Cosmological Perturbations seeded by Topological Defects:
Setting the Initial Conditions

Nathalie Deruelle$^{1,2}$, David Langlois$^1$ and Jean-Philippe Uzan$^1$

$^1$ Département d’Astrophysique Relativiste et de Cosmologie, UPR 176 du Centre National de la Recherche Scientifique, Observatoire de Paris, 92195 Meudon, France

$^2$ DAMTP, University of Cambridge, Silver Street, Cambridge, CB3 9EW, England

July 9th, 1997

Pacs Numbers : 98.80.Cq, 98.70.Vc

Abstract

We consider a perfectly homogeneous and isotropic universe which undergoes a sudden phase transition. If the transition produces topological defects, which we assume, perturbations in the geometry and the cosmic fluid also suddenly appear. We apply the standard general relativistic junction conditions to match the pre- and post- transition eras and thus set the initial conditions for the perturbations. We solve their evolution equations analytically in the case when the defects act as a coherent source and their density scales like the background density. We show that isocurvature as well as adiabatic perturbations are created, in a ratio which is independent of the detailed properties of the defects. We compare our result to the initial conditions currently used in the literature and show how the cosmic fluid naturally “compensates” for the presence of the defects.
1. Introduction

The question of the generation of the small cosmological inhomogeneities at the origin of the large scale structures observed in the universe is still open. Two main approaches, each with its troop of specific models, are currently in competition. On one hand the inflationary scenario (see e.g. [1]) explains cosmological inhomogeneities by the amplification, due to accelerated cosmic expansion, of inescapable quantum fluctuations. On the other hand, the topological defect scenario, which is based on the idea of spatially differentiated spontaneous symmetry breaking (see e.g. [2]), explains these inhomogeneities by the appearance of topological defects which drive fluctuations in other types of matter. Future observations of the small scale anisotropies of the CMBR (Cosmic Microwave Background Radiation), in particular by the planned MAP and PLANCK satellite missions, should discriminate between the two scenarios.

In the present work, we shall be interested in the second one. Whereas the calculation of cosmological perturbations in inflationary models, at least the simplest ones, seems to have now reached a stage of maturity and clarity, there is still some confusion in the literature on perturbations seeded by topological defects, even on the question of how to set the initial conditions. By initial conditions, we mean, as is usual in cosmology, the state of cosmological perturbations for the various matter species at a past epoch during the radiation era when all scales of cosmological interest today were larger than the Hubble radius. Although far in our past, this initial epoch is taken in general very long after the phase transition supposed to have given birth to the topological defects. The reason is that it is difficult to trace numerically the evolution of perturbations on a very long duration. The problem is thus to translate the “starting” conditions imposed by the phase transition into “initial” conditions at the time the numerical computations begin.

At this “initial” time, the topological defects are thus supposed to already exist. The main difficulty is then to set the initial value of the perturbations of the cosmological fluids which are compatible with the distribution of the defects. Several methods have been developed to determine and then implement these compatibility conditions: “integral constraints”, “compensation”, “pseudotensor”... Unfortunately, these methods are rather intricate and confusing...

The purpose of the present paper is to present a self-contained, and hopefully clear, analytical derivation for the setting of these initial conditions. To do so, we start our analysis before the phase transition, when the universe is remarkably simple since it is supposed to be strictly homogeneous and isotropic. Starting from this pre-transition state, one should, in principle, study the detailed evolution of the scalar field at the origin of the defects. An important simplification arises from the fact that the defects can be considered as perturbations and their evolution be obtained by solving their equations of motion in the background geometry. However the problem remains very complicated and requires heavy numerics. What will be retained for our purpose is that the quantities describing the defects can be seen as “external” sources for the evolution equations of the perturbations of the cosmic fluids.

In order then to relate the pre-transition Friedmann Lemaitre Robertson Walker
(FLRW) spacetime to the subsequent perturbed universe containing defects, we shall assume a sudden phase transition where the defects are instantaneously “turned on”. Einstein equations then impose some matching conditions between the perfectly homogeneous spacetime and the defect populated one. These matching conditions provide constraints but are not sufficient in themselves to determine completely the subsequent state of the perturbations.

If one wishes to say more about the post-transition perturbations, one must make some assumptions about the defects. The emergence of defects can be seen as a random process and the information on defects will thus be of a statistical nature. We shall assume that the sources satisfy the properties of causality, scaling and coherence. Causality is physically required and simply states that the sources are uncorrelated on scales larger than the Hubble radius. Scaling means that the statistical properties are invariant in time up to a rescaling with respect to the Hubble radius. This property is a simplifying one but can be justified by the convergence towards scaling observed in numerical simulations. Here, we assume scaling immediately (that is within one Hubble time) after the phase transition. Finally, coherence is a very stringent assumption, because it means that all the statistical properties of the sources are reduced to their statistics at a given time, the time evolution being completely deterministic. However, to consider coherent sources is more general than it seems because any source for our purpose can be decomposed into a sum of coherent ones.

The advantage of considering coherent sources is that, knowing the time dependence of their correlators from scaling and causality, one can then solve for the evolution of the perturbations by a simple use of Einstein’s equations. In the long wavelength limit, i.e. for scales larger than the Hubble radius, one obtains the solution explicitly as a sum of power-law terms.

From our explicit analytical solutions, we are able to justify some of the statements which can be found in the literature. In particular, the solutions for the perturbations, which can be decomposed into terms driven by the sources and “homogeneous” (in the sense of differential equations) terms, are shown to be dominated by the source driven terms. We also recover on our solutions the so-called phenomenon of “compensation”. Another conclusion of our work, which is original to our knowledge, is that, when one allows for cold dark matter, the sources will drive the long-wavelength perturbations into a combination of adiabatic and isocurvature perturbations, the relative ratio being a universal constant.

The paper is organized as follows. In section 2, we write the linearised Einstein equations. Our formalism is based on the usual cosmological perturbation theory using Bardeen type gauge-invariant quantities, where one distinguishes between scalar, vector and tensor perturbations. In addition to the usual geometrical and perfect fluid type matter variables, one must introduce the energy momentum tensor of the defects. This tensor is automatically gauge invariant because, as stated before, the defects are considered as perturbations. In Section 3, we describe the phase transition. In Section 4 we review the matching conditions on a constant energy density surface and apply them to the case of a sudden phase transition in the universe. They enable us to make the link between the unperturbed universe and the post-phase transition perturbed universe, or rather they give
constraints that physically admissible configurations must satisfy. Section 5 deals with the statistical properties of the sources and the notions of coherence, scaling and causality are introduced in detail. In section 6, we give the behaviour of perturbations larger than the Hubble radius after the transition, which will constitute the “initial” conditions. Finally, section 7 comments on the obtained solutions.

2. Linearised Einstein equations

2.1 The background

The universe at large appears to be remarkably homogeneous and isotropic and governed by the gravitational force created by its material content, to wit a mixture of radiation and dust. It is therefore well described by a Robertson-Walker geometry whose time evolution satisfies Friedmann’s equations. Phase transitions [3] [2] can have occurred in the very early universe when it was pure radiation and its spatial curvature negligible. We thus take the background line element in that era to be:

\[ ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j) \]  \hspace{1cm} (1)

where \( x^0 \equiv \eta \) is conformal time, \( x^i, i = 1, 2, 3 \), three cartesian coordinates, and \( a(\eta) \) the scale factor. The Friedmann equations are:

\[ \mathcal{H}' = -\mathcal{H}^2 , \quad \kappa \rho a^2 = 3\mathcal{H}^2 \]  \hspace{1cm} (2)

where a prime denotes a derivative with respect to conformal time, where \( \mathcal{H} \equiv a'/a \), where \( \rho \) is the radiation energy density and \( \kappa \equiv 8\pi G \) is Einstein’s constant. The solution of (2) is:

\[ \mathcal{H} = \frac{1}{\eta} \Rightarrow a = a_0 \eta , \quad \kappa \rho a^2 = \frac{3}{\eta^2} \Rightarrow \frac{\rho'}{\rho} = -\frac{4}{\eta} \]  \hspace{1cm} (3)

where \( a_0 \) is an integration constant which can be chosen so that \( a = 1 \) today.

2.2 Scalar perturbations

Following Bardeen [4] we split the perturbations of the geometry and the matter variables into “scalar”, “vector” and “tensor” parts. In this paragraph we define the scalar part and write the linearised Einstein equations for the gauge invariant scalar perturbations (for reviews of this formalism, see e.g. [5-7]).

