized functions

As we have mentioned in the Introduction, the Fourier transform (2.24) transfers positive spaces in chain (1.6) into some spaces from $L^2(H_{-2}, d\rho(x))$, and we can take this space to be the space of test functions of the variable $x \in H_{-2}$ with the pairing defined by $d\rho(x)$. For an investigation of properties of such test and generalized functions in some important cases, it is convenient to apply some results from [4, 5]. We reformulate these results in a suitable form and then describe briefly these applications.

Let Q be some separable metric space and $(P_n(x))_{n=0}^\infty$ be a fixed sequence of vector-valued functions $Q \ni x \mapsto P_n(x) \in \mathcal{F}_n(H_{-1})$, $P_0(x) = 1$. Suppose that a “generating function” exists: $\exists R_1 > 0$ for which the series

$$\sigma(x, \lambda) = \sum_{n=0}^\infty \frac{1}{n!} \langle \lambda^{\otimes n}, P_n(x) \rangle, \quad \lambda \in B_1 = \{\lambda \in H_1 \mid \|\lambda\|_{H_1} \leq R_1\} \quad (3.1)$$

is convergent $\forall x \in Q$ uniformly with respect to $\lambda \in B_1$ and its sum, $\sigma(x, \lambda)$, is a continuous function in x , locally uniformly bounded in $\lambda \in B_1$. Consider some Borel probability measure ρ on Q , $(L^2) = L^2(Q, d\rho(x))$. This measure is supposed to satisfy the following factorial estimate:

$$\exists C > 0 : \quad \| \| P_n(\cdot) \|_{\mathcal{F}_n(H_{-1})} \|_{(L^2)} \leq C^n n!, \quad n \in \mathbb{Z}_+. \quad (3.2)$$

We will use some special riggings of Fock space (1.6) by weighted spaces. Namely, for fixed rigging (1.4) with $T = \mathbb{N}$ and quasinuclear embedding $O_1 : H_1 \hookrightarrow H$, we introduce the rigging (1.6) with the weights

$$p(q) = (p_n(q))_{n=0}^\infty, \quad p_n(q) = (n!)^2 K^{q_n}, \quad q \in \mathbb{N}, \quad (3.3)$$

and fixed constant $K > \|O_1\|_{HS}^2$.

To construct the space $\mathcal{H}_+$, which will be used later, is convenient to set $p = (K^n)_{n=0}^\infty$. Then the embedding $\mathcal{F}(H_1, p) \hookrightarrow \mathcal{F}(H)$ is quasinuclear (it follows from (1.7)) and we take $\mathcal{H}_+ = \mathcal{F}(H_1, p)$. Of course, $\mathcal{H}_+ \supset \mathcal{F}(H_\tau, p(q))$, $\tau, q \in \mathbb{N}$.

As a result, we will get the following special two-parameter rigging of type (1.6) (with the parameters τ, q)

$$\begin{aligned} \mathcal{F}'_{\text{fin}}(\Phi) \supset \mathcal{F}(H_{-\tau}, p^{-1}(q)) \supset \mathcal{F}(H) \supset \mathcal{F}(H_1, p) \supset \mathcal{F}(H_\tau, p(q)) \supset \mathcal{F}_{\text{fin}}(\Phi), \quad \tau, q \in \mathbb{N}. \\ (\mathcal{F}(H_1, p) = \mathcal{H}_+). \end{aligned} \quad (3.4)$$

By using rigging (3.4), it is possible to construct positive spaces $H(\tau, q)$ of test functions from (L^2) following [4, 5]:

$$\begin{aligned} H(\tau, q) &= \left\{ \varphi(x) = \sum_{n=0}^{\infty} \langle \varphi_n, P_n(x) \rangle, \varphi_n \in \mathcal{F}_n(H_\tau) \mid \|\varphi\|_{H(\tau, q)}^2 \right. \\ &= \left. \sum_{n=0}^{\infty} \|\varphi_n\|_{\mathcal{F}_n(H_\tau)}^2 (n!)^2 K^{qn} < \infty \right\}, \quad \tau, q \in \mathbb{N}. \end{aligned} \quad (3.5)$$

Note that for existence of the embedding $H(\tau, q) \subset (L^2)$, density of $H(\tau, q)$ in (L^2) , etc. it is necessary to impose some additional restrictions on the sequence $(P_n(x))_{n=0}^{\infty}$, see [4, 5], but, in the interesting for us case of sequence (2.27), these properties will be fulfilled automatically.

In [5] the following result was proved.

The space $H(1, 1)$ consists of continuous locally bounded (i.e. bounded on any ball) functions on Q . For every ball $U \subset Q$,

$$\exists A_u > 0 : \quad |\varphi(x)| \leq A_u \|\varphi\|_{H(1, 1)}, \quad x \in U, \quad \varphi \in H^p(1, 1). \quad (3.6)$$

For the negative spaces, constructed by using (3.5) and (L^2) , we get the corresponding chain:

$$H(-\tau, -q) \supset (L^2) \supset H(\tau, q), \quad H(-\tau, -q) = (H(\tau, q))', \quad \tau, q \in \mathbb{N}. \quad (3.7)$$

Inequality (3.6) shows that the δ -function δ_x , concentrated at point $x \in Q$, belongs to $H(-1, -1)$ and its norm, $\|\delta_x\|_{H(-1, -1)}$, is locally bounded with respect to $x \in Q$.

Articles [4, 5] contain a more precise description of the negative spaces (3.7) which is based on the notion of the Appell co-character. In the case under consideration, we can not introduce these objects because we have only the system $(P_n(x))_{n=0}^{\infty}$ and there are no generalized translation operators etc. But, for sequence (2.27), orthogonality relations (2.30), (2.33) hold, and this gives a possibility to repeat corresponding parts of the construction in [5] and to get a similar description with the co-characters (which, roughly speaking, equal now to the functions $P_n(x)$). These results and their applications to the construction in a general situation of S -transform, Wick multiplication etc will be given in a forthcoming publication.

