NUCLEON STRUCTURE FUNCTIONS FROM A CHIRAL
SOLITON IN THE INFINITE MOMENTUM FRAME*

L. GAMBERG\textsuperscript{a)}, H. REINHARDT\textsuperscript{b)} and H. WEIGEL\textsuperscript{b)}

\textsuperscript{a)} Department of Physics and Astronomy
University of Oklahoma
440 West Brooks Avenue
Norman, Oklahoma 73019–0225, USA

\textsuperscript{b)} Institute for Theoretical Physics
Tübingen University
Auf der Morgenstelle 14
D-72076 Tübingen, Germany

ABSTRACT

We study the frame dependence of nucleon structure functions obtained within a chiral soliton model for the nucleon. Employing light cone coordinates and introducing collective coordinates together with their conjugate momenta, translational invariance of the solitonic quark fields (which describe the nucleon as a localized object) is restored. This formulation allows us to perform a Lorentz boost to the infinite momentum frame of the nucleon. The major result is that the Lorentz contraction associated with this boost causes the leading twist contribution to the structure functions to properly vanish when the Bjorken variable $x$ exceeds unity. Furthermore we demonstrate that for structure functions calculated in the valence quark approximation to the Nambu–Jona–Lasinio chiral soliton model the Lorentz contraction also has significant effects on the structure functions for moderate values of the Bjorken variable $x$.

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1. Introduction

Many model calculations of nucleon structure functions are plagued by the model being neither translationally invariant nor exhibiting Lorentz covariance. The reason simply is that the nucleon is described as a localized (non–relativistic) object. Such examples are the MIT–bag [1], the center of mass bag [2, 3] the Skyrme soliton [4, 5] and Nambu–Jona–Lasinio (NJL[6]) chiral soliton [7, 8] models. In the context of model structure function calculations this problem manifests itself as the “support problem” [3, 9, 10, 11, 12, 13, 14]. This refers to the fact that in these models the structure functions do not vanish in the region with Bjorken–x larger than unity. A number of techniques have been developed to address this problem. Sometime ago Jaffe developed a technique that restores Lorentz covariance in the 1+1 bag model while at the same time projecting both quark and nucleon states onto good momentum [14]. The merit of this approach is that while solving the support problem (by simultaneously restoring both translational invariance and Lorentz covariance) the corresponding sum rules are preserved\(^a\)[17, 18].

In this paper we extend this technique to a 3 + 1 dimension venue within the NJL chiral soliton model of the nucleon [7, 8]. In applying this “projection” technique to the NJL chiral soliton one finds the major difference to the bag model is that the soliton does not have a boundary where all densities vanish discontinuously; rather the densities decay exponentially in case the pion mass is non–zero thus enabling a straightforward application of this projection technique.

Generally speaking, denoting by Φ(\(x\)) a localized field configuration which, for example solves the classical equations of motion in a chiral soliton model, the above mentioned symmetries can formally be restored by defining a projected configuration

\[
Ψ_p(x) = \frac{1}{\mathcal{N}} \int d^4 y \ e^{-ip y} \ S(\bar{\Lambda}) \Phi \left( \bar{\Lambda}^{-1}(x - y) \right)
\]

(0.1)

where \(\bar{\Lambda}\) refers to a general Lorentz transformation and \(S(\bar{\Lambda})\) denotes the group operator associated with the representation \(\Phi\). \(\mathcal{N}\) represents a suitable normalization constant. Eq (0.1) essentially gives the relativistic generalization of the (ordinary) non–relativistic projection. The new element is the Lorentz boost \(\bar{\Lambda}\), which restores covariance. Furthermore we emphasize that \(Ψ_p(x)\) in eq (0.1) has the correct transformation properties of a field with good four–momentum \(p\). Further we note that if \(Φ(x)\) is a solution to the full time–dependent equations of motion in a translationally and Lorentz invariant model the configuration \(S(\bar{\Lambda})Φ \left( \bar{\Lambda}^{-1}(x - y) \right)\) will be a solution as well.

In case the localized field configuration is only a solution to the static equations of motion (\(i.e.\) time–translational invariance is not violated) the integral in eq (0.1) will only involve the spatial components.

\(^a\)Other (non–relativistic) projection techniques, which attempt to generate states with good momentum like Peierls–Yoccoz or Peierls–Thouless have been employed in structure function calculations [11, 12, 13, 15]. However, recently it has been stated [16] that these techniques yield badly normalized quark distributions which may be in contradiction with established sum rules.
An alternative path to restore the translational symmetry for a static localized configuration is to use the collective coordinate method of Gervais, Jevicki and Sakita [19] introducing a time–dependent collective coordinate $x_0(t)$ which parameterizes the spatial position of the localized field configuration

$$\Phi(x_0(t)) = \Phi(x - x_0(t)) . \quad (0.2)$$

Adopting this field configuration yields a classical Lagrange function $L(x_0, \dot{x}_0)$ for the collective coordinate $x_0(t)$. This enables one to extract the conjugate momentum $p = \partial L(x_0, \dot{x}_0)/\partial \dot{x}_0$, which is treated as a quantum variable by imposing canonical commutation relations, i.e. $[\{x_0\}, \{p_j\}] = i\delta_{ij}$. The wave–function of a nucleon with three–momentum $p$ will, for example in the case of a hedgehog chiral soliton, be considered a function of the collective coordinates $\varphi_p(x_0, t) = \langle x(t), A(t) | N(p) \rangle$ (0.3) together with the on–shell condition $p_0 = \sqrt{p^2 + m^2}$. The $D$ function denotes the projection onto states with good spin $(J, J_3)$ and isospin $(I = J, I_3)$ which is also required for chiral solitons as discussed in section 3. The corresponding collective coordinates are comprised in the SU(2) matrix $A(t)$ (see section 3 for a brief discussion in the context of the NJL model). These nucleon states are normalized like Fock states: $\langle p, I = J | p', I' = J' \rangle = \left( \frac{2\pi}{3} \right)^3 2^{3/2} p_0 \delta_{J, J'} \delta_{I, I'}$. One may then incorporate Lorentz covariance for nucleon matrix elements by appropriately transforming the operator under consideration $O(x - x_0, t)$ to a suitable frame. This transformation is defined by the boost $\Lambda_{\mu \nu}^p$

$$O \rightarrow S(\Lambda_p) \mathcal{O}(x' - x'_0, t') S^{-1}(\Lambda_p) \quad \text{where} \quad x''_{\mu} = (\Lambda_p^{-1})^\nu_\mu x'_{\nu} . \quad (0.4)$$