The line element of a perturbed Robertson-Walker space time reads, when the perturbations are scalar and the background given by (1-3):

\[ ds^2 = a_0^2 \eta^2 \left[-(1 + 2A)d\eta^2 + 2\partial_i Bdx^i d\eta + \{(1 + 2C)\delta_{ij} + 2\partial_{ij} E\} dx^i dx^j\right] \]  \hspace{1cm} (4)

where \( A, B, C, E \) are four “small” functions of space and time. Under an infinitesimal coordinate transformation

\[ \eta \rightarrow \eta + T , \quad x^k \rightarrow x^k + \partial^k L \]  \hspace{1cm} (5)
where $T$ and $L$ are two arbitrary first order functions of $\eta$ and $x^i$, the four scalar metric perturbations $A, B, C, E$ transform as

$$ A \rightarrow A + T' + \frac{T}{\eta}, \quad B \rightarrow B - T + L', \quad C \rightarrow C + \frac{T}{\eta}, \quad E \rightarrow E + L. \quad (6) $$

One can thus introduce two gauge independent scalar metric perturbations, for example:

$$ \Psi = -C - \frac{1}{\eta}(B - E'), \quad \Phi = A + \frac{1}{\eta}(B - E') + (B - E'). \quad (7) $$

The energy-momentum tensor of the matter content of this perturbed universe can be written as:

$$ T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} + \Theta_{\mu\nu}. \quad (8) $$

$\bar{T}_{\mu\nu}$ is the energy-momentum tensor of the homogeneous and isotropic radiation background; $\delta T_{\mu\nu}$ is its perturbation: its scalar components can be expressed in terms of two scalar quantities, $\delta \equiv \delta \rho/\rho$, the density contrast, and $v$, the velocity perturbation, as (see e.g. [5-7]):

$$ \kappa \delta T_{00}^S = \frac{3}{\eta^2} (\delta + 2A), \quad \kappa \delta T_{0i}^S = -\frac{3}{\eta^2} \partial_i \left( B + \frac{4}{3} v \right), \quad (9) $$

$$ \kappa \delta T_{ij}^S = \frac{1}{\eta^2} \left[ \delta_{ij} (2C + \delta) + 2 \partial_{ij} E \right]. $$

(Note that we describe the material content of the universe as a single radiation fluid (with no anisotropic stresses). This is justified since at the era of the phase transition all matter is highly relativistic.)

In the coordinate transformation (5) $\delta$ and $v$ transform as: $\delta \rightarrow \delta - 4T/\eta$, $v \rightarrow v - L'$, so that two gauge invariant scalar perturbations for the radiation fluid can be constructed, e.g.:

$$ \delta^\flat = \delta + 4C, \quad v^\flat = v + E'. \quad (10) $$

Instead of $\delta^\flat$ one can also use :

$$ \delta^\sharp = \delta - \frac{4}{\eta} (B + v) \quad \text{or} \quad \delta^\circ = \delta - \frac{4}{\eta} (B - E'). \quad (11) $$

$\delta^\flat$, $\delta^\sharp$ and $\delta^\circ$ are the density contrasts which are respectively defined in the comoving gauge where $\delta T_{0i}^0 = 0$, the newtonian (or longitudinal) gauge, and in the flat slicing gauge. They are related by:

$$ \delta^\flat = \delta^\sharp - \frac{4}{\eta} v^\flat \quad \text{and} \quad \delta^\circ = \delta^\sharp - 4\Psi. \quad (12) $$

Finally $\Theta_{\mu\nu}$ is the energy-momentum tensor of the scalar field at the origin of the topological defects. We suppose that it is a small perturbation which does not contribute
to the background (this is the so-called “stiff approximation” (see e.g. [8])). We decompose its scalar components as:

\[
\Theta^S_{00} = \rho^s, \quad \Theta^S_{0i} = -\partial_i v^s, \quad \Theta^S_{ij} = \delta_{ij} \left( P^s - \frac{1}{3} \Delta \Pi^s \right) + \partial_{ij} \Pi^s.
\]  

(13)

The four source functions \( \rho^s, P^s, v^s, \Pi^s \) will be discussed later. They are gauge invariant since \( \Theta_{\mu\nu} \) is a tensor which vanishes in the unperturbed background.

Having defined gauge invariant scalar perturbations for the metric (eq (7)), for the radiation fluid (eq (10-12)) and for the sources (eq (13)), we now write their evolution equations. We shall write them in Fourier space, the Fourier transform of any function \( f(x^i, \eta) \) being defined as

\[
\hat{f}(k^i, \eta) \equiv \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-i k^i x^i} f(x^i, \eta) \quad \Leftrightarrow \quad f(x^i, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i k^i x^i} \hat{f}(k^i, \eta).
\]  

(14)

The conservation equations for the radiation fluid, when the background is governed by equations (2-3), can be cast under the form (see ref [5-7]):

\[
\delta^{\♭} \dot{\delta}^{\♭} = \frac{4}{3} k^2 \hat{v}^s
\]  

(15a)

\[
k_i \left( \hat{\dot{v}}^s + \Phi + \frac{1}{4} \hat{\delta}^s \right) = 0.
\]  

(15b)

The conservation equations for \( \Theta_{\mu\nu} \) read:

\[
\hat{\dot{\rho}}^s + \frac{1}{\eta} (\hat{\dot{\rho}}^s + 3 \hat{\dot{P}}^s) - k^2 \hat{v}^s = 0
\]  

(15c)

\[
k_i \left( \hat{\dot{v}}^s + \frac{2}{\eta} \hat{v}^s + \hat{\dot{P}}^s - \frac{2}{3} k^2 \hat{\Pi}^s \right) = 0.
\]  

(15d)

For all but the \( k = 0 \) mode, the linearised Einstein equations read:

\[
\hat{\Phi} = \kappa \hat{\Pi}^s
\]  

(15e)

\[
-k^2 \hat{\Psi} = \frac{3}{2\eta^2} \hat{\delta}^s + \frac{\kappa}{2} \left( \hat{\rho}^s - \frac{3}{\eta} \hat{v}^s \right)
\]  

(15f)

\[
\hat{\Psi} + \frac{1}{\eta} \hat{\Phi} = -\frac{2}{\eta^2} \hat{\delta}^s - \frac{\kappa}{2} \hat{v}^s
\]  

(15g)

\[
\hat{\Psi}'' + \frac{4}{\eta} \hat{\Psi}' + \frac{1}{3} k^2 \hat{\Psi} = \kappa \left( -\frac{1}{3} k^2 \hat{\Pi}^s + \frac{\hat{\dot{P}}^s}{\eta} + \frac{1}{2} \hat{\dot{P}}^s - \frac{1}{6} \hat{\rho}^s \right).
\]  

(15h)
In section 6 we shall solve this set of eight equations for the four unknowns \( \hat{\Psi}, \hat{\Phi}, \delta^{\natural}, \hat{v}^{\natural} \): the source functions \( \hat{\rho}^{s}, \hat{P}^{s}, \hat{v}^{s}, \hat{\Pi}^{s} \), subject to the constraints (15c) and (15d) being known, eq (15h) will give the metric perturbation \( \hat{\Psi} \). Then \( \hat{\Phi} \) is given by (15e), \( \delta^{\natural} \) by (15f) and \( \hat{v}^{\natural} \) by (15g). The last two equations, (15a) and (15b) are redundant (Bianchi identity) and can be used as a check of the calculation.

### 2.3 Vector perturbations

The line element for “vector” perturbations reads

\[
ds^2 = a_0^2 \eta^2 \left[ -d\eta^2 + 2 \bar{B}_i dx^i d\eta + \{ \delta_{ij} + 2 \partial_{(i} \bar{E}_{j)} \} dx^i dx^j \right],
\]

where \( \bar{B}^i \) and \( \bar{E}^i \) are small functions of space and time subject to the condition : \( \partial_i \bar{B}^i = \partial_i \bar{E}^i = 0 \). (Henceforth all “barred” quantities \( \bar{V}^i \) will be divergenceless vectors : \( \partial_i \bar{V}^i = 0 \).)

Under the infinitesimal coordinate transformation

\[
\eta \to \eta, \quad x^i \to x^i + \bar{L}^i \tag{17}
\]

where \( \bar{L}^i \) is an arbitrary first order divergenceless vector, the four components of the two vector perturbations \( \bar{B}_i \) and \( \bar{E}_i \) transform as

\[
\bar{B}_i \to \bar{B}_i + \bar{L}'_i, \quad \bar{E}_i \to \bar{E}_i + \bar{L}_i, \tag{18}
\]

so that the two components of the vector perturbations

\[
\bar{\Phi}_i = \bar{E}_i' - \bar{B}_i \tag{19}
\]

are gauge invariant.

The non-zero vector components of \( \delta T_{\mu\nu} \), the perturbation of the energy-momentum tensor of the radiation fluid, can be expressed in terms of \( \bar{v}^i \), the vector perturbation of the fluid velocity, as (see ref [5-7])

\[
\kappa \delta T^V_{0i} = -\frac{1}{\eta^2} (3 \bar{B}_i + 4 \bar{v}_i), \quad \kappa \delta T^V_{ii} = \frac{2}{\eta^2} \partial_{(i} \bar{E}_{j)} \tag{20}
\]

In a coordinate transformation : \( \bar{v}^i \to \bar{v}^i \to \bar{v}^{i'} \), so that

\[
\bar{v}^s_i = \bar{v}_i + \bar{B}_i \tag{21}
\]

is a gauge invariant quantity.

As for the non-zero vector components of the energy-momentum tensor of the scalar field at the origin of the topological defects we write them as

\[
\Theta^V_{0i} = -\bar{v}^s_i \quad \text{and} \quad \Theta^V_{ij} = 2 \partial_{(i} \bar{\Pi}^{s}_{j)} \tag{22}
\]

where the four source functions \( \bar{v}^s_i \) and \( \bar{\Pi}^s_i \) are gauge invariant and will be discussed in section 5.
The gauge invariant vector perturbations for the metric (eq (19)), for the radiation fluid (eq (21)) and for the sources (eq (22)) having thus being defined, we now write their evolution equations.

