I. INTRODUCTION

The duality properties of superstring and super p-brane theories have led to hints that there may be hidden timelike dimensions in the underlying theory [1]-[8]. Unifying dualities, supersymmetry and Lorentz invariance suggests a structure in 14 dimensions with signature (11,3) including three timelike dimensions [9]-[12]. For simplicity one may concentrate on two timelike dimensions since the conceptual jump from one to two timelike dimensions can be easily extended to three. At issue is how to make sense of the extra timelike dimensions?

A main feature of this setting is also that the supersymmetry deviates substantially from the standard one, while its structure is intimately related to the answers we seek. By constructing explicit models with the new supersymmetry, and with two timelike dimensions, progress was made in relating these concepts to the familiar one-time world with standard supersymmetry.

In the present paper more progress is reported by constructing superparticle and superstring models in (d – 2,2) dimensions. In the first part of the paper we will discuss a general setup for discussing the two time issues for two groups of particles or p-branes, with interactions within each group. In the second and main part of the paper we will construct actions for the individual superparticles or superstrings as a member of either group.

In recent papers it has been shown that it is possible to formulate an action principle for particles [13], extended objects [14], and field theories [15], with two timelike dimensions, without the traditional problems. The main new ingredient is a gauge symmetry that permits the removal of one timelike dimension for each particle (or point in an extended object), while maintaining covariance in the higher space including the extra timelike dimension.

The timelike dimension that is removed is not the same one for the entire system. In the simplest case there are two particles, labelled #1 and #2, with timelike (or lightlike) momenta \( p^\mu_1, p^\mu_2 \) respectively, where \( \mu = 0', 1, 2, \ldots, (d - 2) \), is an index in \((d - 2,2)\) dimensions. The position coordinates are \( x^\mu_1, x^\mu_2 \). Each coordinate has two timelike dimensions labelled by \( \mu = 0', 0 \). The gauge invariance gives the constraint \( p_1 \cdot p_2 = 0 \), and removes the timelike dimensions \( p_1 \cdot x_2 \) and \( p_2 \cdot x_1 \) in a way that does not break the \( \text{SO}(d - 2,2) \) symmetry of the total system. Then particle #1 moves effectively with one timelike dimension, and similarly particle #2 moves effectively with another orthogonal timelike dimension.

The definition of canonical variables, and quantization, is straightforward because the formulation involves a single worldline parameter \( \tau \) that appears in \( x^\mu_1 (\tau), x^\mu_2 (\tau) \). One should not be confused about trying to introduce two Hamiltonians. It was shown in [13] that \( \text{SO}(d - 2,2) \) covariant quantization is carried out by simply applying the constraints on states.

It was suggested in [13], that a cosmological scenario is possible such that, starting at the Big Bang, the particles of type #2 remain in compactified dimensions, including
one timelike dimension, while particles of type $#1$ move in four dimensions in the observed expanding universe described by an effective theory with a single timelike dimension.

So far the two-times mechanism has been discussed in simple examples without interactions. In section 2 we expand the scope of this approach by showing that it also applies to two systems, called $#1$ and $#2$, each containing any number of interacting particles or p-branes, with two timelike dimensions. The total momenta of each system are $P_{1\mu}, P_{2\nu}$ respectively. We show that these replace the momenta $p^1_{\mu}, p^2_{\nu}$ of individual free particles or p-branes in the mechanism that removes the extra timelike dimensions. Furthermore, previous discussions had given the unintended impression that the approach required a new timelike dimension for each particle or p-brane in the total system. In the present version, it becomes evident that we can discuss many particles or p-branes with two timelike dimensions, without needing to add a new timelike dimension for each new particle or p-brane.

In sections 3–5 we will discuss superparticle and superstring actions that are supersymmetric under the “new superalgebra”

\[
\{Q_\alpha, Q_\beta\} = \gamma^{\mu\nu}_{\alpha\beta} P_1\mu P_2\nu, \quad P_1 \cdot P_2 = 0. \quad (1)
\]

This is the simplest form for a superalgebra in 12D, but it could also be realized in lower dimensions. There is also a new local kappa supersymmetry, in a Green-Schwarz formulation, that parallels the global supersymmetry of this type. We will show that there are classical superstring models in $(d-2,2)$ dimensions with type IIA, type IIB or heterotic supersymmetry, that provide realizations of the new supersymmetry.

The combined superparticle plus superstring system has $SO(d-2,2)$ invariance and a number of gauge invariances. After fixing a subset of gauges and using a subset of equations of motion, if the particle is frozen to a fixed momentum $p^1_{\mu}$, which is either timelike or lightlike depending on the dimension, then it can be shown that the string reduces to the standard classical supersymmetric string in 3,4,6,10 dimensions. If the particle is not frozen, the system is more general and maintains the $SO(d-2,2)$ invariance.

In the quantum theory of the combined superparticle and superstring system, the critical dimension is $(10,2)$ when the superparticle is massless, and $(9,2)$ when the superparticle is massive (assuming there are no spectator degrees of freedom).

II. TWO SYSTEMS AND TWO TIMES

In [14] the general scheme for constructing the action with two or more timelike dimensions was given. The discussion involved two or more particles (or p-branes) such that each additional particle (or p-brane) introduced an additional timelike dimension. In this section we extend this construction to any number of interacting particles (or p-branes) in system $#1$ plus any number of interacting particles (or p-branes) in system $#2$, but without needing to introduce additional timelike dimensions beyond two. It is straightforward to add system $#3$ with a third timelike dimension, but here we will limit ourselves to only two timelike dimensions.

The generalized multiparticle prescription for the action with two timelike coordinates is similar to the previous discussion

\[
S = S_1(\lambda_2) + S_2(\lambda_1) + \lambda_1 \cdot \lambda_2. \quad (2)
\]

However, now $S_1(\lambda_2)$ describes many particles or p-branes in system $#1$, moving in a background provided by system $#2$, where the background is represented by the moduli $\lambda_{1i}$. This is a constant vector constructed from canonical variables of system $#2$, and is independent of worldsheet or worldline parameters. Similarly, $S_2(\lambda_1)$ describes particles or p'-branes in system $#2$, moving in a background provided by system $#1$, represented by the moduli $\lambda_{2i}$. The constituents of system $#1$, labelled by $i = 1, 2, \cdots, N$, may be different “beasts” than the constituents of system $#2$ labelled by $j = 1, 2, \cdots, M$, and each system may include any set of interacting p-branes.