The important issue now is to identify the relevant frame. When e.g. computing nucleon form factors the Breit (or brick–wall) frame is preferred [22]. Here we propose that the infinite–momentum–frame (IMF [23]) on the light cone is most suited to study deep–inelastic–scattering (DIS). In the context of DIS one needs to evaluate nucleon matrix elements of the form

$$f_{\Gamma}(x) = \lim_{b \to 0} P_{\mu \nu}^{(\Gamma)} \int \frac{d^4 \xi}{4\pi} \exp(\text{i}q \cdot \xi) \langle N | [J_\mu(\xi), J_\nu(0)] | N \rangle . \quad (0.5)$$

Here $\Gamma$ refers to an appropriate spin–flavor matrix, $P_{\mu \nu}^{(\Gamma)}$ is the associated projector and $J_\mu(x)$ denotes the hadronic current $J_\mu = \bar{\Psi} \Gamma \gamma_\mu \Psi$. The leading twist contribution to the nucleon structure functions can be extracted by assuming the Bjorken limit

$$-q^2 \rightarrow \infty \quad \text{with} \quad x = \frac{-q^2}{2p \cdot q} \text{ fixed} , \quad (0.6)$$

\[b\]Cf. ref. [20] for a covariant approach.
where \( q \) denotes the momentum transferred to the nucleon target with momentum \( p \). In light cone coordinates the IMF is characterized by \( p^+ = (p^0 + p^3)/\sqrt{2} \to \infty \).

2. Relevance of the Infinite–Momentum–Frame

To begin, we briefly review the definition of light–cone coordinates with the \( x^3 \)–direction being distinct. Taking an arbitrary four–vector \( x^\mu \) the light–cone coordinates are defined as

\[
x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^3) = x_\mp \quad \mathbf{x}_\perp = (x^1, x^2) .
\]

A scalar product of two four–vectors \( x \) and \( y \) simply reads

\[
x \cdot y = x^- y^+ + x^+ y^- - \mathbf{x}_\perp \cdot \mathbf{y}_\perp .
\]

When computing structure functions in the Bjorken limit a frame may be chosen wherein the spatial components of both the nucleon and the photon are along the \( x^3 \)–direction, \( i.e. \)

\[
q_\perp = 0 \quad \text{and} \quad p_\perp = 0 .
\]

In this frame the Bjorken limit (0.6) becomes rather simple:

\[
q^- \to \infty \quad \text{with} \quad x = -\frac{q^+}{p^+} \quad \text{finite}
\]

as long as the reference frame is characterized by \( |p^+| > |p^-| \). A special frame of reference, which satisfies this condition, is the IMF defined by

\[
|p| = p^3 \to \infty ,
\]

\( i.e. \) the nucleon is moving in the positive \( x^3 \)–direction. It is crucial to note that only in the limit \( |p| \to \infty \) the parton model interpretation of the structure functions is completely consistent. The reason being that within the parton model the masses of the partons are neglected. This, of course, can only be made consistent with the kinematical conditions if the momenta of both the partons and the nucleon are large.

In the Bjorken limit the free field anti–commutation relations

\[
\left\{ \bar{\Psi}(\xi \), \Psi(0) \right\} = \frac{1}{2\pi} \partial \epsilon(\xi_0) \delta(\xi^2)
\]

are adopted to compute the commutator in eq (0.5). Although the spinor fields \( \Psi \) undergo some (complicated) non–perturbative interaction this approximation is well justified because in the Bjorken limit the intermediate quarks hit by the virtual photon are highly off–shell and hence not at all sensitive to the small momenta associated with this interaction. The typical momentum scale of this interaction is given by the binding of
the nucleon. Eq (0.12) yields expressions for the structure functions which are particularly simple in the light cone formulation (This result can be derived in analogy to the calculation presented in section 4 below) [24, 13]

\[ f_\Gamma(x) = \frac{\sqrt{2}}{4\pi} \int d\xi^+ \exp(-ixp^+ \xi^-) \langle N|\Psi_+(\xi^-)T^2\Psi_+(0)|N\rangle_{\xi^+=0,\xi^-=0}, \tag{0.13} \]

where \( \Psi_\pm = \frac{1}{2} (\gamma^0 \pm \gamma^3) \Psi \) and \( p^+ \) is the light–cone component of the nucleon momentum \( p \). Here \( \xi_\perp = 0 \) results from both the choice \( q_\perp = 0 \) for the momentum of the virtual photon and in addition from the restriction \( \xi^+ = 0 \), which is enforced by eq (0.12). The fact that the hyperplane \( \xi^+ = 0 \) is distinct indicates that when computing \( f_\Gamma(x) \) in a model which does not exhibit translational invariance a frame with \( \xi^+ = 0 \) is preferred. We will see in section 4 that the boost to the IMF enforces this condition. Furthermore it should be noted that in deriving the bilinear expression (0.13) a derivative of \( \langle N|\Psi(\xi)T^2\Psi(0)|N\rangle \) with respect to \( \xi^+ \) has been omitted. As the leading twist contribution of this matrix element is a polynomial in \( \xi \cdot p \) this approximation is well justified in the IMF since therein \( p^- = 0 \).