The equation of conservation for the radiation fluid is (in Fourier space)

\[ \hat{v}_i^{s'} = 0, \]  

(23a)

and is

\[ \hat{\dot{v}}_i^s + \frac{2}{\eta} \hat{\dot{v}}_i^s - k^2 \hat{\Pi}_i^s = 0 \]  

(23b)

for the defects. The Einstein equations split into

\[ -k^2 \hat{\dot{\Phi}}_i = -\frac{8}{\eta^2} \hat{\dot{v}}_i^s - 2\kappa \hat{\dot{v}}_i^s \]  

(23c)

\[ \hat{\Phi}_i' + \frac{2}{\eta} \hat{\Phi}_i = 2\kappa \hat{\Pi}_i^s. \]  

(23d)

The four source functions $\hat{v}_i^s$ and $\hat{\Pi}_i^s$ being known and subject to the constraint (23b), eq (23d) and (23c) will yield the metric perturbation $\hat{\Phi}_i$ and the velocity perturbation $\hat{v}_i$. Eq (23a) is redundant.

2.4 Tensor perturbations

The tensor perturbations of the geometry are defined by

\[ ds^2 = a_0^2 \eta^2 [-d\eta^2 + (\delta_{ij} + 2 \bar{E}_{ij})dx^i dx^j], \]  

(24)

with $\partial_i \bar{E}_{ij} = \bar{E}_i^i = 0$. $\bar{E}_{ij}$ is gauge invariant. The non-zero tensorial components of the perturbation of the energy-momentum tensor of the radiation fluid reduce to :

\[ \kappa \delta T^T_{ij} = \frac{2}{3\eta^2} \bar{E}_{ij} \]  

(25)

and the evolution of $\bar{E}_{ij}$ is given in Fourier space by (see ref [5-7])

\[ \hat{\dot{\bar{E}}}_{il} + \frac{2}{\eta} \hat{\dot{\bar{E}}}_{il} + k^2 \hat{\bar{E}}_{il} = 2\kappa \hat{\Pi}_{il}^s, \]  

(26)

where $\hat{\Pi}_{il}^s$ is the tensorial part of the energy-momentum tensor of the scalar field giving rise to the topological defects.

3. The phase transition

Topological defects are formed in the very early universe during a phase transition when some scalar field acquires different vacuum expectation values in different regions of space [2] [3].
Before the phase transition, the field is zero everywhere and its “false vacuum” potential energy is constant. Its energy-momentum tensor $\Theta_{\mu\nu}$ can then be modelled by a small cosmological constant. Hence, the only non vanishing source functions are

$$\rho^s = \lambda \eta^2, \quad P^s = -\lambda \eta^2,$$

where $\lambda$ is a positive constant related to the false vacuum energy. They satisfy the constraint (15c) (eq (15d) is empty since $P^s$ does not depend on the space coordinates).

Before the phase transition then, the universe is strictly homogeneous and isotropic. Hence the separation made here between the background governed by the radiation fluid and the perturbations caused by the scalar field in its false vacuum state is artificial. Indeed a strictly homogeneous and isotropic perturbation can always be absorbed into a redefinition of the background scale factor. We introduce it however for the sake of clarity.

As the temperature of the radiation fluid drops below a given critical temperature the field rolls down or tunnels to a “true vacuum” state of zero energy. Bubbles of true vacuum are formed, grow and percolate. If the true vacuum manifold is degenerate topological defects appear at the boundaries of bubbles characterized by different true vacuum values of the field [2-3].

The precise energy-momentum tensor of the field after this phase transition depends on the nature of the defects formed and can be obtained solely by means of heavy numerical calculations (see e.g. [9-11]). However some of its statistical properties can be inferred from general arguments (see below Section 5).

The duration $\Delta \eta$ of the phase transition itself must be brief (if it is delayed for too long the universe may enter an inflationary phase [12] [1], and we assume that this does not happen). In fact we assume that it is less or of the order of one Hubble time

$$\Delta \eta \simeq 1/\mathcal{H} \quad \text{with} \quad \mathcal{H} = \frac{1}{\eta_{PT}}$$

where $\eta_{PT}$ is the epoch of the phase transition. Hence the phase transition will look instantaneous to perturbations evolving on time scales greater than $\Delta \eta$ that is to modes $k$ such that

$$\frac{1}{k} \gg \Delta \eta \quad \Leftrightarrow \quad k \eta_{PT} \ll 1$$

that is to modes which are larger than the horizon at the epoch of the transition. Now all modes of interest today were larger than the horizon in the early universe. It is therefore justified to describe a transition which lasts less than a Hubble time as instantaneous.

Thus, we only need to match the spacetime geometries and the matter variables on the surface of transition that is the surface $\Sigma$ of constant temperature (or constant density). This will be done in the next section by imposing that the induced three metric on $\Sigma$ and the extrinsic curvature of $\Sigma$ must be continuous (see e.g [13]).

4. Matching conditions on a constant energy density surface
Following [14], we first write the matching conditions in a gauge where \( \Sigma \) is a constant time hypersurface and then translate them in an arbitrary coordinate system. (The difference with [14] is that here the background geometry evolves smoothly during the transition so that \( \mathcal{H} \) and \( \mathcal{H}' \) are continuous.)

4.1 Scalar modes

In an arbitrary coordinate system in which the line element is given by (4) the surface of transition \( \Sigma \) is defined by \( q(\eta, x^k) \equiv q_0(\eta) + \delta q = \text{Const} \) where \( q \) is a scalar (we shall take \( q \) to be the total energy density). In the coordinate transformation (5) \( \delta q \rightarrow \delta q + T q_0' \). Choosing \( T = -\delta q / q_0' \) (and \( L \) arbitrary) therefore takes us to the gauge where the surface \( \Sigma \) is a constant time surface. In that gauge, that we shall label with a tilde, the unit normal vector to the constant time hypersurfaces is

\[
n_0 = -a(1 + \tilde{A}) \quad \text{and} \quad n_i = 0. \quad (30)
\]

The induced metric and the extrinsic curvature are defined by

\[
\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_\nu, \quad \text{and} \quad K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}, \quad (31)
\]

where \( \mathcal{L} \) stands for the Lie derivative. Their non vanishing scalar components are given by

\[
\gamma_{ij} = a_0^2 \eta^2 \left[ (1 + 2\tilde{C}) \delta_{ij} + 2\partial_i \partial_j \tilde{E} \right] \quad (32)
\]

At first order, when one shifts back to the original, arbitrary gauge (using (6)), the continuity of \( \gamma_{\mu\nu} \) and \( K_{\mu\nu} \) then imposes

\[
\left[ C - \frac{1}{\eta_{PT} q_0'} \frac{\delta q}{q_0} \right]_\pm = 0, \quad \left[ \partial^i \partial_j (E + L) \right]_\pm = 0
\]

\[
\left[ \partial^i \partial_j \left( E' - B - \frac{\delta q}{q_0} \right) \right]_\pm = 0, \quad \left[ -\frac{1}{\eta_{PT}} A + C' + \frac{2}{\eta_{PT}^2} \frac{\delta q}{q_0} \right]_\pm = 0, \quad (33)
\]

where \( [F]_\pm \) is defined as \( [F]_\pm = \lim_{\epsilon \to 0^+} [F(\eta_{PT} + \epsilon) - F(\eta_{PT} - \epsilon)] \) and where \( \eta_{PT} \) is the conformal time at which the transition occurs and \( q \) is the surface of constant total density, so that

\[
q'_0 = \rho', \quad \delta q = \rho \delta + \rho^s / a^2. \quad (34)
\]

The second condition in (33) is empty since one can always choose \( L \) such that it is fulfilled. As for the other three, they can be rewritten in terms of scalar gauge invariant quantities and in Fourier space, as

\[
\left[ \frac{3}{\eta_{PT}^2} \hat{\delta}^s + \kappa \hat{\rho}^s \right]_\pm = 0, \quad \left[ k^i k_j \left( \frac{3}{\eta_{PT}^2} \hat{\delta}^s + \kappa \hat{\rho}^s \right) \right]_\pm = 0
\]

\[
\left[ \frac{1}{\eta_{PT}} \hat{\Phi} + \hat{\Psi}' + \frac{1}{2\eta_{PT}} \left( \hat{\delta}^s + \frac{1}{3} \kappa \hat{\rho}^s \eta_{PT}^2 \right) \right]_\pm = 0 \quad (35)
\]

10
and it is an easy exercise (which makes use of the linearised equations (15)) to show that
the third is redundant.

To summarize, the two independent matching conditions for the scalar perturbations

\[ \left[ \frac{3}{\eta_{PT}} \delta^b + \kappa \rho^s \right]_{\pm} = 0, \quad \left[ k^i k_j \hat{\Psi} \right]_{\pm} = 0. \]  

\[ (36) \]

4.2 Vector and tensor modes

The normal vector to the constant time hypersurfaces does not have any vector nor
tensor component and thus,

\[ n_0 = -a \quad \text{and} \quad n_i = 0. \]  

\[ (37) \]

This leads to the following expression for the vector components of the induced three
metric and the extrinsic curvature,

\[ \perp_{ij} = a_0^2 \eta^2 [\delta_{ij} + 2 \partial_i (\tilde{E}_j)] , \quad \delta K^j_i = \frac{1}{2a_0 \eta} (\partial_i \tilde{\Phi}^j + \partial^j \tilde{\Phi}_i). \]  

\[ (38) \]

As for their tensor components they are

\[ \perp_{ij} = 2a_0^2 \eta^2 \tilde{E}_{ij} , \quad \delta K_{ij} = \frac{1}{a_0 \eta} \tilde{E}^{ij}_i. \]  

\[ (39) \]

The matching conditions then reduce to, in Fourier space

\[ [k_i (\hat{\Phi}_j)]_{\pm} = 0 \quad \text{and} \quad [\hat{E}_{ij}]_{\pm} = 0 \quad ; \quad [\hat{E}^i_{ij}]_{\pm} = 0. \]  

\[ (40) \]

4.3 The case of a phase transition

In the particular case of the phase transition described in Section 3, the perturbations
are strictly homogeneous and isotropic before the transition so that their Fourier trans-
form is a zero mode proportional to the Dirac distribution \( \delta(k) \). After the transition, all
perturbations depend on space (apart from the zero mode which can be absorbed into a
redefinition of the background). In Fourier space then the matching conditions are, for all
modes apart the strictly \( k = 0 \) one :

\[ \left[ \frac{3}{\eta_{PT}} \delta^b + \kappa \rho^s \right]_{\eta_{PT}} = 0, \quad \left[ \hat{\Psi} \right]_{\eta_{PT}} = 0 \]

\[ [\hat{\Phi}_i]_{\eta_{PT}} = 0 \quad , \quad [\hat{E}_{ij}]_{\eta_{PT}} = 0 \quad , \quad [\hat{E}^i_{ij}]_{\eta_{PT}} = 0. \]  

\[ (41) \]

There are two conditions for the scalar modes, one for the vector ones and two for the
tensor ones, which is the required number since the equations of evolution for \( \Psi \) and \( \tilde{E}_{ij} \)
are second order and the one for $\Phi_i$ is first order. Note that these conditions, because we assumed that the universe was strictly homogeneous and isotropic before the transition, do not depend on the physics before the transition, for example on the value of the false vacuum energy.