Note that there arises a natural question in this connection, — whether there exists, for given $(P_n(x))_{n=0}^{\infty}$, a family of generalized translation operators, for which $P_n(x)$ are the corresponding Appell characters. For the model one-dimensional case (i.e. for $H = \mathbb{R}^1$), the answer is given in [6].

Let us go back and to apply the above described scheme to the polynomials of the first kind (2.27). So, we have a field (2.2) and construct its spectral representation using chain

(2.9) with the space $\mathcal{H}_+$ from (3.4). Fourier transform (2.24) transfers the positive spaces $\mathcal{F}(H_\tau, p(q))$ onto the spaces $H(\tau, q) \subset (L^2)$ (3.5) with sequence $(P_n(x))_{n=0}^\infty$ of polynomials of the first kind given by (2.27). Here $x \in H_{-2} = Q$, the pairing measure is equal to the spectral measure ρ .

Proposition 3.1 *Let, for the polynomials of the first kind $P_n(x)$, $x \in H_{-2} = Q$, generating function (3.1) exist, i.e. the series (3.1) converge in the required sense.*

Then it is possible to construct chain (3.7) of positive spaces (3.5) and the corresponding negative spaces of functions on H_{-2} . The space $H(1, 1)$ consists of continuous locally bounded functions, $H(-1, -1)$ contains δ_x , $x \in H_{-2}$; the function $H_{-2} \ni x \mapsto \|\delta_x\|_{H(-1, -1)}$ is locally bounded.

Proof. It is necessary only to check estimate (3.2). Let $(e_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty, |\alpha|=n}$ be a BON of type (1.8) in the space $\mathcal{F}_n(H_1)$ constructed from an orthonormal basis $(e_j)_{j=1}^\infty$, $e_j \in \Phi$. By using (2.24), (2.30), we get

$$\begin{aligned} \left\| \|P_n(\cdot)\|_{\mathcal{F}_n(H_{-1})}^2 \right\|_{(L^2)}^2 &= \int_{H_{-2}} \sum_{\alpha \in \mathbb{Z}_{+,0}^\infty, |\alpha|=n} |\langle P_n(x), e_\alpha \rangle|^2 d\rho(x) \\ &= \int_{H_{-2}} \sum_{\alpha \in \mathbb{Z}_{+,0}^\infty, |\alpha|=n} |\hat{e}_\alpha(x)|^2 d\rho(x) \\ &= \sum_{\alpha \in \mathbb{Z}_{+,0}^\infty, |\alpha|=n} \|e_\alpha\|_{\mathcal{F}_n(H)}^2 = \\ &= \|S_n\|_{HS}^2 \leq \|O_1\|_{HS}^{2n}, \end{aligned} \tag{3.8}$$

where S_n is the embedding operator $\mathcal{F}_n(H_1) \hookrightarrow \mathcal{F}_n(H)$ and $O_1 : H_1 \hookrightarrow H$.

The Schwarz inequality and (3.8) give:

$$\left\| \|P_n(\cdot)\|_{\mathcal{F}_n(H_{-1})} \right\|_{(L^2)} \leq \left(\left\| \|P_n(\cdot)\|_{\mathcal{F}_n(H_{-1})}^2 \right\|_{(L^2)} \right)^{1/2} \leq \|O_1\|_{HS}^{n/2}, \quad n \in \mathbb{Z}_+.$$

□

It is fairly difficult to verify that generating function (3.1) for the polynomials of the first kind exists, and therefore the use of Proposition 3.1 is rather limited. Nevertheless, it is possible to prove a similar result without imposing such a restriction but on the functions that are defined on some subset $Q \subset H_{-2}$ such that the measure $\rho(H_{-2} \setminus Q)$ is small.

Theorem 3.1 *For an arbitrary $\varepsilon > 0$ there exists a Borel subset $Q = Q_\varepsilon \subset H_{-2}$ such that $\rho(H_{-2} \setminus Q) < \varepsilon$ and the claim of Proposition 3.1 is still true for the restrictions of the corresponding functions from (3.5) on Q , which is understood as a metric space with the metric induced by the space H_{-2} .*

More exactly, for the positive spaces $H(\tau, q)_Q$ defined to be the sets of restrictions of functions from $H(\tau, q)$ (3.5) on Q and for the corresponding negative spaces $H(-\tau, -q)_Q$ with respect to the space $(L^2)_Q = L^2(Q, d\rho(x))$, the previous results still hold. In particular, the space $H(1, 1)_Q$ consists of continuous locally bounded functions on Q , $H(-1, -1)_Q$ contains δ_x , $x \in Q$; the function $Q \ni x \mapsto \|\delta_x\|_{H(-1, -1)_Q}$ is locally bounded.

Proof. As it follows from the general Theorem 1.1, for ρ -almost every $x \in H_{-2}$, the sequence $P(x) = (P_n(x))_{n=0}^\infty$ belongs to the space $\mathcal{H}_- = (\mathcal{H}_+)' = \mathcal{F}(H_{-1}, p^{-1})$, $p^{-1} = (K^{-n})_{n=0}^\infty$, and therefore,

$$\|P(x)\|_{\mathcal{H}_-}^2 = \sum_{n=0}^{\infty} \|P_n(x)\|_{\mathcal{F}_n(H_{-1})}^2 K^{-n} = M_x^2 < \infty. \quad (3.9)$$

According to (3.9), the Borel sets $Q_M = \{x \in H_{-2} \mid \|P(x)\|_{\mathcal{H}_-} < M\}$, $M \in \mathbb{N}$, tend to H_{-2} in measure ρ for $M \rightarrow \infty$, and hence, for $\varepsilon > 0$ there is $M = M_\varepsilon$ such that $\rho(H_{-2} \setminus Q_M) < \varepsilon$. Set $Q_\varepsilon = Q_{M_\varepsilon}$.