Consider now the special case where \( \Gamma \) is a projector, i.e. \( \Gamma^2 = \Gamma \). When inserting a complete set of states \( |n\rangle \) and assuming translational invariance the expression (0.13) may be rewritten as

\[ f_\Gamma(x) = \frac{1}{\sqrt{2}} \sum_n \frac{dp_n^3}{4\pi p_n^0} \delta \left(p_n^+ - (1-x)p^+\right) \left| \langle n|\Psi_+(0)\Gamma|N\rangle \right|^2, \tag{0.14} \]

where \( p_n \) is the eigenvalue of the momentum in state \( |n\rangle \) which possess diquark quantum numbers. From the \( \delta \)-function in eq (0.14) one recognizes that because \( p_n^+ > 0 \) for massive intermediate states the structure functions will vanish in region \( x \approx 1 - \epsilon \) unless \( p_+ \to \infty \). Apparently this just restates that the parton model is only well defined in the IMF. Stated otherwise, the numerical integration in (0.14) may be difficult because sizable contributions may stem from the region \( p_n^+ \to \infty \).

From the preceding discussion on the parton model interpretation of the nucleon structure functions we are inclined to conclude that the IMF is indeed singled out as the distinct frame to study nucleon structure functions in DIS.

3. The NJL–Model Chiral Soliton

Before continuing with the discussion of the IMF we will briefly review the subject of the chiral soliton in the NJL model. In particular the origin of eq (0.3) will be discussed here. Starting point is the bosonized version of the NJL–model action [25]

\[ A = \text{Tr}_A \log(iD) + \frac{1}{4G_{NJL}} \int d^4x \, \text{tr} \left( m^0 \left( M + M^\dagger \right) - MM^\dagger \right), \tag{0.15} \]

\[ D = i\partial - (M + M^\dagger) - \gamma_5 \left( M + M^\dagger \right), \tag{0.16} \]
where $M = S + iP$ comprises composite scalar ($S$) and pseudoscalar ($P$) meson fields. The model parameters (cut-off $\Lambda$, coupling constant $G_{NJL}$ and current quark mass matrix $m^0$) are fixed to reproduce the meson properties, in particular those of the pion.

The chiral soliton is given by the hedgehog configuration of these meson fields

$$M_H(x) = m \exp (i \tau \cdot \hat{x} \Theta(r)) .$$

(0.17)

In order to compute the functional trace in eq (0.15) for this static configuration a Hamilton operator, $h$ is extracted from the Dirac operator (0.16), i.e. $D = i\gamma_0 (\partial_t - h)$ with

$$h = \alpha \cdot p + m \exp (i \gamma_5 \tau \cdot \hat{x} \Theta(r)) .$$

(0.18)

We denote the eigenvalues and eigenfunctions of $h$ by $\epsilon_\mu$ and $\Psi_\mu$, respectively. In the proper time regularization scheme the NJL model energy functional is found to be [26, 8]

$$E[\Theta] = \frac{N_C}{2} \epsilon_v (1 + \text{sign}(\epsilon_v)) + \frac{N_C}{2} \int_1^\infty \frac{ds}{\sqrt{4\pi s^3}} \sum_\nu \exp (-s \epsilon_\nu^2)
+ m^2 \int d^3r (1 - \cos \Theta(r)),
$$

(0.19)

with $N_C = 3$ being the number of color degrees of freedom. The subscript “$v$” denotes the valence quark level. This state is the distinct level bound in the soliton background, i.e. $-m < \epsilon_v < m$. The chiral angle, $\Theta(r)$, is obtained by self–consistently extremizing $E[\Theta]$ [7]. States possessing nucleon quantum numbers are generated by taking the large amplitude fluctuations (translation and rotation) to be time dependent, cf. eq (0.2),

$$M(x, t) = A(t) M_H(x - x_0(t)) A^\dagger(t) ,$$

(0.20)

which introduces the collective coordinates discussed in the introduction. Upon substitution of the ansatz (0.20) into the action functional (0.15) and subsequently expanding to quadratic order in the time derivatives of the collective coordinates the Lagrange function for these coordinates is extracted. It is then straightforward to canonically quantize this system. This introduces the linear momentum $p$, the spin $J$ and isospin $I$ as the quantities canonical to the collective coordinates. In particular the wave–function (0.3) is obtained as the eigenfunction of the Hamilton operator in the space of the collective coordinates.

Expectation values of quark–bilinears as in eq (0.13) are expressed as (regularized) sums over bilinear combinations of all eigenfunctions $\Psi_\mu$. In practice, however, it turns out that the dominant contributions to physical quantities stems from the distinct valence level $\Psi_v$ [8]. It is therefore reasonable to approximate these bilinears by their valence quark contribution. In order to obtain nucleon rather than soliton structure functions the cranking piece to the wave–function, which is induced by the collective rotation $A(t)$,
must be included. That is, the valence quark wave–function employed to approximate the bilinears in the structure functions reads

$$\Psi_v(x, t) = e^{-i\epsilon_v t} A(t) \left\{ \Psi_v(x) + \frac{1}{2} \sum_{\mu \neq v} \Psi_{\mu}(x) \frac{\langle \mu | \tau \cdot \Omega | v \rangle}{\epsilon_v - \epsilon_{\mu}} \right\},$$

(0.21)

where the $\Psi_{\mu}$, which appear within the curly parenthesis, are eigenfunctions of the Dirac Hamiltonian (0.18). Upon canonical quantization the angular velocity $\Omega$ is substituted by the nucleon spin operator $J = \alpha^2 \Omega$, with $\alpha^2$ being the moment of inertia [26, 8]. The matrix elements of the collective rotations are obtained via $\langle N | \text{tr}(\tau_i A \tau_j A^\dagger) | N \rangle = -(8/3) \langle N | I_i J_j | N \rangle$ [5]. This can easily be verified using the wave–function (0.3). In what follows we will denote by $\varphi_p(x_0, t)$ only that part of the collective wave–function which depends on the translational collective coordinate $x_0$. The (iso)rotational degrees of freedom are supposed to be contained in the state vector $|N\rangle$.