What needs to be done now is to propagate the matching conditions (41) to a much later time in the radiation era, in order to set analytically the effective initial conditions which must be taken in the numerical integration of the evolution equations for the perturbations. In order to do that we need to specify the energy-momentum tensor of the sources (Section 5) and analytically integrate the evolution equations from the epoch of the phase transition to the time when the numerical integration starts (Section 6).

5. The energy-momentum tensor $\Theta_{\mu\nu}$ of the sources

5.1 Coherent vs independent sources

When the phase transition occurs the scalar field settles randomly in its true vacuum state in uncorrelated spatial domains. The distribution however is statistically homogeneous and isotropic because the background is so and the physics of the transition is supposed to obey the cosmological principle.

The ten components of the energy-momentum tensor $\Theta_{\mu\nu}(\eta, x^i)$ of the topological defects created are therefore ten random fields, as well as their ten Fourier transforms $\hat{\Theta}_{\mu\nu}(\eta, k^i)$ (which are complex but such that $\hat{\Theta}_{\mu\nu}^*(\eta, k^i) = \hat{\Theta}_{\mu\nu}(\eta, -k^i)$). From now on, we ignore the $(k = 0)$ mode which can be absorbed in the background.

The statistical properties of these ten random fields, that we shall denote collectively by $S_a(\eta, x^i)$ or $\hat{S}_a(\eta, k^i)$, are determined by the values of their correlators. Since we have in view the computation of the two-point correlator of the temperature anisotropies of the microwave background, the quantities we need to know are the unequal time two-point correlators of the sources, that is

$$\langle S_a(\eta, x^i)S_{a'}(\eta', x'^i) \rangle \equiv C_{a,a'}(\eta, \eta', r)$$

(42)

where $\langle ... \rangle$ means an ensemble average on a large number of different realisations of the transition, and where the correlator $C_{a,a'}$ depends only on $r \equiv |\vec{x} - \vec{x}'|$ because of the homogeneity and isotropy of the distribution. The power spectrum of the correlator $C_{a,a'}$ is defined as

$$P_{a,a'}(\eta, \eta', k) \equiv (2\pi)^{3/2}\hat{C}_{a,a'}(\eta, \eta', k)$$

(43)

where a hat denotes a Fourier transform (see (14)) and where the dependence in $k \equiv |\vec{k}|$, as well as the fact that $P_{a,a'}(\eta, \eta', k)$ is real are again due to the homogeneity and isotropy of the distribution. The power spectrum is related to the correlators in Fourier space by

$$\langle \hat{S}_a^*(\eta, k^i)\hat{S}_{a'}(\eta', k'^i) \rangle = \delta(k^i - k'^i)P_{a,a'}(\eta, \eta', k).$$

(44)

As shown by Turok [15], a clever way to look at the power spectra $P_{a,a'}(\eta, \eta', k)$ is to see their ensemble as a matrix $M$ where the columns are labelled by the indices $(a, \eta)$.
and the rows by the indices \((a', \eta')\). Because the power spectra are real this matrix is symmetric. Symmetric matrices can be diagonalised (we ignore here the fact that \(M\) is of infinite dimension) so that we have

\[
P_{a,a'}(\eta, \eta', k) = \sum_a \int_{\tilde{\eta}} d\tilde{\eta} \tilde{p}_{a\eta}(k) \lambda_{\tilde{\eta}\eta} \tilde{p}_{a'\eta'}(k)
\]  

(45)

where \(\lambda_{\tilde{\eta}\eta}\) are the infinite number of possibly degenerate and positive eigenvalues of the matrix \(M\).

Depending on the numerical value of the eigenvalues \(\lambda_{\tilde{\eta}\eta}\) and the associated (normalised) eigenvectors \(\tilde{p}_{a\eta}(k)\), the sources can be called coherent, independent, or mixed.

Perfect “time coherence” of a given random field \(S_a\), say, means that its power spectrum factorizes, that is

\[
P_{a,a}(\eta, \eta', k) = p_a(\eta, k)p_a(\eta', k).
\]  

(46)

(This means that in (45) \(\lambda_{\tilde{\eta}\eta}\) is of the form \(\lambda_{\tilde{\eta}\eta} = \lambda \delta_{aa} \delta(\tilde{\eta} - \eta)\), and \(p_a(\eta, k) \equiv \sqrt{\lambda} p^{\eta}_{a\eta}(k)\) \(p_a(\eta, k)\) is the square root of the power spectrum of the equal time auto-correlator of \(S_a\) : 

\[
\langle \hat{S}_a(\eta, k^i) \hat{S}_a(\eta, k^{i'}) \rangle = \delta(k^i - k^{i'}) (p_a(\eta, k))^2.
\]  

(45)

In contrast, perfect “time independence” means that

\[
P_{a,a}(\eta, \eta', k) = \delta(\eta - \eta') P_a(\eta, k).
\]  

(47)

(This means that in (45) \(\lambda_{\tilde{\eta}\eta} = \lambda \delta_{aa}\), \(p_{\eta\eta}(k) = p_{\eta\eta}(\delta(\tilde{\eta} - \eta)\), and that \(P_a(\eta, k) \equiv \lambda (p_{\eta\eta}(k))^2\).)

Textures, which evolve fairly smoothly, tend to be time coherent, whereas local cosmic strings, which undergo complex processes of intercommutation, tend on the other hand to be time independent (see e.g. [16]). It is clear that the evolution of time independent sources is more difficult to describe since at each moment a new realization of the random fields is drawn. In the following we shall consider time coherent sources only.

Time coherence (or independence), which concerns autocorrelators only, does not however characterize completely the statistical properties of the sources. Let us then turn to cross correlators.

A given subset \(\{S_a, a \in A\}\) of the ten sources is statistically coherent if

\[
P_{a,a'}(\eta, \eta', k) = p_a(\eta, k)p_{a'}(\eta', k).
\]  

(48)

(This means that in (45) the eigenvalues \(\lambda_{\tilde{\eta}\eta}\) are zero for all \(\tilde{\eta}\) not in the subset \((a, a')\) and all \(\tilde{\eta}\) not equal to some \(\tilde{\eta}\).) (In keeping with the terminology used in quantum mechanics, such sources \(S_a\) can also be called “pure”, see [15]). Note that statistical coherence of the random fields among themselves, that is property (48), implies time coherence, that is property (46).

Knowing the unequal time two-point correlators (42) or (44), subject to the coherence condition (48), is not enough a priori to specify completely the random fields \(S_a(\eta, x^i)\) of the subset. However a large class of random fields satisfying (48) is

\[
\tilde{S}_a(\eta, k^i) = p_a(\eta, k) e_A(k^i)
\]  

(49)
where $e_A(k^i)$ is a normalized complex random variable characterising the subset $A$ : $(e_A^*(k^i)e_A(k^{i*})) = \delta(k^i - k^{i*})$. It is clear that as long as one computes two-point correlators only the two definitions (48) and (49) are equivalent. (Beware of the fact that the left-hand side of (49) is not some “typical” realisation of the random field but the random field itself, as should be clear from the explicit introduction of the random variable $e_A(k^i)$ in the right-hand side. Omitting $e_A(k^i)$ on grounds of short-hand notations could for example induce into thinking that $\hat{S}_a(\eta, k^i)$ is a real quantity !)

Consider now two disjoint subsets, $\{S_a, a \in A\}, \{S_b, b \in B\}$, each being coherent, that is such that $S_a$ is given by (49) and $S_b$ by a similar expression : $\hat{S}_b(\eta, k^i) = p_b(\eta, k) \ e_B(k^i)$, $e_B(k^i)$ being the random variable characterising the subset $B$, with $(e_B^*(k^i)e_B(k^{i*})) = \delta(k^i - k^{i*})$.

The two subsets $A$ and $B$ are said to be statistically “independent” if $(e_A^*(k^i)e_B(k^{i*})) = \delta_{AB}\delta(k^i - k^{i*})$.