[3], but only for a frozen constant lightmomentum $P^1_{\mu}$ which breaks the $(10,2)$ covariance. These realizations can be interpreted to be a Kaluza-Klein reduction in the coordinate $x^\mu_{\alpha}$ of a bi-local theory. In contrast, an operator version of both $P^1_{\mu}, P^2_{\mu}$ in the form of derivatives was realized in bi-local supersymmetric field theory [15], but in a compactified version such that $P^1_{\mu}, P^2_{\mu}$ point in orthogonal directions. In the present paper both $P^1_{\mu}, P^2_{\mu}$ are quantum operators, and the full SO $(d-2,2)$ covariance is maintained.
The vectors $\lambda_1^\mu$, $\lambda_2^\mu$ are directions in which timelike coordinates will be removed by a gauge principle that is analogous to the gauged WZW mechanism discussed in [13,14]. This implies covariant derivatives on the branes $x_{1i}^i(\tau, \sigma), x_{2j}^j(\tau, \sigma)$

\[
D_\alpha x_{1i}^i = \partial_\alpha x_{1i}^i - \lambda_1^\mu A_{1i\alpha}, \\
D_\alpha x_{2j}^j = \partial_\alpha x_{2j}^j - \lambda_2^\mu A_{2\alpha j},
\]

where $\partial_\alpha, A_{1\alpha}, A_{2\alpha}$ are one-forms on the brane worldvolumes. The gauge invariance for system #1 is

\[
\delta x_{1i}^i = \lambda_2^\mu A_{1i\alpha}, \quad \delta A_{1\alpha i} = \partial_\alpha A_{1i},
\]

and similarly for system #2. Note there is a single $\lambda_2^\mu$ independent of $i$, and a single $\lambda_1^\mu$ independent of $j$.

There may be unspecified interactions within each system independently of the other, but the only interaction considered so far between system #1 and system #2 is the term $\lambda_1 \cdot \lambda_2$. The moduli $\lambda_1^\mu, \lambda_2^\mu$ (which are independent of $\tau, \sigma$) are integrated in the path integral, like other dynamical variables. This implies that the classical action is minimized with respect to the moduli, resulting in the equations of motion for $\lambda_1^\mu, \lambda_2^\mu$,

\[
\lambda_1^\mu = -\frac{\partial S_1(\lambda_2)}{\partial \lambda_2^\mu}, \quad \lambda_2^\mu = -\frac{\partial S_1(\lambda_1)}{\partial \lambda_1^\mu}.
\]

It will now be shown that $\lambda_1^\mu$ ($\lambda_2^\mu$) is proportional to the total momentum $P_1^\mu$ ($P_2^\mu$) of system #1 (#2), for any interactions among the constituents within each system, and that

\[
P_1 \cdot P_2 = 0,
\]

as in the single constituent cases discussed in [13] [14].

To do this use the chain rule\footnote{Here we assume that the only $\lambda_1^\mu, \lambda_2^\mu$ dependence in $S_{1,2}(\lambda_{2,1})$ occurs through the covariant derivatives in the purely bosonic theory. This is not the case in the supersymmetric theory, as seen in the next section. However, the result is still the same as it will be shown there (see discussion following eq. (24)).}

\[
-\frac{\partial S_1(\lambda_2)}{\partial \lambda_2^\mu} = \sum_i \int d\tau d\sigma \frac{\partial S_1(\lambda_2)}{\partial (D_\alpha x_{1i}^i)} A_{1i\alpha}(\tau, \sigma).
\]

Then use the equation of motion for $A_{1i\alpha}(\tau, \sigma)$

\[
\lambda_2^\mu \frac{\partial S_1(\lambda_2)}{\partial (D_\alpha x_{1i}^i)} = 0 \rightarrow \lambda_2 \cdot D^\alpha x_{1i} = 0,
\]

and combine it with (5,7) to show

\[
\lambda_1 \cdot \lambda_2 = 0.
\]

From (3,8) one obtains

\[
\lambda_2^2 A_{1i\alpha}(\tau, \sigma) = \lambda_2 \cdot \partial_\alpha x_{1i}.
\]

Using the gauge freedom $A_{1\alpha}(\tau, \sigma)$ one may choose the gauge

\[
\lambda_2 \cdot x_{1i}(\tau, \sigma) = a_1 \lambda_2^2 \tau
\]

where $a_1$ is a constant independent of the particle $i$. In this gauge $A_{1i\alpha}(\tau, \sigma)$ simplifies

\[
A_{1i\alpha}(\tau, \sigma) = a_1 \partial_\alpha \tau = a_1 \partial_\sigma.
\]

Inserting this form in (7) shows that only the $\alpha = \tau$ term survives and $\lambda_1^\mu$ becomes\footnote{A gauge independent way of proving the same result uses the equations of motion in any gauge, and combines it with the periodicity of the closed p-branes in the worldsheet variables $\sigma$.}

\[
\lambda_1^\mu = a_1 \sum_i \int d\tau d\sigma p_{1i}^\mu(\tau, \sigma) = a_1 \int d\tau \sum_i p_{1i}^\mu(\tau)
\]

\[
= a_1 P_1^\mu \int_0^T d\tau = a_1 T P_1^\mu.
\]

where the definitions of canonical momenta $p_{1i}^\mu(\tau, \sigma) = \partial S_1 / \partial (D_\alpha x_{1i}^i)$, center of mass momentum $p_{11}^\mu(\tau) = \int d\sigma p_{1i}^\mu(\tau, \sigma)$ and total conserved momentum $P_1^\mu = \sum_i p_{1i}^\mu(\tau)$ have been used. Note that even if there are interactions within system #1, the total $P_1^\mu$ is conserved due to translation invariance, although the center of mass momenta of the various p-branes $p_{1i}^\mu(\tau)$ may be time dependent. Thus, $\lambda_1^\mu$ is proportional to the total $P_1^\mu$, and similarly $\lambda_2^\mu$ is proportional to $P_2^\mu$. These are orthogonal to each other $P_1 \cdot P_2 = 0$ because of (9), as promised above.

There is an analog of the positive energy condition for the combined system. Note that the sign of

\[
P_1^\mu P_2^\rho - P_2^\mu P_1^\rho
\]

cannot change under SO($d - 2, 2$) transformations. Combined with the condition $P_1 \cdot P_2 = 0$, this implies that the signs of $P_1^\mu, P_2^\mu$ are correlated with each other in definite representations of supersymmetry and SO($d - 2, 2$). Therefore, if $\lambda_1^\mu \lambda_2^\rho - \lambda_2^\mu \lambda_1^\rho$ is taken positive in the original action (2) then the combined system is considered to be in the “particle-particle sector” (as opposed to particle-antiparticle sector) as explained in [13].

This result extends the scope of the approach in [14] to interacting systems involving any number of constituents within each system #1,#2. The essential points made in [13] about the meaning of two timelike dimensions, and the cosmological scenario, remain the same as in the simple two particle case discussed there.
It is straightforward to extend this discussion to three systems #1, #2, #3, with three timelike dimensions, as in [14], by allowing many interacting constituents in each system. More than three timelike dimensions are à priori possible, but are not needed in our program since minimal unification of supersymmetry and duality is possible in 14D with signature (11,3) [9]. Furthermore, more than three timelike dimensions seem to be forbidden in some supersymmetric dynamics [10].