4. Boosting to the Infinite–Momentum–Frame

Here we present the major topic of this paper, namely the calculation of the chiral soliton model structure function in the IMF. This calculation is motivated by Jaffe’s conjecture [14]; namely that the expressions for structure functions with proper support, obtained in a covariant 1+1–dimensional model, can formally be transferred to a realistic 3 + 1–dimensional model, which describes the nucleon as an extended object. When addressing this conjecture within the context of a chiral soliton model it is crucial to note that, as the soliton is a continuous field configuration, problems originating from the existence of the bag boundary (e.g. periodic boundary conditions and a fluctuating bag boundary [14, 11]) do not occur.

In light–cone coordinates the boost from the rest frame to the IMF in $x^3$–direction may be parameterized in terms of the rapidity $\Omega$:

$$p^+ = e^{+\Omega} \left( \frac{m}{\sqrt{2}} \right), \quad p^- = e^{-\Omega} \left( \frac{m}{\sqrt{2}} \right), \quad p_\perp = 0,$$

(0.22)

where $m$ refers to the invariant mass of the nucleon. Apparently the IMF is characterized by the limit $\Omega \to \infty$. In this limit the transformation matrix for Dirac spinors becomes

$$S(\Lambda) = \sqrt{\frac{p^0 + m}{2m}} 1 + \sqrt{\frac{p^0 - m}{2m}} \alpha_3,$$

$$\text{IMF} \quad S(\Lambda_0) + \mathcal{O} \left( \frac{m}{p^+} \right) \quad \text{with} \quad S(\Lambda_0) = \frac{1}{2} \exp \left( \frac{\Omega}{2} \right) (1 + \alpha_3),$$

(0.23)

where $\alpha_3$ denotes a Dirac matrix. In what follows we will consider localized quark spinors, which result from some static soliton calculation [8], boosted to the IMF

$$\Psi(\xi) \rightarrow S(\Lambda) \Psi \left( \Lambda^{-1}(\xi - x_0) \right),$$

$$\text{IMF} \quad \frac{1}{2} (1 + \alpha_3) \Psi \left( e^{\Omega} (\xi^- - x_0^-), \xi_\perp - x_{0\perp}, (\xi^+ - x_0^+) = 0 \right) \exp \left( \frac{\Omega}{2} \right),$$

(0.24)
Note that the $\xi^-$ coordinate acquires the factor $\exp(\Omega)$ due to the appearance of $\Lambda^{-1}$ in the argument. Here the collective coordinate $x_0^\mu = (0, x_0^a)$, which labels the position of the soliton has been introduced, cf eq (0.2). As noted before, this collective coordinate serves to generate states of good momentum $p$ from a configuration which is not translationally invariant. Eventually an integration over this coordinate is performed. Furthermore we have written the rapidity to the right of the wave–function. Later this ordering will be important when elevating $p^+$ as an operator in the space of the collective coordinates.

In the discussion above we emphasized that the frame with $\xi^+ = 0$, i.e. the null plane, was distinct for structure function calculations. It is hence the IMF ($p^+ \to \infty$) which is preferred, i.e. to leading order in $m/p^+$, this condition is satisfied because $\xi''^+ = e^{-\Omega} \xi^+ \to 0$. This has already been used in eq (0.24).

Assuming again the free field commutation relations (0.12) for the Dirac fields in the defining equation for the hadronic tensor (0.5) we generalize the rest frame expression of the forward moving intermediate quark to the structure function $F_1^{(+)}(x)$ [9, 10] by substituting the boosted quark wave function (0.24) (see appendix A for more details)\textsuperscript{c},

$$F_1^{(+)}(x, q^2) = - \lim_{q^- \to -\infty} \frac{q^0}{4\pi} \int d\xi^+ d\xi^- d\xi_\perp d^3x_0 \exp \left[ i \left( q^+ \xi^+ + q^- \xi^- \right) \right] \frac{\delta(\xi^2) \epsilon(\xi^+ + \xi^-)}{2\pi}$$

$$\times \langle p^+, p^- = \frac{m}{\sqrt{2p^+}}, p_\perp = 0 \mid x_0 \rangle$$

$$\times \langle N \mid S(\Lambda \Omega) \Psi \left( e^{\Omega}(\xi^- - x_0^-), \xi_\perp - x_{0\perp}, (\xi^+ - x_0^+) = 0 \right) \rangle^\dagger \Gamma^\rho$$

$$\times S(\Lambda \Omega) \Psi \left( e^{\Omega}(-x_0^-), -x_{0\perp}, x_0^+ = 0 \right) \mid N \rangle$$

$$\times \langle x_0 \mid p^+, p^- = \frac{m}{\sqrt{2p^+}}, p_\perp = 0 \rangle.$$  

(0.25)

The contribution of the backward moving quark, $F_1^{(-)}(x, q^2)$, comes with the opposite sign and the arguments of the spinors exchanged. For the ongoing discussion it is, however, sufficient to only consider $F_1^{(+)}(x, q^2)$. Here we treat $x_0 = \hat{x}_0$ as an operator in the space of the collective coordinates in the sense that $f(\hat{x}_0)\langle x_0 \rangle = f(x_0)\langle x_0 \rangle$. Of course, we should take $x_0^- = -x_0^3/\sqrt{2}$ as there is no collective coordinate in the time component since the soliton preserves time–translational invariance (when employing light cone coordinates we omit the vector notation).