It follows from (3.9) that $\forall n \in \mathbb{Z}_+$

$$\|P_n(x)\|_{\mathcal{F}_n(H_{-1})} \leq M_\varepsilon K^{n/2}, \quad x \in Q_\varepsilon. \quad (3.10)$$

Estimate (3.10) shows that, for $x \in Q_\varepsilon$, series (3.1) is convergent uniformly with respect to these x and λ from an arbitrary fixed ball $B_1 \subset H_1$. Its sum, $\sigma(x, \lambda)$, is continuous with respect to $x \in Q_\varepsilon$ (because every $P_n(x)$, $x \in H_{-2}$, is continuous) and bounded ($\lambda \in B_1$). So, the generating function $\sigma(x, \lambda)$, $x \in Q_\varepsilon$, $\lambda \in B_1$, exists.

Estimate (3.2) gives a similar estimate, in which (L^2) is replaced with $(L^2)_Q$ because $\|\cdot\|_{(L^2)_Q} \leq \|\cdot\|_{(L^2)}$. The spaces $H(\tau, q)_Q$ are well defined by restrictions of (3.5) to Q_ε , and $H(-\tau, -q)_Q$ are dual spaces with respect to $(L^2)_Q$. The last assertion of the theorem follows, as in Proposition 3.1, from existence of the generating function in our case. $\square$

4 Some examples

In what follows, we give a scheme for constructing three simple enough examples of Jacobi fields and construct their spectral representations. For more details concerning the first two classical examples see [7, 9, 10, 15, 17], and for the third one — [10, 15].

4.1 Classical field and Gaussian measure

The classical creation operator $a_+(h)$ is defined, for an arbitrary $h \in H$, on every n -particles subspace $\mathcal{F}_n(H)$, $n \in \mathbb{Z}_+$, by the formula

$$\mathcal{F}_n(H) \ni f_n \mapsto a_+(h)f_n = \sqrt{n+1}h \widehat{\otimes} f_n \in \mathcal{F}_{n+1}(H) \quad (4.1)$$

and, as it is seen straight away, it is bounded. The corresponding annihilation operator, $a_-(h)$, is defined by the formula $a_-(h) = (a_+(h))^* : \mathcal{F}_{n+1}(H) \mapsto \mathcal{F}_n(H)$. Of course these operators are also bounded if considered as operators acting between the spaces $\mathcal{F}_n(H_1)$ and $\mathcal{F}_{n+1}(H_1)$, where H_1 is taken from chain (2.1) (we suppose it to be fixed), and $h \in H_1$.

The field of operators in $\mathcal{F}(H)$ (2.2), where $\forall \varphi \in \Phi$,

$$\begin{aligned} \mathcal{F}_n(H_1) \ni f_n \mapsto a_n(\varphi)f_n &= a_+(\varphi)f_n \in \mathcal{F}_{n+1}(H_1), \quad b_n(\varphi) = 0, \\ \mathcal{F}_{n+1}(H_1) \ni f_{n+1} \mapsto c_n(\varphi)f_{n+1} &= a_-(\varphi)f_{n+1} \in \mathcal{F}_n(H_1), \quad n \in \mathbb{Z}_+, \end{aligned} \quad (4.2)$$

is equal to the classical free field. It is known that the operators $A(\varphi)$ are essentially self-adjoint and commuting.

From formula (2.7) and formulas (4.2), (4.1), it follows that, in our case,

$$V_{n,n} = \sqrt{n!} \text{Id}, \quad n \in \mathbb{Z}_+, \quad (4.3)$$

where we understand the operator Id as the embedding operator $\mathcal{F}_n(H_2) \hookrightarrow \mathcal{F}_n(H_1)$.

Fix an orthonormal basis $(e_j)_{j=1}^\infty$, $e_j \in \Phi$, and construct the corresponding BON $(e_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty}$. By using equation (2.25) and the form of the operators $A(\varphi)$, it is easy to calculate that, for the functions (2.31), the following representation holds:

$$\begin{aligned} (\hat{e}_\alpha)(x) &= \langle e_\alpha, P_{|\alpha|}(x) \rangle = h_{\alpha_1}((x, e_1)_H) \cdots h_{\alpha_n}((x, e_n)_H), \\ x &\in H_{-2}, \quad \alpha = (\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in \mathbb{Z}_{+,0}^\infty, \end{aligned} \quad (4.4)$$

where $(h_k(\lambda))_{k=0}^\infty$ is the sequence of orthonormal Hermite polynomials on $\mathbb{R}^1$ with respect to the Gaussian measure $d\gamma_{1/2}(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda$.

By using (4.4), we can write, in our case, a representation of Fourier transform (2.24) via Hermite polynomials. The spectral measure $d\rho(x)$ now is the canonical Gaussian measure:

$$d\rho(x) = (d\gamma_{1/2}(x_1)) \times (d\gamma_{1/2}(x_2)) \times \cdots, \quad x \in H_{-2}, \quad x_n = (x, e_n)_H, \quad n \in \mathbb{N}. \quad (4.5)$$

4.2 Perturbed classical field and Poisson measure

Introduce the field of operators $A(\varphi)$, $\varphi \in \Phi$, which are defined as in Section 4.1 but with another, not equal to zero, main diagonal $(b_n(\varphi))_{n=0}^\infty$.