III. SUPERPARTICLES IN (D-2,2) DIMENSIONS

We are interested in constructing models with the new supersymmetry of eq. (1) in various dimensions. Massless superparticles in more than eleven dimensions have been discussed recently in a formalism involving multiple worldline parameters \( \tau_i \) [10]. To avoid inconsistencies in the transformation laws with multiple \( \tau_i \), and also be consistent with our generalized scheme (2), we present the following formulation involving a single worldline parameter \( \tau \). Although there is considerable overlap with (2) for the single particle in a background of the other, there are differences in the supersymmetry transformation rules. Furthermore, we also discuss massless as well as massive superparticles, superparticles in lower dimensions, and superparticles with multiple supersymmetries.

A. Massless superparticle in \((d-2,2)\)

Consider a massless superparticle \( #1 \) in the background of system \( #2 \). The background consists of \( \lambda_2^\alpha \) that can be shown to be proportional to the center of mass momentum of system \( #2 \), as in the previous section, and below. The \( \tau \) reparametrization invariant action is

\[
S_1 = \int d\tau \, L_1(\tau),
\]

where

\[
L_1(\tau) = \frac{1}{2} \varepsilon_1^{-1} \pi_1^\mu \pi_1^\nu \eta_{\mu\nu},
\]

(15)

where

\[
\pi_1^\mu = \pi_0^\mu - \lambda_2^\alpha A_1 + \bar{\theta}_1 \gamma^{\mu\nu} \theta_1 \lambda_2^\nu.
\]

(16)

The signature is \((d-2,2)\), \( \eta_{\mu\nu} = (-,+,+\cdots,+), \) and labelled by the index \( \mu = 0',0,1,\cdots,d-2 \), on \( \pi_1^\mu(\tau) \). The spinor representation for \( \text{SO}(d-2,2) \) is labelled by the index \( \alpha \) on the fermion \( \theta_1^\alpha(\tau) \). We consider the dimensions \( d \) and the spinor representations in which \( \gamma^{\mu\nu}_{\alpha\beta} \) is symmetric in \( (\alpha\beta) \), since this is needed for the new supersymmetry (1). The following discussion is valid for all dimensions \( d \) that satisfy these criteria**. The main interest in this paper is in \((10,2)\) and \((9,2)\) dimensions since these will turn out to be the critical dimensions for the string (but not for the particle by itself). Note in particular that for \((10,2)\) the Majorana-Weyl spinor with 32 real components satisfies this property. The same real spinor is also the Dirac spinor in \((9,2)\) dimensions.

The global supersymmetry transformations that are consistent with the new superalgebra (1) have the form

\[
\delta_1 \varepsilon_1 = \varepsilon_1, \quad \delta_1 x_1^\mu = -\varepsilon \gamma^{\mu\nu} \theta_1 \lambda_2^\nu, \quad \delta_1 A_1 = 0.
\]

(17)

Under these

\[
\delta_1, \pi_1^\mu = 0, \quad \delta_1 \varepsilon_1 = 0, \quad \delta_1 L_1 = 0.
\]

(18)

There is also local kappa supersymmetry given by

\[
\delta_{\kappa_1} \varepsilon_1 = \gamma^{\mu\nu} \kappa_1 \pi_1^\mu \lambda_2^\nu, \quad \delta_{\kappa_1} x_1^\mu = \delta_{\kappa_1} \gamma^{\mu\nu} \theta_1 \lambda_2^\nu, \quad \delta_{\kappa_1} A_1 = 2 \lambda_2 \cdot \pi_1 \kappa_1 \theta_1, \quad \delta_{\kappa_1} \varepsilon_1^{-1} = 4 \lambda_2^2 \kappa_1 \theta_1 \varepsilon_1^{-1}.
\]

(19)

Under the local \( \kappa_1(\tau) \) supersymmetry the Lagrangian is invariant. Note that we have used \( \delta_{\kappa_1} \lambda_2^\alpha = \delta_{\kappa_1} \lambda_2^\alpha = 0 \) in proving that \( S_1(\lambda_2) \) is supersymmetric. Similarly, in verifying the supersymmetry of system \#2, \( S_2(\lambda_1) \), we need to use \( \delta_{\kappa_2} \lambda_1^\alpha = \delta_{\kappa_2} \lambda_1^\alpha = 0 \). Since we expect that \( \lambda_1 \sim p_1^\mu \) after using the equations of motion, we need to check if \( \varepsilon_1 p_1^\mu = 0 \) on shell, for consistency. This will be verified following eq.(25).

The momentum is

\[
p_1^\mu = \varepsilon_1^{-1} \pi_1^\mu.
\]

(20)

In the first order formalism the action may be rewritten as

\[
S_1 = \int d\tau \left( -\frac{1}{2} p_1^\mu \cdot \dot{x}_1 - p_1 \cdot \lambda_2 A_1 + \bar{\theta}_1 \gamma^{\mu\nu} \theta_1 p_1^\mu \lambda_2^\nu - \frac{1}{2} \varepsilon_1 p_1^2 \right).
\]

(21)

The supersymmetry of this form may be shown by deriving the transformation laws for the momentum from its

**As in footnote (†), in \( d = (3,4,5) \text{mod}(8) \) dimensions, with \( \mu = 0,1,2,\cdots,0' \), the real (Weyl) spinor has dimension 2\(^{[\frac{d+1}{2}]}\), and the real gamma matrices are given by \( \gamma^\alpha = (\gamma^m, \gamma^\alpha) \), where the last one is proportional to the identity \( (\gamma^\alpha)^{\alpha\beta} \delta = \delta^\gamma \gamma^\alpha, \) while the other \( \gamma^\alpha \) are real and traceless. For \( d = 6,10 \text{mod}(8) \) the real Dirac spinor has dimension 2\(^{d/2}\) and the real gamma matrices in these dimensions can again be taken as above, with \( \gamma^\alpha = 1 \). Then \( \gamma^{m\alpha} = (\gamma^m, \gamma^0) \) are defined by \( \gamma^{m\alpha} = \gamma^m \) and \( \gamma^{m\alpha} = \frac{1}{2} [\gamma^m, \gamma^\alpha] \). These form the algebra of \( \text{SO}(d-2,2) \). Explicitly, for \( d = 3 : (\gamma^m)^{\alpha\beta} = (\sigma_2 \sigma_3, \sigma_3 \sigma_1) ; \) for \( d = 4 : (\gamma^m)^{\alpha\beta} = (i \sigma_2, \sigma_1, \sigma_3, \sigma_1, 1) ; \) for \( d = 5 : (\gamma^m)^{\alpha\beta} = (i \sigma_2 \sigma_1, \sigma_1 \sigma_3, \sigma_3 \sigma_1, 1) ; \) when multiplied by the antisymmetric charge conjugation matrix \( C = \gamma^\alpha \) to lower the indices, \( C \gamma^m \) and \( C \gamma^{m\alpha} \) are symmetric. These symmetric \( \gamma^m, \gamma^{m\alpha} \) are the ones that appear in the superalgebra, as well as in expressions such as \( \varepsilon \gamma^{m\nu} \theta_1 \), etc.. The structure is similar for the other dimensions.
definition. We find that $\delta_{e_1}p_1^\mu = 0$, but $\delta_{\kappa}p_1^\mu$ is non-zero off shell (unlike [10])