Now it is appropriate to express the arguments of the spinors in terms of integrals over $\delta$–functions and Fourier–expand the latter. This introduces dummy variables $\zeta_\mu$ and $\zeta' \mu$ as well as their conjugates $\alpha$ and $\alpha'$

$$F_1^{(+)}(x, q^2) = - \lim_{q^- \to -\infty} \frac{q^0}{8\pi^2} \frac{m^2}{2} \exp(\Omega)$$

$$\times \int \frac{dx_0^-}{\sqrt{2}} d^2x_{0\perp} \int d\xi^+ d\xi^- d\xi_\perp \exp \left[ i \left( q^+ \xi^+ + q^- \xi^- \right) \right] \delta(\xi^2) \epsilon(\xi^+ + \xi^-)$$

\textsuperscript{c}Here the projector simply is $P_{\mu\nu} = g_{\mu\nu}$.
Here we have made explicit the integration over the collective coordinate $x_0$ as well as the Lorentz boost $S(\Lambda\Omega)$. Also we have split the nucleon state in a piece containing the collective momentum $|p\rangle$ and the remaining degrees of freedom $|N\rangle$ like e.g. isospin, see also eq (0.3). In addition we have introduced the ordering

$$\Lambda = -x_0^- p^+ = -\frac{1}{2} \left( x_0^- p^+ + p^+ x_0^- \right).$$

When implementing Lorentz covariance in the $x^3$–direction, which is sufficient since we consider nucleon states with $p_\perp = 0$, this object as well as the momentum $p^+$ have to be considered as operators in the space of the collective coordinates. For this reason we have not only taken the Hermitian ordering but also carefully treated the ordering of the collective wave–functions $\langle x_0 | p \rangle$. As $p^+$ and $x_0^-$ are conjugate to each other we have the commutation relations

$$[x_0^-, p^+] = -i \quad \text{and} \quad [\Lambda, p^+] = ip^+. \quad (0.28)$$

Imposing these relations should be considered as the semiclassical quantization of the translational degrees of freedom for the classical soliton configuration. It is the analogue of the cranking approach to generate states of good spin and isospin [5] for the chiral soliton, which is nothing but the Lagrange form of the collective coordinate method [19]. Care has to be taken when computing the structure function (0.26) because these operators act on the collective wave–function $\langle x_0 | p \rangle$. At this point also the ordering chosen in (0.24) is crucial. Having put $\exp(\Omega/2) = p^+/2\sqrt{2}m$ to the right of the quark wave–function not only ensures that the boost indeed goes with the nucleon momentum in the IMF but also provides consistent normalization of quark and nucleon wave–functions. We already made use of this ordering when simply writing $\exp(\Omega)$ in eq (0.26).

On repeated application of the commutation relations (0.28) one finds the operator identities

$$\exp \left[ i\alpha \left( p^+ \xi^- + \Lambda \right) \right] = \exp \left( ip^+ \xi^- \right) \exp (i\alpha \Lambda) \exp \left( -ip^+ \xi^- \right). \quad (0.29)$$

$$\exp \left( -ip^+ \xi^- e^{-\alpha} \right) = \exp (i\alpha \Lambda) \exp \left( -ip^+ \xi^- \right) \exp ( -i\alpha \Lambda). \quad (0.30)$$

We could as well have chosen an ordering wherein $\exp(\Omega/2)$ would have been to the left of the quark wave–function in eq (0.24). In that case a consistent normalization would require to introduce a scale dependent mass $\exp(\Omega/2) = p^+/2\sqrt{2} \exp(\alpha^\mu) m$. The final result for the structure functions would remain unchanged. This consideration, however, also indicates that the restoration of Lorentz covariance can only be accomplished in the subspace of the nucleon ground state. A simultaneous treatment of excited baryons does not seem to be feasible at present.
It is very instructive to discuss the physical content of these equations. First we note that the unitary operator \( \exp(i p^+ \xi^-) \) generates translations on the light cone, i.e. \( x_0^- \), by the amount of \( \xi^- \). As the boost operator \( \Lambda \) is linear in \( x_0^- \) any function \( f(\Lambda) \) transforms as

\[
\exp(i p^+ \xi^-) f(\Lambda) \exp(-i p^+ \xi^-) = f(\Lambda + p^+ \xi^-) .
\]

Eq (0.29) corresponds to \( f(\Lambda) = \exp(i \alpha \Lambda) \). The relation (0.29) furthermore ensures that a shift \( (\delta \xi_\mu) \) in the coordinate \( \xi_\mu \) just adds the phase \( \exp(i p^+ \delta \xi^-) \). Of course, this shows that we have restored translational invariance in the subspace under consideration which is characterized by \( p_\perp = 0 \). Similarly the operator \( \exp(-i \alpha \Lambda) \) generates boosts with the rapidity \( \alpha \):

\[
\exp(-i \alpha \Lambda) g(p^+) \exp(i \alpha \Lambda) = g(e^\alpha p^+) .
\]

Eq (0.30) is obtained for \( g(p^+) = \exp(-i p^+ \xi^-) \). Hence by imposing the commutation relations (0.28) in the space of the collective coordinates we have implemented the correct transformation properties of the localized Dirac spinor for the problem at hand.