Namely, let now chain (2.1) have the form

$$S' \supset S_{-2} \supset S_{-1} \supset L^2(\mathbb{R}^1, dt) \supset S_1 \supset S_2 \supset S$$

where $S = \Phi$ is the Schwarz space of rapidly decreasing, infinite differentiable functions on $\mathbb{R}^1$ and $S_1 = H_1$, $S_2 = H_2$ are the corresponding Sobolev spaces with quasinuclear embeddings $S_1 \hookrightarrow L_2(\mathbb{R}^1, dt) = H$, $S_2 \hookrightarrow S_1$. For $\varphi \in S$, we define the operator $b_n(\varphi)$, $n \in \mathbb{N}$, by the formula

$$\begin{aligned} \mathcal{F}_n(H_1) \ni f_n &\mapsto b_n(\varphi) f_n = \\ &= \left(b(\varphi) \otimes \text{Id} \otimes \cdots \otimes \text{Id} + \cdots + \text{Id} \otimes \cdots \otimes \text{Id} \otimes b(\varphi) \right) f_n \in \mathcal{F}_n(H_1), \end{aligned} \quad (4.6)$$

where $b(\varphi)$ is defined in H as the operator of multiplication on $\varphi: H \ni f(\cdot) \mapsto b(\varphi)f(\cdot) = \varphi(\cdot)f(\cdot) \in H$; $b_0(\varphi) = 0$ (in other words $b_n(\varphi)$ is equal to the differential second quantization of the operator $b(\varphi)$, restricted to $\mathcal{F}_n(H_1)$).

The operators $a_n(\varphi)$ and $c_n(\varphi)$ are the same as in (4.2). So, our field of operators $A(\varphi)$ have the form

$$A(\varphi) = A_{\text{free}}(\varphi) + B(\varphi), \quad \varphi \in S = \Phi, \quad (4.7)$$

where $A_{\text{free}}(\cdot)$ is the free field, introduced by (4.2), and the neutral $B(\varphi)$ is its diagonal perturbation, which is given by (4.6).

It is possible to prove [15] that operators (4.7) are essentially self-adjoint and commuting. Operators $V_{n,n}$ have the form as in (4.3) because their definition does not depend on $B(\varphi)$, see (2.7). The spectral measure ρ now is equal to the Poisson centered measure π on $\mathcal{S}'$ which is given by its Fourier transform: $\forall \varphi \in \mathcal{S}$

$$\int_{\mathcal{S}'} e^{i(x,\varphi)_H} d\pi(x) = \exp \left[\int_{\mathbb{R}^1} (e^{i\varphi(t)} - 1 - i\varphi(t)) dt \right]. \quad (4.8)$$

Note that measure π , defined by (4.8), cannot be represented, similar to (4.5), as a product measure. The corresponding polynomials of the first kind (2.27) can be represented via Charlier polynomials [17].

4.3 Field, constructed by means of Jacobi matrices

We now describe a general procedure for constructing field (2.2) by using a sequence of classical Jacobi matrices. In this case, the spectral measure ρ is a product measure of type (4.5).

Let us consider a sequence of infinite Jacobi matrices $(J_j)_{j=1}^\infty$ of the form

$$J_j = \begin{pmatrix} b_0(j) & a_0(j) & 0 & 0 & 0 & \dots \\ a_0(j) & b_1(j) & a_1(j) & 0 & 0 & \dots \\ 0 & a_1(j) & b_2(j) & a_2(j) & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}, \quad (4.9)$$

$$a_i(j) > 0, \quad b_i(j) \in \mathbb{R}^1, \quad i \in \mathbb{Z}_+.$$

In the usual manner, every Jacobi matrix J_j determines, in the Hilbert space H_j spanned by an orthonormal basis $(k_i^{(j)})_{i=0}^\infty$, a linear operator A_j with the domain $\text{Dom}(A_j) = \text{l.s.}(k_i^{(j)})_{i=0}^\infty$.

The operators A_j are clearly Hermitian. In addition we suppose that all the operators A_j are essentially selfadjoint (for conditions on elements of matrix (4.9) that ensure its essential selfadjointness, see, e.g., in [1]).

By using results of Section 1.4, we construct the infinite tensor product $\mathcal{H}^\otimes = \bigotimes_{j=1; k}^\infty H_j$ with a stabilizing sequence $k = (k_0^{(j)})_{j=1}^\infty$. Every operator A_j defines, by formula (1.11), an operator $\mathcal{A}_j$ in the space $\mathcal{H}^\otimes$:

$$\mathcal{A}_j = 1 \otimes \dots \otimes 1 \otimes A_j \otimes 1 \otimes 1 \otimes \dots. \quad (4.10)$$

On the other hand, introduce the Fock space $\mathcal{F}(H)$ by using some fixed real Hilbert space H . The operator K , introduced by (1.10), is unitary and maps $\mathcal{H}^\otimes$ onto $\mathcal{F}(H)$: $K \mathcal{H}^\otimes = \mathcal{F}(H)$. Using K , we construct the images of operators (4.10) which act in $\mathcal{F}(H)$:

$$A(e_j) = K \mathcal{A}_j K^{-1}, \quad j \in \mathbb{N}. \quad (4.11)$$

It should be explained that we have introduced, into the Fock space $\mathcal{F}(H)$, the BON (1.8), where $(e_j)_{j=1}^\infty$ is a fixed orthonormal basis in H . We use in (4.11) convenient notations $A(e_j)$.

As it follows from (1.13), $\text{Dom}(\mathcal{A}_j) = \text{l.s.}(k_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty}$, and therefore, according to (4.11),

$$\text{Dom}(A(e_j)) = \text{l.s.}(e_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty}, \quad j \in \mathbb{N}. \quad (4.12)$$

By Theorem 1.2, the operators $\mathcal{A}_j$ are essentially selfadjoint and their closures commute. Therefore due to (4.11), the operators $A(e_j)$, $j \in \mathbb{N}$, in the space $\mathcal{F}(H)$ have the same properties.