$$\delta_{\kappa}p_1^\mu = 2\kappa_1^\mu \left( \lambda_2^\mu + \gamma \cdot p_1 \right) \hat{\theta}_1. \quad (22)$$

The constraints and equations of motion that follow from (21) are

$$p_i^2 = 0, \quad \gamma \cdot p_1 = 0, \quad \gamma \cdot \dot{\theta}_1 p_{1\mu} \lambda_{2\nu} = 0.$$ \quad (23)

and the equation of motion for $\lambda_2$ in (2) gives

$$\lambda_2^\mu = \int d\tau \left( A_1 p_1^\mu + \bar{\theta}_1 \dot{\gamma}^{\mu\nu} \bar{\gamma} p_{1\nu} \right). \quad (24)$$

The first term is proportional to $p_1^\mu$, which is consistent with $\lambda_2^\mu \sim p_1^\mu$, and the second term is not obviously so (see footnote (1)). The general solution of the equation of motion for $\theta_1$ is

$$\dot{\theta}_1 = (\gamma^{\mu\nu} \psi) p_{1\mu} \lambda_{2\nu}, \quad (25)$$

where $\psi(\tau)$ is any spinor. Inserting this in (24) and using the constraints, it follows that the second term in (24) is also proportional to $p_1^\mu$, for any $\psi(\tau)$. Since $\psi(\tau)$ is independent it can be pulled out of the integral to show once again that $\lambda_2^\mu \sim p_1^\mu$, on shell, as expected.

Now we can verify if $\delta_{\kappa_1} \lambda_2^\mu = 0$ is consistent with $\lambda_2^\mu \sim p_1^\mu$, on shell. In the off shell expression $\delta_{\kappa_1} p_1^\mu$ in (24) we insert the general form (25) and use the constraints. The result is $\delta_{\kappa_1} p_2^\mu = 0$ on shell, consistent with the original assumption $\delta_{\kappa_1} \lambda_2^\mu = 0$.

The superparticles with the new supersymmetry (1,17,19) are consistent both at the classical and quantum levels, in any number of dimensions $d$ for which $\gamma^{\mu\nu}$ are symmetric.

Furthermore, it is possible to have any number of supersymmetries by appending an index on $\theta_1 A (\tau)$ and on $A_1 A_1 (\tau)$, $A = 1, 2, \cdots, N$. There are no restrictions on these generalizations, just as for standard supersymmetry.

The action (15) or (21) has one more bosonic local symmetry with parameter $\omega(\tau)$

$$\delta_\omega \theta = \omega \dot{\theta}, \quad \delta_\omega x^\mu = -\omega \dot{\theta} \gamma^{\mu\nu} \dot{\theta} \lambda_\nu, \quad \delta_\omega e = 0, \quad \delta_\omega A_1 = 0, \quad (26)$$

which gives $\delta_\omega \pi^\mu = 0$. However this symmetry has no additional implications for the on shell theory other than those of $\kappa$ and reparametrization invariances.

### B. Massive superparticle in $(d-2,2)$

In the discussion above the superparticle was massless according to the constraint $p_1^2 = 0$. It is easy to obtain an action for the massive superparticle by doing Kaluza-Klein reduction. For example if the massless superparticle is in $(10,2)$ we reduce the 11th dimension to arrive to a massive superparticle in $(9,2)$. For this we distinguish one spacelike component, which will be denoted $\mu = 11$ in every dimension (instead of calling it $\mu = d-1$). Then replace the corresponding momentum with the constant mass parameter $p_1^{11} \equiv m_1$, while throwing away the corresponding component in $\lambda_2^{11} = 0$. We thus have one less spacelike dimensions as compared to the massless particle case discussed above. The first order action (21) reduces to

$$S_1 = \int d\tau \left( \frac{1}{2} p_{1\mu} \dot{\xi}_{1\mu} - \frac{1}{2} \epsilon_1 p_{1\mu}^2 \right), \quad (27)$$

where $\mu, \nu = 0', 0, 1, \cdots, d-2$. We have dropped $x_1^1 m_1$ since it is a total derivative. Thus, the compactified coordinate $x_1^{11}$ drops out. The supersymmetry transformations follow from the ones in (17,19) by dropping the direction $\mu = 11$ in $x_1^{11}, \lambda_2^{11}$ and using $\delta m_1 = 0$. Note however that now the $\kappa$ transformation has one more term proportional to the mass

$$\delta_\kappa \theta_1 = \gamma^{\mu\nu} \kappa_1 \pi_{1\mu} \lambda_{2\nu} + \gamma^{11\nu} \kappa_1 \lambda_{2\nu} m_1 e_1. \quad (28)$$

Thus, from the massless superparticle action in (10,2) we derive the massive superparticle action in (9,2), and similarly for other values of $d$. Using the equations of motion it is seen that the superparticle is massive.

If two massive superparticles are coupled as in the usual scheme (2), and then we keep the components $\lambda_2^{11} \sim m_{1,2}$ for both particles in doing the reduction (note $\lambda_2^{11} = 0$ above), then the massive Dirac operator for the combined system becomes

$$\left( \gamma^\mu p_{1\mu} + \gamma^{11} m_1 \right) \left( \gamma^\nu p_{2\nu} + \gamma^{11} m_2 \right), \quad (29)$$

with the constraint

$$p_1 \cdot p_2 + m_1 m_2 = 0, \quad (30)$$

while the massive Klein Gordon operator that is obtained by squaring the Dirac operator is, $(p_1^2 + m_1^2)(p_2^2 + m_2^2)$.

There are two global supersymmetries, since the total action is invariant under two separate parameters $\epsilon_{1,2}$ as in (17). After using $\lambda_2^{11} \sim m_{1,2}$ one finds the $N = 2$ superalgebra by applying (17) twice, i.e. $\delta_{e_i} \delta_{e_j} x_1^{11},$ for $i, j = 1, 2$ gives

$$\{ Q_{\alpha\beta}, Q_{\beta\gamma} \} = \delta_{ij} \gamma^{\mu\nu} \pi_{1\mu} \pi_{2\nu}. \quad (31)$$

When particle #2 is frozen, and rotated to a standard superparticle and superstring system discussed in the following sections.

