Radiative Correction to the Nuclear-Size Effect and Hydrogen-Deuterium Isotopic Shift

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Abstract

The radiative correction to the nuclear charge radius contribution to the Lamb shift of order \( \alpha(Z\alpha)^5m_e^2 < r^2 > \) is calculated. In view of the recent high precision experimental data, this theoretical correction produces a significant contribution to the hydrogen-deuterium isotopic shift.
I. INTRODUCTION

Spectacular experimental results were obtained in recent years for the intervals between levels with different principal quantum numbers and for the ground state $1S$ Lamb shift in hydrogen and deuterium [1–4]. Experimental error for the isotopic shift between the hydrogen and deuterium in the $1S − 2S$ transition was reduced to 0.2 kHz (see [5] and a private communication from Hänsch, cited in [6]).

These experimental achievements have created new challenges for the hydrogen-deuterium isotopic shift theory, which now should account at least for all corrections to the order of one tenth of kHz. In a surge of theoretical activity there appeared a number of papers [7–14,6,15,16], where corrections to the isotopic shift induced by the nuclear charge radius and polarizablity were considered, and a set of old results in the field [17,18] were either rederived or improved. However, to the best of our knowledge the radiative correction of order $\alpha (Z\alpha)^5 m_e^3 < r^2 >$ to the main nuclear radius contribution was not taken into account in discussion of the recent experimental data. As we will show below this correction is about 0.6 kHz for the hydrogen-deuterium isotopic shift and should be considered on par with the other contributions.

II. RADIATIVE CORRECTION TO THE FINITE SIZE EFFECT

The radiative correction to the finite size effect was first discussed in [19], where a very large contribution was obtained. The situation was almost immediately clarified in [20], where it was shown that the respective contribution is generated by the region with large intermediate momenta and should actually be a small correction of order $\alpha (Z\alpha)^5 m_e^3 < r^2 >$. Relying on the estimate of [20] the authors of [21] anticipated a contribution of about 10 Hz for the $2S$ state in hydrogen. The error in [19] was connected with an erroneous extrapolation of a nonrelativistic approximation to the relativistic momenta. This led to an overestimate of the resulting contribution.

Let us consider briefly calculation of the leading nuclear size contribution to the energy shift. It is generated by the slope of the nuclear formfactor in its low-momentum expansion

$$F(k^2) \approx 1 - \frac{k^2}{6} < r^2 >,$$

where the dimensionless momentum $k = |k|$ is measured in the units of the electron mass. The momentum squared in the second term above cancels the $1/k^2$ in the Coulomb photon propagator and leads to a momentum independent perturbation potential

$$\Delta V = \frac{2\pi (Z\alpha)}{3} < r^2 >.$$

Then we immediately obtain

$$\Delta E = \frac{2\pi (Z\alpha)}{3} < r^2 > |\psi(0)|^2 = \frac{2(Z\alpha)^4}{3n^3} m_e^3 < r^2 >.$$

Any radiative correction behaves as $k^2$ at small exchanged momenta, and the presence of such a correction pushes the significant integration momenta to the relativistic region for the
electron. The skeleton integrand approach (see, e.g., [22,23]) is ideally suited for calculation of such corrections.

The calculation essentially coincides with the calculation of corrections of order $\alpha^2(Z\alpha)^5$ to the Lamb shift in the skeleton integral framework [23–28] but is technically simpler due to the simplicity of the nuclear formfactor slope contribution in eq.(1). The skeleton contribution to the Lamb shift, induced by the diagrams with two external photons, has the form [23]

$$\Delta E_{\text{skel}} = -\frac{16(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m}\right)^3 m \int_0^\infty dk \frac{R(k)}{k^2},$$  \hspace{1cm} (4)

where $R(k)$ is the factor describing the radiative and the nuclear structure insertions.

There are two kinds of radiative insertions: one-loop polarization insertion in one of the external Coulomb lines, and one-loop radiative insertions in the electron line. It is clear that the magnitude of the radiative corrections to the nuclear size contribution grows with the nuclear charge as $Z^5$, and their importance increases for the highly charged ions. The respective results for the highly charged ions are well known (see, e.g., [29] and references therein). The radiative correction to the nuclear size effect in hydrogen induced by the polarization operator insertion was calculated in [30], and to the best of our knowledge the respective correction induced by the radiative insertions in the electron line was never discussed in the literature. We will now calculate both of these corrections.