As a result, we have constructed a family of operators $(A(e_j))_{j=1}^\infty$ but we needed to construct a field of operators $A(\varphi)$, where φ runs over the linear space Φ from some rigging (2.1). To do that, we will view the space H as the space l_2 using the above introduced orthonormal basis $(e_j)_{j=1}^\infty$, and construct rigging (2.1) as a rigging of the space l_2 . Every $\varphi \in \Phi$ has the form $\varphi = \sum_{j=1}^\infty \varphi_j e_j$ with some conditions on the coordinates $\varphi_j \in \mathbb{C}^1$ and it is natural to put

$$A(\varphi) = A\left(\sum_{j=1}^\infty \varphi_j e_j\right) = \sum_{j=1}^\infty \varphi_j A(e_j). \quad (4.13)$$

Remark that a simple calculation gives the following formula for the action of $A(e_j)$ on BON (1.8): $\forall \alpha \in \mathbb{Z}_{+,0}^\infty$,

$$\begin{aligned} A(e_j)e_\alpha &= a_{\alpha_j-1}(j)e_{\alpha-1_j} + b_{\alpha_j}(j)e_\alpha + a_{\alpha_j}(j)e_{\alpha+1_j}, \\ \alpha \pm 1_j &= (\alpha_1, \dots, \alpha_{j-1}, \alpha_j \pm 1, \alpha_{j+1}, \dots), \quad a_{-1}(j) = 0, \quad j \in \mathbb{N}. \end{aligned} \quad (4.14)$$

Theorem 4.1 *Let*

$$d_j = \max_{i=0,\dots,j} \{a_i(j), |b_i(j)|\}, \quad j \in \mathbb{Z}_+. \quad (4.15)$$

If the coordinates $\varphi_j \in \mathbb{C}^1$ satisfy the condition

$$\sum_{j=1}^\infty |\varphi_j| d_j < \infty, \quad (4.16)$$

then the operator $A(\varphi)$ of form (4.13) is well defined in the Fock space $\mathcal{F}(H)$, and it is essentially self-adjoint in the domain (4.12). The closures of these operators commute, are defined on $\mathcal{F}_{\text{fin}}(H)$ and continuous as operators $\mathcal{F}_{\text{fin}}(H) \rightarrow \mathcal{F}(H)$. The operator $A(\varphi)$ has Jacobi structure (2.2) in $\mathcal{F}(H)$.

Proof. We will use the following result [2]. Let $\mathcal{B}(\mathbb{R}^\infty)$ be a Borel σ -algebra of sets from $\mathbb{R}^\infty$ (with coordinate-wise convergence) and $\mathcal{B}(\mathbb{R}^\infty) \ni \alpha \mapsto \mathcal{E}(\alpha)$ be the joint resolution of identity of the commuting family $(\tilde{\mathcal{B}}_j)_{j=1}^\infty$ of self-adjoint operators in the space $\mathcal{H}^\otimes$ (see Theorem 1.2). Fix $\varphi_j \in \mathbb{C}^1$ such that

$$\sum_{j=1}^\infty |\varphi_j| \|B_j k_0^{(j)}\|_{H_j} < \infty. \quad (4.17)$$

Then the function

$$\mathbb{R}^\infty \ni x = (x_j)_{j=1}^\infty \mapsto L(x) = \sum_{j=1}^\infty \varphi_j x_j$$

is well defined and finite for $\mathcal{E}$ -almost all $x \in \mathbb{R}^\infty$, so that we can define a selfadjoint operator

$$\begin{aligned} \tilde{\mathcal{L}}(\varphi) &= \int_{\mathbb{R}^\infty} L(x) d\mathcal{E}(x), \\ \text{Dom}(\tilde{\mathcal{L}}(\varphi)) &= \left\{ f \in \mathcal{H}^\otimes \mid \int_{\mathbb{R}^\infty} |L(x)|^2 d(\mathcal{E}(x)f, f)_{\mathcal{H}^\otimes} < \infty \right\}. \end{aligned} \quad (4.18)$$

The operator $\tilde{\mathcal{L}}(\varphi)$ coincides with the closure of the essentially selfadjoint operator $\mathcal{L}(\varphi)$ defined by

$$\mathcal{L}(\varphi)f = \sum_{j=1}^\infty \varphi_j \mathcal{B}_j f, \quad f \in \text{Dom}(\mathcal{L}(\varphi)) = a. \bigotimes_{j=1; k}^\infty \text{Dom}(\mathcal{B}_j), \quad (4.19)$$

where the series converges in $\mathcal{H}^\otimes$.

Apply this result to the family $(\tilde{A}_j)_{j=1}^\infty$. Condition (4.17) now has the form

$$\sum_{j=1}^\infty |\varphi_j| \|A_j k_0^{(j)}\|_{H_j} = \sum_{j=1}^\infty |\varphi_j| (a_0^2(j) + b_0^2(j))^{1/2} < \infty$$

and follows from assumption (4.16). Therefore the operator $\mathcal{L}(\varphi)f = \sum_{j=1}^\infty \varphi_j \mathcal{A}_j$ is well defined and essentially selfadjoint on domain (4.19), which in our case is equal to l.s. $(k_\alpha)_{\alpha \in \mathbb{Z}_{+,0}^\infty}$.

By using K , we can pass from the space $\mathcal{H}^\otimes$ to $\mathcal{F}(H)$ and from the operators $\mathcal{A}_j$ to $A(e_j)$. The image of the operator $\mathcal{L}(\varphi)$ is equal to (4.13) and its domain is mapped onto (4.12). Thus the assertions of the theorem follows from the cited result in [2] with an exception of the last three facts in its formulation.

To finish the proof, it is necessary to check that $A(\varphi)$ is defined on $\mathcal{F}_{\text{fin}}(H)$ and continuously maps, $\forall n \in \mathbb{Z}_+$, the space $\mathcal{F}_n(H)$ into the orthogonal sum $\mathcal{F}_{n-1}(H) \oplus \mathcal{F}_n(H) \oplus \mathcal{F}_{n+1}(H)$ ($\mathcal{F}_{-1}(H) = 0$).