In constructing field theories that describe the combined system #1 and #2 (see e.g. [15]) these structures need to be taken into account.
IV. SUPERSTRINGS IN (9,2) DIMENSIONS

We will discuss the system of one superparticle and one superstring as an example of the more general scheme (2). Since the superparticle is discussed in the previous section we will concentrate on the superstring action. To keep the notation simple we will omit the index $#2$ on the string supercoordinates $x^i_0(\tau, \sigma), \theta_{0i}(\tau, \sigma)$, and the index $#1$ on $\lambda^i$. We first discuss the string in the background of a massive superparticle, thus $\lambda^\mu \sim p^i_1$, and therefore $\lambda^\mu$ is timelike. The following discussion applies to $d = 4, 5, 11$ with signature $(d - 2, 2)$, but we are mainly interested in $(9,2)$. The spacetime index takes the values $\mu = 0, 1, \cdots, d - 2, \cdots$. The dimension of the spinor is $2^{[(d - 1)}/2]\right)$ and the gamma matrices $\gamma^\mu$ live in this space as described in footnote (**). There is an additional gamma matrix $\gamma^{(d - 1)}$ which will be denoted $\gamma^{11}$ in every dimension for convenience of notation, as in the previous section on massive superparticles. Note that $\gamma^{11}$ is a scalar under $\text{SO}(d - 2, 2)$. Furthermore, while $\gamma^{11}$ is traceless for the cases $d = 5, 11$, it is equal to 1 for the case $d = 4$ since the spinor is already Weyl projected.

The action we propose is

$$S_2(\lambda) = \frac{1}{2} \int d^2 \sigma \sqrt{-g} g^{ij} \pi_i^\mu \pi_j^\nu \eta_{\mu\nu} - \int d^2 \sigma \pi^{ij} \delta x^{ij} \theta_{\mu\nu\gamma^{11}} \partial \theta_{\lambda \nu} \gamma^{\nu} \lambda^\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{\nu} \lambda^\nu \quad (32)$$

where

$$\pi^\mu_i = \delta x^\mu - \lambda^\mu_i A_i + \bar{\theta} \gamma^{\mu\nu} \partial \theta_{\lambda \nu} \lambda^\nu . \quad (33)$$

Evidently there is reparametrization invariance, and the following gauge invariance

$$\delta x^\mu = \lambda^\mu \Lambda_2(\tau, \sigma), \quad \delta A_i = \partial A_2(\tau, \sigma) . \quad (34)$$

Under the global supersymmetry transformations with the new superalgebra (1) one has

$$\delta_\epsilon \theta = \epsilon, \quad \delta_\epsilon x^\mu = -\epsilon \gamma^{\mu\nu} \theta \lambda^\nu, \quad \delta_\epsilon A_i = 0, \quad \delta_\epsilon g_{ij} = 0 . \quad (35)$$

This gives

$$\delta_\epsilon \pi_i^\mu = 0, \quad (36)$$

showing that first term of $S_2(\lambda)$ is supersymmetric. The remainder of $S_2(\lambda)$ gives

$$\delta_\epsilon S_2 = \int d^2 \sigma \epsilon^{ij} \left( \epsilon \gamma^{\mu\nu} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{\nu} \lambda^\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{\nu} \lambda^\nu \right) \lambda^\nu . \quad (37)$$

The integrand of the first term can be rewritten as

$$\frac{1}{3} \epsilon^{ij} \left( 2 \epsilon \gamma^{\mu\nu} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu - \epsilon \gamma^{\mu\nu} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu \right) \lambda^\nu \lambda^\nu \quad (38)$$

$$+ \frac{1}{3} \epsilon^{ij} \partial_\nu \left( \epsilon \gamma^{\mu\nu} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu \right) \lambda^\nu \lambda^\nu \quad (39)$$

Since the last term is a total derivative it is dropped in $\delta_\epsilon S_2$. The second term in $\delta_\epsilon S_2$ can be written similarly by dropping total derivatives. Then one gets the integrand for the second term

$$\sqrt{-g} \epsilon^{ij} \left( 2 \epsilon \gamma^{\mu\nu} \partial_\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu - \epsilon \gamma^{\mu\nu} \partial_\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu \right) \lambda^\nu \lambda^\nu \quad (40)$$

One can use (2,2), (3,2) and (9,2) gamma matrix identities to show that the two integrands combined gives zero for any timelike $\lambda^\nu$. To prove it we take advantage of the $\text{SO}(d - 2, 2)$ symmetry of the combined string and particle system (2), and choose a Lorentz frame for the particle by taking $\lambda^\nu = (0, 0, \lambda^\nu)$. The little group is $\text{SO}(d - 2, 1)$. It is useful to define chiral spinors of the little group in the remaining (2,1), (3,1) or (9,1) dimensions by $\theta = \frac{1}{2}(\pm \gamma^3)^n \lambda^\nu \text{ (note that for (2,1) case } \lambda^\nu \text{ is equal to 0 since } \gamma^{11} = 1 \text{ for } d = 4 \text{ in the Weyl sector, as explained above).}$. In this frame the total integrand becomes

$$\sqrt{-g} \epsilon^{ij} \left( 2 \epsilon \gamma^{\mu\nu} \partial_\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu - \epsilon \gamma^{\mu\nu} \partial_\nu \gamma^{11} \partial \theta_{\lambda \nu} \gamma^{11} \partial \theta_{\lambda \nu} \lambda^\nu \right) \lambda^\nu \lambda^\nu \quad (41)$$

where $m$ is an index in the remaining $(d - 2, 1)$ little group dimensions. Now we can use the well known gamma matrix identities for symmetrized indices $(\alpha, \beta, \gamma, \delta), (\gamma^{\alpha\beta\gamma\delta})_{(\alpha, \beta, \gamma, \delta)} = 0$, valid in (2,1), (3,1) and (9,1) dimensions, to show that the integrand (40) is a total derivative, as in the usual Green-Schwarz superstring in these little group dimensions. We have thus shown that (37) vanishes in any frame only for $d = 4, 5, 11$. So, the action is supersymmetric for any timelike $\lambda^\nu$ with appropriate positive energy conditions$^{11}$ in these dimensions.