### A. Polarization Correction

For calculation of the polarization correction we have to insert in the integral in eq.(4) the polarization operator

$$\frac{\alpha}{\pi} I_1(k) = \int_0^1 dv \frac{v^2(1-v^2/3)}{4 + (1-v^2)k^2},$$  \hspace{1cm} (5)

instead of the function $R(k)$, and also insert in the integrand the nuclear slope contribution from eq.(1). Then the respective contribution to the energy shift has the form

$$\Delta E_{\text{pol}} = \frac{32\alpha(Z\alpha)^5 \langle r^2 \rangle}{3\pi^2 n^3} > m^2 \left(\frac{m_r}{m}\right)^3 m \int_0^\infty dk I_1(k),$$  \hspace{1cm} (6)

where we have inserted an additional factor 4 in the integral in order to take into account all possible ways to insert the polarization operator and the slope of the nuclear formfactor in the Coulomb photons.

After an easy analytic calculation we obtain, in complete agreement with [30],

$$\Delta E_{\text{pol}} = \frac{1}{2} m_r \langle r^2 \rangle > \frac{\alpha(Z\alpha)^5}{n^3}.$$  \hspace{1cm} (7)
B. Electron-Line Correction

For calculation of the electron-line correction we have to insert in the integral in eq.(4) the electron line factor $L(k)$ [25,31] instead of the function $R(k)$, and also insert in the integrand the nuclear slope contribution from eq.(1). Then the respective contribution to the energy shift has the form

$$\Delta E_{\text{e-line}} = \frac{16\alpha(Z\alpha)^5}{3\pi^2n^3}\frac{<r^2>}{m^3}\frac{m_r^3}{m}\int_0^\infty dkL(k),$$

where we have inserted an additional factor 2 in the integral in order to take into account all possible ways to insert the slope of the nuclear formfactor in the Coulomb photons. After numerical calculation we obtain

$$\Delta E_{\text{e-line}} = -1.985(1)m_r^3 <r^2> \frac{\alpha(Z\alpha)^5}{n^3}.$$  \hspace{1cm} (8)

In principle, this integral also admits an analytic evaluation in the same way as it was done for a more complicated integral in [31].

III. DISCUSSION OF RESULTS

The total radiative correction to the nuclear size effect is given by the sum of contributions in eq.(7) and eq.(9)

$$\Delta E = -1.485(1)m_r^3 <r^2> \frac{\alpha(Z\alpha)^5}{n^3}.$$  \hspace{1cm} (9)

Numerically the contribution to the hydrogen-deuterium isotopic shift for the $1S - 2S$ interval is equal to

$$\Delta E(1S - 2S)_{D-H} = -0.616 \text{ kHz},$$  \hspace{1cm} (10)

where we have used $<r^2>^\frac{1}{2}/D = 2.128(11)$ fm [32] for the deuteron radius, and $<r^2>^\frac{1}{2}/H = 0.862(12)$ fm [33] for the proton radius.

This correction is clearly phenomenologically relevant on the background of the experimental uncertainty which is now about 0.2 kHz for the isotopic shift.

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Notes Added in Proof. The vacuum polarization correction to the finite size effect was considered earlier by J. L. Friar, Z. Physik, A 292, 1 (1979); (E) A 303, 84 (1981). For the spherical distribution of charge the result of this calculation coincides with our Eq.(7) and the result of Ref.[30]. We are grateful to J. L. Friar for bringing his work to our attention.
The radiative correction to the nuclear size effect was considered earlier by K. Pachucki, Phys. Rev. A48, 120 (1993) as a radiative correction to the electron charge density. Restoring the missing factor $\pi$ in Eq.(88) in this work (an apparent misprint) one obtains from Eq.(85) there the value 1.431 instead of the numerical coefficient 1.485 in our Eq.(10) above. In the calculations of K. Pachucki the contributions of the vacuum polarization and the electron factor were not separated, so we were unable to find out the source of the 0.05 discrepancy in the value of the coefficient in Eq.(10). We are grateful to S. Karshenboim who attracted our attention to this work by K. Pachucki.
REFERENCES