We are now in a position to calculate the integrals over the dummy variables \( d\zeta^-,..., d\alpha' \). We find

\[
\int d\zeta^- d\zeta^+ d\zeta_\perp d\zeta'_\perp \frac{d\alpha}{2\pi} \frac{d\alpha'}{2\pi} \delta^2(\zeta_\perp - x_{0\perp} - \zeta'_\perp) \delta^2(-x_{0\perp} - \zeta_\perp) 
\times \exp \left[ i (\alpha' - \alpha) \right] \varphi_p^+ (x_0^-, x_{0\perp}) \exp \left[ i (\alpha' - \alpha) \right] \varphi_{p^+} (x_0^-, x_{0\perp}) \exp \left[ i \epsilon (\zeta'_0 - \zeta_0) - \frac{m}{\sqrt{2}} (\alpha' \zeta' - \alpha \zeta) \right] (0.31)
\]

Here \( \epsilon \) denotes the energy eigenvalue of \( \Psi \) which is determined from the static Dirac equation. Once again we make use of \( \Lambda \) being the boost operator in the space of the collective coordinate to perform the integral over the collective coordinate \( x_0 \)

\[
\int \frac{dx_0^-}{\sqrt{2}} d^2 x_{0\perp} \varphi_p^+ (x_0^-, x_{0\perp}) \exp \left[ i (\alpha' - \alpha) \right] \varphi_{p^+} (x_0^-, x_{0\perp}) \delta^2(\zeta_\perp - x_{0\perp} - \zeta'_\perp) 
\times \exp \left[ i (\alpha' - \alpha) \right] \varphi_p^+ (x_0^-, x_{0\perp}) \exp \left[ i (\alpha' - \alpha) \right] \varphi_{p^+} (x_0^-, x_{0\perp}) \exp \left[ i \epsilon (\zeta'_0 - \zeta_0) - \frac{m}{\sqrt{2}} (\alpha' \zeta' - \alpha \zeta) \right] (0.32)
\]

This \( \delta \)-function enforces \( \alpha = \alpha' \) which removes the \( (c-) \) number ambiguity stemming from the various definitions of the boost operator (0.27). Substituting the above results into eq (0.26) yields

\[
F^{(+)pr}_1(x,q^2) = - \lim_{q^- \to \infty} \frac{q^0 m p^+}{4 \sqrt{2 \pi}} \int d\xi^- d\xi^+ d^2 \xi_\perp \exp \left[ i (q^+ \xi^- + q^- \xi^+) \right] \delta(\xi^-) \epsilon(\xi^+ + \xi^-) 
\times \int dz^- d^2 z_\perp d\gamma^- d^2 \gamma_\perp \frac{d\alpha}{2\pi} \delta^2(\gamma_\perp - x_\perp) \exp \left[ i \frac{\sqrt{2}}{\sqrt{2}} (\epsilon - m \alpha) \gamma^- \right] 
\times \exp \left[ i p^+ z^- (1 - e^{-\alpha}) \right] \langle N | \Psi^+_\perp \left( z + \frac{\gamma}{2} \right) \Gamma_\rho \Psi_+ \left( z - \frac{\gamma}{2} \right) | N \rangle , (0.33)
\]
where we have changed the dummy variables to $\gamma = \zeta' - \zeta$ and $z = (\zeta + \zeta')/2$. The expression (0.33) can further be simplified by introducing the Fourier transform of the Dirac matrix $\Gamma$

$$\Psi(y_\perp, y_3 = -\frac{y}{\sqrt{2}}) = \int \frac{d^2 k_\perp dk_3}{2\pi^2} \exp \left[ i \left( \frac{k_3 y_3}{\sqrt{2}} - k_\perp \cdot y_\perp \right) \right] \tilde{\Psi}(k_\perp, k_3) . \quad (0.34)$$

This now allows us to integrate over the dummy variable $z$

$$F_1^{(+)}(x, q^2) = -\lim_{q^- \to \infty} \frac{q^3 mp^+}{\pi} \int d\xi^- d\xi^+ d^2\xi_\perp \exp \left[ i \left( q^+ \xi^++ q^- \xi^- \right) \right] \delta(\xi^2) \epsilon(\xi^+ + \xi^-)$$

$$\times \int \frac{d\gamma^- d^2\gamma_\perp}{2\pi^2} \exp \left[ i p^+ x^- \left( 1 - e^{-\alpha} \right) \right] \delta^2(\gamma_\perp - x_\perp) \quad (0.35)$$

$$\times \int \frac{d^2 k_\perp dk_3}{2\pi^2} \exp \left[ i \sqrt{2} (\epsilon - m\alpha - k_3) \gamma^- + i \gamma_\perp \cdot k_\perp \right] \langle N | \tilde{\Psi}^\dagger_+(k) \Gamma^+ \tilde{\Psi}_+(k) | N \rangle .$$

The additional factor $1 - e^{-\alpha}$ in the exponential is nothing but the Lorentz contraction associated with the boost into the IMF. Its appearance is crucial and may be interpreted as an effect associated with the relativistic recoil.

We may now integrate over the coordinate $\xi$ by treating $\delta(\xi^2)$ in the well–known manner (cf. ref. [24]). This enforces $\xi^+ = 0$ as well as $\xi^- = 0$. The latter than allows us to also perform the $\gamma^-\perp$ integral

$$F_1^{(+)}(x, q^2) = -mp^+ \int \frac{d\alpha}{2\pi} d\xi^- \exp \left[ -ip^+ \xi^- \left( x - 1 + e^{-\alpha} \right) \right]$$

$$\times \int d\gamma^- d^2\gamma_\perp \frac{d^2 k_\perp dk_3}{2\pi^2} \exp \left[ i \sqrt{2} (\epsilon - m\alpha - k_3) \gamma^- \right] \langle N | \tilde{\Psi}^\dagger_+(k) \Gamma^+ \tilde{\Psi}_+(k) | N \rangle . \quad (0.36)$$

Here we have also introduced the Bjorken variable via $q^+ = -xp^+$ in the IMF, see eq (0.10). At this point we recognize that the integration over $\xi^-$ will yield $\delta(x - 1 + e^{-\alpha})$ which guarantees that $F_1^{(+)}(x, q^2)$ vanishes for $x > 1$. It is important to note that this $\delta$–function will also appear in case the IMF condition $p^+ \to \infty$ is relaxed. However, the remainder of the matrix element will be more complicated and the $\alpha$–integration, which we perform in the next step, might become infeasible. Performing now the $\alpha$–integration yields

$$F_1^{(+)}(x, q^2) = \frac{m}{1-x} \int d\gamma^- \frac{d^2 k_\perp dk_3}{2\pi^2} \exp \left[ i \sqrt{2} (m \ln(1-x) + \epsilon - k_3) \gamma^- \right]$$

$$\times \langle N | \tilde{\Psi}^\dagger_+(k) \Gamma^+ \tilde{\Psi}_+(k) | N \rangle . \quad (0.37)$$