Next we consider local $\kappa$ supersymmetry with

$$\delta_\kappa x^\mu = \delta_\kappa \theta^\mu \lambda^\nu, \quad \delta_\kappa \theta = k^i \pi_i^\mu \gamma_{\mu\nu} \lambda^\nu, \quad (42)$$

$$\delta_\kappa A_i = 2 k^i (\pi_i \lambda^\nu) \left( \partial \gamma_{\nu\mu} \frac{g_{ij} \gamma_{\mu\nu} \gamma_{\nu\mu}}{\sqrt{-g}} \partial \theta_{\lambda \nu} \lambda^\nu \right), \quad (43)$$

$$\delta_\kappa (\sqrt{-g} g^{ij} = 2 \lambda^2 \left( k^i \left( \sqrt{-g} g^{ij} = e^{i\gamma_{\mu\nu} \gamma_{\nu\mu} \gamma_{\nu\mu}} \partial \theta_{\lambda \nu} \lambda^\nu \right) + k^i \left( \sqrt{-g} g^{ij} = e^{i\gamma_{\mu\nu} \gamma_{\nu\mu} \gamma_{\nu\mu}} \partial \theta_{\lambda \nu} \lambda^\nu \right) \right) . \quad (44)$$

$^{11}$Note that in the special frame we have used $\sqrt{-g} = \left| \lambda^\nu \right| = \lambda^\nu$ by recalling the positive energy condition (14) and the fact that we wish to describe a superstring with positive energy $P^\mu > 0$. More generally in the total action (2) that includes (32) one may multiply $\sqrt{-g}$ by the sign of $\lambda^0_0 \lambda^0_2 - \lambda^2_0 \lambda^0_2$ to insure the desired result.
They are constructed so that the two fermi terms in the \( \kappa \) variation of the action vanish. The four fermi terms in the variation are
\[
- \epsilon^{ij} \left( 2\delta_\lambda \theta^{\mu \nu} \gamma^{11} \partial_\lambda \theta^{ij} + \partial_\lambda \theta^{\rho \sigma} \gamma^{11} \partial_\lambda \theta^{ij} \right) \lambda^\nu \lambda^\rho + \frac{\sqrt{-\lambda^2}}{2} \epsilon^{ij} \left( 2\delta_\lambda \theta^{\mu \nu} \gamma^{11} \partial_\lambda \theta^{ij} + \partial_\lambda \theta^{\rho \sigma} \gamma^{11} \partial_\lambda \theta^{ij} \right) \lambda^\nu.
\]

The structure of these terms are similar to those of (38,39) except for the insertions of \( \gamma^{11} \) being in different places. By using gamma matrix identities in the special dimensions \((2,2),(3,2),(9,2)\) it can be shown that this is a total derivative for any timelike \( \lambda^\mu \). To see this, we can use once again the special frame and reduce these four fermi terms to the same structure as (40) except for replacing \( \delta_\varepsilon \theta \) instead of \( \varepsilon \). The argument for the vanishing of \( \delta_\varepsilon S_2 (\lambda) = 0 \) is then similar to \( \delta_\varepsilon S_2 (\lambda) = 0 \). So, the action is \( \kappa \) supersymmetric for any timelike \( \lambda^\mu \) (with appropriate positive energy conditions) in dimensions \( d = 4, 5, 11 \).

Next we wish to analyze the degrees of freedom described by the string action. After using a subset of equations of motion and fixing a subset of gauges, it can easily be shown that the action (32) reduces to the string actions in \((2,1),(3,1)\) and \((9,1)\) dimensions with a single timelike coordinate and standard supersymmetry. To see this, we again choose the special frame for the particle \( \lambda^\mu = (0,0,\lambda^0) \) so that (see footnote (*)
\[
\gamma_\mu \lambda^\mu = -\lambda^0, \quad \gamma_\mu \lambda^\nu = -\lambda^\nu \gamma_\mu
\]
\[
\pi_i^0 = \partial_i x^0 - \lambda^0 \xi_i, \quad \pi_i^m = \partial_i x^m - \lambda^0 \gamma^m \partial_i \theta
\]
where \( m \) is an index in \( d - 1 \) dimensions with signature \((d - 2,1)\). Furthermore, by a gauge choice and solving the corresponding constraint we can eliminate \( x^1 \) and \( A_1 \). Then \( \pi_i^0 = 0 \). Defining chiral spinors in \( d - 1 \) dimensions \( \theta_\pm = \frac{1}{2} (1 \pm \gamma^{11}) \theta \), one can write the first term in the action (32) in terms of \( \pi_i^m \) as
\[
\pi_i^m = \partial_i x^m - \lambda^0 \left( \theta_+ \gamma^m \partial_i \theta_+ - \theta_- \gamma^m \partial_i \theta_- \right)
\]
(recall \( \theta_- = 0 \) in the case of \( d = 4 \)). Similarly, the second term in the action (32) becomes
\[
\lambda^0 \epsilon^{ij} \partial_i x^m \left( \theta_+ \gamma_\mu \partial_i \theta_- - \theta_- \gamma_\mu \partial_i \theta_- \right).
\]

Although not needed, \( \lambda^0 \) can be absorbed into a renormalization of the spinors. The remaining string action is recognized as the Green-Schwarz action in \((2,1),(3,1),(9,1)\) dimensions (see e.g. [16]). This gives the full content of the degrees of freedom in the special frame. The \( SO(d - 2,2) \) covariance of the total system of particle and string is valid with these degrees of freedom, as can be shown by constructing the generators of \( SO(d - 2,2) \) with methods similar to those given in [14].

V. SUPERSTRINGS IN \((10,2)\) DIMENSIONS

As usual we have in mind the total system described by (2). To keep the notation simple we will again omit the index \#2 on the supercoordinates \( x_5^\alpha (\tau, \sigma), x_5^2 (\tau, \sigma) \) and the index \#1 on \( \lambda^\nu \). The following discussion applies to \( d = 5, 12 \) with signature \((d - 2,2)\), but we will concentrate mainly on \((10,2)\). The spinor is real and has dimension 4, 32 for \( d = 5, 12 \) respectively.