The difference between coherent and independent sources is most strikingly seen from the power spectrum of the sum of two different random fields. In the first case ($a$ and $a' \in A$)

$$\langle[\hat{S}_a^*(\eta, k^i) + \hat{S}_{a'}^*(\eta, k^i)][\hat{S}_a(\eta, k^{i*}) + \hat{S}_{a'}(\eta, k^{i*})]\rangle = [p_a(\eta, k) + p_{a'}(\eta, k)]^2 \delta(k^i - k^{i*})$$

wheras in the second ($a \in A, b \in B$)

$$\langle[\hat{S}_a^*(\eta, k^i) + \hat{S}_b^*(\eta, k^i)][\hat{S}_a(\eta, k^{i*}) + \hat{S}_b(\eta, k^{i*})]\rangle = \left([p_a(\eta, k)]^2 + [p_b(\eta, k)]^2\right) \delta(k^i - k^{i*}).$$

In the following we shall consider statistically independent subsets ($A, B, ...$) of statistically (and time) coherent sources ($S_a, S_b, ...$). Indeed, we shall see that causality imposes that the sources be divided into statistically independent subsets and we shall assume that within each subset the sources are coherent. This covers a large class of sources. Indeed, according to (45), the correlators of more complex, “mixed” (or partially coherent) sources can be decomposed into a sum of correlators of coherent sources.

### 5.2 Scaling properties

In statistical physics a dynamical system described by an order parameter $\psi(\eta, x^i)$ is said to follow a scaling law if the time evolution of its statistical properties depends only on a time-dependent length scale $L(\eta)$, so that its equal time auto-correlation function can be written in the form

$$\langle\psi(\eta, x^i)\psi(\eta, x'^i)\rangle = F(r/L(\eta)).$$

(The order parameter is supposed to be an homogeneous and isotropic random field.) In Fourier space the scaling law (52) translates as

$$\langle\hat{\psi}(\eta, k^i)\hat{\psi}(\eta, k^{i*})\rangle = \delta(k^i - k^{i*})(2\pi)^{3/2}L^3(\eta)\hat{F}(kL(\eta)).$$

In a cosmological context a natural length scale is the Hubble radius : $L(\eta) = H^{-1} = \eta$. Dimensionless “order” parameters $\psi(\eta, x^i)$ describing the network of topological defects can
be constructed using background quantities, such as $\Theta_{\mu\nu}(\eta, x^i)/\rho$, where $\rho \propto 1/\eta^4$ is the background energy density. Numerical simulations of texture (see e.g. [9] [10]) and cosmic string network (see e.g. [11]) evolution (as well as qualitative arguments (see e.g. [7])) have indicated that indeed these dimensionless random fields (but not the random fields themselves) do obey, soon after the transition, a scaling law (in fact, if they did not they would be either irrelevant or catastrophic) with the Hubble radius as length scale.

We therefore have

$$\langle \Theta_{\mu\nu}(\eta, x^i)\Theta_{\mu\nu}(\eta, x'^i) \rangle = \frac{1}{\eta^4} F_{\mu\nu}(r/\eta)$$  \hspace{1cm} (54)

so that the power spectra are of the form

$$\langle \hat{\Theta}^*_{\mu\nu}(\eta, k^i)\hat{\Theta}_{\mu\nu}(\eta, k'^i) \rangle = \delta(k^i - k'^i)(2\pi)^{3/2}\frac{1}{\eta} \hat{F}_{\mu\nu}(k\eta).$$  \hspace{1cm} (55)

For statistically coherent sources described by (49) the scaling property (54-55) translates as

$$\hat{S}_a(\eta, k^i) = \frac{1}{\sqrt{\eta}} f_a(k\eta) e^{A(k^i)}.$$  \hspace{1cm} (56)

What remains to be determined then is the behaviour of the ten functions $f_a(k\eta)$, as well as the subsets $A$.

### 5.3 Causality constraints

The detailed structure of each realisation of $\Theta_{\mu\nu}(\eta, x^i)$ is of course very complex. On scales less than the correlation length (of the order of the Hubble radius at the epoch of transition), $\Theta_{\mu\nu}(\eta, x^i)$ is almost zero since the scalar field is there in a true vacuum state. In fact, it is non zero only around the location of the topological defects, decaying more or less quickly to zero away from them, depending on whether the defects are local or global [2]. Numerical simulations have shown that, as time passes, local cosmic strings interconnect and produce loops which decay by emission of gravitational radiation, in such a way that the energy distribution scales like the background density (see previous &). In the case of textures, previously uncorrelated regions become causally connected and the scalar field tends to take the same true vacuum value in each region, so that the number density of textures decreases inversely to the horizon volume; their energy is also redshifted away so that, again, the energy density distribution scales like the background density.

Now, even if each realisation of the phase transition produces a network of defects whose typical length scale is, at all times, greater than the horizon, their ensemble is, because the network appeared at a definite time, that is for causality reasons, completely uncorrelated on scales larger than the horizon. Therefore, as stressed e.g. by Turok [17], the unequal time correlators in two points $P(\eta, x^i)$ and $P'(\eta', x'^i)$ are strictly zero if the past light-cones of $P$ and $P'$ do not intersect on the surface of the phase transition:

$$\langle \Theta_{\mu\nu}(\eta, x^i)\Theta_{\lambda\rho}(\eta', x'^i) \rangle = 0 \quad \text{if} \quad |\vec{x} - \vec{x}'| > \eta + \eta'.$$  \hspace{1cm} (57)
Property (57) translates in Fourier space into the fact that the power spectra are white noise on super horizon scales (that is for \(k\eta \ll 1\)). Indeed, because the correlators (57) have compact supports their Fourier transforms are analytic in \(k^i\).

Let us first consider the spatial components of the correlators. They can be written, in the small \(k\) limit, as

\[
\langle \hat{\Theta}_{ij}^*(\eta, k^i) \hat{\Theta}_{kl}(\eta', k'^l) \rangle = \delta(k^i - k'^i) \left[ A \delta_{ij} \delta_{kl} + B (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] + \ldots, \tag{58a}
\]

where the coefficients \(A\) and \(B\) are independent of \(k^i\) (but may depend on \(\eta\) and \(\eta'\)). The right hand side term was obtained by requiring a constant tensor (with respect to \(k^i\)) which respects the index symmetries of the correlator (see Turok et al. \[18\]). More generally, the correlators are of the form

\[
\langle \hat{\Theta}_{ij}^*(\eta, k^i) \hat{\Theta}_{kl}(\eta, k'^l) \rangle = \delta(k^i - k'^i) \left( \epsilon_t t_{ij} t_{kl} + 2\epsilon_u u_{(i}(k^i u_{l)}j) \right), \tag{58b}
\]

with

\[
t_{ij} = t_0 \delta_{ij} + t_1 k_i k_j, \quad u_{ij} = u_0 \delta_{ij} + u_1 k_i k_j, \tag{58c}
\]

where \(t_0, t_1, u_0\) and \(u_1\) are analytic functions of \(k^2\) (\(\epsilon_t = \pm 1, \epsilon_u = \pm 1\)). This is the only decomposition which is compatible with the homogeneity and isotropy of the distribution.

In Fourier space, any rank 2 symmetric tensor can be decomposed into scalar, vector and tensor parts, respectively:

\[
\hat{\Theta}_{ij} = \hat{\Theta}_{ij}^S + \hat{\Theta}_{ij}^V + \hat{\Theta}_{ij}^T \tag{59a}
\]

with

\[
\hat{\Theta}_{ij}^S \equiv \left( \frac{1}{2} P_{ij} P^{kl} + L_i^k L_j^l \right) \hat{\Theta}_{kl}, \tag{59b}
\]

\[
\hat{\Theta}_{ij}^V \equiv \left( P_i^k L_j^l + L_i^k P_j^l \right) \hat{\Theta}_{kl}, \tag{59c}
\]

\[
\hat{\Theta}_{ij}^T \equiv \left( P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl} \right) \hat{\Theta}_{kl}, \tag{59d}
\]

where the complementary projection operators \(P_{ij}\) and \(L_{ij}\) are defined as

\[
P_{ij} = \delta_{ij} - k_i k_j / k^2, \quad L_{ij} = k_i k_j / k^2. \tag{60}
\]

As a exercise, one can easily check that the pressure \(P^s\) and anisotropic stress \(\Pi^s\) defined in (13) are simply given in Fourier space by the expressions

\[
\hat{P}^s = \delta^{ij} \hat{\Theta}_{ij} / 3, \quad k^2 \hat{\Pi}^s = \frac{1}{2} \left( P^{ij} - 2L^{ij} \right) \hat{\Theta}_{ij}. \]

It can first be shown, using the decomposition (59) with the generic term (58b-c), that any correlator between scalar and vector, or vector and tensor, or tensor and scalar spatial quantities vanishes (this in fact is a consequence of the isotropy of the distribution).

16
Let us then consider first the correlators between scalar spatial quantities. Using the above expressions, one finds easily

\[ \langle \hat{P}^{ss} \hat{P}^s \rangle = (A + 2B/3), \quad \langle \hat{\Pi}^{ss} \hat{\Pi}^s \rangle = 3Bk^{-4}, \quad \langle \hat{P}^{ss} \hat{P}^s \rangle = \mathcal{O}(k^0), \]  

(61)

where \( \langle \hat{P}^{ss} \hat{P}^s \rangle \) etc stand for \( \langle \hat{P}^{ss}(\eta, k^i) \hat{P}^s(\eta', k'^j) \rangle \) and where it is understood that all the correlators are proportional to \( \delta(k^i - k'^j) \).