From (4.14) and (4.9), it follows that such a property holds for every operator $A(e_j)$, $j \in \mathbb{N}$, if it is extended by continuity to $\mathcal{F}_n(H)$, and that the following estimate for the corresponding norm takes place:

$$\|A(e_j) \upharpoonright \mathcal{F}_n(H)\| \leq 3d_n, \quad n \in \mathbb{Z}_+ \quad (4.20)$$

(one should use definition (4.15) and that, for $\alpha \in \mathbb{Z}_{+,0}^\infty$, $|\alpha| = n$, e_α form an orthonormal basis in $\mathcal{F}_n(H)$). Therefore definition (4.13), estimate (4.20), and condition (4.16) show that this property holds for $A(\varphi)$. $\square$

Construct rigging (2.1) in terms of coordinates in H with respect to the orthonormal basis $(e_j)_{j=1}^\infty$:

$$\begin{aligned} \mathbb{R}^\infty \supset l_2(p_{(2)}^{-1}) \supset l_2(p_{(1)}^{-1}) \supset l_2 \supset l_2(p_{(1)}) \supset l_2(p_{(2)}) \supset l_{2,0} \\ (l_2 = H, \quad l_2(p_1) = H_1, \quad l_2(p_2) = H_2, \quad l_{2,0} = \Phi), \end{aligned} \quad (4.21)$$

where $p_{(1)} = (p_{(1),j})_{j=1}^{\infty}$, $p_{(2)} = (p_{(2),j})_{j=1}^{\infty}$, $p_{(2),j} \geq p_{(1),j} \geq 1$ are two weights with the conditions

$$\sum_{j=1}^{\infty} p_{(1),j}^{-1} < \infty, \quad \sum_{j=1}^{\infty} p_{(1),j} p_{(2),j}^{-1} < \infty, \quad (4.22)$$

$l_{2,0}$ is the space of finite sequences and $l_2(p)$ denotes the weighted by $p = (p_j)_{j=1}^{\infty}$ space l_2 ($l_2(p) = \mathcal{F}(\mathbb{R}^1, p)$). Conditions (4.22) are equivalent to the fact that the embeddings $H_1 \hookrightarrow H$ and $H_2 \hookrightarrow H_1$ are quasinuclear.

Assume also that the growth of $p_{(1),j}$, when $j \rightarrow \infty$, is so large that

$$\sum_{j=1}^{\infty} d_j^2 p_{(1),j}^{-1} < \infty. \quad (4.23)$$

Then, for $\varphi = (\varphi_j)_{j=1}^{\infty} \in H_1$,

$$\sum_{j=1}^{\infty} |\varphi_j| d_j \leq \left(\sum_{j=1}^{\infty} |\varphi_j|^2 p_{(1),j} \right)^{1/2} \left(\sum_{j=1}^{\infty} d_j^2 p_{(1),j}^{-1} \right)^{1/2} < \infty,$$

i.e. the vectors φ from H_1 satisfy condition (4.16), and therefore, by Theorem 4.1, the operators $A(\varphi)$, $\varphi \in H_1$, are well defined and have the corresponding properties of self-adjointness, commutativity and continuity.

Theorem 4.2 *Let $(J_j)_{j=1}^{\infty}$ be a sequence of Jacobi matrices J_j generating a selfadjoint operator A_j in $H = l_2$. Introduce the numbers d_j by (4.15) and assume that the weights $p_{(1)} = (p_{(1),j})_{j=1}^{\infty}$, $p_{(2)} = (p_{(2),j})_{j=1}^{\infty}$, $p_{(2),j} \geq p_{(1),j} \geq 1$ satisfy the conditions*

$$\sum_{j=1}^{\infty} d_j^2 p_{(1),j}^{-1} < \infty, \quad \sum_{j=1}^{\infty} p_{(1),j} p_{(2),j}^{-1} < \infty. \quad (4.24)$$

Construct chain (4.21), i.e. (2.1).

Then the family of essentially selfadjoint commuting operators

$$A(\varphi) = A\left(\sum_{j=0}^{\infty} \varphi_j e_j\right) = \sum_{j=0}^{\infty} \varphi_j A(e_j), \quad \varphi \in \Phi = l_{2,0}, \quad (4.25)$$

defined in $\mathcal{F}(H)$ on $\mathcal{F}_{\text{fin}}(\Phi)$ forms a Jacobi field and axioms [F.1] and [F.2] hold.

Proof. First of all note that $\exists c > 0 : d_j \geq c > 0, j \in \mathbb{N}$. Therefore, conditions (4.24) imply (4.22), (4.23), and we can construct chain (4.21), i.e. (2.1).

As it was explained in Theorem 4.1, operators (4.25) are essentially selfadjoint, commuting, act continuously on $\mathcal{F}_{\text{fin}}(H)$, and have Jacobi structure (note that $\varphi \in H_1 \supset \Phi$ satisfies (4.16), $\mathcal{F}_{\text{fin}}(\Phi)$ is equal to (4.12)).

Of course $A(\varphi)$, $\varphi \in \Phi$, have also the Jacobi structure (2.2) with respect to $\mathcal{F}(H_1)$. The entries $a_n(\varphi)$, $b_n(\varphi)$ and $c_n(\varphi)$ of matrix (2.2) act continuously in the corresponding spaces $\mathcal{F}_j(H_1)$. The proof of this assertion follows from the corresponding part of the proof

of Theorem 4.1. It is necessary only to note that, in the estimate of type (4.20), the quantity $3d_n$ will be replaced by another because it is necessary to estimate the norm of the operator $A(e_j)$ acting from $\mathcal{F}_n(H_1)$ into $\mathcal{F}_{n-1}(H_1) \oplus \mathcal{F}_n(H_1) \oplus \mathcal{F}_{n+1}(H_1)$ and H_1 differs from H by the weight $p_{(1)}$. But now $\varphi \in \Phi = l_{2,0}$, i.e. it is finite, and such a change of $3d_n$ does not influence the continuity of entries of $A(\varphi)$.

Thus, axiom [F.1] is satisfied.

Axiom [F.2] is fulfilled by the construction. $\square$

Here we will not investigate neither the fulfilment of axiom [F.3] nor the conditions which would imply its fulfilment. Such a study is rather cumbersome for it is necessary to write the operator $V_{n,n}$ in the BON of the spaces $\mathcal{F}_n(H_1)$ and $\mathcal{F}_n(H_2)$.