A. Heterotic String in \((d,2,2)\)

The action we propose is
\[
S_2^{het} (\lambda) = \frac{1}{2} \int d^2 \sigma \sqrt{-g} g^{ij} \pi_i^\mu \pi_j^\nu \eta_{\mu \nu} - \int d^2 \sigma \epsilon^{ij} \partial_i x^m \bar{\theta} \gamma_{\mu \nu} \partial_j \theta \lambda^\nu \lambda^\rho,
\]
where \( \pi_i^\mu \) is given by (33). As before, there is reparametrization invariance, gauge invariance as in (34), and new supersymmetry under which \( \delta \pi_i^\mu = 0 \), as in (35). The first term of \( S_2^{het} \) is supersymmetric. The remainder of \( S_2 \) gives
\[
\delta S_2^{het} = \int d^2 \sigma \epsilon^{ij} \left( \bar{\xi} \xi_{\mu \nu} \partial_i \theta \bar{\xi} \gamma_{\mu \nu} \partial_j \theta \lambda^\nu \lambda^\rho, \right)
\]
The integrand can be written as
\[
\frac{1}{3} \epsilon^{ij} \left( 2 \xi_{\mu \nu} \partial_i \theta \xi_{\rho \sigma} \partial_j \theta \lambda^\rho \lambda^\sigma \right) \lambda_\mu \lambda_\nu \lambda^\rho \lambda^\sigma,
\]
by dropping a total derivative term. One can use 5D or 12D gamma matrix identities to show that its integral is zero for \( \lambda^2 = 0 \). To prove it we will choose a Lorentz frame in 5D or 12D, \( \lambda^\mu = (\lambda^0,0,0,\lambda^{11}) \). Using footnote (*) one finds
\[
\gamma^{\mu \nu} \lambda_\nu = \left( -\lambda^{11} \gamma^{11}, \gamma^m \left( \lambda^{11} \gamma^{11} - \lambda^0 \right), -\lambda^{11} \gamma^{11} \right).
\]
where \( m \) is in 3D or 10D. In this frame (48) becomes
\[
\frac{1}{3} \left( \lambda^0 \gamma_\mu - \lambda^2 \right) \epsilon^{ij} \left( 2 \xi_{\mu \nu} \partial_i \theta \xi_{\rho \sigma} \partial_j \theta \lambda^\rho \lambda^\sigma \right) \lambda_\mu \lambda_\nu \lambda^\rho \lambda^\sigma
\]
\[
+ \epsilon^{ij} \left( 2 \xi_{\mu \nu} \left( \lambda^{11} \gamma^{11} \gamma^{11} - \lambda^0 \right) \partial_i \theta \xi_{\rho \sigma} \partial_j \theta \lambda^\rho \lambda^\sigma \right) - \xi_{\mu \nu} \left( \lambda^{11} \gamma^{11} \gamma^{11} - \lambda^0 \right) \partial_i \theta \xi_{\rho \sigma} \partial_j \theta \lambda^\rho \lambda^\sigma \right).
\]
The 4 or 32 components of \( \theta \) in (3,2) or (10,2), can be rewritten in terms of 2 or 16 dimensional (chiral) spinors \( \theta_\pm = \frac{1}{2} (1 \pm \gamma^{11}) \theta \) of 3D or 10D, respectively. Then, \( \theta_\pm \gamma_\mu \partial_\mu \theta_\pm = \partial_\mu \theta_\pm \theta_\pm \) is a total derivative, and the term proportional to \( \left( \lambda^0 \gamma_\mu - \lambda^2 \right) \) drops out. Furthermore, \( \partial_\gamma_\mu \partial_\mu \theta_\pm = \partial_\gamma_\mu \partial_\mu \theta_\pm + \partial_\gamma_\mu \partial_\mu \theta_\pm \), and \( \theta_\gamma_\mu \gamma^{11} \partial_\mu \theta = \theta_\gamma_\mu \gamma^{11} \partial_\mu \theta_\pm + \partial_\gamma_\mu \partial_\mu \theta_\pm \)
\( \bar{\theta} \gamma_{m} \partial_{\theta_{+}} - \bar{\theta} \gamma_{m} \partial_{\theta_{-}} \), and similarly for terms involving \( \varepsilon_{\pm} \). The terms with all spinors of same chirality vanish by virtue of 3D or 10D gamma matrix identities for symmetrized indices \( (\alpha \beta \gamma \delta) \), \( (\gamma^{m})_{(\alpha \beta)(\gamma \delta)} = 0 \). The remaining cross terms are

\[
\left( \lambda_{5}^{3} - \lambda_{1}^{2} \right) \sum_{ij} \epsilon^{ij} \left( 2\varepsilon_{\pm} \gamma^{m} \theta_{\pm} \bar{\theta}_{\mp} \gamma_{m} \partial_{\theta_{\mp}} - \varepsilon_{\pm} \gamma^{m} \theta_{\pm} \bar{\theta}_{\mp} \gamma_{m} \partial_{\theta_{\mp}} \right). \tag{51}
\]

The only solution for \( \delta_{c} S (\lambda) = 0 \), is \( \lambda_{5}^{3} - \lambda_{1}^{2} = -\lambda^{2} = 0 \).

The \( \kappa \) symmetry transformations have the same structure as (19) except for replacing the insertion \( \gamma^{11} \) by the identity. The two fermi terms in the \( \kappa \) variation of the action vanish. The four fermi terms are

\[
- \int d^{2}\sigma \epsilon^{ij} \left( \frac{2}{\Lambda} \kappa^{2} \epsilon^{ij} \partial_{\theta_{-}} \partial_{\theta_{+}} \gamma^{m} \partial_{\theta_{m}} \partial_{\theta_{+}} - \varepsilon^{ij} \gamma^{m} \partial_{\theta_{m}} \partial_{\theta_{+}} \right) \lambda^{g} \lambda_{\pm}. \tag{52}
\]

This has the same structure as (48), and thus is zero for \( \lambda^{2} = 0 \). Therefore the four fermi terms vanish and the action we proposed for the (3, 2) or (10, 2) string is \( \kappa \) symmetric for \( \lambda^{2} = 0 \).

To determine the degrees of freedom of the string we will partially choose gauges and use a subset of equations of motion to show that the action (46) reduces to the 3D string and heterotic string in 10D for \( d = 5, 12 \) respectively, provided \( \lambda^{2} = 0 \). For this we choose a Lorentz frame \( \lambda^{\mu} = \left( \lambda^{0}, 0, \theta_{i}, \lambda^{11} \right) \) with \( \lambda^{0} = \left| \lambda^{11} \right| \), and use lightcone components involving the extra timelike coordinate \( \lambda^{\pm} = \left( \lambda^{0} \pm \lambda^{11} \right) \) so that all components of \( \lambda^{\mu} \) except for \( \lambda^{0} \) are zero, \( \lambda^{0} = \lambda^{+} \delta_{+}^{0} \). Then

\[
\gamma_{\mu} \lambda^{\mu} = \gamma^{+} \gamma^{+} = \gamma^{+} (1 + \gamma^{11}), \quad \gamma_{\pm} = \pm 1 + \gamma^{11}, \quad \gamma^{+} = \gamma^{11}, \quad \gamma_{m+} = \gamma_{m} (1 + \gamma^{11}), \quad \gamma^{m+} = \gamma^{m} (1 + \gamma^{11}),
\]

\[
\pi^{+}_{m} = \partial_{x^{m}} + \theta \gamma_{m} \partial_{\theta}, \quad \pi^{-}_{m} = \partial_{x^{m}} - \theta \gamma_{m} \partial_{\theta}, \quad \pi^{+}_{m} = \partial_{x^{m}} + \theta \gamma_{m} (1 + \gamma^{11}) \partial_{\theta}, \quad \pi^{-}_{m} = \partial_{x^{m}} - \theta \gamma_{m} (1 + \gamma^{11}) \partial_{\theta}, \quad \lambda^{+} \lambda^{+},
\]

where \( m \) is an index in (2, 1) or (9, 1) dimensions. The second term in \( S_{2} \) becomes

\[
- \epsilon^{ij} \partial_{\theta^{j}} \gamma^{m} \partial_{\theta_{m}} \partial_{\theta_{j}} \lambda^{\nu} = - \epsilon^{ij} \partial_{\theta_{m}} \gamma^{m} \partial_{\theta_{m}} \partial_{\theta_{j}} \lambda^{+} + \epsilon^{ij} \partial_{\theta_{m}} \gamma^{m} \partial_{\theta_{j}} \lambda^{+} \lambda^{+}. \tag{54}
\]