The correlators between the other scalar terms are obtained by introducing the correlators mixing time and spatial indices, namely

\[ \langle \hat{\Theta}_{00}^s \hat{\Theta}_{00} \rangle = C, \quad \langle \hat{\Theta}_{00}^s \hat{\Theta}_{ij} \rangle = D_0 \delta_{ij} + D_1 k_i k_j, \quad \langle \hat{\Theta}_{0i}^s \hat{\Theta}_{0j} \rangle = E_0 \delta_{ij} + E_1 k_i k_j, \]  

(62)

\[ \langle \hat{\Theta}_{00}^s \hat{\Theta}_{00} \rangle = -iFk_i, \quad \langle \hat{\Theta}_{0k}^s \hat{\Theta}_{ij} \rangle = ik_k (G_0 \delta_{ij} + G_1 k_i k_j) + 2iG_2 k_i \delta_{ij} k_j, \]  

(63)

the nine functions \( (C, D_0, D_1, E_0, E_1, F, G_0, G_1, G_2) \) being analytic in \( k^2 \). Performing the scalar, vector tensor decomposition we obtain

\[ \langle \hat{\rho}^{ss} \hat{\rho}^s \rangle = C, \quad \langle \hat{\rho}^{ss} \hat{P}^s \rangle = D_0 + D_1 k^2 / 3, \quad \langle \hat{\rho}^{ss} \hat{\Pi}^s \rangle = -D_1, \quad \langle \hat{\rho}^{ss} \hat{\Pi}^s \rangle = E_0 k^{-2} + E_1, \]  

(64)

\[ \langle \hat{\rho}^{ss} \hat{\rho}^s \rangle = F, \quad \langle \hat{\nu}^{ss} \hat{P}^s \rangle = (G_0 + G_2 / 3) + G_1 k^2 / 3, \quad \langle \hat{\nu}^{ss} \hat{\Pi}^s \rangle = -G_1 - 2G_2 / k^2. \]  

(65)

Similarly, one finds for the nonvanishing vector correlators

\[ \langle \hat{\Pi}_{i}^{ss} \hat{\Pi}_{j}^{s} \rangle = B k^{-2} P_{ij}, \quad \langle \hat{v}_{i}^{ss} \hat{v}_{j}^{s} \rangle = E_0 P_{ij}, \quad \langle \hat{v}_{i}^{ss} \hat{\Pi}_{j}^{s} \rangle = G_2 P_{ij} \]  

(66)

and for the tensor correlators

\[ \langle \hat{T}_{ij}^{ss} \hat{T}_{kl}^{s} \rangle = B (P_{ik} P_{jl} + P_{il} P_{jk} - P_{ij} P_{kl}). \]  

(67)

These results imply that, as a consequence of causality requirements, the quantities (56) describing the sources can be divided into three statistically independent subsets as defined in subsection 5.1: a scalar subset including \( \rho^s, P^s, \Pi^s \) and \( v^s \), characterised by a random variable \( e_S(k^i) \); a disjoint, independent vector subset including \( \bar{v}_i^s \) and \( \bar{\Pi}_i^s \); characterised by \( e_V(k^i) \); finally a tensor subset containing \( \bar{\Pi}_{ij}^s \) and characterised by \( e_T(k^i) \).

### 5.4 Conservation equations

The last general property satisfied by the sources is the conservation equations. These equations can constrain the correlators obtained in the previous subsection. Indeed, if one multiplies eq (15d) by \( \hat{v}^{ss} \) and then takes the correlator, one finds that the leading term of \( E_0 \) must decay as \( \eta^{-4} \). In the following, we shall neglect this decaying term which turns out to be negligible with respect to the scaling solution (see below) and thus take \( E_0 \propto k^2 \). As a consequence,

\[ \langle \hat{v}^{ss} \hat{\nu}^s \rangle = \mathcal{O}(k^0), \quad \langle \hat{v}_{i}^{ss} \hat{v}_{j}^{s} \rangle = \mathcal{O}(k^2) P_{ij}. \]  

(68)
In order to maintain statistical coherence among the scalar subset of variables, we shall now restrict our study to the case where the dominant \( k^{-4} \) dependence of the autocorrelator of \( \hat{\Pi}^s \) (see eq. (61)) vanishes (otherwise, \( \hat{\Pi}^s \) would go like \( k^{-2} \) while \( \hat{\rho}^s \) and \( \hat{P}^s \) go like \( k^0 \), which would imply, assuming coherence, that the cross-correlators of \( \hat{\Pi}^s \) with \( \hat{\rho}^s \) or \( \hat{P}^s \) would go like \( k^{-2} \) in contradiction with (61) and (64). Thus, taking into account the next order terms in the analytic expansion, we shall assume

\[
\langle \hat{\Pi}^{ss}\hat{\Pi}^s \rangle = \mathcal{O}(k^0), \quad \langle \hat{\Pi}^{ij}_{ij}\hat{\Pi}^s_{kl} \rangle = \mathcal{O}(k^4) (P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}).
\] (69)

Combining our results with the scaling property (55-56) completely determines on super-horizon scales the auto-correlators (48) or, equivalently for our purposes, the random fields (49). Introducing the scalar, vector and tensor decomposition performed in section 2, the random fields describing statistically coherent sources that scale, satisfy the causality and conservation constraints finally are:

\[
\hat{\rho}^s = \frac{1}{\sqrt{\eta}} f_1(k\eta)e_S(k^i), \quad \hat{P}^s = \frac{1}{\sqrt{\eta}} f_2(k\eta)e_S(k^i),
\]

\[
\hat{v}^s = -\sqrt{\eta} f_3(k\eta)e_S(k^i), \quad \hat{\Pi}^s = \eta^{3/2} f_4(k\eta)e_S(k^i),
\]

\[
\hat{v}_i^s = k\sqrt{\eta} g_{1i}(k\eta)e_V(k^i), \quad \hat{\Pi}_i^s = \eta^{3/2} g_{2i}(k\eta)e_V(k^i), \quad \hat{\Pi}_{ij}^s = k^2 \eta^{3/2} h_{ij}(k\eta)e_T(k^i)
\] (71)

where the ten\(^1\) functions \( f_a(k\eta), g_{ai}(k\eta), h_{ij}(k\eta) \) tend to constants on super horizon scales. On smaller scales \( (k\eta \gg 1) \) they tend to zero (in an oscillatory fashion) at a rate which depends on whether the defects are local or global and which can be determined precisely solely by means of numerical simulations. However, since our purpose is to set initial conditions at a time when all relevant scales are outside the horizon, we shall not need to know these functions for \( k\eta > 1 \).

6. The long wavelength solution after the transition

We shall now construct the long-wavelength solutions, that is valid as long as \( k\eta \ll 1 \), of the equations of motion written in Fourier space for the scalar (eq (15)), the vector (eq (23)) and the tensor (eq (26)) perturbations, subject to the matching conditions (41) at \( \eta = \eta_{PT} \). The source terms will be supposed to be given by Eq (70-71) where the functions \( f_a(k\eta), g_{ai}(k\eta), h_{ij}(k\eta) \) can be expanded in power series of \( k\eta \) and tend to constants when \( k\eta \to 0 \):

\[
f_a(k\eta) \to A_a \quad a = 1, 2, \ldots 4, \quad g_{ai}(k\eta) \to B_{ai} \quad a = 1, 2, \quad h_{ij}(k\eta) \to C_{ij} \quad \text{when} \quad k\eta \to 0.
\] (72)

\(^1\) In fact there are six such functions only: see “Cosmic microwave background anisotropies seeded by coherent topological defects: a semi-analytical approach” by J.P. Uzan, N. Deruelle and A. Riazuelo
The long wavelength description of the topological defects is thus reduced to the giving of ten constants.

6.1 Scalar modes

The four scalar source terms \( \hat{\rho}^s, \hat{P}^s, \hat{v}^s, \hat{\Pi}^s \) are subject to the two constraints (15c-d). In the long-wavelength limit (72), these constraints read:

\[
A_1 = -6A_2, \quad A_3 = \frac{2}{5}A_2. \tag{73}
\]

The general long wavelength solution of Eq (15h) for \( \hat{\Psi} \) is (taking into account (73))

\[
\hat{\Psi} = \Psi_0 + \frac{\Psi_1}{\eta^3} + \frac{2}{9}\kappa\eta^{3/2}(A_4 + A_2) \tag{74}
\]

where \( \Psi_0 \) and \( \Psi_1 \) are two constants of integration and where, from now on, we omit writing the normalised random field \( e_s(k^i) \) to which each scalar perturbation is proportional to. Eq (15e-f-g) then give the other perturbations

\[
\hat{\Phi} = \Psi_0 + \frac{\Psi_1}{\eta^3} + \frac{1}{9}\kappa\eta^{3/2}(2A_2 - 7A_4), \quad \hat{\delta}^\flat = \frac{8}{5}\kappa\eta^{3/2}A_2, \tag{75}
\]

\[
\hat{\delta}^\sharp = \frac{\eta^2}{2}\Psi_0 + \frac{\Psi_1}{\eta^2} - \frac{2\kappa\eta^{5/2}}{45}(4A_2 - 5A_4). \tag{76}
\]

Equation (12) then gives:

\[
\hat{\delta}^\flat = -6\Psi_0, \quad \hat{\delta}^\sharp = -2\Psi_0 + \frac{4\Psi_1}{\eta^3} + \frac{8}{9}\kappa\eta^{3/2}(A_4 + A_2). \tag{77}
\]

(One can check that the Bianchi identities (15a-b) are identically satisfied by the solution (74-75).) Finally the constants \( \Psi_0 \) and \( \Psi_1 \) are determined by the matching conditions (41):

\[
\Psi_0 = -\frac{1}{3}\kappa\eta_{PT}^{3/2}A_2, \quad \Psi_1 = \frac{1}{9}\kappa\eta_{PT}^{3/2}(-2A_4 + A_2). \tag{77}
\]

One therefore sees that deep in the radiation era but long after the transition, when \( \eta \gg \eta_{PT} \), all terms proportional to \( \Psi_0 \) and \( \Psi_1 \) in the expressions for \( \hat{\Psi}, \hat{\Phi}, \hat{\delta}^\flat, \hat{\delta}^\sharp, \hat{v}^\flat \) and \( \hat{\delta}^\sharp \) can be neglected. We shall comment on \( \hat{\delta}^\flat \) in the paragraph on compensation.