Note, however, that in the case of matrices J_j such that, for every $j \in \mathbb{N}$, $a_n(j) = \sqrt{n+1}$, $n \in \mathbb{Z}_+$, axiom [F.3] is satisfied.

In fact, if also $\forall j \in \mathbb{N}$, $b_n(j) = 0$, $n \in \mathbb{Z}_+$, then, as it is easy to calculate, $A(\varphi)$ is a free field $A_{\text{free}}(\varphi)$, and therefore $V_{n,n} = \sqrt{n!} \text{Id}$, see (4.3). The diagonals $(b_n(j))_{n=0}^\infty$ form the neutral operator $B(\varphi)$ which perturbs $A_{\text{free}}(\varphi)$ and does not influence $V_{n,n}$. So, in our case $V_{n,n} = \sqrt{n!} \text{Id}$, $n \in \mathbb{Z}_+$.

For field (4.25), $\Phi' = \mathbb{R}^\infty$, $H_{-2} = l_2(p_{(2)}^{-1})$ and $H_1 = l_2(p_{(1)})$. It is possible to calculate that in this case the Fourier transform in the form (2.34) can be expressed via polynomials of the first kind $(P_k(\lambda, j))_{k=0}^\infty$ generated by the matrix J_j , $j \in \mathbb{N}$. Namely, we have

$$\begin{aligned} (\hat{e}_\alpha)(x) &= \langle e_\alpha, P_{|\alpha|}(x) \rangle = P_{\alpha_1}(x_1, 1) \cdots P_{\alpha_n}(x_n, n), \\ x &\in H_{-2} = l_2(p_{(2)}^{-1}), \quad \alpha = (\alpha_1, \dots, \alpha_n, 0, 0, \dots) \in \mathbb{Z}_{+,0}^\infty. \end{aligned} \tag{4.26}$$

Formula (4.4) is, of course, a particular case of (4.26).

The spectral measure ρ for field (4.25) is the product

$$\rho = \rho_1 \times \rho_2 \times \cdots, \tag{4.27}$$

where ρ_j is the spectral measure connected with the matrix J_j .

The proof of (4.27) follows from formula (4.18) written for the field $\tilde{A}(\varphi) = \tilde{\mathcal{L}}(\varphi)$: the resolution of identity $\mathcal{E}$ is equal to the tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \cdots$, where $\mathcal{E}_j$ is the resolution of identity of the operator $\mathcal{A}_j$ (4.10) (recall that the operators $\mathcal{A}_j$ (4.10) act on “different variables”).

References

- [1] Yu. M. Berezansky, *Expansion in Eigenfunctions of Selfadjoint Operators*, AMS, Providence, R. I., 1968 (Russian edition: Naukova Dumka, Kiev, 1965).
- [2] Yu. M. Berezansky, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, AMS, Providence, R. I., 1986 (Russian edition: Naukova Dumka, Kiev, 1978).

- [3] Yu. M. Berezansky, *Spectral approach to white noise analysis*, Proceedings of Symposium "Dynamics of Complex and Irregular Systems", 16–20 December 1991, Germany. In: Bielefeld Encounters in Mathematics and Physics VIII, 1993, World Scientific, Singapore–New Jersey–London– Hong Kong, 131–140.
- [4] Yu. M. Berezansky, *Infinite-dimensional non-Gaussian analysis and generalized translation operators*, Funkts. Anal. Prilozhen. **30**, no. 4 (1996), 61–65.
- [5] Yu. M. Berezansky, *Infinite dimensional analysis connected with generalized translation operators*, Ukr. Mat. Zh. **49**, no. 3 (1997), 364–409.
- [6] Yu. M. Berezansky, *Construction of the operators of generalized translation by Appel characters*, Proceedings of works, devoted to S. G. Krein, World Scientific, 1997 (in print).
- [7] Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral Methods in Infinite Dimensional Analysis*, v. 1, 2, Kluwer, Dordrecht–Boston– London, 1995 (Russian edition: Naukova Dumka, Kiev, 1988).
- [8] Yu. M. Berezansky and V. D. Koshmanenko, *An axiomatic field theory in terms of operator Jacobian matrices*, Soviet Physics. Doklady **14** (1968/70), 1064–1066.
- [9] Yu. M. Berezansky, V. O. Livinsky, and E. W. Lytvynov, *Spectral approach to white noise analysis*, Ukr. Mat. Zh. **46**, no. 3 (1994), 177–197.
- [10] Yu. M. Berezansky, V. O. Livinsky, and E. W. Lytvynov, *A generalization of Gaussian white noise analysis*, Methods Funct. Anal. Topology **1**, no. 1 (1995), 28–55.
- [11] E. Brüning, *On Jacobi-fields*, Colloquia Mathematica Societatis Janos Bolyai, Random Fields (1979), 171–196.
- [12] T. Hida, H.-H. Kuo, J. Potthof, and L. Streit, *White Noise. An Infinite Dimensional Calculus*, Kluwer, Dordrecht–Boston–London, 1993.
- [13] M. G. Krein and M. A. Krasnoselskii, *Basic theorems on extensions of Hermitian operators and certain their applications to the theory of orthogonal polynomials and moment problem*, Uspekhi Mat. Nauk **2**, no. 3 (1947), 60–106.
- [14] E. W. Lytvynov, *Spectral approach to multiple Wiener integrals*, Doklady Akademii Nauk Ukrainy (1994), no. 11, 29–32.
- [15] E. W. Lytvynov, *Multiple Wiener integrals and non-Gaussian white noise: a Jacobi field approach*, Methods Funct. Anal. Topology **1**, no. 1 (1995), 61–85.
- [16] L. Nachbin, *Topology on Spaces of Holomorphic Mappings*, Springer, Berlin–Heidelberg–New York, 1969.
- [17] G. F. Us, *Dual Appel systems in Poisson analysis*, Methods Funct. Anal. Topology **1**, no. 1 (1995), 93–108.