The 4 or 32 components of \( \theta \) in (3, 2) or (10, 2) can be rewritten in terms of 2 or 16 dimensional (chiral) spinors \( \theta_{\pm} = \frac{1}{2} \left( 1 \pm \gamma^{11} \right) \theta \) of 3D or 10D, respectively. Then \( \theta_{11} \theta_{\partial} \partial_{\theta_{\partial}} \partial_{\theta_{\partial}} \) is a total derivative, and the second term above drops out.

Now we choose gauges and use equations of motion as follows: let \( x^{+} = 0 \) by a gauge choice, \( \pi^{+} = 0 \) by equation of motion \( \delta A_{1} \), and \( A_{1} \) is determined by the equation of motion of \( \delta x^{1} \). The remaining action is the 3D superstring or 10D heterotic superstring respectively. We have shown that they are embedded covariantly in (3, 2), (10, 2) dimensions respectively provided the particle to which it is coupled is massless, since \( \lambda^{2} = 0 \).

**B. Type IIA and IIB superstrings in (10, 2)**

Type IIA, B superstrings are constructed by doubling the fermions \( \theta_{A}, A = 1, 2 \). The discussion applies in \( d = 5, 12 \) so that each spinor is real and has dimension 4.32 respectively. For type-IIB the 12D chirality of the two spinors is the same while for type IIA they are opposite. The covariant momentum is

\[
\pi^{\mu}_{\pm} = \partial_{\pm} x^{\mu} - \lambda^{\nu} A_{1} + \bar{\theta} A_{\nu} \partial_{\theta} \lambda^{\nu} \lambda_{\mu}, \tag{55}
\]

and the action is

\[
S^{II} (\lambda) = \int \frac{1}{2} d^{2} \sigma \sqrt{-g} \epsilon^{ij} \bar{\theta} A_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} - \int d^{2} \sigma \epsilon^{ij} \partial_{i} x^{\mu} (\partial_{j} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu}) \tag{56}
\]

\[
+ \int d^{2} \sigma \epsilon^{ij} \partial_{i} A_{\nu} \partial_{j} A_{\nu} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} \lambda_{\mu}.
\]

The supersymmetry transformation has parameters \( \varepsilon^{A} \), and is given by (35), except that the fermions should be doubled. As usual \( \delta \pi^{\mu}_{\pm} = 0 \), and the first term of the action is invariant. The second term gives

\[
\delta_{\varepsilon} S_{II}^{II} = \int d^{2} \sigma \epsilon^{ij} \left( \frac{1}{\sqrt{-g}} \epsilon^{ik} \epsilon_{kl} \right) \left[ \bar{\theta} A_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} + \bar{\theta} A_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} \right]. \tag{57}
\]

This vanishes since it has the same structure as in the heterotic case (47), provided \( \lambda^{2} = 0 \).

The local \( \kappa \) supersymmetry transformations are given by

\[
\delta_{\varepsilon} x^{\mu} = \delta_{\varepsilon} \lambda_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu}, \quad \delta_{\varepsilon} \lambda_{\nu} = \kappa^{A} \pi^{\mu}_{\pm} \gamma_{m} \lambda^{\nu}, \quad \delta_{\varepsilon} A_{\nu} = \kappa^{A} \pi^{\mu}_{\pm} \gamma_{m} \lambda^{\nu}, \quad \delta_{\varepsilon} \lambda_{\nu} = \kappa^{A} \pi^{\mu}_{\pm} \gamma_{m} \lambda^{\nu}.
\]

The \( \kappa^{A} \) parameters are anti-self-dual for \( A = 1 \) and self-dual for \( A = 2 \) [16]

\[
\kappa^{1} = P_{+}^{ij} \kappa^{j}, \quad \kappa^{2} = P_{+}^{ij} \kappa^{j}. \tag{60}
\]

The \( \kappa \) transformations are designed so that two fermi terms in the variation of the action vanish. The four fermi terms take the form

\[
\epsilon^{ij} \left( - \delta_{\varepsilon} \lambda_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} + \delta_{\varepsilon} \lambda_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} + \delta_{\varepsilon} \lambda_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu}
\]

\[
- \delta_{\varepsilon} \lambda_{\nu} \gamma_{m} \partial_{\theta} \gamma_{m} \partial_{\theta} \lambda^{\nu} \right) \lambda^{\nu} \lambda_{\mu}. \tag{61}
\]
This has the same structure as (48) and thus vanishes provided \( \lambda^2 = 0 \).

To analyze the degrees of freedom we choose gauges and use equations of motion as in the heterotic case. For \( \lambda^2 = 0 \) in \((10,2)\), this procedure gives the 10D superstrings, but now with the fermions doubled, so that \( \theta_{\pm} = \frac{1}{2} (1 \pm \gamma^{11}) \theta_{1,2} \) are 16 dimensional chiral spinors in \((9,1)\). When the 12D chiralities of \( \theta_{1,2} \) are the same/opposite then the 10D chiralities are also the same/opposite. Therefore we have obtained the IIA,B superstring theory embedded covariantly in \((10,2)\) provided the particle to which it is coupled is massless, since \( \lambda^2 = 0 \).

The discussion above was at the classical level. The quantum theory can be analyzed either in SO\((d-2,2)\) covariant quantization or lightcone quantization. The methods are discussed in detail in [14]. However, we can quickly determine the outcome since we have already identified the degrees of freedom and we know the critical dimensions of the sub-theories. On this basis, we can determine that the type IIA,B theories (massless particle plus string) are critical and quantum consistent in \((10,2)\) dimensions. Similarly, the massive particle plus string theory is critical and quantum consistent in \((9,2)\) dimensions.

The remaining theories in lower dimensions and the heterotic case become quantum consistent when additional degrees of freedom are added so that the Virasoro algebra becomes critical.