6.2 Vector and tensor modes

The two vector source terms \( \hat{\bar{v}}^s_i, \hat{\bar{\Pi}}^s_i \), given by (71-72) are constrained by the conservation equation (23b). In the long wavelength limit this leads to \( \bar{B}_{1i} = 0 \). At next order one gets

\[
\bar{v}^s_i = \frac{2}{9}\eta^{5/2}k^3\bar{B}_{2i}, \tag{78}
\]
where, again, the normalised random field $e_V(k^i)$ to which each vector perturbation is proportional has been omitted. The general long wavelength solution of equation (23d) is

$$\hat{\Phi}_i = \frac{\Phi_{0i}}{\eta^2} + \frac{4}{9} \kappa k \bar{B}_{2i} \eta^{5/2},$$  \hspace{1cm} (79)

where $\Phi_{0i}$ is a constant of integration which is determined by the matching condition (41) :

$$\frac{\Phi_{0i}}{\eta_{PT}^2} = -\frac{4}{9} \kappa k \bar{B}_{2i} \eta_{PT}^{5/2}. \hspace{1cm} (80)$$

Finally (23c) yields

$$\hat{v}_i = \frac{k^2}{8} \bar{\Phi}_{0i} \hspace{1cm} (81)$$

(so that the Bianchi identity (23a) is satisfied).

The general solution of equation (26) for the tensorial modes, for a source given by (71)(72), is

$$\tilde{E}_{kl} = \left[ \tilde{A}_{kl} + \frac{\bar{B}_{kl}}{\eta} + \frac{8}{63} \kappa k^2 \bar{C}_{kl} \eta^{7/2} \right] e_T(k^i), \hspace{1cm} (82)$$

where $\tilde{A}_{kl}$ and $\bar{B}_{kl}$ are two integration constants determined by the matching conditions (41) :

$$\tilde{A}_{kl} = -\frac{4}{7} \kappa k^2 \bar{C}_{kl} \eta_{PT}^{7/2}, \hspace{1cm} \frac{\bar{B}_{kl}}{\eta_{PT}} = \frac{4}{9} \kappa k^2 \bar{C}_{kl} \eta_{PT}^{7/2}. \hspace{1cm} (83)$$

One sees again that deep in the radiation era but long after the transition, all terms proportional to the integration constants can be neglected.

### 6.3 Initial conditions for Cold Dark Matter

We assumed that at the epoch of the transition the matter content of the universe was a radiation fluid consisting of photons, neutrinos and highly relativistic particles, some coupled to the photons, some not (being “WIMPS”, see e.g. [19]). Thus the density contrasts that we have introduced ($\delta^\natural$, $\delta^\sharp$, $\delta^\natural$) are density contrasts of the radiation fluid. Now as the universe expands and its temperature drops, the WIMPS become non relativistic and turn into cold dark matter (CDM). In this paragraph we discuss the evolution of the perturbations of this pressureless component of the cosmic fluid, when the universe is still radiation dominated.

The background energy momentum tensor of CDM simply is : $T_{\mu\nu} = \rho_c u_\mu u_\nu$ with $u_0 = -1, u_i = 0$ (a being the scale factor) and where $\rho_c$ is the energy density. Since CDM, by definition, interacts only gravitationaly with the radiation fluid, $T_{\mu\nu}$ is conserved so that $\rho_c' / \rho_c = -3/\eta$.

The perturbations of $T_{\mu\nu}$ can be decomposed into scalar vector and tensor components which can be written as

$$\delta T_{00}^S = a^2 \rho_c (\delta_c + 2A), \hspace{1cm} \delta T_{0i}^S = -a^2 \rho_c \partial_i (B + v_c), \hspace{1cm} \delta T_{0i}^V = -a^2 \rho_c (\bar{B}_i + \bar{v}_c) \hspace{1cm} (84)$$
(the other components being zero) where \( \delta_c, v_c \) and \( \bar{v}_i^c \) are the density contrast and the velocity components of the perturbations of the CDM fluid. In the coordinate transformation (5) \( \delta_c \rightarrow \delta_c - 3T/\eta \). As for \( v_c \) and \( \bar{v}_i^c \) they transform like the radiation velocities. Various gauge invariant perturbations can therefore be built, such as:

\[
\delta_c^{\hat{\flat}} = \delta_c + 3C, \quad v_c^{\hat{\flat}} = v_c + E', \quad \bar{v}_i^{\hat{\flat}} = \bar{v}_i^c + \bar{B}^i.
\] (85)

Other gauge invariant density contrasts can be introduced, e.g.

\[
\delta_c^{\hat{\natural}} = \delta_c - \frac{3}{\eta} (v + B) \quad \text{or} \quad \delta_c^{\hat{\sharp}} = \delta_c - \frac{3}{\eta} (B - E').
\] (86)

Note that \( \delta_c^{\hat{\natural}} \) is the density contrast in the gauge where the radiation fluid is at rest.

Since radiation still dominates, the density contrast and velocity perturbations of the total fluid are almost equal to the radiation density contrast \( \delta \) and the radiation velocities \( v \) and \( \bar{v}_i \), so that the Einstein equations (15e-h), (23c-d) and (26) remain unchanged. Therefore the solution given in the previous paragraphs for the metric perturbations as well as for the (dominant) radiation component still holds even after the WIMPS have become non relativistic.

As for the evolution of the perturbations of the CDM it is governed by the conservation equations of its energy-momentum tensor, that is (in Fourier space):

\[
\hat{\delta}_c^{\hat{\flat}} = k^2 \hat{v}_c^{\hat{\flat}}, \quad \hat{v}_c^{\hat{\flat}}' + \frac{1}{\eta} \hat{\delta}_c^{\hat{\flat}} = -\hat{\Phi}
\] (87)

\[
\hat{\tilde{v}}_i^{\hat{\flat}} + \frac{1}{\eta} \hat{\tilde{v}}_i^{\hat{\flat}} = 0,
\] (88)

where \( \hat{\Phi} \) is given by (75). The general solution of this system is, on super-horizon scales

\[
\hat{\delta}_c^{\hat{\natural}} = \delta_0, \quad \hat{v}_c^{\hat{\natural}} = \frac{v_0}{\eta} - \frac{\Psi_0}{2 \eta} + \frac{\Psi_1}{\eta^2} - \frac{2}{63} (2A_2 - 7A_4) \kappa \eta^{5/2},
\] (89)

\[
\hat{\tilde{v}}_i^{\hat{\natural}} = \frac{\bar{v}_i^0}{\eta},
\] (90)

where \( \delta_0, v_0 \) and \( \bar{v}_i^0 \) are integration constants, which contrarily to \( \Psi_0 \) and \( \Psi_1 \) are not given by the matching conditions at the time of the transition. Rather they should be deduced from a detailed analysis of the formation of the WIMPS. As for the other density contrasts they are given by

\[
\hat{\delta}_c^{\hat{\sharp}} = \delta_0 + \frac{9}{2} \Psi_0 + \frac{6}{5} A_2 \kappa \eta^{3/2}, \quad \hat{\tilde{v}}_c^{\hat{\sharp}} = \delta_0 + 3 \Psi_0 + \frac{3 \Psi_1}{\eta^3} + \frac{2}{3} (A_2 + A_4) \kappa \eta^{3/2}.
\] (91)

Once more one sees that deep in the radiation era but long after the transition, all terms proportional to \( \Psi_0 \) and \( \Psi_1 \) in the expressions for \( \hat{\delta}_c^{\hat{\natural}}, \hat{v}_c^{\hat{\natural}} \) and \( \hat{\delta}_c^{\hat{\sharp}} \) can be neglected. As
for the terms proportional to the unknown constants \( v_0 \) and \( \bar{v}_i \) they are decaying modes. The only new constant of potential dynamical relevance is therefore \( \delta_0 \).

Let us now compare the density contrasts of CDM and radiation. The isocurvature perturbation is the gauge invariant quantity \( \hat{S} \equiv \hat{\delta}_c - \frac{3}{4} \hat{\delta} \). From (75) and (91) we have that it is given, at lowest order in \( k \), by

\[
\hat{S} = \delta_0 + \frac{9}{2} \Psi_0 \tag{92}
\]

We note that in \( \hat{S} \), just like in the density contrasts \( \hat{\delta}^0 \) and \( \hat{\delta}^\flat \), the terms of the type \( \kappa A_i \eta^{3/2} \) are absent. We must therefore compute them at next order in \( k \). Injecting the asymptotic behaviours (75) and (89) for the velocity perturbations into (15a) and (87), we obtain:

\[
\hat{S} = \delta_0 + \frac{9}{2} \Psi_0 + \frac{8}{245} \kappa A_2 (k \eta)^2 \eta^{3/2}. \tag{93}
\]

In the comoving gauge (in which the radiation fluid is at rest), the ratio of isocurvature to total perturbations is:

\[
\frac{\hat{S}^2}{\delta^2} \to \frac{1}{49} (k \eta)^2, \tag{94}
\]

and does not depend on the detailed structure of the defects [as long as, of course, they can be represented by the random fields (70-71)].