The $\gamma^-$ integral yields yet another $\delta$–function which fixes $k_3 = m\ln(1-x) + \epsilon$. Introducing spherical coordinates in momentum space and making the Dirac matrix $\Gamma^+$ explicit we arrive at

$$F_1^{(+)}(x, q^2) = \frac{m}{\pi(1-x)} \int_{k_{\text{min}}}^{\infty} kdkd\varphi \langle N | \tilde{\Psi}^\dagger_+(k) \left( 1 + \frac{\alpha_3}{\lambda} \right) \tilde{\Psi}_+(k) | N \rangle \mid_{\cos\theta = \frac{m \ln(1-x) + \epsilon}{k}} . (0.38)$$
with \(k_{\text{min}} = |m \ln(1 - x) + \epsilon|\). Application of the parity transformation finally leads to a very familiar expression [14]

\[
F_1^{(+)}(x, q^2) = \frac{m}{\pi(1 - x)} \int_{k_{\text{min}}}^{\infty} k dk d\varphi \langle N | \tilde{\Psi}^\dagger(k)(1 - \alpha_3) \tilde{\Psi}(k) | N \rangle \bigg|_{\cos\Theta = -\frac{m \ln(1 - x) + \epsilon}{k}}
\]

(0.39)

Apparently the leading order in a large \(p^+\) expansion does not depend on \(p^+\) itself. Without the boost into the IMF we had found previously [17]

\[
F_1^{(+)}(x, q^2) = \frac{m}{\pi} \int_{k_{\text{min}}}^{\infty} k dk d\varphi \langle N | \tilde{\Psi}^\dagger(k)(1 - \alpha_3) \tilde{\Psi}(k) | N \rangle \bigg|_{\cos\Theta = \frac{m x - \epsilon}{k}}
\]

(0.40)

together with the lower boundary \(k_{\text{min}} = |m x - \epsilon|\). Comparing eqs (0.39) and (0.40) we finally establish that the structure functions which are calculated from a static localized quark configuration transform as

\[
f_{\text{IMF}}(x) = \Theta(1 - x) \left(1 - x\right) f_{\text{RF}}\left(-\ln(1 - x)\right)
\]

(0.41)

when the quark fields are boosted from the nucleon rest frame (RF) into the IMF. The main result, of course, is that in the large \(p^+\) expansion the structure functions indeed have support only in the interval \(x \in [0, 1]\). The transformation (0.41) was already obtained in ref. [14] for the 1 + 1 dimensional bag model. From the observation that even in the realistic case the DIS is essentially 1 + 1 dimensional, Jaffe then conjectured that the same relation would be true for 3 + 1 dimensions as well. In contrast to the 1 + 1 dimensional model of ref. [14], however, a boost to the IMF must be implemented to satisfy the condition \(x^+ = 0\). In the 1 + 1 dimensional model this is a natural choice.

Finally it is important to note that the transformation (0.41) receives additional (phenomenological) support from the naïve spectator model. In the rest frame one finds for large \(x\) (actually larger than one) \(f_{\text{RF}}(x) \approx \exp(-\kappa N_s x)\) [27]. Hence the transformation (41) leads to structure functions which drop like \(f_{\text{IMF}}(x) \rightarrow (1 - x)^{\kappa N_s - 1}\) as \(x\) approaches unity. The constant of proportionality \(\kappa\) could eventually be determined numerically and compared to the spectator model result \((1 - x)^{2N_s - 1}\) where \(N_s\) refers to the number of spectators. In this respect, in the large \(N_C\) limit \(N_s\) can be identified with the number of colors, \(N_C\). Effectively, this comparison plays the role of a consistency check to the transformation (0.41).

5. Numerical Effects

In the previous section we have verified Jaffe’s conjecture [14] that structure functions, which are calculated from a localized 3 + 1 dimensional quark configuration, get modified according to eq (0.41) when Lorentz covariance is restored in the direction defined by the

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\(^*\) In ref. [11] a comparison of the projection (0.41) and the method of Peierls–Yoccoz has been reported for the case of the 1 + 1–dimensional bag model. Only slight differences originating from the sharp bag boundary were observed.
Figure 0.1: The unpolarized structure functions $F_1(x)$ for deep-inelastic electron nucleon scattering as functions of the Bjorken variable $x$. Left panel: rest frame calculation of ref. [17], right panel: projection according to eq (0.41).

incident (virtual) photon by boosting to the IMF. Here we want to briefly demonstrate with the help of an example that this not only provides the proper support of the structure function but also effects the predicted structure functions at small and moderate $x$. In figure 1 we therefore compare the valence quark approximation to the unpolarized structure function $F_1(x)$ in deep-inelastic electron nucleon scattering as it arises in the NJL chiral soliton model [17]. The full structure function $F_1(x)$ also contains the contribution of the backward moving intermediate quark, $F_1^{(-)}(x)$, which is transformed according to eq (0.41) as well. Apparently the IMF structure functions drop to zero already at $x \approx 0.8$. On the other hand they are more pronounced in the intermediate range $x \approx 0.4$. This is, of course, a consequence of the fact that the transformation (0.41) conserves the area under the curve.