Institute of Mathematics
Ukrainian Academy of Sciences
3 Tereshchinkivs'ka St.,
252601 Kiev,
Ukraine

e-mail: berezan@mathber.carrier.kiev.ua

1991 Mathematics Subject Classification: 60G20, 46F25, 60J30

Preprints are available at: <http://www-sfb288.math.tu-berlin.de>
List of most recent 50 sfb288 preprints

- 209 Ch. Kreft, R. Seiler: Models of the Hofstadter Type
- 210 M. Lüdde: A Cellular Braid Action and the Yang-Baxter Equation
- 211 K. Große-Braukmann, K. Polthier: Constant Mean Curvature Surfaces Derived from Delaunay's and Wente's Examples
- 212 H. Baum, I. Kath: Normally hyperbolic operators, the Huygens property and conformal geometry
- 213 V. Bach, J. Fröhlich, I. Sigal: Quantum Electrodynamics of Confined Non-Relativistic Particles
- 214 H. Baum: The Dirac operator on Lorentzian spin manifolds and the Huygens property
- 215 A.G. Bytsko: Fusion of q-tensor operators: quasi-Hopf-algebraic point of view
- 216 M. Karowski, R. Schrader, E. Vogt: Invariants of 3-manifolds, unitary representations of the mapping class group and numerical calculations
- 217 P. Nurowski: Twistor Bundles, Einstein Equations and real Structures
- 218 U. Eitner: Schlesinger Transformations for Bonnet Surfaces
- 219 A. Schmitt: The Energy Function of the Skyrme Model
- 220 W. Müller, M. Schmidt, R. Schrader: Hyperelliptic Riemann surfaces of infinite genus and solutions of the KdV equation
- 221 J. Brüning: The Signature Theorem for Manifolds with Metric Horns
- 222 J. Brüning, M. Lesch: On the eta-invariant of certain non-local boundary value problems
- 223 J. Dorfmeister, G. Haak: On constant mean curvature surfaces with periodic metric
- 224 C. Figurei de Morisson Faria, A. Fring, R. Schrader: On the Influence of Pulse Shapes on Ionization Probability
- 225 B.D. Dörfel, S. Meißner: Finite-size corrections of an integrable chain with alternating spins
- 226 G. Haak: On a theorem by do Carmo and Dajczer
- 227 B. Schroer: Notes on the Wigner Representation Theory of the Pioncaré Group, Localization and Statistics
- 228 B. Schroer: Motivations and Physical Aims of Algebraic QFT
- 229 J. Kellendonk: Topological Equivalence of Tilings
- 230 Thomas Schmitt: The Cauchy Problem for Abstract Evolution Equations with Ghost and Fermion Degrees of Freedom
- 231 N. Kutz: Free massive fermions inside the quantum discrete sine-Gordon model
- 232 A. Knauf, Y.G. Sinai: Classical Nonintegrability, Quantum Chaos
- 233 M. Pflaum, M. Schottenloher: Holomorphic deformation of Hopf algebras and applications to quantum groups
- 234 F. Nill: On the Structure of Momodromy Algebras and Drinfeld Doubles

- 235 H. Babujian, M. Karowski, A. Zapletal: $SU(N)$ Matrix Difference Equations and a Nested Bethe Ansatz
- 236 H. Babujian, M. Karowski, A. Zapletal: $U(N)$ Matrix Difference Equations and a Nested Bethe Ansatz
- 237 J. Weber: Relation between Morse Theory of closed Geodesics of S^1 and Symplectic Floer Theory of T^*S^1
- 238 A. Bytsko, V. Schomerus: Vertex Operators - From a Toy Model to Lattice Algebras
- 239 U. Hertrich-Jeromin, T. Hoffmann, U. Pinkall: A discrete version of the Darboux transform for isothermic surfaces
- 240 A. Moroianu: On the Infinitesimal Isometries of Manifolds with Killing Spinors
- 241 A. Moroianu: Parallel and Killing Spinors on $Spin^c$ Manifolds
- 242 A. Belavin, A. Fring: On the Fermionic Quasi-particle Interpretation in Minimal Models of Conformal Field
- 243 A. Moroianu: Complex Contact Structures and $Spin^c$ Manifolds
- 244 H. Baumgaertel, F. Lledo: Superselection structures for C^* -algebras with nontrivial center
- 245 M.U. Schmidt, A.P. Veselov: Elliptic Calogero Moser quantum problem and deformations of algebraic surfaces
- 246 C. Binnenhei: O Realized on Bose Fock Space
- 247 K. Mohnke: On Fillings by Holomorphic Discs of the Levels of a Morse Function
- 248 E. Akkermans, J.E. Avron, R. Narevich, R. Seiler: Boundary Conditions for Bulk and Edge States in Quantum Hall Systems
- 249 J.C.A. Barata, F. Nill: Dyonic Sectors and Intertwiner Connections in 2+1-dimensional Lattice ZN-Higgs Models
- 250 H. Baum: Strictly pseudoconvex spin manifolds, Fefferman spaces and Lorentzian twistor spinors
- 251 T. Friedrich, A. Trautman: Clifford structures and spinor bundles
- 252 Piotr G. Grinevich, Martin U. Schmidt: Conformal invariant functionals of immersions of tori R^3
- 253 Ines Kath: Pseudo-Riemannian T-duals of compact Riemannian reductive spaces
- 254 P. G. Grinevich, M. U. Schmidt: Closed curves in R^3 : a characterization in terms of curvature and torsion, the Hasimoto map and periodic solutions of the Filament Equation
- 255 D. Guido, R. Longo, H.-W. Wiesbrock: Extensions of Conformal Nets and Superselection Structures
- 256 F. Duzaar, E. Kuwert: Minimization of Conformally Invariant Energies in Homotopy Classes
- 257 F. Duzaar, A. Gastel: Minimizing p -Harmonic Maps at a Free Boundary
- 258 Y. M. Berezansky: Commutative Jacobi Fields in Fock Space