7. Comments and discussion

In this Section we briefly compare our results to the initial conditions currently used in the literature.

7.1 “Natural” initial conditions

In [22-23], Durrer, Sakellariadou and Gangui studied the perturbations of the radiation and CDM components of the cosmic fluid seeded by the sources (70). They chose “natural” initial conditions, which amounts to setting \( \Psi_0 = \Psi_1 = 0 \) in (74-76) (they considered the scalar perturbations only). Our work justifies their choice since, as we saw, the homogeneous modes (in the sense of differential equations) of the evolution equations become negligible fairly soon after the phase transition. Their solution however differs from ours: indeed they only solve the conservation equations for the radiation and CDM fluids (that is Eq. (15 a-b) and (87)) together with the Einstein equations (15 e-f). They therefore ignore the conservation equations for the sources, that is Eq. (15 g-h), arguing that they do not hold for “incoherent” sources. Their solution hence depends on three constants, \( A_1, A_3 \) and \( A_4 \), instead of two. However it is clear that the conservation equations for the sources (or, equivalently, the Einstein equations (15 g-h)) must be satisfied, whether the sources are described by ordinary functions or random fields, and that \( A_3 \) in [22-23] should be set equal to \(- \frac{1}{15} A_1 \).
7.2 Fixing the initial conditions by means of a “pseudo-tensor”

The evolution of the perturbations in synchronous gauges where $A = B = B_t = 0$, which are used by a number of authors, in particular Pen, Spergel and Turok [9], can be straightforwardly obtained from the definition (10-11) (85-86) of the gauge invariant quantities we introduced and their explicit expressions (74-77) (89-91). For example:

$$\hat{\delta}^{\text{syn}}_c = \delta_0 + \frac{9}{2} \Psi_0 - \frac{3 e_0}{\eta^2} + \frac{6}{7} \kappa \eta^{3/2} A_2 \quad \text{and} \quad \hat{\delta}^{\text{syn}} = -\frac{4 e_0}{\eta^2} + \frac{8}{7} \kappa \eta^{3/2} A_2$$  \hspace{1cm} (95)

where $e_0$ is an extra constant of integration linked to the fact that synchronous gauges are not completely fixed.

Now the solution (95) can of course also be obtained by a direct integration of the perturbation equations (15) written in the synchronous gauge. It is an easy exercise to show from Eq (15) and (87), and (70) (72), that the synchronous gauge density contrasts for the CDM and the radiation fluid satisfy, on super-horizon scales, the following equations:

$$\hat{\delta}^{\text{syn}}_c' = \frac{3}{4} \hat{\delta}^{\text{syn}} + D_0 \quad \text{and} \quad \hat{\delta}^{\text{syn}}'' = \frac{\hat{\delta}^{\text{syn}}'}{\eta} - \frac{4 \hat{\delta}^{\text{syn}}}{\eta^2} = -\frac{2}{3} A_2 \sqrt{\eta}$$  \hspace{1cm} (96)

the general solutions of which are

$$\hat{\delta}^{\text{syn}}_c = \frac{3}{4} \hat{\delta}^{\text{syn}} + D_0 \quad \text{and} \quad \hat{\delta}^{\text{syn}} = D_1 \eta^2 + \frac{D_2}{\eta^2} + \frac{8}{7} A_2 \kappa \eta^{3/2}$$  \hspace{1cm} (97)

where $D_0, D_1, D_2$ are three integration constants. After proper identification of the integration constants one sees that (95) and (97) are indeed equivalent, up however to the growing mode proportional to $D_1 \eta^2$. In order to eliminate this growing mode, a conserved “pseudo-tensor” $\tau_{\mu\nu}$, i.e. such that $\partial_\mu \tau_{\nu} = 0$, was built by Turok et al. [9] [16]. On superhorizon scales the 00 component of this pseudo-tensor is

$$\kappa \tau_{00} = \frac{3 \hat{\delta}^{\text{syn}}}{\eta^2} - \frac{6 \dot{C}'}{\eta} + \frac{\kappa A_1}{\sqrt{\eta}}$$  \hspace{1cm} (98)

and is, because $\tau_{\mu\nu}$ is conserved, constant. Since before the phase transition it was zero, it must be zero at all times:

$$\tau_{00} = 0.$$  \hspace{1cm} (99)

Now, from (15a) and (10), $\dot{C}' = -\hat{\delta}'/4$. Using (97) one then indeed sees that the condition $\tau_{00} = 0$ is equivalent to imposing $D_1 = 0$.

Introducing a pseudo-tensor is therefore a way to insure that the solution (97) is a solution of Einstein’s equations. In fact condition (99) is nothing but the relativistic Poisson equation written in synchronous gauge. It is not a way to choose the four true integration constants of the general solution of Einstein’s equations (15) and (87), that is $\Psi_0$ and $\Psi_1$, $\delta_0$ and $v_0$. These constants are fixed by Turok et al. [9] [17] by choosing the solution of (87) ($\hat{\delta}^{\text{syn}}_c' + 3 \dot{C}' = 0$) to be $\hat{\delta}^{\text{syn}} = -3 \hat{C}$ on the grounds that before the
transition all perturbations were zero, and by choosing “adiabatic” perturbations, that is imposing $\dot{S}_{\text{sym}} = 0$. This amounts to choosing $\Psi_0 = \delta_0 = 0$, which contradicts (77) but, as we saw, this does not really matter since the solution at late time is not sensitive to the matching conditions.

### 7.3 “Integral constraints” and compensation

Trashen, in [24], introduced a new way to extract from Einstein’s equations integral equalities which relate the total matter perturbations on a Roberston-Walker background. They are defined in synchronous gauge and read

$$\int_{\Sigma} (\delta T_{00}^0 - \mathcal{H} \delta T_{0k}^0 x^k) d^3x = \int_{\partial \Sigma} B^l dS_l$$

(100)

$$\int_{\Sigma} \left( \delta T_{0i}^0 + \mathcal{H} \delta T_{li}^0 \left[ \frac{1}{2} \delta^{il} x^p x_p - x^l x^i \right] \right) d^3x = \int_{\partial \Sigma} B^{li} dS_l$$

where $\delta T_{\mu \nu}$ is the perturbation of the total energy-momentum tensor of the defects and the cosmic fluid, where the quantities $B^l$ and $B^{li}$ are linear in the metric perturbations, and where $\Sigma$ is a comoving three-volume, $\partial \Sigma$ being its boundary.

These integral equalities can be interpreted as defining the energy of the perturbations (see [25]) and have been much invoked in the literature on cosmological perturbations seeded by topological defects (see e.g. [8-9] [26-27]). To understand better their role we shall restrict our attention to scalar perturbations and shall rewrite them, in the context here considered of a radiation dominated universe, as (see (4) (9) (10-11) (13))

$$\int_{\Sigma} U d^3x = \int_{\partial \Sigma} C^l dS_l$$

$$\int_{\Sigma} U x^i d^3x = \int_{\partial \Sigma} C^{li} dS_l \quad \text{with} \quad U \equiv \frac{3 \delta \eta^2}{\eta^2} + 3\kappa \left[ \rho^s - \frac{3}{\eta} v^s \right]$$

(101)

where $C^l$ and $C^{li}$ are different from $B^l$ and $B^{li}$ because integration by parts were performed. Written under the form (101) Trashen’s equalities relate gauge invariant quantities.

As always with conservation laws in general relativity, eq (101) give some information about the solution of Einstein’s equations without having to actually solve them. In the case at hand they tell the following: if the sources of the perturbations are localised within one horizon volume or if, as is the case with topological defects, they are randomly distributed on scales larger than the horizon, then the surface integrals on the right-hand side of (101) vanish for $\Sigma$ larger than a horizon volume. The left-hand sides are hence also zero and, for scales larger than the horizon, the integrands being almost constant must also vanish. Now the equation $U = 0$ is just Eq (15f) on super-horizon scales, that is nothing but another rewriting of the relativistic Poisson equation.

Trashen’s equalities were also interpreted in terms of “compensation” [8-9] [21] [27-28]. The idea here is that if the surface integrals in (101) vanish, then, as shown by Abbot and Trashen [29], the Fourier transform of $U$ must be of order $k^2$. Now, as we have seen, the Fourier transforms of the source functions, e.g. $\rho^s$ and $v^s$, are white noise i.e. of order
Consequently, $\delta^k$ must also be white noise, perfectly correlated to the sources so as to “compensate” them. Again this is a rephrasing of the Poisson equation.

Finally, the fact that $\delta^s$ is much smaller than $\Psi$ deep in the radiation era (see eq (74-76)) has also been interpreted by Durrer and Sakellariadou [23] in terms of “compensation” but it is perhaps simpler to say that this property follows from the continuity equation (15a).

In conclusion, we believe that the somewhat heavy terminology used in the literature of “pseudo-tensor”, “integral constraints” or “compensation” is not really required if one straightforwardly solves Einstein’s equations.

Acknowledgments

We are very grateful to Ruth Durrer and Mairi Sakellariadou for numerous and enjoyable discussions. We warmly thank Neil Turok for enlightening explanations and for pointing out a mistake in eq. (65). We thank R. Brandenberger for interesting remarks as well as Robert Caldwell and Joao Magueijo for commenting their work.

References