6. Conclusions

The calculation of nucleon structure functions in the Bjorken limit singles out the hyperplane $\xi^+ = 0$. We have seen that upon transformation to the IMF this condition can be satisfied even for models where the nucleon emerges as a (static) localized object. For quark soliton models this transformation can be performed by introducing a collective coordinate which parameterizes the position of the soliton and subsequently defining a boost in the space of the this coordinate. Elevating this coordinate as well as its conjugate momentum to operators in the framework of the semiclassical quantization not only gener-
states of good momentum but allows one to restore Lorentz covariance, at least in the \( \xi^\pm \) subspace. Fortunately this is sufficient when calculating structure functions because those components of the nucleon momentum which are orthogonal to this subspace may be put to zero. When computing the structure functions in a static soliton model with the quark spinors, which reside in the background of the soliton, boosted to the IMF we have observed that the common problem of improper support for the structure functions, \textit{i.e.} non–vanishing structure functions for \( x > 1 \), is cured along the line suggested by Jaffe \cite{14} some time ago. The reason is that the Lorentz contraction associated with the boost to the IMF maps the infinite line exactly onto the interval \( x \in [0, 1] \). Furthermore for the case of electron–nucleon scattering we have seen that this Lorentz contraction effects the structure functions also at small and moderate \( x \).

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Appendix: Boosting to the Infinite Momentum Frame

In this appendix we detail the two step process of restoring proper support to nucleon structure functions. We recall that the RF contribution of the forward moving intermediate quark to the unpolarized spin structure function, \( F_1^{(+)}(x, q^2) \) is given in terms of rest–frame quark wave functions \cite{9, 10, 17},

\[
F_1^{(+)}(x, q^2) = - \lim_{q^- \to \infty} \frac{q^\rho}{4\pi} \int d\xi^+ d\xi^- d^2\xi_\perp \frac{\delta(\xi^2) \epsilon(\xi^+ + \xi^-)}{2\pi} e^{i(q^- \xi^- + q^+ \xi^+)}
\]

\[
\times \int \frac{dx_0^- dx_0^\perp}{\sqrt{2}} \langle N | \Psi_+^\dagger(\xi - x_0) \Gamma^\rho \Psi_+(-x_0) | N \rangle,
\]

(A.1)

where \( |N\rangle \) is related to the zero momentum nucleon state \cite{9, 10, 22},

\[
|p = 0\rangle = [(2\pi)^3 2m]^{1/2} |N\rangle.
\]

(A.2)

In eq (A.1) the integration over the collective coordinate \( x_0 \) corresponds to averaging over the position of the localized quark fields. In the next step we perform the Lorentz transformation of the quark wave functions from the rest frame to the IMF specified by \( \Lambda \) and eqs (0.22) and (0.23),

\[
\Psi_{x_0}(\xi) \overset{\text{IMF}}{\rightarrow} \Psi e^{\Omega((\xi_0^- - x_0^-), \xi_0^\perp - x_0^\perp, \xi^+ - x_0^+ = 0)} S(\Lambda \Omega).
\]

(A.3)

Substituting this boosted quark wave function into eq (A.1) yields the Lorentz transformed expression of eq (A.1),

\[
F_1^{(+)}(x, q^2) = - \lim_{q^- \to \infty} \frac{q^\rho}{4\pi} \int d\xi^+ d\xi^- d^2\xi_\perp \frac{dx_0^- dx_0^\perp}{\sqrt{2}} \frac{\delta(\xi^2) \epsilon(\xi^+ + \xi^-)}{2\pi} e^{i(q^- \xi^- + q^+ \xi^+)}
\]

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\[ \langle p^+, p^- = m \frac{\sqrt{2p^+}}{m}, p^\perp = 0 | x_0 \rangle \]
\[ \times \langle N | \left[ \Psi \left( e^\Omega (\xi^-, -x_0^-), \xi^\perp - x_0^\perp, \xi^+ - x_0^+ = 0 \right) S(\Lambda_\Omega) \right]^\dagger \Gamma^p \]
\[ \times \Psi \left( e^\Omega (-x_0^-), -x_0^\perp, x_0^+ = 0 \right) S(\Lambda_\Omega) | N \rangle \]
\[ \times \langle x_0 | p^+, p^- = m \frac{\sqrt{2p^+}}{m}, p^\perp = 0 \rangle , \]
which yields eq (0.25). In addition we have identified the nucleon wave function in terms of the collective coordinate \( x_0 \). It is most convenient to treat the collective coordinate with the help of Dirac–\( \delta \) functions as it allows us to extract the dependence on the rapidity \( e^\Omega = \sqrt{2p^+}/m \)
\[ \Psi \left( \frac{\sqrt{2p^+}}{m} (\xi^- - x_0^-), \xi^\perp - x_0^\perp, \xi^+ - x_0^+ = 0 \right) \]
\[ = \frac{m}{\sqrt{2} p^+} \int \frac{d\xi^-}{\sqrt{2}} \frac{d\xi^\perp}{\sqrt{2}} \delta \left( \xi^- - x_0^- - \frac{m}{\sqrt{2} p^+} \zeta^- \right) \]
\[ \times \delta^2 (\xi^\perp - x_0^\perp - \zeta^\perp) S(\Lambda_\Omega) \Psi (\zeta) \]
\[ = \frac{m}{\sqrt{2}} \int \frac{d\zeta^-}{\sqrt{2}} \frac{d\zeta^\perp}{\sqrt{2}} \frac{d\alpha}{2\pi} \exp \left[ i\alpha \left( p^+ \xi^- + \Lambda - \frac{m}{\sqrt{2}} \zeta^- \right) \right] \]
\[ \times \delta^2 (\xi^\perp - x_0^\perp - \zeta^\perp) S(\Lambda_\Omega) \Psi (\zeta) . \]

Apparently the dependence on the nucleon momentum \( p^+ \) has disappeared from the argument of the quark wave–function \( \Psi (\zeta) \). This technique has repeatedly been used in section 4.

References